

ON THE FATOU THEOREM FOR $\bar{\partial}_J$ -SUBSOLUTIONS IN WEDGES

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(Received 7 February 2022; first published online 17 August 2022)

Abstract We prove a version of the Fatou theorem for bounded functions with a bounded $\bar{\partial}_J$ part of the differential on wedge-type domains in an almost complex manifold.

Keywords: almost complex manifold; $\bar{\partial}$ -operator; pseudoholomorphic disc; wedge-type domain; totally real manifold; the Fatou theorem

2020 *Mathematics subject classification* Primary 32H02; 53C15

1. Introduction

The present paper is a continuation of the work [13]. Our goal is to study the boundary behaviour of certain classes of functions on almost complex manifolds with boundary. It is well known that non-constant holomorphic functions do not exist (even locally) on an almost complex manifold (M, J) (of complex dimension > 1) with an almost complex structure J in general position. This makes it natural to study the functions satisfying suitable assumptions on the $\bar{\partial}_J$ -part of their differential: indeed, in this case, the problem of existence does not arise. Various aspects of the boundary behaviour of functions in \mathbb{C}^n whose $\bar{\partial}$ (with respect to the standard complex structure) part of the differential is of some prescribed growth, have been explored by several authors [2, 5, 8–10]. Their results admit important applications in several complex variables.

We extend some of the well-known results on boundary values of bounded holomorphic functions (see [1, 2, 4, 11]) of several complex variables to the almost complex case. Note that our main results are new also in the case of the space \mathbb{C}^n equipped with the standard complex structure. The main result is Theorem 3.2 establishing a Fatou type theorem for domains with generic corners (wedges). I also mention that, despite the fact the obtained results concern the classes of functions much larger than the holomorphic ones, the presence of an (almost) complex structure is crucial. In particular, this is due to the fact that we are working with low-dimensional submanifolds of the boundary which are transverse to an almost complex structure (totally real manifolds). Also, pseudoholomorphic curves (introduced in [3]) are our main technical tool. We note that the main difficulty of the

proof is that in the case of wedge-type domains the Chirka–Lindelöf principle does not assure a non-tangential convergence. This obstacle is a principal difference with respect to the case of smooth boundaries and it considerably complicates the proof. This is the main motivation for the present paper.

The paper is organized as follows. Section 2 is preliminary and contains a brief presentation of the theory of almost complex manifolds and their properties. In § 3, we present our main result. Section 4 contains its proof.

2. Preliminaries: almost complex manifolds and their maps

Here, we briefly recall basic notions concerning almost complex manifolds; a detailed presentation is contained for example in [13]. Everywhere through this paper, we assume that manifolds and almost complex structures are of class C^∞ (the word ‘smooth’ means the regularity of this class); we notice however that the main results remain true under considerably weaker regularity assumptions.

Let M be a smooth manifold of real dimension $2n$. An almost complex structure J on M is a smooth map which associates to every point $p \in M$ a linear isomorphism $J(p) : T_pM \rightarrow T_pM$ of the tangent space T_pM such that $J(p)^2 = -Id$; here Id denotes the identity map of T_pM . Thus, every linear map $J(p)$ is a complex structure (in the usual sense of Linear Algebra) on a real vector space T_pM . A couple (M, J) is called an almost complex manifold of complex dimension n .

A basic example is given by the standard complex structure $J_{st} = J_{st}^{(2)}$ on $M = \mathbb{R}^2$; it is represented in the canonical coordinates of \mathbb{R}^2 by the matrix

$$J_{st}^{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

More generally, the standard complex structure J_{st} on \mathbb{R}^{2n} is represented by the block diagonal matrix $J_{st} = \text{diag}(J_{st}^{(2)}, \dots, J_{st}^{(2)})$ (here and below, we drop the notation of dimension). Putting $iv := Jv$ for $v \in \mathbb{R}^{2n}$, we identify $(\mathbb{R}^{2n}, J_{st})$ with \mathbb{C}^n ; we use the notation $z = x + iy = x + Jy$ for the standard complex coordinates $z = (z_1, \dots, z_n) \in \mathbb{C}^n$.

Let (M, J) and (M', J') be smooth almost complex manifolds. A C^1 -map $f : M' \rightarrow M$ is called (J', J) -complex or (J', J) -holomorphic if it satisfies the Cauchy–Riemann equations

$$df \circ J' = J \circ df. \tag{2}$$

For example, a map $f : \mathbb{C}^n \rightarrow \mathbb{C}^m$ is (J_{st}, J_{st}) -holomorphic if and only if each component of f is a usual holomorphic function. In this special case, Equation (2) coincides with the usual Cauchy–Riemann equations in their real form. Note that in general, the first-order PDE elliptic system (2) does not split on independent equations for components of f .

Every almost complex manifold (M, J) can be viewed locally as the Euclidean unit ball \mathbb{B}^n (or any other domain) in \mathbb{C}^n equipped with a small (in any C^m -norm) almost complex deformation of J_{st} . The following well-known statement is often very useful.

Lemma 2.1. *Let (M, J) be an almost complex manifold of complex dimension n . Then, for every point $p \in M$, every $m \geq 0$ and $\lambda_0 > 0$ there exists a neighbourhood U of p and a coordinate diffeomorphism $z : U \rightarrow \mathbb{B}^n$ such that $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$, and the direct image $z_*(J) := dz \circ J \circ dz^{-1}$ satisfies $\|z_*(J) - J_{st}\|_{C^m(\overline{\mathbb{B}^n})} \leq \lambda_0$.*

A simple proof is contained for example in [13].

