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Negative energy standing wave instability in the presence of flow

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We study standing waves on the surface of a tangential discontinuity in an incompressible plasma. The plasma is moving with constant velocity at one side of the discontinuity, while it is at rest at the other side. The moving plasma is ideal and the plasma at rest is viscous. We only consider the long wavelength limit where the viscous Reynolds number is large. A standing wave is a superposition of a forward and a backward wave. When the flow speed is between the critical speed and the Kelvin–Helmholtz threshold the backward wave is a negative energy wave, while the forward wave is always a positive energy wave. We show that viscosity causes the standing wave to grow. Its increment is equal to the difference between the negative energy wave increment and the positive energy wave decrement.

Key words: plasma flows, plasma instabilities, plasma waves

1. Introduction

The concept of negative energy waves is very useful in studying stability. It was first proposed by L. Chua in 1951 (Pierce 1974) in application to electron beams. Later this concept became very popular in plasma physics (e.g. Kadomtsev, Mikhailovskii & Timofeev 1965; Mikhailovskii 1974; Nezlin 1976). In hydrodynamics the concept of negative energy waves was first used in the pioneering article by Benjamin (1963). However this concept became popular in hydrodynamics much later, not least owing the article by Cairns (1979) where it was applied to the stability of shear flows. An excellent review of the theory of negative energy waves in hydrodynamics is given by Ostrovskii, Rybak & Tsimring (1986) (see also Stepanyants & Fabrikant 1989; Fabrikant & Stepanyants 1998). To our knowledge, Ryutova (1988) was the first to consider negative energy waves in magnetohydrodynamics. She studied the propagation of kink waves along thin magnetic tubes in the presence of a homogeneous parallel flow outside the tube and applied the results to space physics. After that the application of the theory of negative energy waves to problems of space physics were considered by many authors (e.g. Joarder, Nakariakov & Roberts (1997), Ruderman & Wright (1998), Andries & Goossens (2001); see also reviews

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by Ruderman & Belov (2010) and Taroyan & Ruderman (2011)). The main property of negative energy waves is that they become unstable in the presence of dissipation.

To our knowledge all studies of negative energy waves dealt with propagating waves. A simple general rule was obtained to recognise that a propagating wave becomes a negative energy wave in a particular reference frame. It is a negative energy wave when, due to the effect of flow, its phase speed changes the sign. Then a question arises: If the negative energy wave instability exits in a bounded wave guides where the waves are standing rather than propagating? The answer to this question is not so obvious. Consider a simplest case of a wave guide where there are only two wave modes that propagate in the opposite direction in the absence of flow. Assume now that there is a siphon flow in this wave guide. If the flow velocity is high enough, both waves propagate in the same direction, and the wave that propagates in the direction opposite to the flow direction in the reference frame moving together with the flow becomes a negative energy wave. The wave propagating in the flow direction is a positive energy wave. If there is dissipation in the system then the negative energy wave starts to grow, but the positive energy wave starts to decay. However the two waves cannot do this independently because the ratio of their amplitudes in the standing wave remains constant. Hence, the standing wave will grow or decay depending on which of the two effects dominate, the negative energy wave growth or the positive energy wave decay.

An example of a bounded wave guide with siphon flow is a coronal magnetic loop with the footpoints frozen in the dense photospheric plasma. Usually the velocity of siphon flows observed in coronal loops is below the negative energy wave instability threshold if we assume that the magnetic field lines in the loop are not twisted. However, the magnetic twist can substantially decrease the instability threshold.

In this paper we do not consider any applications. Our aim is to answer the main question: whether the negative energy wave instability occurs in bounded wave guides. For this we consider a very simple problem of stability of the magnetohydrodynamic (MHD) tangential discontinuity in an incompressible plasma viscous at one side of the discontinuity and inviscid at the other side. The article is organised as follows. In the next section the problem is formulated and the main equations and boundary conditions are given. In § 3 the stability of standing waves is studied. Section 4 contains the summary of the obtained results and conclusion.

