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Total disconnectedness of Julia sets of random quadratic polynomials

KRZYSZTOF LECH and ANNA ZDUNIK

Institute of Mathematics, University of Warsaw, ul. Banacha 2, 02-097 Warsaw, Poland
(e-mail: K.Lech@mimuw.edu.pl, A.Zdunik@mimuw.edu.pl)

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Abstract. For a sequence of complex parameters (c_n) we consider the composition of functions $f_{c_n}(z) = z^2 + c_n$, the non-autonomous version of the classical quadratic dynamical system. The definitions of Julia and Fatou sets are naturally generalized to this setting. We answer a question posed by Brück, Büger and Reitz, whether the Julia set for such a sequence is almost always totally disconnected, if the values c_n are chosen randomly from a large disc. Our proof is easily generalized to answer a lot of other related questions regarding typical connectivity of the random Julia set. In fact we prove the statement for a much larger family of sets than just discs; in particular if one picks c_n randomly from the main cardioid of the Mandelbrot set, then the Julia set is still almost always totally disconnected.

Key words: random dynamics, holomorphic dynamics, Julia sets, connectivity
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1. *Introduction*

We consider non–autonomous compositions of quadratic polynomials $f_c = z^2 + c$ where at each step c is chosen randomly from some bounded Borel $V \subset \mathbb{C}$ (e.g. the disc $\mathbb{D}(0, R)$). Let us introduce the parameter space $\Omega = V^{\mathbb{N}}$. The space Ω is equipped with a natural left shift map σ. Namely, for every $\omega = (c_0, c_1, c_2, \ldots) \in \Omega$, put

$$\sigma(\omega) = (c_1, c_2, \ldots).$$

Next, for every $\omega \in \Omega$, $\omega = (c_0, c_1, \ldots)$ denote by f_ω the map f_{c_0}. Then the non-autonomous composition f_ω^n is given by the formula

$$f_\omega^n := f_{c_{n-1}} \circ f_{c_{n-2}} \circ \cdots \circ f_{c_0}.$$

The global dynamics can be described as a skew product $F : \Omega \times \mathbb{C} \to \Omega \times \mathbb{C}$,

$$F(\omega, z) = (\sigma(\omega), f_\omega(z)).$$

Then, for all $n \in \mathbb{N}$, we have that

$$F^n(\omega, z) = (\sigma^n(\omega), f_\omega^n(z)).$$

So, every sequence $\omega \in \Omega$ determines a sequence of non-autonomous iterates: $(f_\omega^n)_{n\in\mathbb{N}}$.

Let μ be a Borel probability measure on V. We denote by $\mathbb{P}$ the product distribution on Ω generated by μ. Then $(\Omega, \mathbb{P})$ becomes a measurable space, and $\sigma : \Omega \to \Omega$ is an ergodic measure-preserving endomorphism.

Analogously to the autonomous case, it is natural to consider the following objects:

- the *escaping set*, or *basin of infinity*,

$$\mathcal{A}_\omega = \{z \in \mathbb{C} : f_\omega^n(z) \xrightarrow[n\to\infty]{} \infty\};$$

- the non-autonomous Julia set

$$J_\omega = \{z \in \mathbb{C} : \text{for every open set } U \ni z \text{ the family } f^n_{\omega\,|U} \text{ is not normal}\}; \quad (1)$$

- the non-autonomous filled-in Julia set

$$K_\omega = \mathbb{C} \setminus \mathcal{A}_\omega. \quad (2)$$

The following proposition, which can be found in [4, Theorem 1] is analogous to the autonomous case.

PROPOSITION 1. *Let $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$. Then*

$$J_\omega = \partial\mathcal{A}_\omega.$$

Let us also note the following straightforward observations:

PROPOSITION 2. *For every $\omega \in \Omega$:*

- $J_{\sigma\omega} = f_\omega(J_\omega)$;
- $\mathcal{A}_{\sigma\omega} = f_\omega(\mathcal{A}_\omega)$.

The study of iterates of non-autonomous and random rational maps, and, in particular, non-autonomous and random polynomials, originated from the seminal paper [7] by Fornæss and Sibony. It has since been developed by many authors.