In what follows we often denote the direct image $z_*(J)$ of J again by J , viewing it as a matrix representation of J in the local coordinate system (z) . Of course, the coordinate map z is $(J, z_*(J))$ -biholomorphic. However, in general, $z_*(J)$ does not coincide with J_{st} in a neighbourhood of the origin in \mathbb{C}^n . Recall that an almost complex structure J is called *integrable* if (M, J) is locally biholomorphic in a neighbourhood of each point to an open subset of (\mathbb{C}^n, J_{st}) . In the case of complex dimension 1, every almost complex structure is integrable. In the case of complex dimension > 1 integrable almost complex structures form a highly special subclass in the space of all almost complex structures on M . An efficient criterion of integrability is provided by the classical theorem of Newlander–Nirenberg [6]: the entries of J must satisfy some PDE system.

In the special case where M' has the complex dimension 1, the solutions f of (2) are called *J -complex (or J -holomorphic or pseudoholomorphic) curves*. Note that we view here the curves as maps, i.e., we consider parametrized curves. We use the notation $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ for the unit disc in \mathbb{C} (i.e., \mathbb{B}^1) always assuming that it is equipped with the standard complex structure J_{st} . Considering Equation (2) with $M' = \mathbb{D}$, we call such a map f a *J -complex disc* or a *pseudoholomorphic disc* or just a *holomorphic disc* when a structure J on the target space is fixed. If a disc f is continuous up to the boundary $b\mathbb{D}$ of \mathbb{D} , then the restriction of f on $b\mathbb{D}$ is called *the boundary* of f . Let γ be a non-empty subset of $b\mathbb{D}$ and let K be a subset of M . If $f(\gamma) \subset K$, we say that f is *attached or glued* to K along γ . In this paper, γ usually will be the upper half-circle.

A fundamental fact is that pseudoholomorphic discs always exist in a suitable neighbourhood of any point of $p \in M$; furthermore, one can choose such a disc tangent to any prescribed direction $v \in T_p M$. These discs depend smoothly on the deformation of J , p and v . Furthermore, one can view them as a small deformation of discs in usual complex lines. This is the classical Nijenhuis–Wolf theorem (see [7]). For the proof and other applications, it is convenient to rewrite Equation (2) in local coordinates similarly to the complex version of the usual Cauchy–Riemann equations.

Our considerations are local, so assume that we are in a neighbourhood Ω of 0 in \mathbb{C}^n with the standard complex coordinates $z = (z_1, \dots, z_n)$. We assume that J is an almost complex structure defined on Ω and $J(0) = J_{st}$. Let a C^1 -map

$$\begin{aligned} z &: \mathbb{D} \rightarrow \Omega, \\ z &: \zeta \mapsto z(\zeta) \end{aligned}$$

be a J -complex disc. Equation (2) can be rewritten in the equivalent form

$$z_{\bar{\zeta}} - A(z)\bar{z}_{\bar{\zeta}} = 0, \quad \zeta \in \mathbb{D}. \quad (3)$$

where we use the notation $z_{\bar{\zeta}} = \partial z / \partial \bar{\zeta}$. Here, a smooth map $A : \Omega \rightarrow \text{Mat}(n, \mathbb{C})$ is defined by the equality $L(z)v = A\bar{v}$ for any vector $v \in \mathbb{C}^n$ and L is an \mathbb{R} -linear map defined by

$L = (J_{st} + J)^{-1}(J_{st} - J)$. It is easy to check that the condition $J^2 = -Id$ is equivalent to the fact that L is \mathbb{C} -linear. The matrix $A(z)$ is called *the complex matrix* of J in the local coordinates z (see [14]). Locally the correspondence between A and J is one-to-one. Note that the condition $J(0) = J_{st}$ means that $A(0) = 0$.

If t are other local coordinates and A' is the corresponding complex matrix of J in the coordinates t , then, as it is easy to check, we have the following transformation rule:

$$A' = (t_z A + t_{\bar{z}})(\bar{t}_{\bar{z}} + \bar{t}_z A)^{-1} \tag{4}$$

(see the proof in [14]).

For the convenience of readers, I sketch here the proof of the above-mentioned Nijenhuis–Wolf theorem because this standard construction will be used below in the proof of the main results. Recall that for a complex function f the Cauchy–Green transform Tf is defined by

$$Tf(\zeta) = \frac{1}{2\pi i} \int_{\mathbb{D}} \frac{f(\omega) d\omega \wedge d\bar{\omega}}{\omega - \zeta} \tag{5}$$

This classical integral operator has the following properties:

- (i) $T : C^r(\mathbb{D}) \rightarrow C^{r+1}(\mathbb{D})$ is a bounded linear operator for every non-integer $r > 0$ (a similar property holds in the Sobolev scale). Here, we use the usual Hölder norm on the space $C^r(\mathbb{D})$.
- (ii) $(Tf)_{\bar{\zeta}} = f$, i.e., T solves the $\bar{\partial}$ -equation in the unit disc.
- (iii) the function Tf is holomorphic on $\mathbb{C} \setminus \bar{\mathbb{D}}$.

Now, fix a real non-integer $r > 1$. Let $z : \mathbb{D} \rightarrow \mathbb{C}^n$, $z : \mathbb{D} \ni \zeta \mapsto z(\zeta)$ be a J -complex disc. Since the operator

$$\Psi_J : z \longrightarrow w = z - TA(z)\bar{z}_{\bar{\zeta}}$$

takes the space $C^r(\bar{\mathbb{D}})$ into itself, we can write equation (3) in the form

$$\Psi_J(z) = h$$

where h is an arbitrary holomorphic (with respect to J_{st}) vector-function. Thus, the disc z is J -holomorphic if and only if the disc $h = \Psi_J(z) : \mathbb{D} \rightarrow \mathbb{C}^n$ is J_{st} -holomorphic. When the norm of A is small enough (which is assured by Lemma 2.1), then the operator Ψ_J is a small deformation of the identity and by the implicit function theorem, this operator is invertible. Hence we obtain a one-to-one correspondence between J -holomorphic discs and usual J_{st} -holomorphic discs. This easily implies the existence of a J -holomorphic disc in a given tangent direction through a given point of M (choosing a suitable complex linear disc as h), as well as a smooth dependence of such a disc on a deformation of a point or a tangent vector, or on an almost complex structure; this also establishes the interior elliptic regularity of discs.

