

ON UNIVERSAL OPERATORS AND UNIVERSAL PAIRS

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Abstract We study some basic properties of the class of universal operators on Hilbert spaces, and provide new examples of universal operators and universal pairs.

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1. Introduction

Let $\mathcal{L}(H)$ be the algebra of bounded linear operators on the separable infinite-dimensional Hilbert space H . Recall that the operators $T_1 \in \mathcal{L}(H_1)$ and $T_2 \in \mathcal{L}(H_2)$ are similar if there exists a linear isomorphism $J : H_1 \rightarrow H_2$ such that $T_1 = J^{-1}T_2J$. The operator $U \in \mathcal{L}(H)$ is called *universal* if for any $T \in \mathcal{L}(H)$ there exist a closed U -invariant subspace $M \subset H$, i.e. $U(M) \subset M$, and a constant $c \neq 0$ such that the operators $U|_M : M \rightarrow M$ and $cT : H \rightarrow H$ are similar. The concept of a universal operator was introduced by Rota [28], who showed that the backward shift of infinite multiplicity is an explicit example of these seemingly strange objects. The invariant subspace problem provides motivation for studying concrete universal operators, namely, every operator in $\mathcal{L}(H)$ has a non-trivial invariant subspace if and only if the minimal non-trivial invariant subspaces of (any) universal operator are one dimensional. General references for this approach and results about universal operators include [3, Chapter 8] and the survey [6].

Later, Caradus [1] (see also [3, Theorem 8.1.3]) described a simple sufficient condition for an operator to be universal, namely,

(C) *If $U \in \mathcal{L}(H)$ is such that the kernel $\text{Ker}(U)$ is infinite dimensional and its range $\text{Ran}(U) = H$, then U is universal for H .*

However, Caradus's condition and its recent generalization by Pozzi [26] are very far from being necessary. In § 2 of this paper, we look more closely at some fundamental properties

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of the class of universal operators, as well as some of its subclasses. In particular, we derive spectral theoretic consequences of universality which can be used to verify that a given operator is not universal.

Owing to its simplicity, condition (C) is often used to obtain examples of universal operators, even though the desired properties can be difficult to verify for many concrete operators. By a celebrated example of Nordgren *et al.* [24] from 1987, the operators $C_\varphi - \lambda I$ are universal on the Hardy space $H^2(\mathbb{D})$ whenever the composition operator $f \mapsto C_\varphi f = f \circ \varphi$ is induced by a hyperbolic automorphism φ of the unit disc \mathbb{D} and λ belongs to the interior of the spectrum of C_φ . In this case, the infinite-dimensionality of $\text{Ker}(C_\varphi - \lambda I)$ is verified by explicit computation, but the original argument for the surjectivity relies on fairly sophisticated properties of multiplication operators induced by certain Blaschke products in $H^2(\mathbb{D})$. Only very recently, an alternative argument for the universality of $C_\varphi - \lambda I$ on $H^2(\mathbb{D})$ was given [8]. For other concrete examples of universal operators, see [5, 7, 25, 26]. Moreover, the connection between the invariant subspace problem and universality has motivated recent work on the lattice of invariant subspaces of C_φ on $H^2(\mathbb{D})$ for hyperbolic automorphisms φ , e.g. [18] and [12].

In § 3 we show that the adjoint $C_\varphi^* - \bar{\lambda}I$ is universal on $S^2(\mathbb{D})$, the Hilbert space consisting of analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $f' \in H^2(\mathbb{D})$, whenever λ is an interior point of the spectrum of C_φ on $S^2(\mathbb{D})$. It follows from known results that $C_\varphi - \lambda I$ is not a universal operator on $S^2(\mathbb{D})$ for any $\lambda \in \mathbb{C}$, which suggests that universality passes to the adjoint for small enough spaces in the scale of weighted Dirichlet spaces of analytic functions on \mathbb{D} .

Recently, Müller [22] introduced the notion of universality for commuting n -tuples of operators and obtained versions of the sufficient condition (C) in this setting. However, examples of universal commuting n -tuples are rather more difficult to exhibit compared with the case of a single operator, and in § 4 we discuss new concrete examples of universal commuting pairs $(U_1, U_2) \in \mathcal{L}(H)^2$. In particular, we show that certain pairs (L_A, R_B) of left and right multiplication operators on the ideal of the Hilbert–Schmidt operators are universal, and consider the case of universal Nordgren–Rosenthal–Wintrobe pairs $(C_\varphi - \lambda I, C_\psi - \mu I)$ in $\mathcal{L}(H^2(\mathbb{D}))^2$.

2. Structure of the class of universal operators

The main interest has been in exhibiting and analysing concrete examples of universal operators belonging to various classes of operators, and less attention has been paid to general properties of the full class

$$\mathcal{U}(H) = \{U \in \mathcal{L}(H) : U \text{ is universal}\}.$$

In this section, we systematically consider $\mathcal{U}(H)$ and some of its subclasses. Clearly, $\mathcal{U}(H_1)$ and $\mathcal{U}(H_2)$ are related by similarity whenever H_1 and H_2 are separable infinite-dimensional Hilbert spaces, so the particular realization of the Hilbert space H is immaterial. We will use the notation $B_\infty : (\oplus_{\mathbb{Z}_+} \ell^2)_{\ell^2} \rightarrow (\oplus_{\mathbb{Z}_+} \ell^2)_{\ell^2}$ for Rota’s universal model operator,

$$B_\infty x = B_\infty(x_0, x_1, \dots) = (x_1, x_2, \dots),$$

for $x = (x_n)_{n \in \mathbb{Z}_+} \in (\oplus_{\mathbb{Z}_+} \ell^2)_{\ell^2}$, where $x_n \in \ell^2$ for any $n \geq 0$. The universality of the backward shift B_∞ of infinite multiplicity on $(\oplus_{\mathbb{Z}_+} \ell^2)_{\ell^2}$ is immediate from (C), but the original argument by Rota [28] is quite direct.

It was pointed out in [6, p. 44] that the precise relationship between universality and condition (C) is somewhat circular: $U \in \mathcal{U}(H)$ if and only if there is a U -invariant infinite-dimensional subspace $M \subset H$ so that the restricted operator $U|_M : M \rightarrow M$ satisfies condition (C). This can be seen by recalling that the restriction of any $U \in \mathcal{U}(H)$ to some invariant subspace is similar to cB_∞ for some $c \neq 0$, combined with an observation of Pozzi recalled separately as Proposition 2.1 below. To state the proposition in a convenient form, we write operators $V \in \mathcal{L}(H)$ with respect to direct sum decompositions $H = M \oplus M^\perp$ as vector-valued operator matrices

$$V = \begin{pmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{pmatrix}$$

in the obvious fashion. Thus, $V_{1,1} = PV|_M$, $V_{1,2} = PV|_{M^\perp}$, $V_{2,1} = QV|_M$ and $V_{2,2} = QV|_{M^\perp}$, where $P : H \rightarrow M$ and $Q = I - P : H \rightarrow M^\perp$ are the related orthogonal projections. The following fact allows us to construct examples of universal operators having different properties on direct sums $H \oplus H$.

Proposition 2.1 (Pozzi [26, Remark 1.4]). *Let $H = M \oplus M^\perp$, where $M \subset H$ is an infinite-dimensional subspace, and let*

$$V = \begin{pmatrix} U & A \\ 0 & B \end{pmatrix} \in \mathcal{L}(H)$$

as above. Suppose that $U \in \mathcal{L}(M)$ is a universal operator for M . Then $V \in \mathcal{U}(H)$ for any operators A and B .

