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LINEAR MAPPINGS PRESERVING SQUARE-ZERO MATRICES

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Let sl_n denote the set of all $n \times n$ complex matrices with trace zero. Suppose that $\phi : sl_n \longrightarrow sl_n$ is a bijective linear mapping preserving square-zero matrices. Then ϕ is either of the form $\phi(A) = cUAU^{-1}$ or $\phi(A) = cUA^tU^{-1}$ where U is an invertible $n \times n$ matrix and c is a nonzero complex number. The same result holds if we assume that ϕ is a linear mapping preserving square-zero matrices in both directions. Applying this result we prove that a linear mapping ϕ defined on the algebra of all $n \times n$ matrices is an automorphism if and only if it preserves zero products in both directions and satisfies $\phi(I) = I$. An extension of this last result to the infinite-dimensional case is considered.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

The problem of characterising linear operators on matrix algebras that leave invariant certain functions, subsets or relations has attracted the attention of many mathematicians in the last few decades [4]. It seems that the systematic study of such mappings begins with the paper of Marcus and Moyls [5]. They proved that every linear operator defined on a matrix algebra that preserves the spectrum is either an automorphism or an antiautomorphism. This result was extended by Howard [3] who obtained the general form of bijective linear mappings that preserve the set of all matrices annihilated by a given polynomial with at least two distinct roots. Let a be an arbitrary complex number and p a polynomial given by $p(\lambda) = (\lambda - a)^k$ for some positive integer k, $k \geqslant 2$. If ϕ is a linear mapping on M_n that preserves the set of all matrices annihilated by p then the linear transformation φ defined by $\varphi(A) = \phi(A + aI) - aI$ preserves nilpotent matrices of nilindex not more than k. So, in order to get the complete description of all linear mappings that preserve the set of all matrices annihilated by a given polynomial, it suffices to study linear mappings preserving nilpotents of nilindex at most k, $2 \leqslant k \leqslant n$. We denote by sl_n the subspace of all matrices with trace zero. It is easy to see that the linear span of all nilpotent matrices is sl_n [1]. Thus, when studying linear preservers of nilpotents of a bounded nilindex, we assume that these mappings are defined on sl_n.

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The case $k = n$ was treated in [1] where it was proved that a linear bijective mapping on sl_n that preserves nilpotent matrices differs from an automorphism or an antiautomorphism by a multiplicative constant. It is the aim of this note to solve the above mentioned problem in the case $k = 2$.

A matrix A is square-zero if $A^2 = 0$. A mapping $\phi : sl_n \longrightarrow sl_n$ preserves square-zero matrices if $A \in sl_n$ and $A^2 = 0$ imply $\phi(A)^2 = 0$. We say that a mapping ϕ preserves square-zero matrices in both directions if for every $A \in sl_n$ the matrix $\phi(A)$ is square-zero if and only if $A^2 = 0$. Our first result is

THEOREM 1. *Let $\phi : sl_n \longrightarrow sl_n$ be a linear mapping preserving square-zero matrices in both directions. Then ϕ is either of the form*

$$\phi(A) = cUAU^{-1} \tag{1}$$

or

$$\phi(A) = cUA^tU^{-1} \tag{2}$$

where $U \in M_n$ is an invertible matrix and $c \in \mathbb{C}$ is a nonzero number.

It follows from [2, Lemma 1] that if a linear bijective mapping $\phi : sl_n \longrightarrow sl_n$ preserves square-zero matrices in one direction, then it preserves them in both directions. This yields together with Theorem 1 the following result.

COROLLARY 2. *Assume that $\phi : sl_n \longrightarrow sl_n$ is a bijective linear mapping preserving square-zero matrices. Then ϕ is either of the form (1) or of the form (2).*

In the above result the bijectivity assumption is indispensable. Namely, let $V \subset sl_n$ be a subalgebra of square zero (that is, $A, B \in V$ implies $AB = 0$). Then every linear mapping $\phi : sl_n \longrightarrow V \subset sl_n$ preserves square-zero matrices.

A mapping ϕ on M_n preserves zero products in both directions if for any two matrices $A, B \in M_n$ the relation $\phi(A)\phi(B) = 0$ holds if and only if $AB = 0$. Another simple consequence of Theorem 1 is:

COROLLARY 3. *Let $\phi : M_n \longrightarrow M_n$ be a linear mapping preserving zero products in both directions. Assume also that $\phi(I) = I$. Then ϕ is an automorphism of the algebra M_n.*

The same result is not valid in the infinite-dimensional case. Let H be an infinite-dimensional Hilbert space and $B(H)$ the algebra of all bounded linear operators. The mapping $\phi : B(H) \longrightarrow B(H \oplus H)$ given by $\phi(T) = T \oplus T$ preserves zero products in both directions and satisfies $\phi(I) = I$. However, it is not an automorphism. In order to extend Corollary 3 to the infinite-dimensional case we have to assume that the mapping ϕ is also surjective.