Now, we can define the $\bar{\partial}_J$ -operator on an almost complex manifold (M, J) . Consider first the situation when J be an almost complex structure defined in a domain $\Omega \subset \mathbb{C}^n$; one can view this as a local coordinate representation of J in a chart on M .

A C^1 function $F : \Omega \rightarrow \mathbb{C}$ is (J, J_{st}) -holomorphic if and only if it satisfies the Cauchy–Riemann equations

$$F_{\bar{z}} + F_z A(z) = 0, \quad (6)$$

where $F_{\bar{z}} = (\partial F / \partial \bar{z}_1, \dots, \partial F / \partial \bar{z}_n)$ and $F_z = (\partial F / \partial z_1, \dots, \partial F / \partial z_n)$ are viewed as row-vectors. Generally, the only solutions to (6) are constant functions unless J is integrable (then A vanishes identically in suitable coordinates). Note also that (6) is an overdetermined linear PDE system while (3) is a quasilinear PDE for a vector function on \mathbb{D} .

Every 1-differential form ϕ on (M, J) admits a unique decomposition $\phi = \phi^{1,0} + \phi^{0,1}$ with respect to J . In particular, if $F : (M, J) \rightarrow \mathbb{C}$ is a C^1 -complex function, we have $dF = dF^{1,0} + dF^{0,1}$. We use the notation

$$\partial_J F = dF^{1,0} \text{ and } \bar{\partial}_J F = dF^{0,1} \quad (7)$$

To write these operators explicitly in local coordinates, we find a local basis in the space of (1,0) and (0,1) forms. We view $dz = (dz_1, \dots, dz_n)^t$ and $d\bar{z} = (d\bar{z}_1, \dots, d\bar{z}_n)^t$ as vector-columns. Then, the forms

$$\alpha = (\alpha_1, \dots, \alpha_n)^t = dz - Ad\bar{z} \text{ and } \bar{\alpha} = d\bar{z} - \bar{A}dz \quad (8)$$

form a basis in the space of (1,0) and (0,1) forms respectively. Indeed, it suffices to note that for 1-form β is (1,0) (respectively (0, 1)) for if and only if for every J -holomorphic disc z the pull-back $z^* \beta$ is a usual (1,0) (respectively (0, 1)) form on \mathbb{D} . Using the equations, (3) we obtain the claim.

Now, we decompose the differential $dF = F_z dz + F_{\bar{z}} d\bar{z} = \partial_J F + \bar{\partial}_J F$ in the basis $\alpha, \bar{\alpha}$ using (8) and obtain the explicit expression

$$\bar{\partial}_J F = (F_{\bar{z}}(I - \bar{A}A)^{-1} + F_z(I - A\bar{A})^{-1}A)\bar{\alpha} \quad (9)$$

It is easy to check that the holomorphy condition $\bar{\partial}_J F = 0$ is equivalent to (6) because $(I - A\bar{A})^{-1}A(I - \bar{A}A) = A$. Thus

$$\bar{\partial}_J F = (F_{\bar{z}} + F_z A)(I - \bar{A}A)^{-1}\bar{\alpha}$$

We note that the term $(I - A\bar{A})^{-1}$ as well as the forms α affect only the non-essential constants in local estimates of the $\bar{\partial}_J$ -operator near a boundary point which we will perform in the next sections. So, we may assume that in local coordinates this operator is simply given by the left-hand expression of (6).

3. Main result

First, we introduce the main class of domains for this paper.

Let M be an almost complex manifold. A generic manifold E of real codimension k in M can be defined as

$$E = \{p \in M : \rho_j(p) = 0, j = 1, \dots, k\} \tag{10}$$

where $\rho_j : M \rightarrow \mathbb{R}$ are smooth real functions satisfying

$$\bar{\partial}_J \rho_1 \wedge \dots \wedge \bar{\partial}_J \rho_k \neq 0 \tag{11}$$

near E . This condition means that the complex linear parts (with respect to J) of differentials of ρ_j are \mathbb{C} -linearly independent. Precisely as in the case of an integrable structure, this means that the tangent space $T_p E$ spans $T_p M$, i.e., the complex hull of $T_p E$ coincides with $T_p M$. If additionally $k = n$ (the maximal possible value of k compatible with the assumption (11)), then E is *totally real*. It is equivalent to the fact that for every p the holomorphic tangent space $T_p E \cap J(T_p E)$ is trivial.

A domain $W = W(E)$ of the form

$$W(E) = \{p \in M : \rho_j(p) < 0, j = 1, \dots, k\} \tag{12}$$

is called a wedge with the edge E . Of course, if $k = 1$ we have the usual smoothly bounded domains.

Let Ω be a bounded domain in an almost complex manifold (M, J) . We always assume that Ω is a wedge $W(E)$ of type (12) with the edge E .

Fix a hermitian metric on M compatible with J ; a choice of such metric will not affect our results because it changes only constant factors in estimates. We measure all distances and norms with respect to the chosen metric.

Let $p \in E$ be a point of the edge. Fix local coordinates z on M near p such that $p = 0$ in these coordinates. A *cone* $K \subset \Omega$ with the vertex p is defined as the set of $z \in \Omega$ such in the above local coordinates K is a usual circular cone with vertex at the origin and directed by some ray $l \subset \Omega$.

A *non-tangential approach* to E at p can be defined as the limit along the sets K . Clearly, this notion is independent of the choice of local coordinates.