Proof. If $T \in \mathcal{L}(H)$ is given, there are, by assumption, a U -invariant subspace $N \subset M$ and $c \neq 0$ such that $U|_N : N \rightarrow N$ and $cT : H \rightarrow H$ are similar. Fix an isometry $J_0 : M \rightarrow H$. We have that $U|_N$ is similar to $cJ_0^{-1}TJ_0$. Since $N \subset M$, we get that N is V -invariant, and $V|_N = U|_N$ is similar to $cJ_0^{-1}TJ_0$ and consequently also to cT . □

We are interested in conditions that enable us to decide whether a given concrete operator is universal or not. We first consider spectral criteria. Let $\sigma(S; H)$ denote the spectrum of $S \in \mathcal{L}(H)$. The spectrum of a diagonal sum of operators on $H_1 \oplus H_2$ satisfies

$$\sigma \left(\begin{pmatrix} U & 0 \\ 0 & B \end{pmatrix}; H_1 \oplus H_2 \right) = \sigma(U; H_1) \cup \sigma(B; H_2)$$

for any $U \in \mathcal{L}(H_1)$ and $B \in \mathcal{L}(H_2)$. It follows from Proposition 2.1 that there is no general characterization of universal operators purely in terms of their spectra. Nevertheless, the universality of $U \in \mathcal{U}(H)$ does have relevant consequences for various subsets of the

spectrum of U . For this, recall the classes of semi-Fredholm operators

$$\begin{aligned} \Phi_-(H) &= \{S \in \mathcal{L}(H) : \dim(H/\text{Ran}(S)) < \infty\}, \\ \Phi_+(H) &= \{S \in \mathcal{L}(H) : \dim(\text{Ker}(S)) < \infty, \text{Ran}(S) \text{ is closed}\}, \end{aligned}$$

where $\Phi(H) = \Phi_+(H) \cap \Phi_-(H)$ consists of the Fredholm operators. Operators $S \in \Phi_+(H)$ cannot be universal, since clearly $\text{Ker}(S)$ has to be infinite dimensional for S to be a universal operator. We will need the Φ_+ -spectrum of $S \in \mathcal{L}(H)$ defined as

$$\sigma_e^+(S; H) = \{\lambda \in \mathbb{C} : S - \lambda I \notin \Phi_+(H)\}.$$

It is known [21, Chapter III.19] that $\sigma_e^+(S; H)$ is a non-empty compact subset of the essential spectrum

$$\sigma_e(S; H) = \{\lambda \in \mathbb{C} : S - \lambda I \notin \Phi(H)\}$$

of S . Furthermore, let $\sigma_p(S; H)$ denote the point spectrum of S .

It follows from the definition of universality that Riesz operators $S \in \mathcal{L}(H)$ cannot be universal. (Recall that S is a Riesz operator if $\sigma_e(S; H) = \{0\}$.) The following result reveals some further common spectral properties of universal operators.

Theorem 2.2. *Let $U \in \mathcal{U}(H)$ be an arbitrary universal operator. Then the following hold.*

- (i) *There is $r > 0$ such that the open disc*

$$B(0, r) \subset \sigma_p(U; H) \cap \sigma_e^+(U; H) \subset \sigma_e(U; H) \subset \sigma(U; H), \tag{1}$$

and, moreover, any $\lambda \in B(0, r)$ is an eigenvalue of U having infinite multiplicity. In particular, if $U \in \mathcal{U}(H)$, then 0 is an interior point of any of the sets $\sigma_e^+(U; H)$, $\sigma_e(U; H)$ and $\sigma_p(U; H)$, as well as $\sigma(U; H)$.

- (ii) *There is $r > 0$ and a vector-valued holomorphic map $z \mapsto y_z : B(0, r) \rightarrow H$ for which*

$$Uy_z = zy_z, \quad z \in B(0, r).$$

Proof. We first recall some well-known spectral properties of the backward shift B_∞ on the direct sum $H_0 \equiv (\oplus_N \ell^2)_{\ell^2}$. Let $|z| < 1$ and fix the non-zero vector $x_0 \in \ell^2$, whence the sequence $x_z = (z^n x_0)_{n \geq 0} = (x_0, zx_0, \dots) \in H_0$. Clearly

$$B_\infty(x_z) = (zx_0, z^2x_0, \dots) = zx_z,$$

so that any $|z| < 1$ is an eigenvalue of infinite multiplicity for B_∞ , that is, $z \in \sigma_e^+(B_\infty; H_0)$. Moreover, $z \mapsto x_z$ is a (weakly) holomorphic map $B(0, 1) \rightarrow H_0$, since

$$z \mapsto \langle x_z, y \rangle = \sum_{n=0}^{\infty} z^n \langle x_0, y_n \rangle$$

is analytic for any $y = (y_n) \in H_0$, where $\langle \cdot, \cdot \rangle$ denotes the respective inner product.

Let U be an arbitrary universal operator on H . By assumption, there is a constant $c \neq 0$, a U -invariant subspace $M \subset H$ and a linear isomorphism $J : H_0 \rightarrow M$ so that $U|_M = J(cB_\infty)J^{-1}$. Since eigenvalues are preserved under similarity, we get that

$$B(0, |c|) \subset \sigma_p(cB_\infty; H_0) = \sigma_p(U|_M; M) \subset \sigma(U; H).$$

Towards the related claim for $\sigma_e^+(U; H)$, one obtains instead that

$$B(0, |c|) \subset \sigma_e^+(cB_\infty; H_0) = \sigma_e^+(U|_M; M) \subset \sigma_e^+(U; H).$$

For the right-hand inclusion, note for example that $\text{Ker}(\lambda I_M - U|_M) \subset \text{Ker}(\lambda I - U)$, where the left-hand subspace is infinite dimensional by similarity, since $\dim(\text{Ker}(\lambda I - cB_\infty)) = \infty$.

Finally, the above identities $B_\infty x_z = z x_z$ and $cJB_\infty = (U|_M)J$ imply that

$$U(Jx_z) = cJB_\infty(x_z) = czJ(x_z), \quad |z| < 1.$$

It follows that the renormalized holomorphic map $z \mapsto y_z \equiv J(x_z/c)$ satisfies condition (ii) in the disc $B(0, |c|)$. □

By Example 3.4 below, the spectral condition (1) in Theorem 2.2 is not sufficient for the universality of U . We next state some typical applications of the preceding result.

Corollary 2.3. *The operator $T \in \mathcal{L}(H)$ cannot be universal if any of the following conditions holds:*

- (i) *the interior $\text{int}(\sigma_p(T; H)) = \emptyset$,*
- (ii) *the interior $\text{int}(\sigma_e(T; H)) = \emptyset$,*
- (iii) *every non-zero eigenvalue $\alpha \in \sigma_p(T; H)$ has finite multiplicity.*

Another immediate consequence of Theorem 2.2 which will be useful in § 3 is as follows.

Corollary 2.4. *Suppose that $T \in \mathcal{L}(H)$ and $\lambda \in \partial\sigma(T; H)$. Then $T - \lambda I$ cannot be universal. In particular, if $\sigma(T; H) = \partial\sigma(T; H)$, then $T - \lambda I$ is not universal for any $\lambda \in \mathbb{C}$.*

We next consider general properties of the class $\mathcal{U}(H)$ of the universal operators. Recently, Pozzi [26, Theorem 3.8] extended Caradus’s condition (C) as follows.