A systematic study of dynamics on non-autonomous and random dynamics of quadratic polynomials was done by Brück, Büger and Reitz; see [2–4]. Other results related to random polynomial dynamics in general have also been achieved by Comerford in [5, 6]. Finally, in [9], Mayer, Skorulski and Urbanski, among other results, confirm a conjecture by Brück and Büger concerning the typical Hausdorff dimension of a certain random quadratic Julia set.

In [3] the authors focus on the question of the connectedness of the Julia set, giving, among other results, a transparent sufficient and necessary condition for the Julia set to be connected.

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THEOREM 1.1. in [3] *Let $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$, $R > 0$. The Julia set J_ω is disconnected if and only if there exists $k \in \mathbb{N}$ such that*

$$f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty.$$

Note that the point 0 plays a special role, since it is a common critical point of all maps f_c. Recall that in the autonomous case (i.e. the iterates of a single map f_c) the Julia set is disconnected if and only if $f^n_c(0) \xrightarrow[n\to\infty]{} \infty$. Moreover, if the Julia set $J(f_c)$ is disconnected, then it is totally disconnected. The last statement is no longer true in the non-autonomous case considered here; for example, one can easily construct sequences ω for which J_ω is disconnected and has finitely many connected components.

Looking at the above characterization of connected Julia sets J_ω, one may conjecture that the condition

$$f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty \quad \text{for every } k \in \mathbb{N}$$

is the right characterization of totally disconnected Julia sets J_ω.

However, this condition is neither necessary nor sufficient. Indeed, in [3] the authors construct an example of a sequence $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$ such that, for every $k \in \mathbb{N}$, $f^n_{\sigma^k\omega}(0) \to \infty$ as $n \to \infty$, but the Julia set J_ω is not totally disconnected (see [3, Example 4.4]). On the other hand, Example 4.5 in the same paper shows that the Julia set may be totally disconnected even if, for infinitely many $k \in \mathbb{N}$, $f^n_{\sigma^k\omega}(0)$ does not tend to infinity as $n \to \infty$.

Clearly, the behaviour of the (typical) dynamics depends on the domain from which the parameters c_n are chosen. In particular, in case of a disc $\mathbb{D}(0, R)$, the dynamics depend on R.

If $R \le 1/4$ then, for every $\omega \in \mathbb{D}^{\mathbb{N}}$, $\omega = (c_i)_{i=0}^{\infty}$, the Julia set J_ω is connected (see [3, Remark 1.2]). Note that in this case all parameters c_i are chosen from the main cardioid in the Mandelbrot set.

For $R > 1/4$ the situation changes drastically. Indeed, the disc $\mathbb{D}(0, R)$ now contains parameters from the complement of the Mandelbrot set $\mathcal{M}$. So, it is evident that putting, for instance, $\omega = (c, c, c, \ldots)$, where $c \in \mathbb{D}(0, R) \setminus \mathcal{M}$, one obtains a totally disconnected Julia set J_ω.

This motivates the following question, which was raised in [2, 3]: what is a typical behaviour of the Julia set J_ω, in terms of connectedness? More formally, in [2, 3] the authors introduce subsets of $\Omega = \mathbb{D}(0, R)^{\mathbb{N}}$, denoted by $\mathcal{D}$, $\mathcal{D}_N$, $\mathcal{D}_\infty$, $\mathcal{T}$, and described in terms of connectedness:

$$\begin{aligned}
\mathcal{D} &= \{\omega \in \Omega : J_\omega \text{ is disconnected}\},\\
\mathcal{D}_N &= \{\omega \in \Omega : J_\omega \text{ has at least } N \text{ connected components}\},\\
\mathcal{D}_\infty &= \{\omega \in \Omega : J_\omega \text{ has infinitely many connected components}\},\\
\mathcal{T} &= \{\omega \in \Omega : J_\omega \text{ is totally disconnected}\},\\
\mathcal{F} &= \{\omega \in \Omega : \text{ for all } k \in \mathbb{N},\ f^n_{\sigma^k\omega}(0) \xrightarrow[n\to\infty]{} \infty\}.
\end{aligned}$$

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Clearly, for $N > 1$, $\mathcal{D} \supset \mathcal{D}_N \supset \mathcal{D}_\infty \supset \mathcal{T}$. But, as mentioned above, the set $\mathcal{F}$ is neither contained in nor contains $\mathcal{T}$.