Definition 3.1. A function $F : \Omega \rightarrow \mathbb{C}$ admits a non-tangential limit L at $p \in E$ if

$$\lim_{K \ni z \rightarrow p} F(z) = L$$

for each cone $K \subset \Omega$ with vertex at p .

As above, this definition is independent of a choice of local coordinates and metrics.

The main result of the present paper is the following version of the Fatou theorem.

Theorem 3.2. *Let (M, J) be an almost complex manifold of complex dimension $n \geq 1$ and $W(E)$ be a wedge with a totally real edge E in M . Suppose that $F \in L^\infty(W(E))$ is a complex function of class C^1 on $W(E)$ and $\bar{\partial}_J F$ is bounded on $W(E)$. Then, F admits a non-tangential limit almost everywhere on E .*

Of course, the interesting case arises only for $n > 1$. Note that this result is new also in the case where $M = \mathbb{C}^n$ and J coincides with the standard complex structure J_{st} . Note

also that in the case where the edge E is not totally real but only a generic manifold with non-zero tangent space, the convergent regions are tangent to E along the holomorphic tangent space of E , as usual in this type of problems (see [1, 2, 11, 13]). The assumption of the boundedness of $\bar{\partial}_J F$ also may be weakened. We drop the technical details focusing our presentation on the key case.

4. Proof of Theorem 3.2

Our approach is based on the works [1, 4, 11, 13]. The proof of Theorem contains several steps.

4.1. One-dimensional case

Recall some boundary properties of subsolutions of the $\bar{\partial}$ -operator in the unit disc.

Denote by $W^{k,p}(\mathbb{D})$ the usual Sobolev classes of functions admitting generalized partial derivatives up to the order k in $L^p(\mathbb{D})$ (in fact we need only the case $k = 0$ and $k = 1$). In particular $W^{0,p}(\mathbb{D}) = L^p(\mathbb{D})$. We will always assume that $p > 2$.

Denote also by $\|f\|_\infty = \sup_{\mathbb{D}} |f|$ the usual sup-norm on the space $L^\infty(\mathbb{D})$ of complex functions bounded on \mathbb{D} .

Lemma 4.1. *Let $f \in L^\infty(\mathbb{D})$ and $f_{\bar{\zeta}} \in L^p(\mathbb{D})$ for some $p > 2$. Then*

- (a) *f admits a non-tangential limit at almost every point $\zeta \in b\mathbb{D}$.*
- (b) *if f admits a limit along a curve in \mathbb{D} approaching $b\mathbb{D}$ non-tangentially at a boundary point $e^{i\theta} \in b\mathbb{D}$, then f admits a non-tangential limit at $e^{i\theta}$.*
- (c) *for each positive $r < 1$ there exists a constant $C = C(r) > 0$ (independent of f) such that for every $\zeta_j \in r\mathbb{D}$, $j = 1, 2$ one has*

$$|f(\zeta_1) - f(\zeta_2)| \leq C(\|f\|_\infty + \|f_{\bar{\zeta}}\|_{L^p(\mathbb{D})})|\zeta_1 - \zeta_2|^{1-2/p} \tag{13}$$

The proof is contained in [13].

Sometimes it is convenient to apply the part (c) of Lemma on the disc $\rho\mathbb{D}$ with $\rho > 0$. Let $g \in L^\infty(\rho\mathbb{D})$ and $g_{\bar{\zeta}} \in L^p(\rho\mathbb{D})$. The function $f(\zeta) := g(\rho\zeta)$ satisfies the assumptions of Lemma 4.1 on \mathbb{D} . Let $0 < \alpha < \rho$ and let $|\tau_j| < \alpha$, $j = 1, 2$. Set $\zeta_j = \tau_j/\rho$. Then, $|\zeta_j| < r = \alpha/\rho < 1$, $j = 1, 2$. Applying (c) Lemma 4.1 to f , we obtain:

$$|g(\tau_1) - g(\tau_2)| \leq (C(r)/\rho^{1-2/p})(\|g\|_\infty + \rho \|g_{\bar{\zeta}}\|_{L^p(\rho\mathbb{D})})|\tau_1 - \tau_2|^{1-2/p} \tag{14}$$

Note that $C = C(r) = C(\alpha/\rho)$ depends only on the quotient $r = \alpha/\rho < 1$. If r is separated from 1, the value of C is fixed.

4.2. Attaching discs to a totally real manifold

Our proof of Theorem 3.2 uses the properties of a family of pseudoholomorphic discs constructed in [12]. For the sake of completeness, I briefly recall this construction; the proofs are contained in [12]. Note also that this construction is well known in the case of the standard complex structure.

- (a) First consider the model case where $M = \mathbb{C}^n$ with $J = J_{st}$ and $E = i\mathbb{R}^n = \{x_j = 0, j = 1, \dots, n\}$. Denote by W the standard wedge $W_0 = \{z = x + iy : x_j < 0, j = 1, \dots, n\}$.

Consider the family of complex lines in \mathbb{C}^n :

$$l : (c, t, \zeta) \mapsto (\zeta, \zeta t + ic) \in \mathbb{C}^n \tag{15}$$

Here, $\zeta \in \mathbb{C}$; the variables $c = (c_2, \dots, c_n) \in \mathbb{R}^{n-1}$ and $t \in \mathbb{R}_+^{n-1} = \{t = (t_2, \dots, t_n) \in \mathbb{R}^{n-1} : t_j > 0\}$ are viewed as parameters. Hence we write $l(c, t, \zeta) = l(c, t)(\zeta)$. Denote by V the wedge $V = \mathbb{R}^{n-1} \times \mathbb{R}_+^{n-1}$. Also let $II = \{\text{Re } \zeta < 0\}$ be the left half-plane; its boundary bII coincides with the imaginary axis $i\mathbb{R}$. The following properties of the above family are easy to check:

- (a1) the images $l(c, t)(bII)$ form a family of real lines in $i\mathbb{R}^n = E$. For every fixed $t \in \mathbb{R}_+^{n-1}$, these lines are disjoint and

$$\cup_{c \in \mathbb{R}^{n-1}} l(c, t)(bII) = E.$$

In other words, for every t , this family (depending on the parameter c) forms a foliation of E by parallel lines.