2. Problem formulation and governing equations

We consider an incompressible infinitely electrically conducting plasma. In Cartesian coordinates (x, y, z) the unperturbed state is an MHD tangential discontinuity. The unperturbed magnetic field and velocity are $\boldsymbol{B} = B\boldsymbol{e}_x$ and $\boldsymbol{U} = U_0\boldsymbol{e}_x$, where $\boldsymbol{e}_x$ is the unit vector in the x-direction. The equilibrium density ρ and the quantities B and U_0 are given by

$$\rho = \begin{cases} \rho_1, & z < 0, \\ \rho_2, & z > 0, \end{cases} \qquad B = \begin{cases} B_1, & z < 0, \\ B_2, & z > 0, \end{cases} \qquad U_0 = \begin{cases} 0, & z < 0, \\ U, & z > 0. \end{cases} \tag{2.1a–c}$$

The plasma motion is described by the ideal MHD equations in the incompressible plasma approximation,

$$\nabla \cdot \boldsymbol{v} = 0, \quad \nabla \cdot \boldsymbol{b} = 0, \tag{2.2a,b}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + U\frac{\partial \boldsymbol{v}}{\partial x} = -\frac{1}{\rho}\nabla P + \frac{B}{\mu_0 \rho}\frac{\partial \boldsymbol{b}}{\partial x} + \nu_0 \nabla^2 \boldsymbol{v}, \tag{2.2c}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + U\frac{\partial \boldsymbol{b}}{\partial x} = B\frac{\partial \boldsymbol{v}}{\partial x}, \tag{2.2d}$$

where $\boldsymbol{v} = (u, v, w)$ is the velocity perturbation, $\boldsymbol{b} = (b_x, b_y, b_z)$ the magnetic field perturbation, P the total pressure (plasma plus magnetic) perturbation, μ_0 the magnetic permeability of free space and ν_0 the kinematic viscosity equal to ν at $z < 0$ and 0 at $z > 0$.

We write the equation of the perturbed tangential discontinuity as $z = \zeta(t, x, y)$. The perturbations must satisfy the kinematic boundary conditions and the conditions of the stress continuity at $z = 0$:

$$w_1 = \frac{\partial \zeta}{\partial t}, \tag{2.3a}$$

$$w_2 = \frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x}, \tag{2.3b}$$

$$P_2 = P_1 - 2\rho_1 \nu \frac{\partial w_1}{\partial z}, \tag{2.3c}$$

$$\frac{\partial u_1}{\partial z} + \frac{\partial w_1}{\partial x} = 0, \quad \frac{\partial v_1}{\partial z} + \frac{\partial w_1}{\partial x} = 0, \tag{2.3d,e}$$

where the indices 1 and 2 refer to quantities below ($z < 0$) and above ($z > 0$) the discontinuity. In addition, we assume that all perturbations tend to zero as $|z| \to \infty$. Finally, we assume that the magnetic field lines are frozen in the immovable plasma at two planes orthogonal to the x-axis at $x = 0$ and $x = L$. In terms of the velocity and magnetic field perturbation this condition is written as $U\boldsymbol{b}_\perp - B\boldsymbol{v}_\perp = 0$ at $x = 0$, L, where $\boldsymbol{b}_\perp$ and $\boldsymbol{v}_\perp$ are the components of the magnetic field and velocity perturbation orthogonal to the x-axis (see e.g. Ruderman 2010). In particular, it follows that

$$\zeta = 0 \quad \text{at } x = 0, L. \tag{2.4}$$

3. Instability of standing waves

Ruderman & Goossens (1995) used the system of (2.2) and boundary conditions (2.3) to derive the equation describing the propagation of surface waves on the tangential discontinuity. This equation reads

$$(\rho_1 + \rho_2)\frac{\partial^2 \zeta}{\partial t^2} + 2\rho_2 U \frac{\partial^2 \zeta}{\partial t \partial x} + \rho_2(U^2 - U_c^2)\frac{\partial^2 \zeta}{\partial x^2} = 4\nu\rho_1 \nabla^2 \frac{\partial \zeta}{\partial t}, \tag{3.1}$$

where

$$U_c^2 = \frac{\rho_1 V_1^2 + \rho_2 V_2^2}{\rho_1} = \frac{\rho_1 V_{KH}^2}{\rho_1 + \rho_2}, \quad V_{1,2}^2 = \frac{B_{1,2}^2}{\mu_0 \rho_{1,2}}. \tag{3.2a,b}$$