Here, typicality may be understood in a topological or metric sense. The space $\Omega = \mathbb{D}(0, R)^{\mathbb{N}}$ carries the natural product topology induced by the standard topology on $\mathbb{D}(0, R)$. Note that this topology is completely metrizable.

The space Ω also carries the natural product measure $\mathbb{P} := \otimes_{n=0}^{\infty} \lambda_R$ where each λ_R is the normalized Lebesgue measure on $\mathbb{D}(0, R)$. In [3] the authors prove that $\mathbb{P}(\mathcal{D}) = 1$, provided $R > 1/4$ [3, Theorem 2.3]. It can be deduced from the proof, in a rather straightforward way, that $\mathbb{P}(\mathcal{F}) = 1$ and also (although it is not explicitly stated in the paper) that $\mathbb{P}(\mathcal{D}_\infty) = 1$.

The work [2] deals with topological aspects of typicality of the above sets. In particular, the author proves (assuming $R > 1/4$) that:

- the set $\mathcal{T}$ is dense in Ω (Theorem 1.1);
- the set $\mathcal{D}_\infty$ has empty interior in Ω (Theorem 1.2);
- for every $N \in \mathbb{N}$, the set $\mathcal{D}_N$ is an open dense subset of Ω; which immediately implies that
- the set $\mathcal{D}_\infty$ is of the second Baire category.

In [2] the author asked if the set $\mathcal{T}$ is also of the second Baire category. This question was positively answered by Gong, Qiu and Li in [8].

However, the question about metric typicality of $\mathcal{T}$, formulated in [3], remained open until now.

Question. Is it true that $\mathbb{P}(\mathcal{T}) = 1$ provided that $R > 1/4$ is large enough?

In this paper, we answer the above question positively, providing a number of stronger statements. Specifically, we prove the following.

THEOREM A. *Let $R > 1/4$. Consider $\Omega = \mathbb{D}(0, R)^{\mathbb{N}}$ equipped with the product distribution $\mathbb{P} := \otimes_{n=0}^{\infty} \lambda_R$. Let*

$$\mathcal{T} = \{\omega \in \Omega : J_\omega \text{ is totally disconnected}\}.$$

Then $\mathbb{P}(\mathcal{T}) = 1$.

In other words, a typical (metrically) Julia set J_ω is totally disconnected.

One might expect that the phenomenon described in Theorem A is based on the fact that for $R > 1/4$ the disc $\mathbb{D}(0, R)$ intersects the complement of the Mandelbrot set $\mathcal{M}$. However, the following generalization shows that the analogous statement holds true also for domains which are completely contained in the Mandelbrot set. Namely, we have the following generalization of Theorem A.

THEOREM B. *Let V be an open and bounded set such that $\mathbb{D}(0, \frac{1}{4}) \subset V$ and $V \neq \mathbb{D}(0, \frac{1}{4})$. Consider the space $\Omega = V^{\mathbb{N}}$ equipped with the product $\mathbb{P}$ of uniform distributions on V. Then for $\mathbb{P}$-almost every sequence $\omega \in \Omega$ the Julia set J_ω is totally disconnected.*

Theorem B leads immediately to the following corollary.

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COROLLARY. (Theorem 17) *Let $\Omega = B^{\mathbb{N}}$ where B is the main cardioid of the Mandelbrot set, and let Ω be equipped with the product of uniform distributions on B. Then, for almost every sequence $\omega \in \Omega$, the Julia set J_ω is totally disconnected.*

A number of applications of our approach, possible generalization and further results are presented in §6.

2. *Green's function*

Notation. For every $r > 0$, denote $\mathbb{D}_r := \mathbb{D}(0, r)$ and $\mathbb{D}_r^* := \mathbb{C} \setminus \overline{\mathbb{D}}_r$.

We write $\omega = (c_0, c_1 \cdots)$ in various contexts to denote an infinite sequence of parameters, even if no probability distribution is specified. For such a sequence we use both notations:

$$f_\omega^n = f_{c_{n-1},c_{n-2},\ldots,c_1,\ldots c_0}^n = f_{c_{n-1}} \circ f_{c_{n-2}} \circ \cdots \circ f_{c_1} \circ f_{c_0}.$$

2.1. *Green's function on $\mathcal{A}_\omega$.*

We recall the proposition proved in [7], which we state in a slightly different form.