(C₊) *If $U \in \mathcal{L}(H)$ satisfies $\dim \text{Ker}(U) = \infty$ and $\dim(H/\text{Ran}(U)) < \infty$, then $U \in \mathcal{U}(H)$.*

It is helpful for comparative purposes to introduce the subclasses

$$\begin{aligned} \mathcal{UC}(H) &= \{U \in \mathcal{L}(H) : U \text{ satisfies } (C)\}, \\ \mathcal{UC}_+(H) &= \{U \in \mathcal{L}(H) : U \text{ satisfies } (C_+)\} \end{aligned}$$

of $\mathcal{U}(H)$. Observe that $\mathcal{UC}_+(H) = \Phi_-(H) \setminus \Phi_+(H)$. It follows from the classical perturbation theory for semi-Fredholm operators that $\mathcal{UC}_+(H)$ is preserved under sufficiently

small operator norm perturbations as well as compact perturbations (see [2, Theorem 4.2] or [21, Theorems III.16.18 and III.16.19]). In particular, $U + K \in \mathcal{UC}_+(H)$ whenever $U \in \mathcal{UC}(H)$ and K is a compact operator. In the sequel, we denote by $\mathcal{K}(H)$ the closed ideal of $\mathcal{L}(H)$ consisting of compact operators on H . Moreover, in [5, Theorem 2], the authors obtained by direct means a perturbation result for the class $\mathcal{UC}(H)$ which contains more detailed information. We also recall that the universal model operator B_∞ has the stronger property that its restrictions represent suitable multiples cT up to unitary equivalence for any $T \in \mathcal{L}(H)$ (e.g. [9, Theorem 8.1.5] or [27, Chapter 1.5]).

It is evident from Proposition 2.1 that the subclasses $\mathcal{UC}(H)$ and $\mathcal{UC}_+(H)$ are much smaller than $\mathcal{U}(H)$, and that $\mathcal{U}(H)$ contains operators very different from B_∞ . Moreover, $\mathcal{UC}(H)$ is not preserved by compact perturbations. For the record, we include some very simple related examples.

Example 2.5.

(i) Let $U \in \mathcal{U}(H)$, so that

$$V = \begin{pmatrix} U & 0 \\ 0 & B \end{pmatrix}$$

is a universal operator on $H \oplus H$ for any $B \in \mathcal{L}(H)$ by Proposition 2.1. For instance, if $B(H)$ has infinite codimension in H , then $\text{Ran}(V) = U(H) + B(H)$ has infinite codimension in $H \oplus H$, and if $\text{Ran}(B)$ is not closed, then $\text{Ran}(V)$ is not closed either.

(ii) Define $U \in \mathcal{L}(\ell^2)$ by $Ue_{2n} = e_n$ and $Ue_{2n+1} = 0$ for $n \in \mathbb{N}$, so that $U \in \mathcal{UC}(\ell^2)$. Let $K \in \mathcal{K}(\ell^2)$ be the rank-1 operator defined by $Ke_2 = -e_1$ and $Ke_n = 0$ for $n \neq 2$. Since $(U + K)e_2 = 0$, it follows that $e_1 \notin (U + K)(\ell^2)$, so that $U + K \notin \mathcal{UC}(\ell^2)$ (even though $U + K \in \mathcal{UC}_+(\ell^2)$).

More significantly, explicit examples demonstrate similarly that the full class $\mathcal{U}(H)$ of universal operators is neither open in the operator norm nor invariant under compact perturbations.

Example 2.6. Fix a universal operator $U \in \mathcal{U}(H)$. Proposition 2.1 implies that

$$V = \begin{pmatrix} U & I_H \\ 0 & 0 \end{pmatrix}$$

is a universal operator on $H \oplus H$, where I_H is the identity map of H . Consider the sequence $(V_n) \subset \mathcal{L}(H \oplus H)$ defined by

$$V_n = \begin{pmatrix} U & I_H \\ \frac{1}{n}I_H & 0 \end{pmatrix},$$

that is, $V_n(x, y) = (Ux + y, (1/n)x)$ for $(x, y) \in H \oplus H$. Note that V_n is not universal on $H \oplus H$ for any $n \in \mathbb{N}$, since $\text{Ker}(V_n) = \{(0, 0)\}$. In fact, $V_n(x, y) = (Ux + y, (1/n)x) = (0, 0)$ yields that $x = 0$ and $y = -Ux = 0$. Clearly, $\|V_n - V\| = 1/n$ for $n \in \mathbb{N}$, so that $V \in \mathcal{U}(H \oplus H)$ is not an interior point of $\mathcal{U}(H \oplus H)$.

Furthermore, let $K \in \mathcal{K}(H)$ be the diagonal operator defined by $Kf_n = (1/n)f_n$ for $n \in \mathbb{N}$, where (f_n) is some fixed orthonormal basis of H . Consider

$$W = \begin{pmatrix} U & I_H \\ K & 0 \end{pmatrix} = V + \begin{pmatrix} 0 & 0 \\ K & 0 \end{pmatrix} \in \mathcal{L}(H \oplus H),$$

that is, $W(x, y) = (Ux + y, Kx)$ for $(x, y) \in H \oplus H$. Thus, W is a compact perturbation of $V \in \mathcal{U}(H \oplus H)$, but W is not a universal operator, since again $\text{Ker}(W) = \{(0, 0)\}$.

It follows from the algebraic semi-group property of $\Phi_-(H)$ [21, Theorem III.16.5] that the subclass $\mathcal{UC}_+(H)$ is multiplicative in the sense that $UV \in \mathcal{UC}_+(H)$ whenever $U, V \in \mathcal{UC}_+(H)$ (and this property is obvious for $\mathcal{UC}(H)$). Multiplicativity easily fails for the class $\mathcal{U}(H)$. In fact, fix $U_0, V_0 \in \mathcal{U}(H)$. Then

$$U = \begin{pmatrix} U_0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 0 & 0 \\ 0 & V_0 \end{pmatrix}$$

belong to $\mathcal{U}(H \oplus H)$ by Proposition 2.1, but $UV = 0$.

3. Universality of the adjoint $C_\varphi^* - \bar{\lambda}I$ on $S^2(\mathbb{D})$

Recall that Nordgren *et al.* [24] showed that the operators $C_\varphi - \lambda I$ are universal on the Hardy space $H^2(\mathbb{D})$ for any hyperbolic automorphism φ of the unit disc \mathbb{D} and $\lambda \in \text{int}(\sigma(C_\varphi; H^2(\mathbb{D})))$. Here, C_φ is the composition operator $f \mapsto f \circ \varphi$. In this section, we will discuss potential analogues of this result in the scale of weighted Dirichlet spaces $D_\beta(\mathbb{D})$, which are Hilbert spaces of analytic functions defined on the unit disc \mathbb{D} . Our main observation (Theorem 3.2) is that the adjoint $C_\varphi^* - \bar{\lambda}I$ is universal on the space $S^2(\mathbb{D})$, whenever φ is a hyperbolic automorphism of \mathbb{D} and $\lambda \in \text{int}(\sigma(C_\varphi; S^2(\mathbb{D})))$. Here, $S^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that $f' \in H^2(\mathbb{D})$, whence $S^2(\mathbb{D})$ is continuously embedded in the classical Dirichlet space \mathcal{D}^2 .

Recall for $\beta \in \mathbb{R}$ that the weighted Dirichlet space $\mathcal{D}_\beta(\mathbb{D})$ is the Hilbert space of analytic functions $f(z) = \sum_{n=0}^\infty a_n z^n$ that satisfy

$$\|f\|_\beta^2 = \sum_{n=0}^\infty |a_n|^2 (n+1)^{2\beta} < \infty.$$

(These spaces are also special cases of the weighted Hardy spaces.) The Hardy space $H^2(\mathbb{D})$ is obtained for $\beta = 0$, the Bergman space $A^2(\mathbb{D})$ for $\beta = -1/2$, the Dirichlet space \mathcal{D}^2 for $\beta = 1/2$ and $S^2(\mathbb{D})$ for $\beta = 1$ (possibly up to an equivalent norm). We also recall that there is a continuous embedding $D_\beta(\mathbb{D}) \subset D_\alpha(\mathbb{D})$ for $\alpha < \beta$. The reference [9, Chapter 2.1] contains more background about these spaces.