- (a2) one has

$$\cup_{(c,t) \in V} l(c, t)(II) = W_0.$$

- (a3) For every fixed $t \in \mathbb{R}_+^{n-1}$, one has

$$\cup_{c \in \mathbb{R}^{n-1}} l(c, t)(II) = E_t = \{z \in \mathbb{C}^n : \text{Re}(z_j - t_j z_1) = 0, j = 2, \dots, n\} \cap W_0$$

and the union is disjoint. Every E_t is a real linear $(n + 1)$ -dimensional half-space contained in W_0 and $bE_t = E$.

- (a4) the family $(E_t), t \in \mathbb{R}_+^{n-1}$ is disjoint in W_0 and its union coincides with W_0 .

In what follows we will use these properties locally in a neighbourhood of the origin. It is convenient to reparametrize the family of complex half-lines $l(c, t)$ by complex discs. Consider the Schwarz integral:

$$S\phi(\zeta) = \frac{1}{2\pi i} \int_{b\mathbb{D}} \frac{\omega + \zeta}{\omega - \zeta} \phi(\omega) \frac{d\omega}{\omega} \tag{16}$$

For a non-integer $r > 1$ consider the Banach spaces $C^r(b\mathbb{D})$ and $C^r(\mathbb{D})$ (with the usual Hölder norm). It is classical that S is a bounded linear map in these classes of functions. For a real function $\phi \in C^r(b\mathbb{D})$ the Schwarz integral $S\phi$ is a function of class $C^r(\mathbb{D})$ holomorphic in \mathbb{D} ; the trace of its real part on the boundary coincides with ϕ and its imaginary part vanishes at the origin.

To fill W_0 by complex discs glued to $i\mathbb{R}^n$ along the (closed) upper semi-circle $b\mathbb{D}^+ = \{e^{i\theta} : \theta \in [0, \pi]\}$ we have to reparametrize the above family of complex lines. Set also $b\mathbb{D}^- := b\mathbb{D} \setminus b\mathbb{D}^+$. Fix a smooth real function $\phi : b\mathbb{D} \rightarrow [-1, 0]$ such that $\phi|_{b\mathbb{D}^+} = 0$ and $\phi|_{b\mathbb{D}^-} < 0$.

Consider now a real $2n$ -parametric family of holomorphic discs $z^0 = (z_1^0, \dots, z_n^0) : \mathbb{D} \rightarrow \mathbb{C}^n$ with components

$$z_j^0(c, t)(\zeta) = x_j(\zeta) + iy_j(\zeta) = t_j S\phi(\zeta) + ic_j, j = 1, \dots, n \tag{17}$$

Here, $t_j > 0$ and $c_j \in \mathbb{R}$ are parameters

Obviously, every $z^0(c, t)(\mathbb{D})$ is a subset of $l(c, t)(II)$ and $z^0(b\mathbb{D}^+) = l(c, t)(bII)$. Thus, the family $z^0(c, t)$ is a (local) biholomorphic reparametrization of the family $l(c, t)$. As a consequence, the properties (a1)–(a5) also hold for the family $z^0(c, t)$. Notice also the following obvious properties of this family:

- (a6) for every j , one has $x_j|_{b\mathbb{D}^+} = 0$ and $x_j(\zeta) < 0$ when $\zeta \in \mathbb{D}$ (by the maximum principle for harmonic functions).
- (a7) the evaluation map $Ev_0 : (c, t, \zeta) \mapsto z^0(c, t)(\zeta)$ is one-to-one from $V \times \mathbb{D}$ to W_0 .

In the general case consider a totally real manifold E and the wedge $W = W(E)$ given by (12). Applying the implicit function theorem in suitable local coordinates, one can assume that E is defined by the vector equation $x = h(y)$ where $h(0) = 0$, and $dh(0) = 0$. Using the Cauchy–Green operator and the Schwarz integral, one can write a nonlinear integral equation such that its solutions of the form

$$(c, t, \zeta) \mapsto z(c, t)(\zeta) \tag{18}$$

are J -complex discs glued to E along $b\mathbb{D}^+$. Since $h(y) = o(|y|)$, the family $z(c, t)$ is a small deformation of the family $z^0(c, t)$ in any C^m norm, $m > 1$. This follows by the implicit function theorem solving the above-mentioned integral equation (see details in [13]). Hence, the geometric properties of obtained discs remain similar to the above model case: indeed, the properties of linear discs (a1)–(a5) are stable under small perturbations.

- (b) For reader’s convenience, we state explicitly the properties of the family (18). Fix $\delta > 0$. The family $z(c, t) : \mathbb{D} \rightarrow W$ of pseudoholomorphic discs smooth on $\overline{\mathbb{D}}$ and smoothly depending on real parameters $t = (t_2, \dots, t_n)$, $t_j > 0$ and $c \in \mathbb{R}^{n-1}$, satisfies the following properties:
 - (b1) the images $z(c, t)(b\mathbb{D}^+)$ form a family of real curves in E . For every fixed $t \in \mathbb{R}_+^{n-1}$, these curves are disjoint and

$$\cup_{c \in \mathbb{R}^{n-1}} z(c, t)(b\mathbb{D}^+) = E.$$

In other words, for every t , this family (depending on the parameter t) forms a foliation of E . Furthermore, every disc is contained in $W = W(E)$.

