Ergod. Th. & Dynam. Sys. (2021), **41**, 2349–2368
doi:10.1017/etds.2020.45

A new characterization of the Haagerup property by actions on infinite measure spaces

THIEBOUT DELABIE†, PAUL JOLISSAINT‡ and ALEXANDRE ZUMBRUNNEN‡

† *Université Paris-Sud, Faculté des Sciences d'Orsay, Département de Mathématiques, Bâtiment 307, F-91405 Orsay Cedex, France*
(e-mail: thiebout.delabie@gmail.com)

‡ *Université de Neuchâtel, Institut de Mathématiques, E.-Argand 11, 2000 Neuchâtel, Switzerland*
(e-mail: paul.jolissaint@unine.ch, alexandre.zumbrunnen@unine.ch)

(*Received* 3 *May* 2019 *and accepted in revised form* 17 *April* 2020)

Abstract. The aim of the article is to provide a characterization of the Haagerup property for locally compact, second countable groups in terms of actions on σ-finite measure spaces. It is inspired by the very first definition of amenability, namely the existence of an invariant mean on the algebra of essentially bounded, measurable functions on the group.

Key words: locally compact groups, unitary representations, Haagerup property
2020 Mathematics Subject Classification: 22D10, 22D40 (Primary); 28D05 (Secondary)

1. *Introduction*

Throughout this article, G denotes a locally compact, second countable group (lcsc group for short); we assume furthermore that it is non-compact, because the compact case is not relevant to the Haagerup property [**5**], which is the central theme of these notes.

The latter property is often interpreted as a weak form of amenability or as a strong negation of Kazhdan's property (T), depending on the context.

A way to see that it is a weak form of amenability is to consider the following characterization: recall that G has the Haagerup property if and only if there exists a sequence of normalized, positive-definite functions $(\varphi_n)_{n\geq 1}$ on G such that φ_n converges to the constant function 1 uniformly on compact subsets of G as $n \to \infty$, and each $\varphi_n \in C_0(G)$, i.e. $\varphi_n \to 0$ at infinity. In turn, G is amenable if and only if each φ_n can be chosen with compact support.

Nowadays, there are several characterizations of the Haagerup property: apart from the ones presented in the monograph [**5**], we can mention for instance the ones involving actions on median spaces or on measured walls as in [**4**]. In fact, the characterization presented here has no direct relationship with the latter ones; it rests rather on strongly mixing actions on probability spaces: see Theorem 1.6.

The present article has its origin in the characterization of amenability given by its original definition: the lcsc group G is amenable if and only if the algebra $L^\infty(G)$ has a G-invariant mean with respect to the action of G on itself by left translation.

Observe that such an action is a special case of proper actions that we recall now. Let Ω be a locally compact space. Suppose that the lcsc group G acts continuously on Ω. Then the action is *proper* if for all compact subsets $K, L \subset \Omega$, the set

$$\{g \in G : gK \cap L \neq \varnothing\}$$

is relatively compact in G. Then a question arises: which lcsc groups admit proper actions on locally compact spaces Ω equipped with an invariant measure and such that $L^\infty(\Omega)$ has an invariant mean? Here is the answer.

PROPOSITION 1.1. *Let G be a lcsc group and let Ω be a locally compact space on which G acts properly and which admits a G-invariant, regular Borel measure μ. If $L^\infty(\Omega)$ has an invariant mean, then G is amenable.*

Indeed, it is a well-known fact that the existence of an invariant mean on $L^\infty(\Omega)$ is equivalent to the fact that, for every compact set $L \subset G$ and every $\varepsilon > 0$, there exists a continuous function ξ on Ω with compact support such that $\|\xi\|_2 = 1$ and

$$\sup_{g \in L} |\langle \pi_\Omega(g)\xi | \xi\rangle - 1| < \varepsilon,$$

where π_Ω denotes the natural unitary representation of G on $L^2(\Omega)$ associated to the action of G on Ω. As the latter is proper and ξ has compact support, the coefficient function $\varphi = \langle \pi_\Omega(\cdot)\xi | \xi\rangle$ has compact support too and thus the constant function 1 is a uniform limit on compact sets of compactly supported positive-definite functions, which means that G is amenable.

Thus, G is amenable if and only if it admits a proper, measure-preserving action on some locally compact space Ω so that $L^\infty(\Omega)$ admits a G-invariant mean.

As properness of actions is too strong to characterize the Haagerup property, we consider the setting of measure-preserving actions on measure spaces equipped with invariant measures. It turns out that the following property is well adapted to our situation.

Definition 1.2. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space on which a lcsc group G acts by Borel automorphisms which preserve μ. Then we say that the corresponding dynamical system $(\Omega, \mathcal{B}, \mu, G)$ is a *C_0-dynamical system* if, for all $A, B \in \mathcal{B}$ such that $0 \leq \mu(A), \mu(B) < \infty$, one has

$$\lim_{g \to \infty} \mu(gA \cap B) = 0.$$

Remark 1.3. Let $(\Omega, \mathcal{B}, \mu, G)$ be a C_0-dynamical system. Then every G-invariant set $A \in \mathcal{B}$ such that $\mu(A) < \infty$ is automatically of measure zero. In particular, if μ is finite, then it is equal to zero. Moreover, the action of G is not ergodic in general. Indeed, let $\mathbb{Z}$ act by translations on $\mathbb{R}$ equipped with Lebesgue measure μ. Then

$$A := \{x + k : x \in (0, 1/2),\ k \in \mathbb{Z}\} = \bigsqcup_{k \in \mathbb{Z}} (k, k + 1/2)$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

is $\mathbb{Z}$-invariant and $\mu(A) = \mu(A^c) = \infty$. The action is C_0 since it is proper, as it is the restriction to $\mathbb{Z}$ of the action of $\mathbb{R}$ on itself.

Remark 1.4. Let $(\Omega, \mathcal{B}, \mu, G)$ be a (measure-preserving) dynamical system; then it is a C_0-dynamical system if and only if the permutation representation π_Ω of G on $L^2(\Omega)$ is a C_0-representation. Hence, we infer that if G admits a C_0-dynamical system $(\Omega, \mathcal{B}, \mu, G)$ such that $L^\infty(\Omega)$ has a G-invariant mean, then G has the Haagerup property.

Then the goal of the present article is to prove the converse.

THEOREM 1.5. *Let G be a lcsc group which has the Haagerup property. Then there exists a C_0-dynamical system $(\Omega, \mathcal{B}, \mu, G)$ such that $L^\infty(\Omega)$ has a G-invariant mean. More precisely:*

(1) *the measure μ is σ-finite, G-invariant, and the Hilbert space $L^2(\Omega, \mu)$ is separable;*

(2) *for all measurable sets $A, B \in \mathcal{B}$ such that $0 \leq \mu(A), \mu(B) < \infty$, we have*

$$\lim_{g\to\infty} \mu(gA \cap B) = 0;$$

in other words, π_Ω is a C_0-representation;

(3) *there exists a sequence of unit vectors $(\xi_n) \subset L^2(\Omega, \mu)$ such that $\xi_n \geq 0$ for every n and, for every compact set $K \subset G$, one has*

$$\lim_{n\to\infty} \sup_{g\in K} \langle \pi_\Omega(g)\xi_n | \xi_n \rangle = 1;$$

in other words, the dynamical system $(\Omega, \mathcal{B}, \mu, G)$ has an invariant mean.

The proof of Theorem 1.5 will occupy the rest of the article. It relies on the characterization of the Haagerup property stated in [**5**, Theorem 2.2.2] that we recall now.

THEOREM 1.6. [**5**, Theorem 2.2.2] *Let G be a lcsc group. Then it has the Haagerup property if and only if there exists a standard probability space $(S, \mathcal{B}_S, \nu)$ on which G acts by Borel automorphisms which preserve ν, and $(S, \mathcal{B}_S, \nu)$ has the following additional two properties:*

(a) *the action of G on S is strongly mixing, which means that for all $A, B \in \mathcal{B}_S$,*

$$\lim_{g\to\infty} \nu(gA \cap B) = \nu(A)\nu(B);$$

(b) *the action admits a non-trivial asymptotically invariant sequence: there exists a sequence $(A_n)_{n\geq 1} \subset \mathcal{B}_S$ such that $\nu(A_n) = 1/2$ for every n and such that, for every compact set $K \subset G$,*

$$\lim_{n\to\infty} \sup_{g\in K} \nu(gA_n \triangle A_n) = 0.$$

Furthermore, we assume that S is a compact metric space on which G acts continuously, and ν has support S, according to (the proof of) [**1**, Lemma 1.3] that we recall for the reader's convenience.