PROPOSITION 3. *Let V be a bounded Borel subset of $\mathbb{C}$, and put $\Omega = V^{\mathbb{N}}$. Let μ be a Borel probability measure on V, and $\mathbb{P}$ be the product distribution on Ω generated by μ. For every $\omega \in \Omega$, the following limit exists for $g_\omega : \mathcal{A}_\omega \to \mathbb{R}$:*

$$g_\omega(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f_\omega^n(z)|. \tag{3}$$

The function $z \mapsto g_\omega(z)$ is Green's function on $\mathcal{A}_\omega$ with pole at infinity. Putting $g_\omega \equiv 0$ on the complement of $\mathcal{A}_\infty$, g_ω extends continuously to the whole plane. With z fixed, the function $\omega \mapsto g_\omega(z)$ is $\mathbb{P}$– measurable.

This is a generalization of a well-known formula for the autonomous case: for the map $f_c(z) := z^2 + c$ and its basin of infinity $\mathcal{A}_c$, Green's function with a pole at infinity is given by

$$g_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f_c^n(z)|.$$

COROLLARY 4. *We have*

$$g_{\sigma\omega}(f_\omega(z)) = 2g_\omega(z). \tag{4}$$

Proof. This follows directly from formula (3), defining Green's function g_ω. □

Observation: Critical points of g_ω. Writing f_i for f_{c_i}, we see that g_ω has critical points at each point of the following sets:

$$\begin{aligned}
\mathcal{C}_0 &= \{0\},\\
\mathcal{C}_1 &= f_1^{-1}(0),\\
\mathcal{C}_2 &= f_1^{-1} f_2^{-1}(0), \ldots,\\
\mathcal{C}_k &= f_1^{-1} f_2^{-1} f_3^{-1}, \ldots, f_k^{-1}(0), \ldots.
\end{aligned}$$

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Let us note that in the autonomous case the critical points of g_c form a 'tree' (i.e. $\mathcal{C}_k = f^{-1}(\mathcal{C}_{k-1})$), while in a general non-autonomous case the set $\mathcal{C}_k$ is not a preimage of $\mathcal{C}_{k-1}$ under any the maps f_i.

2.2. *Estimates for Green's function.*

PROPOSITION 5. *For every $\varepsilon > 0$, $R > 0$, there exists $R_0 > 0$ such that, for every $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$,*

$$f_\omega(\mathbb{D}^*_{R_0}) \subset \mathbb{D}^*_{2R_0}, \tag{5}$$

$$\mathbb{D}^*_{R_0} \subset \mathcal{A}_\omega, \tag{6}$$

$$|g_\omega(z) - \log|z|| < \varepsilon \quad \text{in } \mathbb{D}^*_{R_0}. \tag{7}$$

Proof. First, since $|c_n| < R$ for all c, one can choose $R_1 > 0$ to ensure

$$f_\omega(\mathbb{D}^*_{R_0}) \subset \mathbb{D}^*_{2R_0} \tag{8}$$

for every $R_0 \geq R_1$. This guarantees (5) and(6).

Let $a_0(z) = \log|z|$, $a_n(z) = (1/2^n)\log|f^n_\omega(z)|$, for $n \geq 1$, and note that we have, for all $n \geq 0$,

$$\begin{aligned} a_{n+1}(z) &= \frac{1}{2^{n+1}}\log|f^{n+1}_\omega(z)| \\ &= \frac{1}{2}\left(\frac{1}{2^n}\log|(f^n_\omega(z))^2 + c_n|\right) \\ &= \frac{1}{2^n}\left(\log|f^n_\omega(z)| + \frac{1}{2}\log\left|1 + \frac{c_n}{(f^n_\omega(z))^2}\right|\right) \\ &= a_n(z) + \frac{1}{2^{n+1}}\left(\log\left|1 + \frac{c_n}{(f^n_\omega(z))^2}\right|\right). \end{aligned}$$

In particular, this means that

$$\begin{aligned} g_\omega(z) = a_0 + \sum_{n=0}^{\infty}(1/2^{n+1})\log|1 + (c_n/(f^n_\omega(z))^2)| = \log|z| \\ + \sum_{n=0}^{\infty}(1/2^{n+1})\log|1 + (c_n/(f^n_\omega(z))^2)|. \end{aligned}$$

Choosing $R_0 \geq R_1$ large enough, we can have

$$\sum_{n=0}^{\infty}(1/2^{n+1})\log|1 + (c_n/(f^n_\omega(z))^2)| < \varepsilon$$

on $\mathbb{D}^*_{R_0}$. This yields

$$|g_\omega(z) - \log|z|| < \varepsilon,$$

which concludes the proof. □

The following is an immediate consequence of item (5) of Proposition 5.