It will be enough for our purposes to consider the normalized hyperbolic automorphisms of \mathbb{D} that have the form

$$\varphi_r(z) = \frac{z+r}{1+rz}, \quad r \in (0, 1). \tag{2}$$

In fact, it is known that all other hyperbolic automorphisms of \mathbb{D} can be conjugated by automorphisms of \mathbb{D} to the preceding normalized form. We will later need the fact that

$$\varphi_r^{-1}(z) = \varphi_{-r}(z) = \frac{z-r}{1-rz}$$

belongs to the same conjugacy class as φ_r , since $\varphi_{-r} = g \circ \varphi_r \circ g$, where $g(z) = -z$ for $z \in \mathbb{D}$. Hence, $C_{\varphi_{-r}} = C_g C_{\varphi_r} C_g$, so that C_{φ_r} and $C_{\varphi_{-r}} = C_{\varphi_r}^{-1}$ are similar operators. For more information on linear fractional transformations in general, see for example [31, Chapter 0], and on composition operators acting on spaces of analytic functions; see [9] or [31].

The composition operators C_{φ_r} are known to be bounded on $\mathcal{D}_\beta(\mathbb{D})$ for all $\beta \in \mathbb{R}$, see [9, Chapter 3.1] and [32] for various ranges of β . We will require the result that

$$\sigma(C_{\varphi_r}; S^2(\mathbb{D})) = \sigma(C_{\varphi_r}; H^2(\mathbb{D})) = \left\{ \lambda \in \mathbb{C} : \left(\frac{1-r}{1+r} \right)^{1/2} \leq |\lambda| \leq \left(\frac{1-r}{1+r} \right)^{-1/2} \right\} \quad (3)$$

for all $0 < r < 1$. We refer to [9, Theorem 7.4] for the Hardy space case and to [13, Theorem 3.9] for the case $S^2(\mathbb{D}) = \mathcal{D}_1(\mathbb{D})$. In the sequel, we will denote the corresponding open annulus, i.e. the interior of the above spectrum, by

$$\mathcal{A}_r := \left\{ \lambda \in \mathbb{C} : \left(\frac{1-r}{1+r} \right)^{1/2} < |\lambda| < \left(\frac{1-r}{1+r} \right)^{-1/2} \right\}.$$

We point out as an initial motivation that $C_{\varphi_r} - \lambda I$ is not universal on any of the small weighted Dirichlet spaces contained in the classical Dirichlet space $\mathcal{D}^2 = \mathcal{D}_{1/2}(\mathbb{D})$.

Example 3.1. Let $0 < r < 1$. Then $C_{\varphi_r} - \lambda I$ is not universal on $\mathcal{D}_\beta(\mathbb{D})$ for any $\beta \geq 1/2$ and any $\lambda \in \mathbb{C}$.

In fact, for $\beta = 1/2$ it is known that $\sigma(C_{\varphi_r}; \mathcal{D}^2) = \mathbb{T}$ by [15, Theorem 3.2]. It follows from Corollary 2.4 that neither $C_{\varphi_r} - \lambda I$ nor its adjoint $C_{\varphi_r}^* - \bar{\lambda} I$ is universal on \mathcal{D}^2 for any $\lambda \in \mathbb{C}$.

For $\beta > 1/2$, it follows from [13, Theorem 3.9] and its proof that in this case the point spectrum $\sigma_p(C_{\varphi_r}; \mathcal{D}_\beta(\mathbb{D})) = \{1\}$, and, moreover, that $\text{Ker}(C_{\varphi_r} - I; \mathcal{D}_\beta(\mathbb{D})) = \mathbb{C}$. Consequently, $\dim \text{Ker}(C_{\varphi_r} - \lambda I; \mathcal{D}_\beta(\mathbb{D}))$ is either 0 (for $\lambda \neq 1$) or 1 (for $\lambda = 1$), so Corollary 2.3 yields that $C_{\varphi_r} - \lambda I$ cannot be universal on the weighted Dirichlet spaces for any $\beta > 1/2$ and $\lambda \in \mathbb{C}$.

As a contrast, we show in the main result of this section that the adjoint of $C_{\varphi_r} - \lambda I$ is universal on $S^2(\mathbb{D})$.

Theorem 3.2. Let φ_r be the hyperbolic automorphism of \mathbb{D} defined by $\varphi_r(z) = ((z+r)/(1+rz))$ for $r \in (0, 1)$. Then $C_{\varphi_r}^* - \bar{\lambda} I$ is universal on $S^2(\mathbb{D})$ for any $\lambda \in \mathcal{A}_r$.

Proof. Let $0 < r < 1$ and write $H^2(\mathbb{D}) = zH^2(\mathbb{D}) \oplus [\mathbf{1}]$, where $[\mathbf{1}]$ denotes the constant functions. The crux of the argument is the fact that the compression

$$P_{zH^2} C_{\varphi_r}^{-1} : zH^2(\mathbb{D}) \longrightarrow zH^2(\mathbb{D})$$

and the restriction of the adjoint

$$C_{\varphi_r}^* : zS^2(\mathbb{D}) \longrightarrow zS^2(\mathbb{D})$$

are similar operators, where the subspace $zS^2(\mathbb{D})$ is invariant under $C_{\varphi_r}^*$. The details of the similarity are explained in Corollary 3.6 and Remark 3.8 in [13], which in turn are based on a duality argument of Hurst [16, Theorem 5].

We first consider the operator $C_{\varphi_r}^{-1} - \lambda I : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$. Since $C_{\varphi_r}^{-1}(\mathbf{1}) = \mathbf{1}$, we may write $C_{\varphi_r}^{-1} - \lambda I$ as the following operator matrix acting on $H^2(\mathbb{D}) = zH^2(\mathbb{D}) \oplus [\mathbf{1}]$:

$$C_{\varphi_r}^{-1} - \lambda I = \begin{pmatrix} P_{zH^2}C_{\varphi_r}^{-1} - \lambda I & 0 \\ P_{\mathbb{C}}C_{\varphi_r}^{-1} & (1 - \lambda) \end{pmatrix}.$$

We claim that the compression $P_{zH^2}C_{\varphi_r}^{-1} - \lambda I : zH^2(\mathbb{D}) \rightarrow zH^2(\mathbb{D})$ satisfies Caradus’s condition (C) for all $\lambda \in \mathcal{A}_r$. Since $C_{\varphi_r}^{-1} = C_{\varphi_{-r}}$, we know that the operator $C_{\varphi_r}^{-1} - \lambda I : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$ is surjective by the proof of [24, Theorem 6.2]. It follows that $(P_{zH^2}C_{\varphi_r}^{-1} - \lambda I)(zH^2(\mathbb{D})) = zH^2(\mathbb{D})$ as well. In fact, if $g \in zH^2(\mathbb{D})$ is arbitrary, then there is $f = f_1 + f_2 \in H^2(\mathbb{D})$, with $f_1 \in zH^2(\mathbb{D})$ and $f_2 \in [\mathbf{1}]$, such that $(C_{\varphi_r}^{-1} - \lambda I)f = g$, that is,

$$\begin{pmatrix} P_{zH^2}C_{\varphi_r}^{-1} - \lambda I & 0 \\ P_{\mathbb{C}}C_{\varphi_r}^{-1} & (1 - \lambda) \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} (P_{zH^2}C_{\varphi_r}^{-1} - \lambda I)f_1 \\ P_{\mathbb{C}}C_{\varphi_r}^{-1}f_1 + (1 - \lambda)f_2 \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix}.$$

In particular, $(P_{zH^2}C_{\varphi_r}^{-1} - \lambda I)f_1 = g$, so that the compression $P_{zH^2}C_{\varphi_r}^{-1} - \lambda I$ is an onto map $zH^2(\mathbb{D}) \rightarrow zH^2(\mathbb{D})$ for $\lambda \in \mathcal{A}_r$. Moreover, it is not difficult to check that since $\lambda \in \mathcal{A}_r$ is an eigenvalue of infinite multiplicity for $C_{\varphi_r}^{-1} : H^2(\mathbb{D}) \rightarrow H^2(\mathbb{D})$, the same fact holds for the compression $P_{zH^2}C_{\varphi_r}^{-1} : zH^2(\mathbb{D}) \rightarrow zH^2(\mathbb{D})$. Consequently, $P_{zH^2}C_{\varphi_r}^{-1} - \lambda I : zH^2(\mathbb{D}) \rightarrow zH^2(\mathbb{D})$ satisfies (C).