THEOREM 4. *Let X be a Banach space, $\dim X > 1$, and $B(X)$ the algebra of all bounded linear operators on X. Then a linear mapping ϕ from $B(X)$ onto $B(X)$*

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is an automorphism if and only if ϕ preserves zero products in both directions and satisfies $\phi(I) = I$.

2. Proofs

In order to prove Theorem 1 we shall need the following Lemma.

Lemma 5. *Let A be a nonzero $n \times n$ complex matrix satisfying $A^2 = 0$. Then the following conditions are equivalent.*

(i) $\operatorname{rank} A = 1$.

(ii) $\dim\left(\operatorname{span}\{B \in sl_n \,:\, B^2 = 0 \text{ and } (A+B)^2 = 0\}\right) \geqslant n(n-2)$.

Proof: In the case $n = 2$ there is nothing to prove. So, let us assume that $n > 2$. Let A be a rank one matrix. We can assume that A is of the form

$$A = \begin{pmatrix} J & 0_{2,n-2} \\ 0_{n-2,2} & 0_{n-2,n-2} \end{pmatrix}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

while $0_{i,j}$ denotes the zero matrix of dimension $i \times j$. The set of all $i \times j$ matrices shall be denoted by $M_{i,j}$. It is easy to see that matrices having the following forms

$$\begin{pmatrix} 0 & C \\ 0_{n-1,1} & 0_{n-1,n-1} \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0_{1,n-2} \\ 0 & 0 & 0_{1,n-2} \\ 0_{n-2,1} & D & 0_{n-2,n-2} \end{pmatrix}, \quad \begin{pmatrix} 0_{2,2} & 0_{2,n-2} \\ 0_{n-2,2} & E \end{pmatrix}$$

where $C \in M_{1,n-1}$, $D \in M_{n-2,1}$, $E \in sl_{n-2}$, belongs to $\operatorname{span}\{B \,:\, B^2 = 0 \text{ and } (A+B)^2 = 0\}$. It follows that the condition (ii) is fulfilled.

In order to prove the reverse implication we consider a square-zero matrix A satisfying $\operatorname{rank} A = k > 1$. Assume with no loss of generality that A is of the form

$$A = \begin{pmatrix} J & 0_{2,2} & \cdots & 0_{2,2} & 0_{2,n-2k} \\ 0_{2,2} & J & \cdots & 0_{2,2} & 0_{2,n-2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2,2} & 0_{2,2} & \cdots & J & 0_{2,n-2k} \\ 0_{n-2k,2} & 0_{n-2k,2} & \cdots & 0_{n-2k,2} & 0_{n-2k,n-2k} \end{pmatrix}.$$

Let us write an arbitrary square-zero matrix B satisfying $(A+B)^2 = 0$ in the form

$$B = \begin{pmatrix} B_{11} & \cdots & B_{1k} & Y_1 \\ \vdots & & \vdots & \vdots \\ B_{k1} & \cdots & B_{kk} & Y_k \\ X_1 & \cdots & X_k & Z \end{pmatrix} \tag{3}$$

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where $B_{ij} \in M_2$, $X_i \in M_{n-2k,2}$, $Y_j \in M_{2,n-2k}$, $i,j = 1,\dots k$, and $Z \in M_{n-2k,n-2k}$. It follows from $A^2 = B^2 = (A+B)^2 = 0$ that $AB + BA = 0$, and consequently, $JB_{ij} + B_{ij}J = 0$, $X_iJ = 0$, and $JY_j = 0$ for all $i,j = 1,\dots,k$. A straightforward computation shows that matrices B_{ij}, X_i, and Y_j have the form

$$B_{ij} = \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & -a_{ij} \end{pmatrix}, \quad a_{ij}, b_{ij} \in \mathbb{C}, \tag{4}$$

$$X_i = \begin{pmatrix} 0_{n-2k,1} & X_i' \end{pmatrix}, \quad X_i' \in M_{n-2k,1}, \tag{5}$$

and

$$Y_j = \begin{pmatrix} Y_j' \\ 0_{1,n-2k} \end{pmatrix}, \quad Y_j' \in M_{1,n-2k}. \tag{6}$$

Let us first consider the case $n > 2k$. Clearly, we have $trB = 0$. Applying (4) we get $trZ = 0$. This yields together with (4), (5), and (6) that

$$\dim\Big(\operatorname{span}\{B \in sl_n \,:\, B^2 = 0 \text{ and } (A+B)^2 = 0\}\Big) \leqslant \\ n^2 - 2k^2 - 2k(n-2k) - 1 = (n-k)^2 + k^2 - 1. \tag{7}$$