LEMMA 1.7. [**1**, Lemma 1.3] *Let X be a standard Borel G-space with a G-invariant probability measure μ. Then there exists a compact metric space Y, on which G acts continuously, and a G-invariant probability measure ν, whose support is Y, such that $L^2(X, \mu)$ and $L^2(Y, \nu)$ are G-isomorphic.*

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

We end this introduction with a brief sketch of the proof of Theorem 1.5. It uses elementary measure theory (see [**3**] on this subject) but it is quite long and involved.

We start with the infinite-product space $X = \prod_{n \geq 1} S$, where S satisfies all properties of Theorem 1.6 and Lemma 1.7. We equip X with the diagonal action of G and with a suitable family $\mathcal{F}$ of subsets containing all sets of the form $B = \prod_n B_n$ such that the infinite product $\prod_n 2\nu(B_n)$ converges, where $B_n \in \mathcal{B}_S$ for every n. We construct a measure μ on $\sigma(\mathcal{F})$, the σ-algebra generated by $\mathcal{F}$, that satisfies $\mu(B) = \prod_n 2\nu(B_n)$ for every B as above.

One of the reasons of the choices of such sets and measure μ is that one can extract from the family (A_n) in condition (b) of Theorem 1.6 a sequence $(X_m)_m \subset \mathcal{F}$ so that the associated sequence of unit vectors $\xi_m = \chi_{X_m}$ satisfies condition (3) of Theorem 1.5. They are unit vectors since $\nu(A_n) = 1/2$ for every n.

In order to prove that condition (2) holds, consider two subsets $A = \prod_n A_n$ and $B = \prod_n B_n$ such that the infinite products $\prod_n 2\nu(A_n)$ and $\prod_n 2\nu(B_n)$ converge. Given $\varepsilon > 0$, choose first N large enough so that

$$\tfrac{1}{2} - \varepsilon < \nu(A_n), \quad \nu(B_n) < \tfrac{1}{2} + \varepsilon$$

for every $n \geq N$. Then, using the strong mixing property of the action of G on S (condition (a) in Theorem 1.6), we choose a suitable positive number $\varepsilon' > 0$, an integer $m > 0$, and a compact set $K \subset G$ such that, for every $g \notin K$, $\nu(gA_n \cap B_n) \leq \nu(A_n)\nu(B_n) + \varepsilon'$ for all $N \leq n \leq N + m$. Hence, we get

$$\begin{aligned}\mu(gA \cap B) &\leq \prod_{n=1}^{N-1} 2\nu(A_n) \cdot \prod_{n > N+m} 2\nu(A_n) \cdot \prod_{n=N}^{N+m} 2(\nu(A_n)\nu(B_n) + \varepsilon') \\ &= \mu(A) \prod_{n=N}^{N+m} \frac{2\nu(A_n)\nu(B_n) + 2\varepsilon'}{2\nu(A_n)} < \varepsilon\end{aligned}$$

since the quotients $(2\nu(A_n)\nu(B_n) + 2\varepsilon')/2\nu(A_n)$ belong to some interval $(0, \delta)$ for a convenient value of $\delta < 1$ such that $\delta^{m+1} < \varepsilon/\mu(A)$.

It turns out that the σ-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$ is too large, so that the dynamical system $(X, \sigma(\mathcal{F}), \mu, G)$ is not σ-finite, and, as observed by A. Calderi and A. Valette (cf. Remark 2.9), the associated representation π_X on $L^2(X, \mu)$ is not continuous.

Thus, we need to divide the proof of Theorem 1.5 into two parts: in the first one, we prove that $(X, \sigma(\mathcal{F}), \mu, G)$ is a C_0-dynamical system as stated in Definition 1.2 on the one hand, and we construct a sequence of unit vectors that satisfy condition (3) in Theorem 1.5 on the other hand. In the case where G is discrete, it is very simple to restrict our dynamical system to a σ-finite one, so that the proof is complete with the former additional assumption. All this is contained in §2.

In the last part of the proof, which is the subject of §3, we define a sub-σ-algebra $\sigma(\mathcal{F}_c)$ of $\sigma(\mathcal{F})$ and a measure μ_c so that, for every $A \in \sigma(\mathcal{F}_c)$, $\mu_c(A) < \infty$, we have $\lim_{g \to e} \mu_c(gA \,\triangle\, A) = 0$. This implies the continuity of the permutation representation $\pi_X : G \to U(L^2(X, \sigma(\mathcal{F}_c), \mu_c))$, and finally the latter property is used to prove that we can restrict our dynamical system to get a σ-finite measure.

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

2. *Proof of Theorem 1.5, Part 1*

For the rest of the article, G denotes a (non-compact) lcsc group with the Haagerup property. According to Theorem 1.6 and Lemma 1.7, let $(S, \mathcal{B}_S, \nu)$ be a compact metric space equipped with a probability measure ν whose support is S, and G acts continuously on S and preserves ν and which satisfies conditions (a) and (b) of Theorem 1.6.

Then put $X = \prod_{n\geq 1} S = \{(s_n)_{n\geq 1} : s_n \in S\ \forall\, n\}$. If $\mathcal{S}$ is any non-empty family of subsets of X, we denote by $\sigma(\mathcal{S})$ the σ-algebra generated by $\mathcal{S}$.

Here is the starting point of our construction.

Definition 2.1. Let X be as above.

(1) We denote by $\mathcal{F}_0$ the family of subsets of X of the form $A = \prod_{n\geq 1} A_n$, where $A_n \in \mathcal{B}_S$ for all n such that the infinite product $\prod_{n=1}^{\infty} 2\nu(A_n)$ exists, i.e. the sequence of partial products $(\prod_{n=1}^{N} 2\nu(A_n))_{N\geq 1}$ converges to some limit in $[0, \infty)$†. We also set

$$\mathcal{F}_{0,+} = \left\{B = \prod_n B_n \in \mathcal{F}_0 : \nu(B_n) > 0\ \forall n\right\}.$$

(2) We define the following sequence $(\mathcal{F}_n)_{n\geq 1}$ of collections of subsets of X by induction: for $n \geq 1$, set $\mathcal{F}_n = \{B \setminus A : A, B \in \mathcal{F}_{n-1}\}$. Finally, we set

$$\mathcal{F} := \bigcup_{n\geq 0} \mathcal{F}_n.$$

Observe that $\varnothing \in \mathcal{F}_0$ and hence also that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for every n.

LEMMA 2.2. *For all $A, B \in \mathcal{F}_0$, one has $A \cap B \in \mathcal{F}_0$.*

Proof. Let $A, B \in \mathcal{F}_0$. Let us write $A = \prod_n A_n$ and $B = \prod_n B_n$ as above. Then $\nu(A_n \cap B_n) \leq \min(\nu(A_n), \nu(B_n))$ for every n. If there exists an integer n such that $\min(\nu(A_n), \nu(B_n)) = 0$, then the product $\prod_{n=1}^{\infty} 2\nu(A_n \cap B_n)$ converges trivially and $A \cap B \in \mathcal{F}_0$. Suppose then that $\nu(A_n) > 0$ for every n and consider the product of conditional probabilities

$$a_N = \prod_{n=1}^{N} \nu(B_n | A_n) = \prod_{n=1}^{N} \frac{\nu(A_n \cap B_n)}{\nu(A_n)}.$$

As $0 \leq \nu(B_n|A_n) \leq 1$ for every n, one has $0 \leq a_{N+1} \leq a_N \leq 1$ for every N, and the bounded, decreasing sequence $(a_N)_{N\geq 1}$ converges, say, to $a \in [0, 1]$. Then the sequence

$$\prod_{n=1}^{N} 2\nu(A_n \cap B_n) = a_N \cdot \prod_{n=1}^{N} 2\nu(A_n)$$

converges to $a \cdot \prod_{n=1}^{\infty} 2\nu(A_n)$. □

LEMMA 2.3. *The family $\mathcal{F}$ is a semiring of subsets of X, i.e.:*

(i) *if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;*

(ii) *if $A, B \in \mathcal{F}$, then $B \setminus A \in \mathcal{F}$.*

† Recall that an infinite product $\prod_n u_n$ of complex numbers *converges* if $\lim_{N\to\infty} \prod_{n=1}^{N} u_n$ exists *and* is different from 0, and it *converges trivially* if the limit equals 0. Thus, we require that the infinite product $\prod_{n=1}^{\infty} 2\nu(A_n)$ either converges or converges trivially.