This equation is only valid in the long wavelength approximation, that is when $\mathrm{Re} = LV_0/\nu \gg 1$ where $V_0 = (V_1 + V_2)/2$. When $U > V_{KH}$ the discontinuity is subject to the Kelvin–Helmholtz instability. Equation (3.1) describes two wave modes propagating in the opposite directions in the absence of flow. They are

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called the forward wave mode and the backward wave mode, respectively. Ruderman & Goossens (1995) showed that the backward mode becomes a negative energy wave when $U > U_c$. This mode is unstable due to the presence of viscosity. The forward mode is always a positive energy wave that decays due to the presence of viscosity. When $U_c < U < V_{KH}$ the increment of the negative energy wave is

$$\gamma_- = \frac{2\rho_1 \nu k^2 C_-}{\sqrt{\rho_1\rho_2(V_{KH}^2 - U^2)}}, \tag{3.3}$$

and the decrement of the positive energy wave is

$$\gamma_+ = \frac{2\rho_1 \nu k^2 C_+}{\sqrt{\rho_1\rho_2(V_{KH}^2 - U^2)}}. \tag{3.4}$$

In these expressions $k^2 = k_x^2 + k_y^2$, k_x and k_y are the x and y-component of the wave vector, $\boldsymbol{C}_\pm = \omega/k_x$ and ω is the wave frequency. The quantity $\boldsymbol{C}_\pm$ is given by

$$\boldsymbol{C}_\pm = \frac{\rho_2 U \pm \sqrt{\rho_1\rho_2(V_{KH}^2 - U^2)}}{\rho_1 + \rho_2}. \tag{3.5}$$

We now consider a solution to (3.1) in the form of standing wave satisfying the boundary conditions (2.4). We assume that $U_c < U < V_{KH}$. We only consider waves standing in the y-direction and take ζ proportional to $\cos(k_y y)$. The characteristic time of variation of the wave amplitude is equal to the characteristic period of oscillations times Re. In accordance with this we introduce the 'slow' time $T = \epsilon t$, where $\epsilon = \mathrm{Re}^{-1}$. We also introduce the scaled shear viscosity $\bar\nu = \epsilon^{-1}\nu$. Then (3.1) is transformed to

$$\begin{aligned}(\rho_1+\rho_2)\frac{\partial^2\zeta}{\partial t^2} + 2\rho_2 U\frac{\partial^2\zeta}{\partial t\partial x} + \rho_2(U^2 - U_c^2)\frac{\partial^2\zeta}{\partial x^2} \\ + 2\epsilon\left((\rho_1+\rho_2)\frac{\partial^2\zeta}{\partial t\partial T} + \rho_2 U\frac{\partial^2\zeta}{\partial x\partial T}\right) - 4\epsilon\bar\nu\rho_1\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)\frac{\partial\zeta}{\partial t} = O(\epsilon^2).\end{aligned} \tag{3.6}$$

We look for the solution to this equation in the form of series

$$\zeta = \zeta_1 + \epsilon\zeta_2 + O(\epsilon^2). \tag{3.7}$$

Substituting this expression in (3.6) and collecting terms of the order of unity yields

$$(\rho_1+\rho_2)\frac{\partial^2\zeta_1}{\partial t^2} + 2\rho_2 U\frac{\partial^2\zeta_1}{\partial t\partial x} + \rho_2(U^2 - U_c^2)\frac{\partial^2\zeta_1}{\partial x^2} = 0. \tag{3.8}$$