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COROLLARY 6. *For every* $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$, $K_\omega \subset \mathbb{D}_{R_0}$.

Determining constants. Now, for every R, we fix some $R_0 > R$ satisfying the conditions formulated in Proposition 5 with $\varepsilon := 1$, in particular,

$$|g_\omega(z) - \log|z|| < 1 \quad \text{in } \mathbb{D}^*_{R_0}. \tag{9}$$

Next, for every $R > 0$, let us fix also some $\tilde{R}_0 \in (R_0, R_0^2 - R)$, say, $\tilde{R}_0 = \frac{1}{2}(R_0 + R_0^2 - R)$. Then, for every $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$, $f_\omega^{-1}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}$. By Proposition 5,

$$G = G(R) := \sup_{|R_0| \le |z| \le \tilde{R}_0, \omega \in \mathbb{D}(0,R)^{\mathbb{N}}} (g_\omega(z)) < \infty, \tag{10}$$

PROPOSITION 7. *For every* $R > 0$, *for every* $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$,

$$\sup_{z \in \mathbb{D}_{R_0}} g_\omega(z) \le \log R_0 + 1.$$

In particular, the function $\mathbb{D}(0, R)^{\mathbb{N}} \ni \omega \mapsto g_\omega(0)$ *is bounded above, and*

$$\sup_{\omega \in \mathbb{D}(0,R)^{\mathbb{N}}} g_\omega(0) \le \log R_0 + 1.$$

Proof. Since $|g_\omega(z)| \le \log|z| + 1$ in $\mathbb{D}^*_{R_0}$, we have, in particular, $g_{\omega|\partial\mathbb{D}_{R_0}} \le \log R_0 + 1$. By the maximum principle, the same estimate holds in the whole disc $\mathbb{D}_{R_0}$, in particular, for $z = 0$. So, $\sup_{\omega \in \mathbb{D}(0,R)^{\mathbb{N}}} g_\omega(0) \le \log R_0 + 1$. □

2.3. *Escape rate of the critical point.* We introduce the following definition.

Definition 2.1. Let $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$. For every $z \in \mathbb{D}(0, R_0)$, we denote by $k(z, \omega)$ the escape time of z from $\mathbb{D}_{R_0}$:

$$k(z, \omega) = \begin{cases} \min\{j : |f_\omega^j(z)| \ge R_0\} & \text{if } z \in \mathcal{A}_\omega, \\ \infty & \text{if } z \in K_\omega. \end{cases} \tag{11}$$

PROPOSITION 8. *For every* $z \in \mathcal{A}_\omega \cap \mathbb{D}_{R_0}$,

$$(\log R_0 - 1)2^{-k(z,\omega)} \le g_\omega(z) \le 2(\log R_0 + 1)2^{-k(z,\omega)}. \tag{12}$$

Proof. Recall that, by (4),

$$g_\omega(z) = g_{\sigma^{k(z,\omega)}\omega}(f_\omega^{k(z,\omega)}(z)) \cdot 2^{-k(z,\omega)}$$

and

$$g_{\sigma^{k(z,\omega)}\omega}(f_\omega^{k(z,\omega)}(z)) = 2g_{\sigma^{k(z,\omega)-1}\omega}(f_\omega^{k(z,\omega)-1}(z)) \le 2(\log R_0 + 1),$$

since $f_\omega^{k(z,\omega)-1}(z)) \in \mathbb{D}_{R_0}$.

On the other hand,

$$g_{\sigma^{k(z,\omega)}\omega}(f_\omega^{k(z,\omega)}(z)) \ge \log|f_\omega^{k(z,\omega)}(z)| - 1 \ge \log R_0 - 1.$$

This implies that (12) holds. □

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Our estimates show that the distribution of the random variable $\log^- g_\omega(0)$ is roughly the same as that of $k(0, \omega)$.