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

In particular, the set of all finite, disjoint unions of elements of $\mathcal{F}$ is a ring of subsets of X. It is the ring generated by $\mathcal{F}$ and is denoted by $\mathcal{R}(\mathcal{F})$. Moreover:

(iii) *for every $A \in \mathcal{F}$, there exists $B \in \mathcal{F}_0$ such that $A \subset B$.*

Proof. (i) We prove by induction on $n \geq 0$ that for all $A, B \in \mathcal{F}_n$, one has $A \cap B \in \mathcal{F}$. The claim is true for $n = 0$ by Lemma 2.2. Thus, let us assume that the claim is true for $n \geq 0$, and let $A, B \in \mathcal{F}_{n+1}$. Then there exist $A_1, A_2, B_1, B_2 \in \mathcal{F}_n$ such that $A = A_1 \setminus A_2$ and $B = B_1 \setminus B_2$. Then, by the induction hypothesis, there exists $m \geq n$ such that $A_1 \cap B_1 \in \mathcal{F}_m$. As

$$A \cap B = (A_1 \cap A_2^c) \cap (B_1 \cap B_2^c) = ((A_1 \cap B_1) \setminus A_2) \setminus B_2,$$

this shows that $A \cap B \in \mathcal{F}_{m+2} \subset \mathcal{F}$ since $(A_1 \cap B_1) \setminus A_2 \in F_{m+1}$.

Assertion (ii) follows readily from the definitions, and (iii) is established by induction on n. □

The next step consists in defining a suitable measure μ on the σ-algebra $\sigma(\mathcal{F}) = \sigma(\mathcal{F}_0)$ generated by $\mathcal{F}$ (or equivalently by $\mathcal{F}_0$).

In order to do that, we associate to every element $B = \prod_n B_n \in \mathcal{F}_{0,+}$ the probability measure $\mathbb{P}_B$ on the σ-algebra $\sigma(\mathcal{C})$ generated by the family $\mathcal{C}$ of all cylinder sets in $X = \prod_n S$. We observe for future use that $\mathcal{F}_0 \subset \sigma(\mathcal{C})$ and hence that $\sigma(\mathcal{F}) \subset \sigma(\mathcal{C})$ as well.

Then $\mathbb{P}_B$ is the product probability measure $\bigotimes_n \nu_{n,B}$, where $\nu_{n,B}$ is the probability measure on $\mathcal{B}_S$ given by

$$\nu_{n,B}(E) := \frac{\nu(E \cap B_n)}{\nu(B_n)} = \nu(E|B_n)$$

for every $E \in \mathcal{B}_S$ and for every n. As is well known, if $C = \prod_n C_n$ with $C_n \subset S$ Borel for every n, then $\mathbb{P}_B(C) = \prod_{n=1}^{\infty} \nu_{n,B}(C_n)$ because $C = \bigcap_N C^{(N)}$, where $C^{(N)} = C_1 \times C_2 \times \cdots \times C_N \times S \times S \times \cdots \in \mathcal{C}$ and $\mathbb{P}_B(C^{(N)}) = \prod_{n=1}^{N} \nu_{n,B}(C_n)$ for all N.

We define now a premeasure μ on $\mathcal{F}$.

Definition 2.4. For $A \in \mathcal{F}_0$, $A = \prod_n A_n$, set $\mu(A) := \prod_{n=1}^{\infty} 2\nu(A_n)$. For $A \in \bigcup_{n\geq 1} \mathcal{F}_n$, let $B \in \mathcal{F}_0$ be such that $A \subset B$; then set

$$\mu(A) = \begin{cases} \mathbb{P}_B(A)\mu(B) & \text{if } B \in \mathcal{F}_{0,+}, \\ 0 & \text{if } \mu(B) = 0. \end{cases}$$

We need to check that μ is well defined.

LEMMA 2.5. *Let $A \in \mathcal{F}$.*

(i) *If there exists $B \in \mathcal{F}_0$ such that $A \subset B$ and $\mu(B) = 0$, then $\mathbb{P}_C(A)\mu(C) = 0$ for every $C \in \mathcal{F}_{0,+}$ such that $A \subset C$.*

(ii) *If $B, C \in \mathcal{F}_{0,+}$ are such that $A \subset B \cap C$, one has*

$$\mathbb{P}_B(A)\mu(B) = \mathbb{P}_C(A)\mu(C). \tag{2.1}$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

Proof. (i) If B and C are as above, then

$$\begin{aligned}0 \leq \mathbb{P}_C(A)\mu(C) &= \mathbb{P}_C(A\cap B)\mu(C)\\ &\leq \mathbb{P}_C(B\cap C)\mu(C)\\ &= \prod_{n=1}^{\infty}\frac{\nu(B_n\cap C_n)}{\nu(C_n)}\cdot\prod_{n=1}^{\infty}2\nu(C_n)\\ &= \lim_{N\to\infty}\prod_{n=1}^{N}\frac{2\nu(B_n\cap C_n)}{2\nu(C_n)}\cdot 2\nu(C_n)\\ &= \prod_{n=1}^{\infty}2\nu(B_n\cap C_n) \leq \prod_{n=1}^{\infty}2\nu(B_n) = 0.\end{aligned}$$

(ii) Observe first that if $B, C \in \mathcal{F}_{0,+}$ are such that $\mu(B) = \mu(C) = 0$, then equality (2.1) holds trivially. By (i), it also holds if $\mu(B)\mu(C) = 0$. Thus, it remains to prove that (2.1) holds when $A \subset B \cap C$ with $B, C \in \mathcal{F}_{0,+}$ and $\mu(B)\mu(C) > 0$.

We assume first that $A = \prod_n A_n \in \mathcal{F}_0$; then

$$\begin{aligned}\mathbb{P}_B(A)\mu(B) &= \prod_{n=1}^{\infty}\frac{\nu(A_n\cap B_n)}{\nu(B_n)}\cdot\prod_{n=1}^{\infty}2\nu(B_n)\\ &= \lim_{N\to\infty}\prod_{n=1}^{N}\frac{2\nu(A_n\cap B_n)}{2\nu(B_n)}2\nu(B_n)\\ &= \prod_{n=1}^{\infty}2\nu(\underbrace{A_n\cap B_n}_{=A_n})\\ &= \mu(A).\end{aligned}$$

Similarly, we get $\mathbb{P}_C(A)\mu(C) = \mu(A)$.

In the last part of the proof, we fix $B, C \in \mathcal{F}_{0,+}$ such that $\mu(B)\mu(C) > 0$, and we define two measures $\mu^{(B)}$ and $\mu^{(C)}$ on the σ-algebra $\sigma(\mathcal{C})$ (which contains $\sigma(\mathcal{F})$) by

$$\mu^{(B)}(E) := \mathbb{P}_B(B\cap E)\mu(B) \quad \text{for all } E \in \sigma(\mathcal{C})$$

and similarly for $\mu^{(C)}$. Then $\mu^{(B)}(X) = \mathbb{P}_B(B)\mu(B) = \mu(B) < \infty$ (respectively $\mu^{(C)}(X) = \mu(C)$), so that they are both finite measures on $\sigma(\mathcal{C})$.

Set $\mathcal{A} := \{A \in \sigma(\mathcal{F}) : \mu^{(B)}(A\cap B\cap C) = \mu^{(C)}(A\cap B\cap C)\}$. Then the second part shows that $\mathcal{F}_0 \subset \mathcal{A}$ and, in particular, since $B\cap C \in \mathcal{F}_0$, one has that $\mu^{(B)}(B\cap C) = \mu^{(C)}(B\cap C)$, which implies that $X \in \mathcal{A}$.

Let us check that if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$: indeed, as $B\cap C = (A^c\cap B\cap C)\sqcup(A\cap B\cap C)$,

$$\begin{aligned}\mu^{(B)}(A^c\cap B\cap C) &= \mu^{(B)}(B\cap C) - \mu^{(B)}(A\cap B\cap C)\\ &= \mu^{(C)}(B\cap C) - \mu^{(C)}(A\cap B\cap C)\\ &= \mu^{(C)}(A^c\cap B\cap C).\end{aligned}$$

Finally, it is straightforward to check that $\mathcal{A}$ is a monotone class. It implies that it is a σ-algebra which contains $\mathcal{F}_0$ and hence $\mathcal{A} = \sigma(\mathcal{F})$. □

As a consequence of Lemma 2.5 and Caratheodory's theorem, we have the following result.

PROPOSITION 2.6. *The premeasure $\mu : \mathcal{F} \to \mathbb{R}_+$ is σ-additive and thus it extends to a measure still denoted by μ on the σ-algebra $\sigma(\mathcal{F})$. However, μ is not σ-finite; in particular, it is infinite.*

Proof. Let $(A^{(k)})_{k\geq 1} \subset \mathcal{F}$ be a sequence of pairwise-disjoint sets such that $A := \bigsqcup_{k\geq 1} A^{(k)}$ still belongs to $\mathcal{F}$. Choose $B \in \mathcal{F}_0$ such that $A \subset B$. Then, as $A^{(k)} \subset B$ for every k, if $\mu(B) = 0$, one has, by Lemma 2.5, $\mu(A^{(k)}) = \mu(A) = 0$ for every k. If $\mu(B) > 0$, one has

$$\mu(A) = \mathbb{P}_B(A)\mu(B) = \sum_k \mathbb{P}_B(A^{(k)})\mu(B) = \sum_k \mu(A^{(k)})$$

since $\mathbb{P}_B$ is σ-additive. Thus, by Caratheodory's theorem (see for instance [**3**, Theorem 11.1]), μ extends to a measure still denoted by μ on $\sigma(\mathcal{F})$.