It follows from (2.4) that ζ_1 must satisfy the boundary conditions

$$\zeta_1 = 0 \quad \text{at } x = 0, L. \tag{3.9}$$

We look for the solution to the boundary value problem constituted by (3.8) and the boundary conditions (3.9) in the form

$$\zeta_1 = A(T)[\cos(\omega t - k_+ x) - \cos(\omega t - k_- x)], \tag{3.10}$$

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where $k_{\pm} = \omega/\boldsymbol{C}_{\pm}$. It is straightforward to verify that this expression satisfies (3.8). It also satisfies the condition $\zeta_1 = 0$ at $x = 0$. The condition $\zeta_1 = 0$ at $x = L$ can be written as

$$\sin\left(\omega t - \frac{L(k_+ + k_-)}{2}\right) \sin \frac{L(k_+ - k_-)}{2} = 0. \tag{3.11}$$

It follows from this equation that $L(k_- - k_+) = 2\pi n$, which leads to

$$\omega = \frac{2\pi n C_+ C_-}{L(C_+ - C_-)} = \frac{\pi n(U^2 - U_c^2)}{LU}, \quad n = 1, 2, \ldots . \tag{3.12}$$

When there is no viscosity equation (3.10) with $A = \text{const.}$ gives an exact solution to (3.1) with $\nu = 0$. The presence of viscosity causes the variation of the wave amplitude A with time. To determine this variation we proceed to the next-order approximation. Collecting terms of the order of ϵ in (3.8) and using (3.10) and (3.12) yields

$$\begin{aligned}
&(\rho_1 + \rho_2)\frac{\partial^2 \zeta_2}{\partial t^2} + 2\rho_2 U \frac{\partial^2 \zeta_2}{\partial t \partial x} + \rho_2(U^2 - U_c^2)\frac{\partial^2 \zeta_2}{\partial x^2} \\
&\quad = \frac{\mathrm{d}A}{\mathrm{d}T}\sqrt{\rho_1\rho_2(V_{KH}^2 - U^2)}[k_+ \sin(\omega t - k_+ x) + k_- \sin(\omega t - k_- x)] \\
&\qquad + 4\bar{\nu}\omega\rho_1 A[(k_y^2 + k_+^2)\sin(\omega t - k_+ x) - (k_y^2 + k_-^2)\sin(\omega t - k_- x)].
\end{aligned} \tag{3.13}$$

It follows from (2.4) that ζ_2 must satisfy the boundary conditions

$$\zeta_2 = 0 \quad \text{at } x = 0, L. \tag{3.14}$$

The left-hand side of (3.13) coincides with (3.8). This implies that ζ_1 is a solution to the homogeneous counterpart of the boundary value problem constituted by (3.13) and the boundary conditions (3.14). Then the inhomogeneous boundary value problem has solutions only if the right-hand side of (3.13) satisfies the compatibility condition, which is the condition that it is orthogonal to ζ_1. To obtain the compatibility condition we multiply (3.13) by ζ_1, integrate with respect to x and use (3.11) and the boundary conditions (3.14). Then, after long but straightforward calculation we obtain

$$\begin{aligned}
&\sin\left(2\omega t - \frac{L(k_+ + k_-)}{2}\right)\sin\frac{L(k_+ + k_-)}{2} \\
&\quad \times \left(\frac{\mathrm{d}A}{\mathrm{d}T}\sqrt{\rho_1\rho_2(V_{KH}^2 - U^2)} - \frac{2\bar{\nu}\omega\rho_1 A(k_- - k_+)(k_- k_+ - k_y^2)}{k_- k_+}\right) = 0.
\end{aligned} \tag{3.15}$$

