Correspondence

DEAR EDITOR,

'Piecing it together'

In an earlier article I argued that the results from a theory associated with Felix Baumgartner's fall from a great height over Roswell in New Mexico in October 2012 showed a good measure of agreement with practice [1], except perhaps in respect of total free fall time. Between acceptance and publication of that article an even greater fall in the same region was undertaken in October 2014 by Alan Eustace. I subsequently analyzed and compared results from both falls recently and proposed also a simple modification to the earlier theory that resulted in an improved agreement with known facts of free fall time. Readers who wish to know more can find the details in the letters column of [2].

References

1. John D Mahony, Piecing it together, *Math. Gaz.*, **99** (March 2015), pp. 40-44.
2. John D Mahony, A Difference in Autumn Falls, *Mathematics Today*, **51** (February 2015), pp. 40-42.

doi:10.1017/mag.2015.56

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Feedback

On Note 93.36: Paul Dale writes: A succinct solution of the difference equation

$$f(k, n + 1) = 2f(k, n) - \binom{n}{k}, n \geqslant 1 \tag{1}$$

and the initial condition $f(k, 1) = 2$ (equation (5) of Martin Griffiths' note) is here derived by a rather less cumbersome method.

For $k = 2$ we find

$$\{f(2, n) : n \geqslant 1\} = \{2, 4, 7, 11, 16, 22, \dots\}$$

$$\therefore \{\Delta f(2, n)\} = \{2, 3, 4, 5, 6, \dots\}$$

or $$\Delta f(2, n) = 1 + n = 1 + n^{\underline{1}}$$

where $n^{\underline{r}}$ denotes the falling factorial power

$$n(n - 1)(n - 2)\dots(n - r + 1).$$

On summing we get

$$f(2, n) = C + n^{\underline{1}} + \frac{n^{\underline{2}}}{2!}.$$

But $f(2, 1) = 2$ therefore $C = 1$, and so

$$f(2, n) = \sum_{r=0}^{2} \binom{n}{r}.$$

Similarly for $k = 3$

$$\{f(3, n) : n \geqslant 1\} = \{2, 4, 8, 15, 26, 42, 64, \ldots\}$$

$$\therefore \{\Delta f(3, n)\} = \{2, 4, 7, 11, 16, 22, \ldots\}$$

$$= \{f(2, n)\}$$

or

$$\Delta f(3, n) = 1 + n^{\underline{1}} + \frac{n^{\underline{2}}}{2!}$$

giving

$$f(3, n) = C' + n^{\underline{1}} + \frac{n^{\underline{2}}}{2!} + \frac{n^{\underline{3}}}{3!}.$$

But $f(3, 1) = 2$ therefore $C' = 1$, and so

$$f(3, n) = \sum_{r=0}^{3} \binom{n}{r}.$$

These results suggest that

$$f(k, n) = \sum_{r=0}^{k} \binom{n}{r}. \tag{2}$$

To prove (2) we show that it satisfies (1).

Proof:

$$f(k, n + 1) = \sum_{r=0}^{k} \binom{n+1}{r}$$

but

$$\binom{n+1}{r} = \binom{n}{r} + \binom{n}{r-1}$$

$$\therefore \quad f(k, n + 1) = \sum_{r=0}^{k} \left\{\binom{n}{r} + \binom{n}{r-1}\right\}$$

$$= \sum_{r=0}^{k} \binom{n}{r} + \sum_{r=1}^{k} \binom{n}{r-1}$$

$$= \sum_{r=0}^{k} \binom{n}{r} + \sum_{r=0}^{k-1} \binom{n}{r}$$

$$= 2\sum_{r=0}^{k} \binom{n}{r} - \binom{n}{r}$$

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$$f(k, n+1) = 2f(k, n) - \binom{n}{r}.$$

As required. Also

$$f(k, 1) = \sum_{r=0}^{k} \binom{1}{r}, \qquad k \geqslant 2$$

$$= \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2,$$

satisfying the initial condition.

On 98.16: Andrew Jobbings writes: I think there is a much simpler proof of part (3) of the author's result than that given on pages 337-338. In particular, I don't think it is necessary to deal with the special case first.

We are given a quadrilateral *ABCD* and a point *M*, from which 'equiangular' lines are drawn to meet the sides *AB*, *BC*, *CD* and *DA* respectively in *E*, *F*, *G* and *H*. Part (1) of the author's result states that the four quadrilaterals *AEMH*, *BFME*, *CGMF* and *DHMG* are cyclic. Part (3) states that the quadrilateral *PQRS*, formed by the circumcentres of the four quadrilaterals in part (1), is similar to *ABCD*.

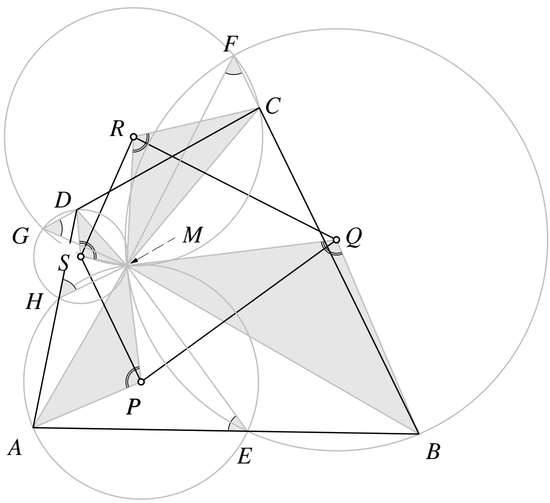

FIGURE 1

To prove part (3), we note that *P*, *Q*, *R* and *S* are the centres of the four circles, and *AM*, *BM*, *CM* and *DM* are chords. Therefore, from the 'angle at the centre' theorem, we have $\angle APM = 2\angle AEM$, $\angle BQM = 2\angle BFM$, and so on. It follows from the 'equiangular' condition that $\angle APM = \angle BQM = \angle CRM = \angle DSM$.