It follows from the similarity stated at the beginning of the argument that the restricted adjoint $C_{\varphi_r}^* - \lambda I : zS^2(\mathbb{D}) \rightarrow zS^2(\mathbb{D})$ also satisfies (C) and is hence universal on $zS^2(\mathbb{D})$. Write $C_{\varphi_r}^* - \lambda I$ on $S^2(\mathbb{D})$ as an operator matrix acting on $zS^2(\mathbb{D}) \oplus [\mathbf{1}]$, that is,

$$\begin{pmatrix} C_{\varphi_r}^* - \lambda I & P_{zS^2}C_{\varphi_r}^* \\ 0 & (1 - \lambda) \end{pmatrix},$$

where we also take into account that $C_{\varphi_r}^*(zS^2(\mathbb{D})) \subset zS^2(\mathbb{D})$. It follows that $C_{\varphi_r}^* - \lambda I : S^2(\mathbb{D}) \rightarrow S^2(\mathbb{D})$ is universal by Proposition 2.1.

Alternatively, in the last step one may also note that if $\lambda \neq 1$, then $C_{\varphi_r}^* - \lambda I : S^2(\mathbb{D}) \rightarrow S^2(\mathbb{D})$ satisfies (C), while $\text{codim Ran}(C_{\varphi_r}^* - I) = 1$ in $S^2(\mathbb{D})$, so that $C_{\varphi_r}^* - I$ satisfies the generalized condition (C₊).

Finally, note that the annulus \mathcal{A}_r is preserved by complex conjugation, so that we may above change $C_{\varphi_r}^* - \lambda I$ to $C_{\varphi_r}^* - \bar{\lambda}I$. □

Heller [14] found a concrete formula for the adjoint $C_{\varphi_r}^*$ on $S^2(\mathbb{D})$ which involves a compact perturbation. This fact leads to a related universal operator. Let M_z be the multiplication operator $f \mapsto zf$ on $S^2(\mathbb{D})$, whose adjoint $M_z^* \in \mathcal{L}(S^2(\mathbb{D}))$ has the form

$$M_z^* \left(\sum_{n=0}^{\infty} a_n z^n \right) = a_1 + \sum_{n=1}^{\infty} a_{n+1} \left(\frac{n+1}{n} \right)^n z^n$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in S^2(\mathbb{D})$.

Corollary 3.3. *Let φ_r be as in Theorem 3.2. Then the operator*

$$C_{\varphi_r} - \frac{r}{1+r^2}(M_z^* + M_z)C_{\varphi_r} - \frac{1-r^2}{1+r^2}\lambda I$$

is universal on $S^2(\mathbb{D})$ for all $\lambda \in \mathcal{A}_r$.

Proof. By [14, Theorem 6.5], we can write

$$(C_{\varphi_r}^{-1})^* - \lambda I = \frac{1+r^2}{1-r^2}C_{\varphi_r} - \frac{r}{1-r^2}(M_z^* + M_z)C_{\varphi_r} - \lambda I + K,$$

where K is a compact operator on $S^2(\mathbb{D})$. Recall that C_{φ_r} and $C_{\varphi_r}^{-1} = C_{\varphi_{-r}}$ are similar operators on $S^2(\mathbb{D})$. From the symmetry of \mathcal{A}_r , we get that Theorem 3.2 holds if we replace $C_{\varphi_r}^*$ by $(C_{\varphi_r}^{-1})^*$. Moreover, the proof of Theorem 3.2 shows that $(C_{\varphi_r}^{-1})^* - \lambda I$ satisfies condition (C_+) on $S^2(\mathbb{D})$ for all $\lambda \in \mathcal{A}_r$. Since the class \mathcal{UC}_+ is preserved by compact perturbations, we deduce that $((1+r^2)/(1-r^2))C_{\varphi_r} - (r/(1-r^2))(M_z^* + M_z)C_{\varphi_r} - \lambda I$ is a universal operator on $S^2(\mathbb{D})$. □

We include an explicit example of a non-universal composition operator, for which the necessary condition in (1) from Theorem 2.2 holds.

Example 3.4. Fix $0 < s < 1$ and let $\phi : \mathbb{D} \rightarrow \mathbb{D}$ be the hyperbolic non-automorphism defined by $\phi(z) = sz + 1 - s$. Then

$$\sigma_e^+(C_\phi; H^2(\mathbb{D})) = \{\lambda \in \mathbb{C} : |\lambda| \leq s^{-1/2}\}, \tag{4}$$

but C_ϕ is not universal on $H^2(\mathbb{D})$.

In fact, the argument in [16, Theorem 8] implies that for each $\lambda = s^{p+iq}$ with $p > -1/2$ and $q \in \mathbb{R}$ (so that $0 < |\lambda| < s^{-1/2}$), the functions $f_n(z) = \exp((p + i(q + 2n\pi t)) \log(1 - z))$, where $t = 1/\log(s)$, are linearly independent in $H^2(\mathbb{D}) \cap \text{Ker}(C_\phi - \lambda I)$ for $n \in \mathbb{Z}$. Here, \log is the principal branch of the natural logarithm. This implies that (4) holds, since $\sigma_e^+(C_\phi; H^2(\mathbb{D}))$ is a closed set. Clearly, C_ϕ is an injective operator and thus not universal. Finally, to obtain an example where 0 belongs to the point spectrum of the operator, one extends

$$C_\phi \text{ to } \begin{pmatrix} C_\phi & 0 \\ 0 & 0 \end{pmatrix} \text{ on } H^2(\mathbb{D}) \oplus [x],$$

where $[x]$ denotes a one-dimensional space.

Recently, composition operators have also been studied on the Hardy space and the weighted Bergman spaces of the upper half-plane $\Pi^+ = \{z \in \mathbb{C} : \text{Im } z > 0\}$, where new phenomena occur (e.g. [10, 11, 19]). Recall that the analytic function $F : \Pi^+ \rightarrow \mathbb{C}$ belongs

to the Hardy space $H^2(\Pi^+)$ if

$$\|F\|_{H^2(\Pi^+)}^2 = \sup_{y>0} \int_{-\infty}^{\infty} |F(x + iy)|^2 dx < \infty,$$

and to the weighted Bergman space $\mathcal{A}_\alpha^2(\Pi^+)$, for $\alpha > -1$, if

$$\|F\|_{\mathcal{A}_\alpha^2(\Pi^+)}^2 = \int_0^\infty \int_{-\infty}^{\infty} |F(x + iy)|^2 y^\alpha \frac{dx dy}{\pi} < \infty.$$

Let τ be a hyperbolic automorphism of Π^+ , that is, $\tau(w) = \mu w + w_0$, where $w_0 \in \mathbb{R}$ and $\mu \in (0, 1) \cup (1, \infty)$. It follows from [10, Theorem 3.1] and [11, Theorem 3.4] that the composition operator C_τ is bounded on $H^2(\Pi^+)$ and on $\mathcal{A}_\alpha^2(\Pi^+)$, respectively, for all $\alpha > -1$. It is natural to ask whether there is an analogue of the theorem of Nordgren, Rosenthal and Wintrobe on these spaces.