Definition 2.2. Let V be a bounded Borel subset of $\mathbb{C}$, $V \subset \mathbb{D}(0, R)$. Let μ be a probability Borel measure on V, and let $\mathbb{P}$ be the product distribution on $V^{\mathbb{N}}$ generated by μ. Fix the values $R_0 = R_0(R)$ and $G = G(R)$ according to (10). We say that the critical point is *typically fast escaping* if there exists $\gamma > 0$ such that

$$\mathbb{P}\left(\left\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\right\}\right) < e^{-\gamma k}. \tag{13}$$

3. *Sufficient condition for total disconnectedness*

Recall that in §2.2 we assigned, for every $R > 0$, the values R_0 and $\tilde{R}_0$.

LEMMA 9. *Choose an arbitrary radius $\rho \in [R_0, \tilde{R}_0]$ and let $D := \mathbb{D}_\rho$. Then the filled-in Julia set K_ω (i.e. the set of points z whose trajectories $f_\omega^n(z)$ do not escape to ∞) can be written as*

$$K_\omega := \bigcap_{k\in\mathbb{N}} (f_\omega^k)^{-1}(D).$$

Proof. Since the trajectory of every point $z \in \bigcap_{k\in\mathbb{N}} (f_\omega^k)^{-1}(D)$ is bounded, it is clear that

$$\bigcap_{k\in\mathbb{N}} (f_\omega^k)^{-1}(D) \subset K_\omega.$$

On the other hand, if $z \notin \bigcap_{k\in\mathbb{N}} (f_\omega^k)^{-1}(D)$ then, for some $k \in \mathbb{N}$, $|f_\omega^k(z)| \geq \rho \geq R_0$, and it follows from the choice of R_0 that

$$f_\omega^n(z) = f_{\sigma^k\omega}^{n-k}(f_\omega^k(z)) \xrightarrow[n\to\infty]{} \infty,$$

so $z \notin K_\omega$. □

Observe that $\bigcap_{k\in\mathbb{N}} (f_\omega^k)^{-1}(D)$ is an intersection of a descending sequence of sets. At each level k the set

$$D^k(\omega) := (f_\omega^k)^{-1}(D)$$

is a union of pairwise disjoint topological discs $D_j^k(\omega)$, each of them being mapped by f_ω^k onto D with some degree $d_j^k \leq 2^k$.

Now put $\rho = \tilde{R}_0$, that is, put $D := \mathbb{D}_{\tilde{R}_0}$. The following proposition formulates, in terms of the degree of the maps $f_\omega^k : D_j^k(\omega) \to D$, a sufficient condition for total disconnectedness of the Julia set J_ω.

PROPOSITION 10. *Let $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$. If there exists $N \in \mathbb{N}$ such that, for infinitely many integers $k \in \mathbb{N}$, for each component $D_j^k(\omega)$ of the set $D^k(\omega) = (f_\omega^k)^{-1}(D)$ the degree of the map*

$$f_\omega^k : D_j^k(\omega) \to D$$

is at most N, then the Julia set J_ω is totally disconnected.

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Proof. In what follows, to simplify the notation we write D_j^k and D^k in place of $D_j^k(\omega)$ and $D^k(\omega)$, respectively. Recall that R_0 and $\tilde{R}_0$ were chosen in §2.2 in such a way that

$$\text{for all } \nu \in \mathbb{D}(0, R)^{\mathbb{N}}, \quad \mathbb{D}^*_{R_0} \subset A_\omega \quad \text{and} \quad f_\nu^{-1}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}. \tag{14}$$

Denote by P the annulus

$$P = \{z : R_0 < |z| < \tilde{R}_0\}.$$

For every $k \in \mathbb{N}$ and for every component D_j^k of D^k, the map $f_\omega^k : D_j^k \to D$ is a proper holomorphic map onto D.

By the assumption there exists an increasing sequence of positive integers $\{k_n\}$ such that the maps

$$f_\omega^{k_n} : D_j^{k_n} \to D$$

have degree at most N for all j.

Now, let us divide the annulus P into N nested geometric annuli with the same modulus M. These N annuli all lie in the intersection of D and all basins of infinity $\mathcal{A}_\nu$, $\nu \in \mathbb{D}(0, R)^{\mathbb{N}}$, by (14).

Let us pick a point z in the Julia set J_ω, and let $D_{j_n}^{k_n}$ be the component of D^{k_n} such that $z \in D_{j_n}^{k_n}$.