- (b2) one has the inclusion

$$W_\delta = \{z : \rho_j - \delta \sum_{k \neq j} \rho_k < 0\} \subset \cup_{(c,t) \in V} z(c, t)(\mathbb{D}).$$

(b3) For every fixed $t \in \mathbb{R}_+^{n-1}$, the union

$$E_t := \cup_{c \in \mathbb{R}^{n-1}} z(c, t)(\mathbb{D}) \subset W$$

is a real generic $(n + 1)$ -dimensional manifold with boundary $bE_t = E$.

(b4) the family (E_t) , $t \in \mathbb{R}_+^{n-1}$ is disjoint and its union contains W_δ .

4.3. Around the Chirka–Lindelöf principle

Here, we introduce an analog of the Chirka–Lindelöf principle [1] for wedges in almost complex manifolds. This is one of the main technical tools of our proof. Note that in [1] the situation is considered in full generality (for integrable complex structures) with minimal assumptions on the regularity of domains. It is observed there that in the case of domains with piecewise smooth boundaries, for a bounded holomorphic function an existence of a boundary limit along a smooth curve (transverse to the boundary) does not imply an existence of a non-tangential limit at a boundary point. This is a serious difference with respect to smoothly bounded domains and one of the main technical obstacles in the proof of our main result. Nevertheless, in the non-smooth case, the convergence along a curve implies an existence of the limit along any curve with the same tangent line at a boundary point.

Let $W = W(E)$ be a wedge with the edge E in an almost complex manifold (M, J) of dimension > 1 and let p be a point of E .

A curve $\gamma : [0, 1] \rightarrow W(E)$ is called *p-admissible* if the following assumptions are satisfied:

- (i) γ of class $C^\infty[0, 1]$ and $p = \gamma(1) \in E$;
- (ii) γ is not tangent to any face $\{\rho_j = 0\}$, $j = 1, \dots, n$ at p . In particular, γ is not tangent to E .

Proposition 4.2. *Let $W(E)$ be a wedge with the edge E in an almost complex manifold (M, J) of dimension > 1 , and let F satisfies assumptions of Theorem 3.2. If F has a limit along a p-admissible curve γ_1 at $p \in E$, then F has the same limit along each curve in $W(E)$ tangent to γ_1 at p .*

Proof. Let γ_2 be another curve satisfying the assumptions (i), (ii) and such that γ_1 and γ_2 have the same tangent line at p . Without loss of generality assume that $p = 0$ (in local coordinates).

It follows by the Nijenhuis–Woelf theorem that there exists a family $z_t(\zeta) : \mathbb{D} \rightarrow \mathbb{C}^n$, of embedded J -holomorphic discs near the origin in \mathbb{C}^n satisfying the following properties:

- (iii) the family z_t is smooth on $\bar{\mathbb{D}} \times [0, 1]$
- (iv) for every $t \in [0, 1]$, the disc z_t transversally intersects each curve γ_j at a unique point corresponding to some parameter value $\zeta_j(t) \in \mathbb{D}$, $t \in [0, 1]$, $j = 1, 2$. In other words $\gamma_j(t) = z_t(\zeta_j(t))$. Furthermore, $\zeta_1(t) = 0$, i.e., this point is the center of the disc z_t .

In the case of the standard complex structure each, such a disc is simply an open piece (suitably parametrized) of a complex line intersecting transversally the both of curves γ_j . Recall that the curves are embedded near the origin and tangent at the origin so such a family of complex lines obviously exist. The J -holomorphic discs are obtained from this family of lines by a small deformation described in the proof of the Nijenhuis–Woolf theorem in § 2.

Furthermore, because of the condition (iii), the restrictions $F \circ z_t$ have $\bar{\zeta}$ -derivatives bounded on \mathbb{D} uniformly with respect to t . Indeed, it follows by the Chain Rule and (3) that

$$(F \circ z)_{\bar{\zeta}} = (F_{\bar{z}} + F_z A) \bar{z}_{\bar{\zeta}}$$

and now we use the assumption that $\bar{\partial}_J F$ is bounded.

Since the curves γ_j are tangent at the origin, we have

$$|\zeta_2(t)| = o(1-t) \tag{19}$$

as $t \rightarrow 1$.

The curve γ_1 is admissible, so we have

$$\text{dist}(\gamma_1(t), bW) = O(1-t)$$

as $t \rightarrow 1$. Hence, there exists $\rho(t) = O(1-t)$ as $t \rightarrow 1$ such that $z_t(\rho(t)\mathbb{D})$ is contained in W . Fix a real $r > 0$ which appears in (14). Applying (14) to the composition $f := F \circ z_t(\zeta)$ on the disc $\rho(t)\mathbb{D}$, we obtain

$$|f(0) - f(\zeta_2(t))| \leq (C/O(1-t)^{1-2/p})(\|f\|_{\infty} + O(1-t) \|f_{\bar{\zeta}}\|_{\infty})o((1-t)^{1-2/p}) \rightarrow 0 \tag{20}$$

as $t \rightarrow 1$. Note that by (19) for every t , the point $\zeta_2(t)$ is contained in $(1/2)\rho(t)\mathbb{D}$; hence, the constant C is independent of t (see remark after (14)). This concludes the proof. \square

4.4. Convergence along families of rays

Under some additional assumptions, one can assure a non-tangential convergence. Fix local coordinates such that $E = i\mathbb{R}^n$, $W = W(E) = \{x_j < 0, j = 1, \dots, n\}$, $J(0) = J_{st}$. Such a change of coordinates is always possible by Lemma 2.1. *Through the remaining part of this paper, we assume that such local coordinates are fixed.*

Let $z(c, t)(\zeta)$ be a J -complex disc constructed in § 4.2. It follows by Lemma 4.1 that the composition $F \circ z(c, t)$ admits non-tangential limits almost everywhere on $b\mathbb{D}^+$.