Proposition 3.5. *The operator $C_\tau - \lambda I$ is not universal on $H^2(\Pi^+)$ or $\mathcal{A}_\alpha^2(\Pi^+)$, where $\alpha > -1$, for any $\lambda \in \mathbb{C}$ and any hyperbolic automorphism τ of Π^+ .*

Proof. The claim follows from Corollary 2.4 and the spectral results

$$\sigma(C_\tau; H^2(\Pi^+)) = \{\lambda \in \mathbb{C} : |\lambda| = \mu^{-1/2}\},$$

[20, Theorem 2.12], and

$$\sigma(C_\tau; \mathcal{A}_\alpha^2(\Pi^+)) = \{\lambda \in \mathbb{C} : |\lambda| = \mu^{-(\alpha+2)/2}\}$$

for $\alpha > -1$ [30, Theorem 1.2]. Here, $\tau(w) = \mu w + w_0$ and $\mu \in (0, 1) \cup (1, \infty)$ as above. \square

4. Examples of universal commuting pairs

Recently, Müller [22] introduced a notion of universality for commuting pairs of operators (and, more generally, for commuting n -tuples). Let H be a separable infinite-dimensional Hilbert space. The commuting pair $(U_1, U_2) \in \mathcal{L}(H)^2$ is said to be *universal* if for each commuting pair $(S_1, S_2) \in \mathcal{L}(H)^2$ there is a constant $c \neq 0$ and a subspace $M \subset H$, invariant for both U_1 and U_2 , so that the pairs $(U_1|_M, U_2|_M)$ and (cS_1, cS_2) are similar, that is, there is an isomorphism $V : H \rightarrow M$ such that $U_1V = cVS_1$ and $U_2V = cVS_2$.

If (U_1, U_2) is a universal commuting pair, then $\dim(\text{Ker}(U_1) \cap \text{Ker}(U_2)) = \infty$, and both U_1 and U_2 must be universal operators for H . Müller [22, Theorem 3] obtained a version of Caradus’s condition for the universality of commuting pairs (U_1, U_2) , which we recall next in the special case where U_1, U_2 are surjections (see [22, Corollary 8]).

(M) *Let $U_1, U_2 \in \mathcal{L}(H)$ be commuting surjections, such that:*

(i) $\dim(\text{Ker}(U_1) \cap \text{Ker}(U_2)) = \infty$, and

(ii) $\text{Ker}(U_1U_2) = \text{Ker}(U_1) + \text{Ker}(U_2)$.

Then $(U_1, U_2) \in \mathcal{L}(H)^2$ is a universal commuting pair.

The following concrete example of a universal commuting pair is contained in [22, Example 9]. Let H be a separable infinite-dimensional Hilbert space and $K = \ell^2(\mathbb{Z}_+^2, H)$,

the space of double-indexed sequences with values in H . Define $U_i \in \mathcal{L}(K)$ by $U_i f(\alpha) = f(\alpha + \beta_i)$ for $\alpha \in \mathbb{Z}_+^2$, $f \in \ell^2(\mathbb{Z}_+^2, H)$ and $i = 1, 2$, where $\beta_1 = (1, 0)$ and $\beta_2 = (0, 1)$ in \mathbb{Z}_+^2 . Then $(U_1, U_2) \in \mathcal{L}(K)^2$ is a universal commuting pair. (Alternatively, $U_i = M_{z_i}^*$ for $i = 1, 2$, where M_{z_i} denotes multiplication by the variable z_i in the vector-valued Hardy space $H^2(\mathbb{D}^2, H)$ for $(z_1, z_2) \in \mathbb{D}^2$.)

In this section, we are mainly interested in obtaining further concrete examples of universal commuting pairs, since it turns out that such examples are rather more difficult to write down explicitly compared with the class $\mathcal{U}(H)$. Our first observations and examples illustrate some of the obstructions, apart from the technical fact that condition (M) requires knowledge of $\text{Ker}(U_1)$ and $\text{Ker}(U_2)$. We begin by noting that there is a kind of algebraic independence between U_1 and U_2 for universal pairs (U_1, U_2) . For this, let $\{T\}' = \{S \in \mathcal{L}(H) : ST = TS\}$ stand for the commutant of T .

Proposition 4.1. *Let H be a separable infinite-dimensional Hilbert space and $T \in \mathcal{L}(H)$.*

- (i) *If $S \in \{T\}'$, then (T, ST) is not a universal commuting pair. In particular, $(T, p(T)T)$ is not a universal commuting pair for any complex polynomial $p(z) = a_1z + \dots + a_nz^n$ satisfying $p(0) = 0$, where $p(T) = a_1T + \dots + a_nT^n$.*
- (ii) *(T^m, T^n) is not a universal commuting pair for any $m, n \in \mathbb{N}$.*

Proof.

- (i) Consider $(0, I_H) \in \mathcal{L}(H)^2$. If (T, ST) is a universal pair, then corresponding to the pair $(0, I_H) \in \mathcal{L}(H)^2$ there is an infinite-dimensional subspace $M \subset H$ invariant under T , and $c \neq 0$, so that $(T|_M, ST|_M)$ and $(0, cI_H)$ are similar. However, the similarity of $T|_M$ and 0 implies that $M \subset \text{Ker}(T)$, so that $ST|_M$ cannot be similar to cI_H .
- (ii) Observe that one may assume $m < n$ by symmetry, and then argue as in part (i). □

The following example looks at simple ways to construct universal pairs starting from given universal operators $U, V \in \mathcal{U}(H)$.

Example 4.2.

- (i) Suppose that $U, V \in \mathcal{L}(H)$ satisfy condition (C), and

$$U_0 = \begin{pmatrix} U & 0 \\ 0 & I_H \end{pmatrix}, \quad V_0 = \begin{pmatrix} I_H & 0 \\ 0 & V \end{pmatrix} \in \mathcal{L}(K),$$

where $K = H \oplus H$. Then U_0 and V_0 are commuting surjections on K , and $\text{Ker}(U_0) = \text{Ker}(U) \times \{0\}$ and $\text{Ker}(V_0) = \{0\} \times \text{Ker}(V)$. Hence, the pair $(U_0, V_0) \in (\mathcal{L}(K))^2$ is not universal, since $\text{Ker}(U_0) \cap \text{Ker}(V_0) = \{(0, 0)\}$. (Note, however, that $\text{Ker}(U_0V_0) = \text{Ker}(U_0) + \text{Ker}(V_0)$ in this case.)

- (ii) Suppose that (U_1, U_2) and (V_1, V_2) are commuting pairs of surjections that satisfy condition (M), and

$$U = \begin{pmatrix} U_1 & 0 \\ 0 & V_1 \end{pmatrix}, \quad V = \begin{pmatrix} U_2 & 0 \\ 0 & V_2 \end{pmatrix} \in \mathcal{L}(K).$$

In this case, $(U, V) \in \mathcal{L}(K)^2$ is a universal commuting pair. In fact, (U, V) also satisfies (M), since it is not difficult to check that

$$\text{Ker}(UV) = (\text{Ker}(U_1) + \text{Ker}(U_2)) \times (\text{Ker}(V_1) + \text{Ker}(V_2)) = \text{Ker}(U) + \text{Ker}(V).$$

We next look for a non-commutative version of the example from [22]. Let H be a separable infinite-dimensional Hilbert space and $\mathcal{C}_2(H)$ the space of Hilbert–Schmidt operators on H equipped with the Hilbert–Schmidt norm $\|\cdot\|_2$. Recall that $T \in \mathcal{C}_2(H)$ if there is an orthonormal basis (f_n) of H such that $\sum_{n=1}^\infty \|Tf_n\|^2 < \infty$, where

$$\|T\|_2 = \left(\sum_{n=1}^\infty \|Tf_n\|^2 \right)^{1/2}$$

is independent of the basis. Then $(\mathcal{C}_2(H), \|\cdot\|_2)$ is a separable Hilbert space, and $\|USV\|_2 \leq \|U\| \cdot \|V\| \cdot \|S\|_2$ for $S \in \mathcal{C}_2(H)$ and $U, V \in \mathcal{L}(H)$. We refer to [23, Chapter 2.4], for example, for more background on the ideal $\mathcal{C}_2(H)$ of $\mathcal{L}(H)$.