Since the degree of $f_\omega^{k_n}$ on $D_{j_n}^{k_n}$ is at most N, one of the N annuli contains no critical values of $f_\omega^{k_n}$; let us choose such an annulus and denote it by P_n. Consider now the (possibly smaller) disc $D' \subset D$, bounded by the outer boundary circle of the annulus P_n, and let $D'^{k_n}_{j_n}$ be the connected component of $(f_\omega^{k_n})^{-1}(D')$, containing the point z. The map $f_\omega^{k_n} : D'^{k_n}_{j_n} \to D'$ is also proper, and the preimage of the annulus P_n under this map, denoted here by P'_n, is again a (topological) annulus. The map $f_\omega^{k_n}$ restricted to P'_n is a covering map of degree at most N, so, the modulus of P'_n is at least M/N.

The point z lies in some connected component of $(f_\omega^{k_n})^{-1}(\mathbb{D}_{R_0})$ contained in $D_{j_n}^{k_n}$, so, in particular, it lies in the bounded component of the complement of the annulus P'_n.

Observation. Now, let us recall that, according to the choice of R_0 and $\tilde{R}_0$, for every $\nu \in \mathbb{D}(0, R)^{\mathbb{N}}$ we have that

$$f_\nu^{-1}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}.$$

So, in particular, for every $k \geq 1$, $f^{-1}_{\sigma^{k-1}\omega}(\mathbb{D}_{\tilde{R}_0}) \subset \mathbb{D}_{R_0}$. This also implies that, for any k and any $\omega \in \mathbb{D}(0, R)^{\mathbb{N}}$, each component of $(f_\omega^{(k+1)})^{-1}(\mathbb{D}_{\tilde{R}_0})$ is contained in some component of $(f_\omega^k)^{-1}(\mathbb{D}_{R_0})$ (since each such component is mapped by f_ω^k onto some component of $f^{-1}_{\sigma^{k-1}\omega}(\mathbb{D}_{\tilde{R}_0})$). Clearly, the same is true with $k+1$ being replaced by any arbitrary integer $m > k$.

We shall apply this observation for $k := k_n$ and $m := k_{n+1}$. As before, for k_{n+1} we find a topological annulus P'_{n+1} of modulus at least M/N, in the connected component of $(f_\omega^{k_{n+1}})^{-1}(\mathbb{D}_{\tilde{R}_0})$ containing the point z, and such that z lies in the bounded component of the complement of the annulus P'_{n+1}.

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Using the above observation, we conclude that the annulus P'_{n+1} is contained in the component of $(f_\omega^{k_n})^{-1}(\mathbb{D}_{R_0})$ containing the point z; in particular, it is contained in the bounded component of the complement of P'_n.

In this way, we obtain a nested infinite sequence of disjoint annuli P'_n, each of modulus at least M/N, all contained in $\mathcal{A}_\omega$, the point z being in the bounded component of the complement of each of them.

Now let us fix n and consider the topological annulus $\mathcal{P}_n$ that is bounded by the boundaries of D and $D_{j_n}^{k_n}$. Since it contains the nested sequence of annuli $P'_1, P'_2, \ldots, P'_n$, each of modulus at least M/N, then, by Grötzsch inequality, it must have modulus at least $n(M/N)$ (see, for example, [**1**, Proposition 5.4] or [**10**, Theorem B5, p. 192]). This in turn means it contains an actual geometric annulus of modulus at least $n(d/N) - C$ (where C is some constant), which separates the components of the boundary of $\mathcal{P}_n$. (see, for example, [**10**, Theorem 2.1]). Since, for every n, the connected component of K_ω containing the point z is contained in the bounded component of the complement of $\mathcal{P}_n$, this implies that the component of K_ω containing z must have arbitrarily small diameter, that is, it is the single point z.

Since the choice of the point z was arbitrary, this finally means the Julia set is totally disconnected, which concludes the proof of Proposition 10. □

4. *Typically fast escaping critical point and total disconnectedness*

In this section we check that the condition formulated in Definition 2.2 is sufficient to prove that the assumptions of Proposition 10 are satisfied for $\mathbb{P}$-almost every ω. More precisely, we prove the following theorem.