In order to prove that μ is not σ-finite, let us choose $A \in \mathcal{B}_S$ such that $\nu(A) = 1/2$, and set $A_0 = A$ and $A_1 = A^c$. Next, for every sequence $\varepsilon := (\varepsilon_n)_{n\geq 1} \subset \{0, 1\}^{\mathbb{N}^*} =: \mathcal{E}$, set $A_\varepsilon := \prod_n A_{\varepsilon_n} \in \mathcal{F}_0$. Then $\mu(A_\varepsilon) = 1$ for every $\varepsilon \in \mathcal{E}$ and $A_\varepsilon \cap A_{\varepsilon'} = \varnothing$ for all $\varepsilon \neq \varepsilon'$. If μ was σ-finite, then $\mathcal{E}$ would be countable: indeed, there would exist an increasing sequence $X_1 \subset X_2 \subset \cdots \subset X$ such that $\bigcup_k X_k = X$ and $\mu(X_k) < \infty$ for every k. Then, for $k \geq 2$, set $\mathcal{E}_k = \{\varepsilon \in \mathcal{E} : \mu(X_k \cap A_\varepsilon) \geq 1/2\}$. Then $|\mathcal{E}_k| \leq 2\mu(X_k)$ is finite. As $\lim_{k\to\infty} \mu(X_k \cap A_\varepsilon) = \mu(A_\varepsilon) = 1$ for every ε, it follows that $\mathcal{E} = \bigcup_k \mathcal{E}_k$ would be countable, which is not the case. □

Now we consider the diagonal action of G on $(X, \sigma(\mathcal{F}), \mu)$, i.e.

$$g \cdot (s_n)_{n\geq 1} = (gs_n)_{n\geq 1}$$

for $g \in G$ and $(s_n)_{n\geq 1} \in X = \prod_n S$.

LEMMA 2.7. *The action of G on X defined above is measurable. In particular, G acts by measurable automorphisms on $(X, \sigma(\mathcal{F}), \mu)$ and it preserves μ.*

Proof. Let $\alpha : G \times X \to X$ be defined by $\alpha(g, (x_n)_{n\geq 1}) = (gx_n)_{n\geq 1}$. To prove that α is measurable, it is sufficient to see that the preimage of any element of $\mathcal{F}_0$ is measurable.

Let $A \in \mathcal{F}_0$ with $A = \prod_{n\geq 1} A_n$. For $m \geq 1$, we define a permutation function $\beta_m : G \times X \to G \times X$ by

$$(g, (x_n)_{n\geq 1}) \mapsto (g, (x_m, x_1, x_2, \ldots, x_{m-1}, x_{m+1}, x_{m+2}, \ldots),$$

which is clearly measurable.

Let us consider now $\alpha^{-1}(A)$; we have

$$\begin{aligned}\alpha^{-1}(A) &= \{(g, (x_n)_{n\geq 1}) : \alpha((g, (x_n)_{n\geq 1})) \in A\}\\ &= \{(g, (x_n)_{n\geq 1}) : (gx_n)_{n\geq 1} \in A\}\\ &= \bigcap_{k\geq 1} \{(g, (x_n)_{n\geq 1}) : gx_k \in A_k\}.\end{aligned}$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

For fixed k, since the action $\gamma : G \times S \to S$ given by $\gamma(g, s) = gs$ is continuous and hence measurable, we have

$$\{(g, (x_n)_{n\geq 1}) : \ gx_k \in A_k\} = \beta_k^{-1}(\gamma^{-1}(A_k) \times S \times S \times \cdots) \in \mathcal{B}(G) \times \sigma(\mathcal{F}).$$

Whence

$$\alpha^{-1}(A) = \bigcap_{n\geq 1} \beta_n^{-1}\bigg(\gamma^{-1}(A_n) \times \prod_{k\geq 2} S\bigg) \in \mathcal{B}(G) \times \sigma(\mathcal{F}).$$

In particular, if $B \in \sigma(\mathcal{F})$ and for $g \in G$ fixed, then

$$gB = (g^{-1})^{-1}B \in \sigma(\mathcal{F}).$$

This ends the proof of the measurability of the action of G.

Finally, if $A = \prod_n A_n \in \mathcal{F}_0$, then $gA = \prod_n gA_n$, and the equality $\mu(gA) = \mu(A)$ holds since the action of G is diagonal and ν is preserved by the action of G on S.

If $A \in \mathcal{F}$, let $B \in \mathcal{F}_{0,+}$ be such that $A \subset B$, so that $\mu(A) = \mathbb{P}_B(A)\mu(B)$, according to Definition 2.4. Then $gA \subset gB$, so that

$$\mu(gA) = \mathbb{P}_{gB}(gA)\mu(gB) = \mathbb{P}_B(A)\mu(B) = \mu(A)$$

for the following reason: for every cylinder set $C = C_1 \times \cdots \times C_N \times S \times \cdots \in \sigma(\mathcal{C})$, one has

$$\mathbb{P}_{gB}(gC) = \prod_{n=1}^{N} \frac{\nu(gC_n \cap gB_n)}{\nu(gB_n)} = \mathbb{P}_B(C).$$

This equality holds first for every element of the algebra generated by cylinder sets and hence for every element $C \in \sigma(\mathcal{C})$ by uniqueness of probability measures which coincide on given algebras of sets. In particular, one has $\mu(gA) = \mu(A)$ for every $A \in \mathcal{F}$ and hence in the algebra $\mathcal{R}(\mathcal{F})$.

Next, the construction of the extension of μ to $\sigma(\mathcal{F})$ is given by

$$\mu(E) = \inf\bigg\{\sum_{m\geq 1} \mu(B_m) : (B_m)_{m\geq 1} \subset \mathcal{R}(\mathcal{F}),\ E \subset \bigcup_m B_m\bigg\}$$

for every $E \in \sigma(\mathcal{F})$. Thus, if $g \in G$ is fixed, one has

$$\mu(gE) = \inf\bigg\{\sum_{m\geq 1} \mu(B_m) : (B_m)_{m\geq 1} \subset \mathcal{R}(\mathcal{F}),\ E \subset \bigcup_m g^{-1}B_m\bigg\} = \mu(E)$$

because the ring $\mathcal{R}(\mathcal{F})$ is G-invariant and every countable covering of $E \in \sigma(\mathcal{F})$ in the definition of $\mu(E)$ above can be taken of the form

$$E \subset \bigcup_m g^{-1}B_m$$

with $(B_m) \subset \mathcal{R}(\mathcal{F})$ by G-invariance of $\mathcal{R}(\mathcal{F})$. □

We are now ready to prove the first part of Theorem 1.5 in the general case, namely the existence of a C_0-dynamical system with almost invariant vectors, and to finish the proof in the case where G is discrete (hence countable).

PROPOSITION 2.8. *The dynamical system* $(X, \sigma(\mathcal{F}), \mu, G)$ *is a* C_0*-dynamical system, namely, for all* $A, B \in \sigma(\mathcal{F})$ *such that* $0 < \mu(A), \mu(B) < \infty$, *one has*

$$\lim_{g\to\infty} \mu(gA \cap B) = 0, \tag{2.2}$$

and there exists a sequence of unit vectors $(\xi_n) \in L^2(X, \sigma(\mathcal{F}), \mu)$ *such that* $\xi_n \geq 0$ *for every n and, for every compact set* $K \subset G$, *one has*

$$\lim_{n\to\infty} \sup_{g\in K} \langle \pi_X(g)\xi_n | \xi_n\rangle = 1. \tag{2.3}$$

Moreover, if G is discrete, there exists a G-invariant subset $\Omega \in \sigma(\mathcal{F})$ *such that the restriction of* μ *to* Ω *is* σ*-finite and the corresponding Hilbert space* $L^2(\Omega, \mu)$ *contains the sequence* (ξ_n) *and is separable.*

Proof. Assume first that $A, B \in \mathcal{F}_0$, and write $A = \prod_n A_n$ and $B = \prod_n B_n$, so that

$$\mu(A) = \prod_{n\geq 1} 2\nu(A_n) \quad \text{and} \quad \mu(B) = \prod_{n\geq 1} 2\nu(B_n).$$

Let $\varepsilon > 0$ be fixed and take $\varepsilon' > 0$ small enough in order that $\delta := 1/2 + \varepsilon' + \varepsilon'/(1/2 - \varepsilon') < 1$.