Hence, the multiplication maps L_U and R_U are bounded operators $\mathcal{C}_2(H) \rightarrow \mathcal{C}_2(H)$, where $L_U(S) = US$ and $R_U(S) = SU$ for any $U \in \mathcal{L}(H)$ and $S \in \mathcal{C}_2(H)$. Clearly, $(L_U, R_V) \in \mathcal{L}(\mathcal{C}_2(H))^2$ is a commuting pair for any $U, V \in \mathcal{L}(H)$, and we are interested in the universality of (L_U, R_V) on $\mathcal{C}_2(H)$. Let B be the standard backward shift on $\ell^2 = \ell^2(\mathbb{Z}_+)$, that is, $B(x_0, x_1, \dots) = (x_1, x_2, \dots)$ for $(x_j) \in \ell^2$. It turns out that (L_B, R_{B^*}) is not a universal pair (see part (ii) of Theorem 4.3); to obtain universal pairs (L_U, R_V) , we will consider the vector-valued direct ℓ^2 -sum $\mathcal{H} = \ell^2(\mathbb{Z}_+, H) = (\oplus_{n \in \mathbb{Z}_+} H)_{\ell^2}$, where H is a fixed separable infinite-dimensional Hilbert space. Let B_∞ be the backward shift of infinite multiplicity on \mathcal{H} , so that B_∞^* is the corresponding forward shift on \mathcal{H} .

The following result contains the main example of this section. We will use $u \otimes v$ for given $u, v \in H$ to denote the rank-1 operator $x \mapsto \langle x, v \rangle u$ on H .

Theorem 4.3.

- (i) $L_B, R_{B^*} \in \mathcal{U}(\mathcal{C}_2(\ell^2))$ and $L_{B_\infty}, R_{B_\infty^*} \in \mathcal{U}(\mathcal{C}_2(\mathcal{H}))$.
- (ii) (L_B, R_{B^*}) is not a universal pair on $\mathcal{C}_2(\ell^2)$.
- (iii) $(L_{B_\infty}, R_{B_\infty^*})$ is a universal pair on $\mathcal{C}_2(\mathcal{H})$.

Proof. (i) We check that L_B and R_{B^*} satisfy condition (C) on $\mathcal{C}_2(\ell^2)$. In fact, if (e_n) is the standard unit vector basis of ℓ^2 , then $L_B(e_0 \otimes e_n) = Be_0 \otimes e_n = 0$ for any $n \in \mathbb{Z}_+$, so that $\text{Ker}(L_B)$ is infinite dimensional. Moreover, $\text{Ran}(L_B) = \mathcal{C}_2(\ell^2)$, since $BB^* = I_H$. The argument for R_{B^*} is similar.

The universality of L_{B_∞} and $R_{B_\infty^*}$ on $\mathcal{C}_2(\mathcal{H})$ follows from part (iii) (alternatively, one may also modify the preceding argument as in the proof of (iii)).

(ii) Recall that $(e_m \otimes e_n)_{m,n \in \mathbb{Z}_+}$ is an orthonormal basis of $\mathcal{C}_2(\ell^2)$. It follows from the identities $L_B(e_m \otimes e_n) = B e_m \otimes e_n$ and $R_{B^*}(e_m \otimes e_n) = e_m \otimes B e_n$ that $\text{Ker}(L_B) = [e_0 \otimes e_n : n \in \mathbb{Z}_+]$ and $\text{Ker}(R_{B^*}) = [e_n \otimes e_0 : n \in \mathbb{Z}_+]$. Here, $[A]$ denotes the closed linear span of the set $A \subset \mathcal{C}_2(\ell^2)$. In particular, $\text{Ker}(L_B) \cap \text{Ker}(R_{B^*}) = [e_0 \otimes e_0]$ is one dimensional, so that (L_B, R_{B^*}) can not be a universal pair.

(iii) We will verify that condition (M) holds. Towards this, note first that L_{B_∞} and $R_{B_\infty^*}$ are surjections on $\mathcal{C}_2(\mathcal{H})$, since $B_\infty B_\infty^* = I_{\mathcal{H}}$ implies that $L_{B_\infty}(B_\infty^* S) = S$ and $R_{B_\infty^*}(S B_\infty) = S$ for any $S \in \mathcal{C}_2(\mathcal{H})$.

To compute the kernels of L_{B_∞} and $R_{B_\infty^*}$, we need the fact that any $S \in \mathcal{C}_2(\mathcal{H})$ is uniquely determined by its operator matrix components $S_{i,j} = P_i S J_j \in \mathcal{C}_2(H)$ for $i, j \in \mathbb{Z}_+$. Here, P_i is the orthogonal projection $\mathcal{H} \rightarrow H$ onto the i th copy of H , and $J_j : H \rightarrow \mathcal{H}$ the canonical inclusion from the j th copy. One deduces from the definition of $S_{i,j}$ and the identity $L_{B_\infty}(v \otimes u) = B_\infty v \otimes u$ for $u, v \in \mathcal{H}$ that

$$L_{B_\infty}(S) = \begin{pmatrix} S_{1,0} & S_{1,1} & \cdots \\ S_{2,0} & S_{2,1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

for $S = (S_{i,j})$, so that by uniqueness

$$\text{Ker}(L_{B_\infty}) = \{S = (S_{i,j}) \in \mathcal{C}_2(\mathcal{H}) : S_{i,j} = 0 \text{ for } i > 0\}. \tag{5}$$

Similarly, the identity $R_{B_\infty^*}(v \otimes u) = v \otimes B_\infty u$ for $u, v \in \mathcal{H}$ yields that

$$R_{B_\infty^*}(S) = \begin{pmatrix} S_{0,1} & S_{0,2} & \cdots \\ S_{1,1} & S_{1,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

whence

$$\text{Ker}(R_{B_\infty^*}) = \{S = (S_{i,j}) \in \mathcal{C}_2(\mathcal{H}) : S_{i,j} = 0 \text{ for } j > 0\}. \tag{6}$$

In particular, we get that

$$\text{Ker}(L_{B_\infty}) \cap \text{Ker}(R_{B_\infty^*}) = \{S = (S_{i,j}) \in \mathcal{C}_2(\mathcal{H}) : S_{i,j} = 0 \text{ for } i > 0 \text{ or } j > 0\}$$

is infinite dimensional, since the operator $S_{0,0} \in \mathcal{C}_2(H)$ can be chosen freely. Finally, we need to verify that

$$\text{Ker}(L_{B_\infty} R_{B_\infty^*}) = \text{Ker}(L_{B_\infty}) + \text{Ker}(R_{B_\infty^*}). \tag{7}$$

However, (7) follows from (5) and (6), the identity $L_{B_\infty} R_{B_\infty^*}(v \otimes u) = B_\infty v \otimes B_\infty u$ for $u, v \in \mathcal{H}$, and the observation that

$$L_{B_\infty} R_{B_\infty^*}(S) = \begin{pmatrix} S_{1,1} & S_{1,2} & \cdots \\ S_{2,1} & S_{2,2} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

as well as the fact that

$$\begin{pmatrix} S_{0,0} & S_{0,1} & \cdots \\ S_{1,0} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} S_{0,0} & S_{0,1} & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots \\ S_{1,0} & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

is the sum of two well-defined Hilbert–Schmidt operators for any $S = (S_{i,j}) \in \mathcal{C}_2(\mathcal{H})$ (see, for example, the proof of [17, 1.c.8]).

We conclude from condition (M) that $(L_{B_\infty}, R_{B_\infty^*})$ is a universal pair. □

Remarks. By a straightforward modification of the argument in part (iii), one may also show that $(L_{(B_\infty)^m}, R_{(B_\infty^*)^n})$ is a universal pair on $\mathcal{C}_2(\mathcal{H})$ for any $m, n \in \mathbb{N}$. We do not know explicit conditions on $(U, V) \in (\mathcal{L}(H))^2$ which ensure that (L_U, R_V) is a universal pair on $\mathcal{C}_2(H)$.