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Hence the triangles ΔAPM, ΔBQM, ΔCRM and ΔDSM are similar, because they are isosceles and have equal 'vertical' angles. As a result, the spiral similarity with centre M, angle $\angle PMA$, and scale factor $\frac{PM}{MA}$ maps the quadrilateral $ABCD$ to $PQRS$.

Remark

The result still holds when some of the intersection points of the 'equiangular' lines and the sides of $ABCD$ lie outside the quadrilateral, such as the points F and G in the figure above. The proof of part (1) can be modified, where necessary, to use 'angles in the same segment' rather than 'exterior angles of a cyclic quadrilateral'.

On 'Cubics whose vertical translates factorise' [1]: Andrew Jobbings writes: Consider the cubic curve whose equation is

$$y = f(x) \equiv ax^3 + bx^2 + cx + d, \qquad (1)$$

where a, b, c and d are integers.

If the curve has a rational root, then some appropriate enlargement will have an integer root. Such an enlargement corresponds to a linear transformation of the coordinates (x, y). Given this, henceforward I shall ignore integer roots and concentrate on whether the roots are rational.

The point of inflexion of the cubic curve given by (1) is a centre of rotational symmetry of order two. Suppose the point of inflexion has coordinates (ξ, η). Then ξ and η are rational, because they are given by the equations $6a\xi + 2b = 0$ (from $y'' = 0$) and $\eta = a\xi^3 + b\xi^2 + c\xi + d$.

Proposition 1: Let α, β and γ be the roots of $f(x) = 0$. Then the roots of $f(x) - 2\eta = 0$ are $2\xi - \alpha$, $2\xi - \beta$ and $2\xi - \gamma$.

Proof: Consider the curve $y = f(x)$ shown in Figure 1, where the dashed line is the image of the x-axis under a rotation through 180° about (ξ, η).

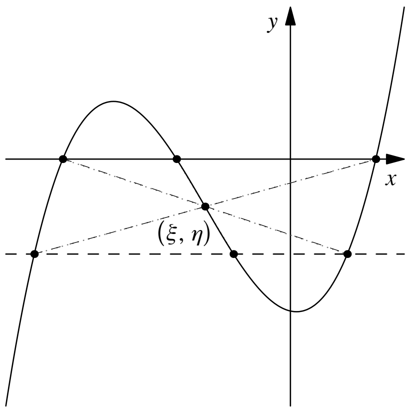

FIGURE 1

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Now the curve $y = f(x) - 2\eta$ is obtained by translating the x-axis to the dashed line.

But from the rotational symmetry, the roots can be placed in pairs, with (ξ, η) as the midpoint of each pair. So any root α of $y = f(x)$ corresponds to a root $2\xi - \alpha$ of $y = f(x) - 2\eta$.

Corollary 1: The roots of $f(x) = 0$ are rational if, and only if, the roots of $f(x) - 2\eta = 0$ are rational.

Examples

We apply the above to the illustrative examples in the Note.

1. The curve $y = 2x^3 + 9x^2 - 2x - 24$ has inflexion point $\left(-\frac{3}{2}, -\frac{15}{2}\right)$, leading to a translation by 15. The original roots are $\frac{3}{2}$, -2 and -4; the translated roots are $-3 - \frac{3}{2} = -\frac{9}{2}$, $-3 - (-2) = -1$ and $-3 - (-4) = 1$.
2. The curve $y = x^3 - 3x^2 + 2x$ has inflexion point $(1, 0)$. So $\eta = 0$ and the two curves are the same.
3. (a) The curve $y = 12x^3 - 56x^2 + 77x - 30$ has inflexion point $\left(\frac{7}{3}, -\frac{136}{243}\right)$, leading to a translation by $\frac{272}{243}$. The roots $\frac{3}{2}, \frac{5}{2}, \frac{2}{3}$ become $\frac{19}{6}, \frac{13}{6}$, 4.
   (b) The curve $y = 12x^3 - 56x^2 + 77x - 26$ has inflexion point $\left(\frac{7}{3}, \frac{836}{243}\right)$ leading to a translation by $\frac{1672}{243}$. The roots $2, \frac{1}{2}, \frac{13}{6}$ become $\frac{8}{3}, \frac{25}{6}, \frac{5}{2}$.
4. (a) The curve $y = x^3 - 21x^2 + 14x + 120$ has inflexion point $(7, -468)$, leading to a translation by 936. The roots -2, 3, 20 become 16, 11, -6.
   (b) The curve $y = x^3 - 21x^2 + 14x + 556$ has inflexion point $(7, -132)$, leading to a translation by 264. The roots -4, 6, 19 become 18, 8, -5.
5. (a) The curve $y = x^3 - 6x^2 + 5x + 3$ has inflexion point $(2, -3)$, leading to a translation by 6. The curves have irrational roots.
   (b) The curve $y = x^3 - 6x^2 + 5x$ has inflexion point $(2, -6)$, leading to a translation by 12. The roots 0, 1, 5 become 4, 3, -1.

Reference

1. Nick Lord, Cubics whose vertical translates factorise, *Math. Gaz.* **98** (July 2014) pp. 349-350.

doi:10.1017/mag.2015.57