Since $0 < \mu(A), \mu(B) < \infty$, there exists N large enough such that

$$\tfrac{1}{2} - \varepsilon' < \nu(A_n), \quad \nu(B_n) < \tfrac{1}{2} + \varepsilon' \quad \text{for all } n \geq N.$$

Since $\delta < 1$, there exists m large enough such that $\delta^{m+1} < (\varepsilon/\mu(A))$. The action of G on (S, ν) being strongly mixing, there exist compact sets $K_n \subset G$ for all $n \in \{N, \ldots, N+m\}$ such that

$$|\nu(gA_n \cap B_n) - \nu(A_n)\nu(B_n)| \leq \varepsilon' \quad \text{for all } g \in G \setminus K_n.$$

Set $K = \bigcup_{n=N}^{N+m} K_n$, which is compact. Then we have for all $g \in G \setminus K$,

$$\begin{aligned}
\mu(gA \cap B) &\leq \prod_{n=1}^{N-1} 2\nu(A_n) \cdot \prod_{n=N}^{N+m} 2(\nu(A_n)\nu(B_n) + \varepsilon') \cdot \prod_{n\geq N+m+1} 2\nu(A_n) \\
&= \mu(A) \cdot \prod_{n=N}^{N+m} \frac{2\nu(A_n)\nu(B_n) + 2\varepsilon'}{2\nu(A_n)} \\
&= \mu(A) \prod_{n=N}^{N+m} \left(\nu(B_n) + \frac{\varepsilon'}{\nu(A_n)}\right) \\
&< \mu(A) \prod_{n=N}^{N+m} \left(\frac{1}{2} + \varepsilon' + \frac{\varepsilon'}{1/2 - \varepsilon'}\right) \\
&= \mu(A)\delta^{m+1} < \varepsilon.
\end{aligned}$$

Thus, we have

$$\lim_{g\to\infty} \mu(gA \cap B) = 0 \quad \text{for all } A, B \in \mathcal{F}_0.$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

The same claim holds for $A, B \in \mathcal{F}$ since there exist $C, D \in \mathcal{F}_0$ such that $A \subset C$ and $B \subset D$ and $\mu(gA \cap B) \leq \mu(gC \cap D) \to 0$ as $g \to \infty$. Moreover, equality (2.2) also holds for all elements of the ring $\mathcal{R}(\mathcal{F})$.

Finally, if $A, B \in \sigma(\mathcal{F})$ are such that $0 < \mu(A), \mu(B) < \infty$, by construction of the measure μ on $\sigma(\mathcal{F})$, if $\varepsilon > 0$ is given, there exist two sequences $(C_k)_{k\geq 1}, (D_\ell)_{\ell\geq 1} \subset \mathcal{R}(\mathcal{F})$ such that

$$A \subset \bigcup_{k\geq 1} C_k \quad \text{and} \quad B \subset \bigcup_{\ell\geq 1} D_\ell$$

and

$$\mu(A) \leq \sum_k \mu(C_k) < \mu(A) + \varepsilon \quad \text{and} \quad \mu(B) \leq \sum_\ell \mu(D_\ell) < \mu(B) + \varepsilon.$$

Choose first N large enough so that $\sum_{\ell > N} \mu(D_\ell) < \varepsilon/3$. Then, as

$$gA \cap B \subset \bigg(\bigcup_{\ell=1}^{N} gA \cap D_\ell\bigg) \cup \bigg(\bigcup_{\ell > N} D_\ell\bigg),$$

we get

$$\mu(gA \cap B) \leq \sum_{\ell=1}^{N} \mu(gA \cap D_\ell) + \varepsilon/3.$$

Choose next M large enough so that $\sum_{k>M} \mu(C_k) < \varepsilon/3N$. Then, as

$$gA \cap D_\ell \subset \bigg(\bigcup_{k=1}^{M} gC_k \cap D_\ell\bigg) \cup \bigg(\bigcup_{k>M} gC_k\bigg)$$

for every $1 \leq \ell \leq N$ and, since μ is G-invariant, we get

$$\mu(gA \cap B) \leq \sum_{\ell=1}^{N} \sum_{k=1}^{M} \mu(gC_k \cap D_\ell) + 2\varepsilon/3$$

for every $g \in G$. By the previous part of the proof, there exists a compact set $K \subset G$ such that

$$\mu(gA \cap B) < \varepsilon \quad \text{for all } g \in G \setminus K.$$

This ends the proof of the first claim of the proposition.

Let us prove now the existence of the sequence $(\xi_n) \subset L^2(X, \sigma(\mathcal{F}), \mu)$ which satisfies (2.3).

The probability standard space (S, ν) contains an asymptotically invariant sequence $(A_n)_{n\geq 1} \subset \mathcal{B}_S$ such that $\nu(A_n) = \frac{1}{2}$ for all n and, for every compact set $K \subset G$,

$$\sup_{g\in K} \nu(gA_n \cap A_n) \to \tfrac{1}{2} \quad \text{as } n \to \infty. \tag{2.4}$$

Since G is a locally compact, second countable group, we choose an increasing sequence of compact sets $(K_n)_{n\geq 1} \subset G$ such that $G = \bigcup_{n\geq 1} K_n$ and such that for every compact set $K \subset G$, there exists $m \geq 1$ such that $K \subset K_m$. Consequently, by (2.4), for all $m, k \geq 1$ there exists an integer $n(k, m)$ such that

$$|\nu(gA_{n(k,m)} \cap A_{n(k,m)}) - \tfrac{1}{2}| \leq \tfrac{1}{2}(1 - e^{-1/m2^k}) \quad \text{for all } g \in K_m.$$

Then we set for all m

$$\xi_m = \chi_{\prod_{k\geq 1} A_{n(k,m)}} \in L^2(X, \mu).$$

By construction, $0 \leq \xi_m \leq 1$ and $\|\xi_m\|_2 = 1$ for all m.

Now let K be a compact subset of G; then, for every integer $m \geq 1$ such that $K \subset K_m$, we have

$$\begin{aligned} 1 \geq \langle \pi_X(g)\xi_m | \xi_m \rangle &= \int_X \chi_{\prod_{k\geq 1} A_{n(k,m)}}(g^{-1}x)\chi_{\prod_{k\geq 1} A_{n(k,m)}}(x)\, d\mu(x) \\ &= \int_X \chi_{\prod_{k\geq 1}(gA_{n(k,m)} \cap A_{n(k,m)})}(x)\, d\mu(x) \\ &= \mu\bigg(\prod_{k\geq 1}(gA_{n(k,m)} \cap A_{n(k,m)})\bigg) = \prod_{k\geq 1} 2\nu(gA_{n(k,m)} \cap A_{n(k,m)}) \\ &\geq \prod_{k\geq 1} 2\bigg(\frac{1}{2} - \frac{1}{2}(1 - e^{-1/m2^k})\bigg) = \prod_{k\geq 1} e^{-1/m2^k} \\ &= e^{-\sum_{k=1}^{\infty}(1/m2^k)} = e^{-1/m} \to_{m\to\infty} 1 \end{aligned}$$

uniformly on K, where the first inequality follows from the Cauchy–Schwarz inequality.

Assume now that G is discrete and hence countable; set

$$\Omega = \bigcup_{g\in G} \bigcup_{m\geq 1} g \cdot X_m,$$

where $X_m = \prod_k A_{n(k,m)}$ is the support of ξ_m for every m. Then Ω is G-invariant, the restriction of μ to Ω is σ-finite, and $L^2(\Omega, \mu)$ is separable since G is countable, and it contains the sequence (ξ_m) by construction. □

Thus, the proof of Theorem 1.5 is complete in the case where G is discrete and hence we will focus on the case where G is not discrete.

Remark 2.9. We are very grateful to Alessandro Calderi and Alain Valette for having kindly communicated the present remark. The associated representation π_X of G on $L^2(X, \sigma(\mathcal{F}), \mu)$ is not continuous in general. Indeed, if it was continuous, then we would have

$$\lim_{g\to e} \langle \pi_X(g)\xi | \xi \rangle = \|\xi\|_2^2$$

for every element $\xi \in L^2(X, \mu)$ and, in particular, $\lim_{g\to e} \mu(gB \cap B) = 1$ for every $B \in \mathcal{F}_0$ with $\mu(B) = 1$. Choose a Borel set $A \subset S$ such that $\nu(A) = 1/2$ and set $B = \prod_n A \in \mathcal{F}_0$. If $g \in G$ is such that $\nu(gA \triangle A) > 0$, we have $\mu(gB \cap B) = 0$ since $\nu(gA \cap A) < 1/2$. If we can make $g \to e$, we have proved that π_X is not continuous. It is the case for $G = S = S^1$ equipped with its normalized Lebesgue measure.