Obviously the first multiplier on the right-hand side of this equation cannot be equal to zero for all values of t. If we assume that the second multiplier is zero, then it follows from this condition and (3.13) that C_+/C_- is a rational number, which is only possible for a countable set of parameters. Eliminating this set from the consideration we obtain that the second multiplier is not equal to zero. Hence, the only possibility left is that the third multiplier is zero, which gives

$$\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}T} = \frac{2\bar{\nu}\omega\rho_1(\boldsymbol{k}_- - \boldsymbol{k}_+)(\boldsymbol{k}_- \boldsymbol{k}_+ - k_y^2)}{\boldsymbol{k}_- \boldsymbol{k}_+ \sqrt{\rho_1\rho_2(V_{KH}^2 - U^2)}}. \tag{3.16}$$

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Now, returning to the original non-scaled time and shear viscosity we obtain after some algebra

$$A = A_0 e^{\gamma t}, \tag{3.17}$$

where A_0 is the initial value of A and

$$\gamma = 4\nu\rho_1 \left(\frac{\pi^2 n^2 (U^2 - U_c^2)}{\rho_2 L^2 U^2} - \frac{k_y^2}{\rho_1 + \rho_2} \right). \tag{3.18}$$

We see that for sufficiently small k_y satisfying

$$k_y^2 < \frac{\pi^2 n^2 (\rho_1 + \rho_2)(U^2 - U_c^2)}{\rho_2 L^2 U^2}, \tag{3.19}$$

$\gamma > 0$ and the amplitude of the standing wave grows exponentially.

As we have already seen the standing wave is a superposition of the negative energy wave with the wave vector $\boldsymbol{k}_-$ and the positive energy wave with the wave vector $\boldsymbol{k}_+$. Using (3.3)–(3.5) to calculate the increment γ_- for the first wave and the decrement γ_+ for the second wave we easily obtain

$$\gamma = \gamma_- - \gamma_+. \tag{3.20}$$

4. Summary and conclusions

In this article we studied the dissipative instability of standing waves caused by the flow. We considered a simple equilibrium where two incompressible plasmas are separated by a tangential discontinuity. There is the plasma flow with constant velocity parallel to the discontinuity at one of its sides while the plasma at the other side is at rest. There is also magnetic field parallel to the flow velocity. The plasma that is at rest is viscous, while the moving plasma is ideal. We only consider long waves satisfying the condition that the Reynolds number is large. In the linear approximation these waves are described by (3.1) derived by Ruderman & Goossens (1995).

When the flow speed U exceeds the Kelvin–Helmholtz threshold V_{KH} the discontinuity is subject to the Kelvin–Helmholtz instability. When $U < V_{KH}$ there are two surface wave modes that can propagate on the surface of the discontinuity. In the absence of flow these modes propagate in the opposite directions. We call the mode propagating in the flow velocity direction forward, and the mode propagating in the opposite direction backward. When $U_c < U < V_{KH}$ both modes propagate in the same direction. In this case the forward mode is a positive energy wave, while the backward mode is a negative energy wave. In the absence of viscosity both modes are neutrally stable. However viscosity causes the forward mode to decay, and the backward mode to grow. This growth of the backward mode is called the negative energy wave instability.

We assumed that the magnetic field lines are frozen in a dense plasma at two planes perpendicular to the flow direction lines. These plane are a distance L from each other. This, in particular, implies that the tangential discontinuity is fixed at two lines perpendicular to the flow direction that are a distance L from each other. Then we consider surface waves that are standing both in the direction of the flow velocity and in direction orthogonal to the flow velocity. A standing wave is a linear superposition of a forward wave and a backward wave. We restricted our analysis to the case where $U_c < U < V_{KH}$ and showed that the standing wave can grow due to the presence of viscosity. However it is only possible when its wavelength in the direction orthogonal to the flow velocity is sufficiently large. The increment of the standing wave is equal to the difference between the increment of the backward wave and the decrement of the forward wave.

We emphasise that our analysis is only valid for slowly growing waves with the increment much smaller than the wave frequency. In the case when the increment is of the order of the wave frequency the stability analysis must be modified substantially.

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