Let $p \in E$. Assume $F \circ z(c, t)$ admits a radial limit at point $\zeta^0 \in b\mathbb{D}^+$ and $z(c, t)(\zeta^0) = p$. The image γ_p of this radial segment by disc $z(c, t)$ in general is not a segment of some real line in \mathbb{C}^n , but only an admissible curve. Hence it follows by Proposition 4.2 that F admits the same limit along the real ray $l_p \subset W$ with vertex at p and tangent to γ_p at p . It is enough to consider the case $p = 0$.

Lemma 4.3. *Let $\Lambda \subset W$ be the set of rays with vertex at 0. Assume that Λ is the uniqueness set for holomorphic (with respect to the standard structure J_{st}) functions. Suppose that $F(z)$ admits a limit along any ray from Λ as $z \rightarrow 0$. Then, F admits a limit along any ray contained in a cone $K \subset W$. If the limits are the same for all rays from Λ , the F admits a non-tangential limit at 0.*

Proof. We present the proof in three steps.

Step 1. Consider a sequence of functions $F_k(z) = F(z/k)$, $k = 1, 2, \dots$. We need the following analog of the Montel compactness principle: \square

Lemma 4.4. *The family (F_k) contains a subsequence converging uniformly on compacts in any cone K to a function F^0 holomorphic with respect to J_{st} .*

This is a consequence of Lemma 4.1. Indeed, it suffices to prove the equicontinuity. By the Nijenhuis–Wolf theorem any ball small enough is foliated by the pseudoholomorphic discs through the center (of course, this foliation is singular at the center of the ball). Such a foliation is a small deformation of the foliation of the ball by complex lines through its center. Thus, we apply Lemma 4.1 to the restriction of F_k on each disc and conclude that the sequence (F_k) is equicontinuous on this ball. This implies the above-mentioned compactness of this family and proves Lemma 4.4

We continue the proof of Lemma 4.3. Passing to the limit we obtain that $\bar{\partial}_{J_{st}}F^0 = 0$ in the sense of distributions which implies that F^0 is a usual holomorphic function (with respect to J_{st}). Indeed we have

$$|F_{\bar{z}}(z) + A(z)F_z(z)| \leq C$$

Hence for $\varepsilon > 0$ we obtain

$$|F_{\bar{z}}(\varepsilon z) + A(\varepsilon z)F_z(\varepsilon z)| \leq C$$

Therefore,

$$|\varepsilon F_{\bar{z}}(\varepsilon z) + A(\varepsilon z)\varepsilon F_z(\varepsilon z)| \leq \varepsilon C$$

and as a consequence

$$|(F(\varepsilon z))_{\bar{z}} + A(\varepsilon z)(F(\varepsilon z))_z| \leq \varepsilon C$$

Since $F(\varepsilon z) \rightarrow F^0(z)$ converges in the sense of distributions as $\varepsilon \rightarrow 0$ and $A(0) = 0$, we obtain that $F_{\bar{z}}^0 = 0$ in the sense of distributions and so F^0 is a usual holomorphic function.

Step 2. Since F admits a limit along any ray from Λ , we obtain that F^0 is constant along such a ray. Thus, given $\alpha > 0$ one has $F^0(\alpha z) = F^0(z)$ for all $z \in \Lambda$. But Λ is a uniqueness set. Therefore, the last identity (with fixed α) holds for all $z \in W$ and F^0 is constant already along any ray in W . This implies that F admits a limit along each ray.

Step 3. If the limits are the same, say, L , along all rays from Λ and Λ is the uniqueness set, we obtain that $F^0 = L$ on Λ and hence $F^0 \equiv L$ on W . This implies Lemma.

4.5. Limits along rays almost everywhere

Now, we study the existence of limits along rays. Fix a family l_z of rays smoothly depending on $z \in E = i\mathbb{R}^n$ such that $l_z \subset W$.

Lemma 4.5. *Let $K_z \subset W$ be a family of open cones smoothly depending on z , with vertex at z and directed by l_z . For almost every $z \in E$ the function F admits a limit along any ray in K_z .*

Note that at this moment, we do not yet claim that the limits are the same independently of a ray with the same vertex.

Proof. We begin with the model flat case where $J = J_{st}$. Consider the family of flat complex discs $z(c, t)$ given by (17). Let \mathcal{S}^{n-1} denote the unit sphere in \mathbb{R}^n . Denote by Σ a countable dense set in $\mathcal{S}^{n-1} \cap \mathbb{R}_+^n$. Set $\Sigma = \cup_{j \in \mathbb{N}} t^j$ and fix some t^j . By Lemma 4.1, the restriction of F on every disc $z(c, t^j)$ admits a non-tangential limit almost everywhere on $b\mathbb{D}^+$, i.e., on the full measure subset Y_j in $b\mathbb{D}^+$. Note that the image of every ray with vertex in $b\mathbb{D}^+$ by $z(c, t^j)$ belongs to $\mathbb{R}^n + ic$ (and its parallel translation to 0 belongs to \mathbb{R}^n). When c runs over a neighbourhood of 0, the images $z(c, t^j)(Y_j)$ sweep a full measure set X_j in $E = i\mathbb{R}^n$. The intersection $X = \cap_j X_j$ of such sets is again a full measure set in $E = i\mathbb{R}^n$. For every point $p \in X$ and every j the function F admits a limit along the ray with vertex at p and parallel to t^j . But Σ is dense in $\mathcal{S}^{n-1} \cap \mathbb{R}_+^n$ so for every fixed $p \in X$, these rays form a uniqueness set for usual holomorphic functions. Therefore, we can apply Lemma 4.3 at p .