For our last examples, we return to the setting (and notations) from §3 related to composition operators on $H^2(\mathbb{D})$ associated with hyperbolic automorphisms of \mathbb{D} . Recall that $C_{\varphi_r} - \lambda I$ and $C_{\varphi_s} - \mu I$ commute for any $0 < r, s < 1$ and $\lambda, \mu \in \mathbb{C}$. This follows from the fact that

$$\varphi_r \circ \varphi_s = \varphi_t = \varphi_s \circ \varphi_r, \tag{8}$$

where $t = (r + s)/(1 + rs)$. The result of Nordgren *et al.* [24] suggests the following natural question.

Problem. *Are there universal pairs of the form*

$$(C_{\varphi_r} - \lambda I, C_{\varphi_s} - \mu I) \in \mathcal{L}(H^2(\mathbb{D}))^2,$$

for some $0 < r, s < 1$, $\lambda \in \mathcal{A}_r$ and $\mu \in \mathcal{A}_s$?

We first note some obvious restrictions in view of Proposition 4.1.

Example 4.4.

- (i) The pair $(C_{\varphi_r} - \lambda I, C_{\varphi_r} - \mu I)$ is not universal for any $0 < r < 1$ and $\lambda \neq \mu$. In fact, in this case, $\text{Ker}(C_{\varphi_r} - \lambda I) \cap \text{Ker}(C_{\varphi_r} - \mu I) = \{0\}$ once $\lambda \neq \mu$.
- (ii) Let $0 < r < 1$. By (8), there is $r_n \in (0, 1)$ such that $\varphi_r^n = \varphi_{r_n}$, where $\varphi_r^n = \varphi_r \circ \dots \circ \varphi_r$ (n -fold composition). Then $(C_{\varphi_r} - \lambda I, C_{\varphi_{r_n}} - \lambda^n I)$ is not a universal pair for any $n \geq 2$ and $\lambda \in \mathcal{A}_r$. Indeed, to see this we write $C_{\varphi_{r_n}} - \lambda^n I = S(C_{\varphi_r} - \lambda I)$, where S commutes with C_{φ_r} , and apply Proposition 4.1.

In our final example, we use the recent approach of Cowen and Gallardo-Gutiérrez [8] to the universality result by Nordgren, Rosenthal and Wintrobe in order to analyse more carefully a pair which shows subtler obstructions related to the existence of Nordgren–Rosenthal–Wintrobe pair.

Example 4.5. There are $r, s \in (0, 1)$ and respective eigenvalues $\lambda \in \mathcal{A}_r$, $\mu \in \mathcal{A}_s$ such that $\text{Ker}(C_{\varphi_r} - \lambda I) \cap \text{Ker}(C_{\varphi_s} - \mu I)$ is infinite dimensional, but condition (M) fails to hold for the pair $(C_{\varphi_r} - \lambda I, C_{\varphi_s} - \mu I)$.

Proof. Fix $0 < r < 1$ and consider the positive eigenvalue $\lambda = ((1 - r)/(1 + r))^{-u} \in \mathcal{A}_r$, where $0 < u < 1/2$. It follows from the proofs of Lemma 7.3 and Theorem 7.4 in [9] that the functions

$$f_n(z) = \exp\left((u + in2\pi a) \log \frac{1 + z}{1 - z}\right),$$

where $a = (\log((1 - r)/(1 + r)))^{-1}$ and $n \in \mathbb{Z}$, forms a linearly independent family of eigenfunctions for C_{φ_r} in $H^2(\mathbb{D})$ associated with the eigenvalue λ . Here, the logarithm refers to the principal branch. Let $r < s < 1$ and consider $\mu = ((1 - s)/(1 + s))^{-u} \in \mathcal{A}_s$. As above,

$$g_m(z) = \exp\left((u + im2\pi b) \log \frac{1 + z}{1 - z}\right),$$

where $b = (\log((1 - s)/(1 + s)))^{-1}$ and $m \in \mathbb{Z}$, are linearly independent eigenfunctions of C_{φ_s} for the eigenvalue μ . Moreover, we select r and s so that

$$\frac{n}{m} = \frac{b}{a} = \frac{\log((1 + r)/(1 - r))}{\log((1 + s)/(1 - s))} \tag{9}$$

holds for infinitely many pairs $n, m \in \mathbb{Z} \setminus \{0\}$. This ensures that $\text{Ker}(C_{\varphi_r} - \lambda I) \cap \text{Ker}(C_{\varphi_s} - \mu I)$ is infinite dimensional by comparing the above eigenfunctions.

Recall from [4, Theorem 5] (see also [8]) that there is an analytic covering map $\psi_r : \mathbb{D} \rightarrow \mathcal{A}_r$, such that $C_{\varphi_r} - \lambda I$ and $T_{\psi_r - \lambda}^*$ are similar operators on $H^2(\mathbb{D})$, where $T_{\psi_r - \lambda} \in \mathcal{L}(H^2(\mathbb{D}))$ is the analytic Toeplitz operator $f \mapsto (\psi_r - \lambda)f$. Indeed,

$$\psi_r(z) = \left(\frac{1 - z}{1 + z}\right)^{it_r/\pi},$$

where $t_r = -\log((1 - r)/(1 + r)) > 0$. Moreover, $C_{\varphi_s} - \mu I$ and $T_{\psi_s - \mu}^*$ are similar operators on $H^2(\mathbb{D})$ for the covering map $\psi_s : \mathbb{D} \rightarrow \mathcal{A}_s$, where $\psi_s(z) = ((1 - z)/(1 + z))^{it_s/\pi}$ and $t_s = -\log((1 - s)/(1 + s))$.

By standard duality, the dual version of part (ii) of condition (M) for the pair $(T_{\psi_r - \lambda}^*, T_{\psi_s - \mu}^*)$ is the requirement that

$$\text{Ran}(T_{\psi_s - \mu} T_{\psi_r - \lambda}) = \text{Ran}(T_{\psi_r - \lambda}) \cap \text{Ran}(T_{\psi_s - \mu}). \tag{10}$$

Note for this that all the ranges are closed, since the adjoints are onto maps by similarity.

We claim that condition (10) does not hold. Towards this, consider the standard factorization $\psi_r - \lambda = B_1 S_1 F_1$ into a Blaschke product B_1 containing the zeroes of the function (counting multiplicity), a singular inner function S_1 and an outer function F_2 . Let $\psi_s - \mu = B_2 S_2 F_2$ be the analogous factorization for $\psi_s - \mu$. It is not difficult to check from (9) and the explicit form of ψ_r and ψ_s that the functions $\psi_r - \lambda$ and $\psi_s - \mu$ have infinitely many simple common zeroes. Let $B_1 = B_0 B_3$ and $B_2 = B_0 B_4$, where B_0 is the Blaschke product which contains the common zeroes.

To conclude the argument, consider the function $g = B_0 B_3 B_4 S_1 F_1 S_2 F_2$. Observe that $g \in H^2(\mathbb{D})$, since the maps $\psi_r - \lambda$ and $\psi_s - \mu$, and hence also $S_1 F_1$ and $S_2 F_2$, belong to H^∞ (see [29, Theorem 17.9], for instance). Clearly, $g \in \text{Ran}(T_{\psi_r - \lambda}) \cap \text{Ran}(T_{\psi_s - \mu})$ by inspection. However, the Blaschke product containing the zeroes of $(\psi_r - \lambda)(\psi_s - \mu)$

already has the form $B_0^2 B_3 B_4$, so that $g \notin \text{Ran}(T_{\psi_s - \mu} T_{\psi_r - \lambda})$ in view of the uniqueness of the factorization. \square

Remarks. One may verify from (9) that the pair $(C_{\varphi_r} - \lambda I, C_{\varphi_s} - \mu I)$ studied in Example 4.5 has the property that there is $p, q \in \mathbb{N}$ with $p < q$, for which $C_{\varphi_r}^q - \lambda^q I = C_{\varphi_s}^p - \mu^p I$. However, we do not have general results which would exclude such a property for universal pairs.

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