We also observe that, in the case where G is countable, provided that we put $X_0 = B = \prod_n A$ as above and we take $\Omega = \bigcup_{g\in G} \bigcup_{m\geq 0} g \cdot X_m$, the unit vector $\chi_B \in L^2(\Omega, \mu)$ satisfies the following condition:

$$\varphi_B(g) := \langle \pi_X(g)\chi_B | \chi_B \rangle = \begin{cases} 1, & g = e, \\ 0, & g \neq e. \end{cases}$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

This means that φ_B is the positive-definite function δ_e whose GNS construction (see [**2**, Theorem C.4.10]) is the (left) regular representation of G. Hence, the left regular representation of G is a subrepresentation of π_Ω.

If G is not discrete, we see no reason why π_Ω should contain the regular representation, as in this case there is no analogue of the function δ_e.

3. *Proof of Theorem 1.5, Part 2*

Before proving the last part of our main theorem, let us make the following comment, which we owe to S. Baaj: recall that if G be a lcsc group that acts measurably on some measure space (Ω, μ), where μ is G-invariant and σ-finite, and if $L^2(\Omega, \mu)$ is separable, then the associated unitary representation π_Ω is automatically continuous. This follows from [**2**, Proposition A.6.1] for instance.

This means that, if we had been able to restrict our action of G to such a measure space, then the proof of Theorem 1.5 would be complete. Unfortunately, we were unable to do that and thus we have to proceed as follows: we define a subfamily $\mathcal{F}_c$ of $\mathcal{F}$ and a measure μ_c on $\sigma(\mathcal{F}_c)$ so that the permutation representation on $L^2(X, \sigma(\mathcal{F}_c), \mu_c)$ is continuous and then, using continuity, we are able to restrict the action of G to a σ-finite measure subspace of X which has all the desired properties.

Before defining the above-mentioned family $\mathcal{F}_c$, we fix an increasing sequence of compact sets $(K_n)_{n\geq 1}$ of G with the following properties: $e \in \mathring{K}_1$, $K_n \subset \mathring{K}_{n+1}$ for every $n \geq 1$, and $G = \bigcup_{n\geq 1} K_n$. We also set $K := K_1$, which is a compact neighbourhood of e.

Definition 3.1.

(1) The family $\mathcal{F}_{c,0}$ is the collection of sets $A = \prod_n A_n \in \mathcal{F}_0$ such that $\mu(A) = \prod_n 2\nu(A_n) = 0$ or that
$$\lim_{N\to\infty} \prod_{n=N}^{\infty} 2\nu(gA_n \cap A_n) = 1$$
uniformly for $g \in K$.

(2) We define the sequence $(\mathcal{F}_{c,n})_{n\geq 1}$ of collections of subsets of X by induction: $\mathcal{F}_{c,n} := \{B \setminus A : A, B \in \mathcal{F}_{c,n-1}\}$, and we set finally $\mathcal{F}_c = \bigcup_{n\geq 0} \mathcal{F}_{c,n}$.

We also denote by $\sigma(\mathcal{F}_c)$ the σ-algebra generated by $\mathcal{F}_c$; it is a sub-σ-algebra of $\sigma(\mathcal{F})$.

We observe that the sequence $(\prod_{k\geq 1} A_{n(k,m)})_{m\geq 1}$ constructed in the proof of Proposition 2.8 is contained in $\mathcal{F}_{c,0}$. Indeed, we have proved that
$$2\nu(gA_{n(k,m)} \cap A_{n(k,m)}) \in [e^{-1/m2^k}, 1] \quad \text{for all } g \in K.$$

Hence, we get
$$\begin{aligned} 1 &\geq \lim_{N\to\infty} \prod_{k=N}^{\infty} 2\nu(gA_{n(k,m)} \cap A_{n(k,m)}) \\ &\geq \lim_{N\to\infty} \prod_{k=N}^{\infty} e^{-1/m2^k} \\ &= e^{-(1/m)\lim_{N\to\infty} \sum_{k=N}^{\infty} (1/2^k)} = 1 \end{aligned}$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

uniformly on K. This proves among others that the associated sequence of vectors $(\xi_m)_{m\geq 1} \subset L^2(X, \sigma(\mathcal{F}_c), \mu)$ is an almost-invariant sequence of unit vectors.

In order to see that $\mathcal{F}_{c,0}$ is stable under finite intersections, we need the following elementary lemma.

LEMMA 3.2. *Let* $(x_k)_{k\geq 1}, (a_k)_{k\geq 1} \subset \mathbb{R}^+$ *be sequences such that*

$$\lim_{N\to\infty} \prod_{k\geq N} x_k = \lim_{N\to\infty} \prod_{k\geq N} (1-a_k) = 1.$$

Then

$$\lim_{N\to\infty} \prod_{k\geq N} (x_k - a_k) = 1.$$

Proof. There exists N_0 so that $x_k > 2/3$ and $0 \leq a_k < 1/2$ for all $k \geq N_0$. This implies immediately that

$$(-2a_k + a_k^2)x_k \leq -\tfrac{3}{2}a_k x_k \leq -a_k$$

for all $k \geq N_0$ and we get for all $N \geq N_0$

$$\prod_{k\geq N} x_k(1-a_k)^2 \leq \prod_{k\geq N} (x_k - a_k) \leq \prod_{k\geq N} x_k,$$

which proves the claim. □

LEMMA 3.3. *Let* $A, B \in \mathcal{F}_{c,0}$. *Then* $A \cap B \in \mathcal{F}_{c,0}$.

Proof. Let $A, B \in \mathcal{F}_{c,0}$, where $A = \prod_n A_n$ and $B = \prod_n B_n$.

If A or B has measure zero, then $\mu(A \cap B) = 0$ and $A \cap B \in \mathcal{F}_{c,0}$. We assume that $\mu(A \cap B) \neq 0$. Thus, we have that the five sequences

$$\prod_{n=N}^{\infty} 2\nu(gA_n \cap A_n), \quad \prod_{n=N}^{\infty} 2\nu(gB_n \cap B_n),$$
$$\prod_{n=N}^{\infty} 2\nu(A_n \cap B_n), \quad \prod_{n=N}^{\infty} 2\nu(A_n), \quad \prod_{n=N}^{\infty} 2\nu(B_n)$$

converge to 1 uniformly on K as $N \to \infty$.

As $\nu(A_n), \nu(B_n) \neq 0$ for every n, we have that

$$\prod_{n=N}^{\infty} \frac{2\nu(gA_n \cap A_n)}{2\nu(A_n)}, \quad \prod_{n=N}^{\infty} \frac{2\nu(gB_n \cap B_n)}{2\nu(B_n)},$$
$$\prod_{n=N}^{\infty} \frac{2\nu(A_n \cap B_n)}{2\nu(A_n)} \quad \text{and} \quad \prod_{n=N}^{\infty} \frac{2\nu(A_n \cap B_n)}{2\nu(B_n)}$$

converge to 1 uniformly on K as $N \to \infty$.

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

Moreover,

$$
\begin{aligned}
1 &\geq \lim_{N\to\infty} \prod_{n=N}^{\infty} \frac{2\nu(g(A_n\cap B_n)\cap(A_n\cap B_n))}{2\nu(A_n\cap B_n)} \\
&\geq \lim_{N\to\infty} \prod_{n=N}^{\infty} \frac{2\nu(A_n\cap B_n) - 2\nu(A_n\cap B_n\cap gA_n^c) - 2\nu(A_n\cap B_n\cap gA_n\cap gB_n^c)}{2\nu(A_n\cap B_n)} \\
&\geq \lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(1 - \frac{2\nu(A_n\setminus gA_n)}{2\nu(A_n\cap B_n)} - \frac{2\nu(B_n\setminus gB_n)}{2\nu(A_n\cap B_n)}\bigg).
\end{aligned}
$$

Using Lemma 3.2, we will show that

$$
\lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(1 - \frac{2\nu(A_n\setminus gA_n)}{2\nu(A_n\cap B_n)} - \frac{2\nu(B_n\setminus gB_n)}{2\nu(A_n\cap B_n)}\bigg) = 1. \qquad (*)
$$

One has to check that:

(1) $\lim_{N\to\infty} \prod_{n=N}^{\infty}(1 - (2\nu(B_n\setminus gB_n)/2\nu(A_n\cap B_n))) = 1$;

(2) $\lim_{N\to\infty} \prod_{n=N}^{\infty}(1 - (2\nu(A_n\setminus gA_n)/2\nu(A_n\cap B_n))) = 1$.