It follows from $k \geqslant 2$ that $n - k \leqslant n - 2$. The assumption $n > 2k$ implies together with $k \geqslant 2$ that $k - n + 2 < 0$. Thus,

$$\dim\Big(\operatorname{span}\{B \in sl_n \,:\, B^2 = 0 \text{ and } (A+B)^2 = 0\}\Big) \\ \leqslant (n-k)(n-2) + k^2 - 1 = n(n-2) + k(k-n+2) - 1 < n(n-2),$$

which completes the proof in the case $n > 2k$. If $n = 2k$ then the matrix Z in the block form (3) of B has dimension 0×0. For $n = 2k > 4$ the left-hand side of (7) becomes $n^2 - 2k^2 = 2k^2 < 4k^2 - 4k = n(n-2)$. So, it remains to consider the case $(n,k) = (4,2)$. A straightforward computation shows that in this case the relation $B^2 = 0$ implies $a_{11} = -a_{22}$. One can now easily complete the proof. ∎

PROOF OF THEOREM 1: First, we shall prove that ϕ is bijective. Assume that A is a nonzero matrix satisfying $\phi(A) = 0$. Let x and y be nonzero column vectors such that $Ax = y$. Now ϕ preserves square-zero matrices in both directions, and consequently, $A^2 = 0$. It follows that x and y are linearly independent. This yields the existence of a column vector z such that $z^tx = 0$ and $z^ty = 1$. Let us define a square-zero matrix $B = xz^t$. It follows from $0 = (\phi(B))^2 = (\phi(A+B))^2$ that $A + B$ is square-zero. On the other hand, we have $(A+B)^2x = x$. This contradiction shows that ϕ is bijective.

It follows from Lemma 5 that the bijective linear mapping $\phi : sl_n \longrightarrow sl_n$ preserves nilpotents of rank one in both directions. It was proved in [1] that such mappings are necessarily of the form (1) or (2). ∎

We have already mentioned that Corollary 2 follows immediately from Theorem 1 and [**2**, Lemma 1]. One can easily prove Corollary 3 using Theorem 1 and the direct sum decomposition $M_n = \mathbb{C}I \oplus sl_n$. We shall omit this proof because Corollary 3 follows also from Theorem 4 as soon as we prove that ϕ is one-to-one. It shall be the first step in the proof of Theorem 4 to show that every linear mapping $\phi : B(X) \longrightarrow B(X)$ preserving zero products in both directions is injective.

Let X be a Banach space. By $F(X)$ we denote the ideal of all operators in $B(X)$ of finite rank. For any $x \in X$ and $f \in X'$ we denote by $x \otimes f$ the linear bounded rank one operator on X defined by $(x \otimes f)y = f(y)x$ for $y \in X$. Note that every operator of rank one can be written in this form.

Proof of Theorem 4: Clearly, every automorphism on $B(X)$ preserves zero products in both directions. In order to prove the reverse implication we assume that $\phi(I) = I$ and that ϕ preserves zero products in both directions. First we shall show that ϕ is injective. Let $A \in B(X)$ be a nonzero operator. It is an easy consequence of the Hahn-Banach theorem that there exists $B \in B(X)$ such that $AB \neq 0$. It follows that $\phi(A)\phi(B) \neq 0$, and consequently, $\phi(A) \neq 0$.

Our next step shall be to show that ϕ preserves idempotent operators in both directions. Indeed, a linear bounded operator P on X is idempotent if and only if $P(I - P) = 0$. According to our assumptions this is equivalent to $\phi(P)(I - \phi(P)) = 0$. This last relation is fulfilled if and only if $\phi(P)$ is idempotent.

Next, we shall show that ϕ preserves idempotents of rank one. Let P be an idempotent operator of rank one such that $\phi(P)$ has rank greater than one. Then we have $\phi(P) = Q_1 + Q_2 = \phi(P_1) + \phi(P_2)$ for some nonzero idempotents Q_1, Q_2, P_1, P_2. This yields $P = P_1 + P_2$ - a contradiction. The same conclusion holds for ϕ^{-1}, and consequently, ϕ preserves projections of rank one in both directions. It follows from [**6**, Proposition 2.6] that there exists either a bounded linear invertible operator U on X such that $\phi(A) = UAU^{-1}$ for all $A \in F(X)$, or a linear bounded invertible operator $V : X' \longrightarrow X$ such that $\phi(A) = VA'V^{-1}$ for all $A \in F(X)$. Here, X' denotes the dual of X and A' denotes the adjoint of A. The second case can not occur. Namely, in this case we would have for an arbitrary pair $A, B \in F(X)$ that $AB = 0$ if and only if $\phi(A)\phi(B) = 0$ which is equivalent to $BA = 0$. But this certainly is not true.

Define $\varphi : B(X) \longrightarrow B(X)$ by $\varphi(A) = U^{-1}\phi(A)U$. Obviously, φ is a bijective linear mapping on $B(X)$ preserving zero products in both directions, and $\varphi(A) = A$ for every $A \in F(X)$. Let A be an arbitrary operator from $B(X)$ and x an arbitrary vector from X. Choose $f \in X'$ such that $f(x) = 1$. We define $y = Ax$. Then we have $(A - y \otimes f)(x \otimes f) = 0$ which implies $(\varphi(A) - y \otimes f)(x \otimes f) = 0$. As a consequence we get $\varphi(A)x = y$. Hence, φ maps every operator from $B(X)$ into itself. This completes the proof. $\square$

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