The proof in the general case of any almost complex structure J follows by a perturbation argument. Indeed, indeed, we fill W by discs $z(c, v)$ of the form (18) constructed in § 4.2. In general, the images of rays by $z(c, v)$ are not rays in W , but they are the admissible curves. By Proposition 4.2, the function F admits a limit along any ray tangent to such curve at its boundary point on E . Then, the above argument goes through literally. This completes the proof. \square

4.6. Non-tangential limits

Here, we conclude the proof of theorem establishing the following

Lemma 4.6. *F admits non-tangential limits almost everywhere on E .*

Proof. Recall again that $E = i\mathbb{R}^n$. Fix a unit vector v such that $z + l$ belongs to W , where $z \in E$ and the ray l is directed by v . Consider a family of cones K_z with the vertex z and directed by the vector l ; we assume that K_z smoothly depends on z . By Lemma 4.5 there exists a full measure subset \tilde{E} of E such that the function F admits a limit F^* along the ray $z + l$ with vertex at $z \in \tilde{E}$. Consider the sequence of functions $f_m(z) = \sup_{0 < t \leq 1/m} |F(z + tv) - F^*(z)|$, where $z \in \tilde{E}$ and $m > 0$, $m \in \mathbb{N}$. This sequence converges to a function 0 for almost every $z \in \tilde{E}$. Applying the Egorov theorem, we conclude that for each $\delta > 0$ there exists a subset $E_\delta \subset \tilde{E}$ such that

- (i) $m(\tilde{E} \setminus E_\delta) < \delta$ (here $m(X)$ denotes the Lebesgue n -measure of the subset $X \subset E = i\mathbb{R}^n$).
- (ii) the sequence (f_m) converges uniformly to 0 on $E \setminus E_\delta$.

In particular, the functions $F(z + (1/m)v)$ are continuous on E and converge to F^* as $m \rightarrow \infty$. Hence, the function F^* is continuous on $\tilde{E} \setminus E_\delta$ as the uniform limit of a sequence of continuous functions. Note also that (by Lemma 4.5) one can assume that the function F admits a limit along any ray (not only l) with vertex at each point z of $\tilde{E} \setminus E_\delta$. Recall that by the Lebesgue theorem almost every point of $\tilde{E} \setminus E_\delta$ is a density point with the density equal to 1. We claim that F admits a non-tangential limit at such a point.

Assume that $0 \in \tilde{E} \setminus E_\delta$ is a density point. Also, we may assume that the limit of F along l at 0 is equal to 0, i.e., $F^*(0) = 0$. Let (z^k) be a sequence in an arbitrary ray l_1

in K_0 , (z^k) is converging to 0. We prove that there exists a subsequence (z^{k_q}) such that $F(z^{k_q}) \rightarrow 0$ as $q \rightarrow \infty$. This obviously implies the claim.

Set $r^k = |z^k|$. Then, the point $\tilde{z} = z^k/r^k$ belongs to l_1 , $|\tilde{z}| = 1$. Passing again to a subsequence, assume that \tilde{z} is independent of k . Consider the sequence of functions $\tilde{F}_k(z) = F(r^k z)$. It follows from Lemma 4.4 that one can extract a subsequence \tilde{F}_{k_q} (in what follows we skip the subindex q) converging uniformly on compacts of K_0 to a function \tilde{F} which is a usual holomorphic function on K_0 . Since F admits a limit along every ray at the origin, the function \tilde{F} is constant along every ray with vertex at the origin (as in the proof of Lemma 4.3). It suffices to show that $\tilde{F} = 0$. Indeed, in this case, we obtain

$$\lim_{k \rightarrow \infty} F(z^k) = \lim_{k \rightarrow \infty} \tilde{F}_k(z^k/r^k) = \tilde{F}(\tilde{z}) = 0.$$

Consider the set Λ formed by the real half-lines $L = p + l$, $p \in \tilde{E} \setminus E_\delta$. Clearly, Λ is contained in the subspace spanned by E and l . Λ is a generic (with respect to J_{st}) half-space contained in W , with the boundary E . Consider a sequence of points (q^k) contained in a ray $l_2 \subset \Lambda$ (with the vertex at the origin) and converging to 0. We choose the sequence (q^k) such that $r^k q^k = p^k + w^k$, where $p^k \in \tilde{E} \setminus E_\delta$ and a vector w^k belongs to the ray l . Since 0 is a density point, the set of rays in Λ admitting such a sequence, is a full measure subset of Λ . The sequence $r^k q^k$ tends to 0, hence the sequences p^k and w^k tend to 0 as well. Therefore,

$$\lim_{k \rightarrow \infty} \tilde{F}_k(q^k) = \lim_{k \rightarrow \infty} F(r^k q^k) = \lim_{k \rightarrow \infty} F(p^k + w^k)$$

In view of the uniform convergence of the sequence of functions (f_k) (introduced at the beginning of the proof of Lemma) on $\tilde{E} \setminus E_\delta$, and the continuity of the function F^* at 0, the last limit is equal to $F^*(0) = 0$. Hence, $\tilde{F} = 0$ on the ray l_2 . Thus, the function \tilde{F} vanishes on a full measure subset of Λ which is the uniqueness set for holomorphic functions. Hence, $\tilde{F} = 0$ on K_0 and F tends to 0 along any ray. By Lemma 4.3 F admits a non-tangential limit 0 at the origin. Hence, F admits a non-tangential limit almost everywhere on $\tilde{E} \setminus E_\delta$ because almost all points are density points (with the density 1).

Finally, consider the sets $E_k = E_\delta$ for $\delta = 1/k$. The intersection $\Sigma = \bigcap_k E_k$ has the measure 0 and F admits limits almost everywhere outside Σ . We conclude that F admits non-tangential limits almost everywhere on E . This completes the proof. \square

Acknowledgements. The author is partially supported by the Labex CEMPI as well as Institut of Mathematics with Computing Centre – Subdivision of the Ufa Research Centre of Russian Academy of Sciences, 45008, Chernyshevsky Str. 112, Ufa, Russia.

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