We prove (1) in detail; as the proof of (2) is similar, we will get $(*)$. One has

$$
\begin{aligned}
&\lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(1 - \frac{2\nu(B_n\setminus gB_n)}{2\nu(A_n\cap B_n)}\bigg) \\
&\quad = \lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(1 - \frac{2\nu(B_n\setminus gB_n)}{2\nu(A_n\cap B_n)}\bigg) \lim_{N\to\infty} \prod_{n=N}^{\infty} \frac{2\nu(A_n\cap B_n)}{2\nu(B_n)} \\
&\quad = \lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(\frac{2\nu(A_n\cap B_n) - 2\nu(B_n\setminus gB_n)}{2\nu(B_n)}\bigg).
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
&\lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(1 - \frac{2\nu(B_n\setminus gB_n)}{2\nu(B_n)}\bigg) \\
&\quad = \lim_{N\to\infty} \prod_{n=N}^{\infty} \bigg(\frac{2\nu(B_n) - 2\nu(B_n\setminus gB_n)}{2\nu(B_n)}\bigg) \\
&\quad = \lim_{N\to\infty} \prod_{n=N}^{\infty} \frac{2\nu(B_n\cap gB_n)}{2\nu(B_n)} = 1.
\end{aligned}
$$

As observed above, this ends the proof. □

Next, exactly the same arguments as those in §2 from Lemma 2.3 to Proposition 2.8 allow us to prove the following facts.

PROPOSITION 3.4. *The family $\mathcal{F}_c$ has the following properties.*

(i) *For all $A, B \in \mathcal{F}_c$, one has $A\cap B \in \mathcal{F}_c$ and $A\setminus B \in \mathcal{F}_c$, so that $\mathcal{F}_c$ is a semiring, and the set of all finite disjoint unions of elements of $\mathcal{F}_c$ is the ring generated by $\mathcal{F}_c$, which is denoted by $\mathcal{R}(\mathcal{F}_c)$.*

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

(ii) *The diagonal action of G on $X = \prod_{n\geq 1} S$ is $\sigma(\mathcal{F}_c)$-measurable.*
(iii) *There exists a measure $\mu_c : \sigma(\mathcal{F}_c) \to [0, \infty]$ which is G-invariant and such that $\mu_c(\prod_n A_n) = \prod_n 2\nu(A_n)$ for every $A = \prod_n A_n \in \mathcal{F}_{c,0}$.*
(iv) *The dynamical system $(X, \sigma(\mathcal{F}_c), \mu_c, G)$ is a C_0-dynamical system.*
(v) *Let $(A_{n(k,m)})_{k,m}$ be the family constructed in the proof of Proposition 2.8 and set $X_m = \prod_k A_{n(k,m)}$ and $\xi_m = \chi_{X_m}$ for every $m \geq 1$. Then $(\xi_m) \subset L^2(X, \sigma(\mathcal{F}_c), \mu_c)$ is a sequence of almost-invariant unit vectors.*

We are ready to prove that the representation π_X on $L^2(X, \sigma(\mathcal{F}_c), \mu_c)$ is continuous. To do this, it suffices to prove that, for every $A \in \sigma(\mathcal{F}_c)$ such that $\mu_c(A) < \infty$, one has

$$\|\pi_X(g)\chi_A - \chi_A\|_2^2 = \int_X |\chi_{gA} - \chi_A|^2 \, d\mu_c = \int_X |\chi_{gA} - \chi_A| \, d\mu_c = \mu_c(gA \triangle A) \to 0$$

as $g \to e$.

PROPOSITION 3.5. *Let $A \in \sigma(\mathcal{F}_c)$ be such that $\mu_c(A) < \infty$. Then*

$$\lim_{g\to e} \mu_c(gA \triangle A) = 0.$$

Proof. Denote by $\mathcal{S}$ the family of sets $A \in \sigma(\mathcal{F}_c)$ such that $\mu_c(A) < \infty$ and that $\lim_{g\to e} \mu_c(gA \triangle A) = 0$. Let us prove the following assertions.

(i) One has $\mathcal{F}_c \subset \mathcal{S}$, i.e. $\lim_{g\to e} \mu(gA \triangle A) = 0$ for every $A \in \mathcal{F}_c$.
Indeed, if $\mu(A) = 0$, the claim is obvious. Thus, assume that $A = \prod_n A_n$ and

$$\lim_{N\to\infty} \prod_{n=N}^{\infty} 2\nu(gA_n \cap A_n) = 1$$

uniformly on K. Let $(g_m)_{m\geq 1}$ be a sequence in K which converges to e. We prove that $\mu_c(g_m A \cap A) \to \mu_c(A)$ as $m \to \infty$; this will prove the claim since we have for every m,

$$\begin{aligned}
\mu_c(g_m A \triangle A) &= \int_X |\chi_{g_m A} - \chi_A| \, d\mu_c \\
&\leq \int_X (\chi_A - \chi_{g_m A\cap A}) \, d\mu_c + \int_X (\chi_{g_m A} - \chi_{g_m A\cap A}) \, d\mu_c \\
&= 2(\mu_c(A) - \mu_c(g_m A \cap A)).
\end{aligned}$$

(Notice that we have used that $\mu_c(A) < \infty$ and that μ_c is G-invariant.)

As $\prod_{n=N}^{\infty} 2\nu(gA_n \cap A_n) \to 1$ uniformly on K, there exists N_0 such that

$$\prod_{n=N_0+1}^{\infty} \frac{2\nu(g_m A_n \cap A_n)}{2\nu(A_n)} \in \left(\sqrt{1-\varepsilon}, \sqrt{1+\varepsilon}\right) \quad \text{for all } m.$$

As the representation of G on $L^2(S, \nu)$ is continuous, one has, for every fixed n,

$$\nu(g_m A_n \cap A_n) \to \nu(A_n) \quad \text{as } m \to \infty.$$

Hence, there exists M such that, for every $n \in \{1, \ldots, N_0\}$,

$$\frac{\nu(g_m A_n \cap A_n)}{\nu(A_n)} \in [(1-\varepsilon)^{1/2N_0}, (1+\varepsilon)^{1/2N_0}] \quad \text{for all } m \geq M.$$

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

Then one has for every $m \geq M$,

$$\frac{\prod_{n\geq 1} 2\nu(g_m A_n \cap A_n)}{\prod_{n\geq 1} 2\nu(A_n)} = \bigg(\prod_{n=1}^{N_0} \frac{2\nu(g_m A_n \cap A_n)}{2\nu(A_n)}\bigg) \cdot \bigg(\prod_{n=N_0+1}^{\infty} \frac{2\nu(g_m A_n \cap A_n)}{2\nu(A_n)}\bigg)$$
$$\in (1-\varepsilon, 1+\varepsilon).$$

This shows that $(\mu_c(g_m A \cap A)/\mu_c(A)) \to 1$ and $\mu_c(g_m A \cap A) \to \mu_c(A)$ as $m \to \infty$.

(ii) If $A, B \in \mathcal{S}$, then $A \cap B \in \mathcal{S}$ and $A \setminus B \in \mathcal{S}$. In particular, $\mathcal{S}$ is a semiring of subsets of X which contains $\mathcal{F}_c$.

Indeed, let $A, B \in \mathcal{S}$ and let $g \in G$. Then

$$\begin{aligned}\mu_c(g(A \cap B) \triangle (A \cap B)) &= \int_X |\chi_{gA}\chi_{gB} - \chi_A\chi_B| \, d\mu_c \\ &\leq \int_X |\chi_{gA}(\chi_{gB} - \chi_B)| \, d\mu_c + \int_X |\chi_B(\chi_{gA} - \chi_A)| \, d\mu_c \\ &\leq \mu_c(gA \triangle A) + \mu_c(gB \triangle B) \to 0\end{aligned}$$

as $g \to e$. This shows that $A \cap B \in \mathcal{S}$. Next,

$$\begin{aligned}\mu_c(g(A \setminus B) \triangle (A \setminus B)) &= \int_X |\chi_{g(A\setminus B)} - \chi_{A\setminus B}| \, d\mu_c \\ &= \int_X |\chi_{gA}(1-\chi_{gB}) - \chi_A(1-\chi_B)| \, d\mu_c \\ &= \int_X |\chi_{gA} - \chi_A - (\chi_{gA}\chi_{gB} - \chi_A\chi_B)| \, d\mu_c \\ &\leq \mu_c(gA \triangle A) + \int_X |\chi_{gA} - \chi_A|\chi_{gB} \, d\mu_c \\ &\quad + \int_X \chi_A|\chi_{gB} - \chi_B| \, d\mu_c \\ &\leq 2\mu_c(gA \triangle A) + \mu_c(gB \triangle B) \to 0\end{aligned}$$

as $g \to e$, showing that $A \setminus B \in \mathcal{S}$. Lemma 3.3 and these facts imply that $\mathcal{F}_c \subset \mathcal{S}$.

(iii) Let $A_1, \ldots, A_n \in \mathcal{S}$. Then their union $\bigcup_{j=1}^n A_j \in \mathcal{S}$. In particular, $\mathcal{S}$ is a ring of subsets of X which contains the ring generated by $\mathcal{F}_c$.

Indeed, by (ii), we can assume that $A_i \cap A_j = \varnothing$ for all $i \neq j$. Setting $A = \bigsqcup_{j=1}^n A_j$, we have

$$\begin{aligned}\mu_c(gA \triangle A) &= \int_X \bigg|\sum_{j=1}^n \chi_{gA_j} - \sum_{j=1}^n \chi_{A_j}\bigg| \, d\mu_c \\ &= \int_X \bigg|\sum_{j=1}^n (\chi_{gA_j} - \chi_{A_j})\bigg| \, d\mu_c \\ &\leq \sum_{j=1}^n \mu_c(gA_j \triangle A_j) \to 0\end{aligned}$$

as $g \to e$. This proves (iii).

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

(iv) Let $A \in \sigma(\mathcal{F}_c)$ be such that $\mu_c(A) < \infty$. Then

$$\lim_{g \to e} \mu_c(gA \triangle A) = 0.$$

Indeed, fix $\varepsilon > 0$. By construction of the measure μ_c on $\sigma(\mathcal{F}_c)$ and since $\mathcal{S}$ is a ring which contains the ring $\mathcal{R}(\mathcal{F}_c)$ generated by $\mathcal{F}_c$,

$$\mu_c(A) = \inf\bigg\{\sum_{i \geq 1} \mu_c(A_i) : A_i \in \mathcal{F}_c \ \forall i,\ A \subset \bigcup_i A_i\bigg\}.$$

Hence, there exists a sequence $(A_i)_{i \geq 1} \subset \mathcal{F}_c$ such that $A \subset \bigcup_i A_i := B$ and

$$\mu_c(A) \leq \sum_i \mu_c(A_i) < \mu_c(A) + \frac{\varepsilon}{5}. \tag{3.1}$$

By (ii), we assume that $A_i \cap A_j = \varnothing$ for all $i \neq j$. Choose then N large enough so that

$$\sum_{i > N} \mu_c(A_i) < \frac{\varepsilon}{5}$$

and let $V \subset G$ be an open neighbourhood of e such that

$$\sum_{i=1}^{N} \mu_c(gA_i \triangle A_i) < \frac{\varepsilon}{5} \quad \text{for all } g \in V.$$

Then we have for every $g \in V$,

$$\begin{aligned}
\mu_c(gA \triangle A) &\leq \int_X |\chi_{gA} - \chi_{gB}|\, d\mu_c + \int_X |\chi_{gB} - \chi_B|\, d\mu_c + \int_X |\chi_B - \chi_A|\, d\mu_c \\
&\leq 2\mu_c(B \setminus A) + \sum_i \mu_c(gA_i \triangle A_i) \\
&\leq 2\mu_c(B \setminus A) + \sum_{i=1}^{N} \mu_c(gA_i \triangle A_i) + 2 \sum_{i > N} \mu_c(A_i) < \varepsilon.
\end{aligned}$$

□

Remark 3.6. Since μ is not σ-finite, there is no reason that it coincides with μ_c on $\sigma(\mathcal{F}_c)$, even if it does on $\mathcal{F}_c \subset \mathcal{F}$. Furthermore, we had to use μ_c instead of μ because of equality (3.1); more precisely, it is not necessarily true that, for $A \in \sigma(\mathcal{F}_c)$, one has

$$\mu(A) = \inf\bigg\{\sum_{i \geq 1} \mu(A_i) : A_i \in \mathcal{F}_c \ \forall i,\ \ A \subset \bigcup_i A_i\bigg\}.$$

LEMMA 3.7. *Let $(\Omega, \mathcal{B}, \rho)$ be a measure space on which a group G acts by measurable automorphisms, and such that $\rho(gB) = \rho(B)$ for every $B \in \mathcal{B}$ such that $\rho(B) < \infty$. Then the action of G preserves ρ, i.e. $\rho(gB) = \rho(B)$ for every $B \in \mathcal{B}$.*

Proof. If there existed $g \in G$ and $B \in \mathcal{B}$ such that $\rho(B) = \infty$ and $\rho(gB) \neq \rho(B)$, then, necessarily, we would have $\rho(gB) < \infty$. But this contradicts our hypothesis since then $\rho(g^{-1}gB) = \rho(B)$ would be finite. □

The next proposition is the last step of the proof of Theorem 1.5.

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

PROPOSITION 3.8. *There exists a G-invariant subset $\Omega \in \sigma(\mathcal{F}_c)$ such that the measure $\rho : \sigma(\mathcal{F}_c) \to [0, \infty]$, defined by $\rho(B) := \mu_c(B \cap \Omega)$ for every $B \in \sigma(\mathcal{F}_c)$, is σ-finite and G-invariant. Furthermore, the Hilbert space $L^2(\Omega, \rho)$ is separable and contains the sequence of unit vectors (ξ_m) of Proposition 3.4.*

Proof. Let $D = \{g_1 = e, g_2, \ldots\} \subset G$ be a countable, dense subset of G. Set $Y = \bigcup_{m\geq 1} X_m$, where $X_m = \prod_{k\geq 1} A_{n(k,m)}$ as in Proposition 3.4, and set

$$\Omega = \bigcup_{h\in D} hY.$$

Recall that the measure ρ on $(X, \sigma(\mathcal{F}_c))$ is defined by

$$\rho(B) = \mu_c(B \cap \Omega)$$

for every $B \in \sigma(\mathcal{F}_c)$, so that ρ is σ-finite. It remains to prove that it is G-invariant. By Lemma 3.7, it suffices to prove that $\rho(gB) = \rho(B)$ for every $g \in G$ and $B \in \sigma(\mathcal{F}_c)$ such that $\rho(B) < \infty$. Then, for every $i \geq 1$, set

$$B_i = (B \cap g_i Y) \Big\backslash \left(\bigcup_{j=1}^{i-1} g_j Y \right),$$

so that $B \cap \Omega = \bigsqcup_i B_i$ and thus $\rho(B) = \sum_i \rho(B_i) < \infty$.

Then

$$\rho(gB) \geq \rho(g(B \cap \Omega)) = \sum_i \rho(gB_i).$$

For fixed i, let $(h_n^{(i)})_{n\geq 1} \subset D$ be such that $h_n^{(i)} \to gg_i$ as $n \to \infty$. Then $\rho(gB_i) \geq \rho(gB_i \cap h_n^{(i)} g_i^{-1} B_i)$ for every n and

$$\rho(gB_i \cap h_n^{(i)} g_i^{-1} B_i) = \mu_c(gB_i \cap h_n^{(i)} g_i^{-1} B_i) \to \mu_c(gB_i) = \mu_c(B_i)$$

as $n \to \infty$; thus $\rho(gB_i) \geq \mu_c(B_i)$ and hence

$$\rho(gB) \geq \sum_i \rho(gB_i) \geq \sum_i \mu_c(B_i) = \mu_c(B) \geq \mu_c(B \cap \Omega) = \rho(B).$$

We also get

$$\rho(B) \leq \rho(gB) \leq \rho(g^{-1}gB) = \rho(B).$$

Finally, separability of $L^2(\Omega, \rho)$ follows from the countability of D and from its density in G.

The proof is now complete. □

Acknowledgements. We very grateful to A. Calderi and A. Valette for the content of Remark 2.9, to S. Baaj for his help on the automatic continuity of the permutation representations on separable L^2-spaces, and to the referee for his/her valuable comments.

https://doi.org/10.1017/etds.2020.45 Published online by Cambridge University Press

REFERENCES

[**1**] S. Adams, G. A. Elliot and T. Giordano. Amenable actions of groups. *Trans. Amer. Math. Soc.* **344** (1994), 803–822.

[**2**] B. Bekka, P. de la Harpe and A. Valette. *Kazhdan's Property* (T). Cambridge University Press, Cambridge, 2008.

[**3**] P. Billingsley. *Probability and Measure*. John Wiley, New York, 1979.

[**4**] I. Chatterji, C. Drutu and F. Haglund. Kazhdan and Haagerup properties from the median viewpoint. *Adv. Math.* **225** (2010), 882–921.

[**5**] P.-A. Cherix, M. Cowling, P. Jolissaint, P. Julg and A. Valette. *Groups with the Haagerup Property (Gromov's a-T-Menability)*. Birkhäuser, Basel, 2001.