THEOREM 11. *Let V be a bounded Borel subset of $\mathbb{C}$, $V \subset \mathbb{D}(0, R)$. Let R_0, G be the values assigned to R as in §2.2. Let μ be a Borel probability measure on V and let $\mathbb{P}$ be the product distribution on $\Omega = V^{\mathbb{N}}$, generated by μ.*

If the critical point 0 *is* typically fast escaping, *that is, if* (13) *holds, then the assumptions of Proposition 10 are satisfied for $\mathbb{P}$-almost every $\omega \in \Omega$. Thus, for $\mathbb{P}$-almost every $\omega \in \Omega$, the Julia set J_ω is totally disconnected.*

Actually, the property from (13) is stronger than necessary, since to apply our proof all that is needed is for the series of probabilities to be convergent. In all our applications the bounds are indeed exponential, nevertheless the reader will soon see that the following remark is also true.

Remark. The statement of Theorem 11 is still true if one replaces (13) with

$$\sum_{k=0}^{\infty} \mathbb{P}\left(\left\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\right\}\right) < \infty.$$

Define the sets

$$A_k = \left\{\omega \in \Omega : g_\omega(0) < \frac{1}{2^k} G\right\}.$$

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Before proving Theorem 11 we explain in the next proposition the role of the sets A_k in possible applications of Proposition 10. We apply the setting and notation of Theorem 11.

PROPOSITION 12. *(a) If*

$$\sigma^i\omega \notin A_{k-i} \quad \text{for all } i = 0, \dots, k-1$$

then, for every connected component D_j^k of the preimage $(f_\omega^k)^{-1}(D)$, the degree of the map

$$f_\omega^k : D_j^k \to D$$

is equal to 1.

(b) If the above holds for all but l indices then, for every connected component of the set $f_\omega^{-k}(D)$, the degree of

$$f_\omega^k : D_j^k \to D$$

is bounded above by $N = 2^l$.

Proof. It follows from (10) that, for every $\nu \in \Omega$ and for every $z \in D$, $g_\nu(z) < G$. Let D_*^k be some component of $(f_\omega^k)^{-1}(D)$. Consider the sequence of maps

$$D_*^k \underset{f_\omega}{\longrightarrow} D_*^{k-1} \underset{f_{\sigma\omega}}{\longrightarrow} D_*^{k-2} \cdots \underset{f_{\sigma^{k-2}\omega}}{\longrightarrow} D_*^1 \underset{f_{\sigma^{k-1}\omega}}{\longrightarrow} D,$$

where we denoted by D_*^{k-i} the consecutive images of D_*^k under the maps f_ω, $f_{\sigma\omega}$, …, $f_{\sigma^{k-1}\omega}$. Note that $f_\omega^k : D_*^k \to D$ is just the composition of the above sequence of maps. If D_*^{k-i} contains the critical point 0 then $f_{\sigma^i\omega} : D_*^{k-i} \to D_*^{k-i-1}$ is a degree-two map; otherwise it is univalent.

Now, if

$$\sigma^i\omega \notin A_{k-i} \tag{15}$$

then $g_{\sigma^i\omega}(0) \geq (1/2^{k-i})G$, while for every $z \in D_*^{k-i}$ we have that

$$g_{\sigma^i\omega}(z) = \frac{1}{2^{k-i}} g(f_{\sigma^i\omega}^{k-i}(z)) < \frac{1}{2^{k-i}} \cdot G.$$

This implies that $0 \notin D_*^{k-i}$ and, consequently, the map $f_{\sigma^i\omega} : D_*^{k-i} \to D_*^{k-i-1}$ is univalent. So, if (15) happens for all $i = 0, \dots k-1$, then the map

$$f_\omega^k : D_*^k \to D$$

is univalent, so of degree one.

If (15) fails to hold for l indices i, then for these indices the degree of the map $f_{\sigma^i\omega} : D_*^{k-i} \to D_*^{k-i-1}$ is equal to one or two, while for all other indices it is equal to one, so that the degree of the composition $f_\omega^k : D_*^k \to D$ is at most $N = 2^l$. Proposition 12 is proved. □

Proof of Theorem 11. We now consider the extended probability space

$$\tilde{\Omega} := V^{\mathbb{Z}},$$

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with product probability, which we denote by $\tilde{\mathbb{P}}$. The left shift σ, considered in $\tilde{\Omega}$, is now a measurable automorphism of the space $\tilde{\Omega}$. There is a natural measurable projection

$$\pi : (\tilde{\Omega}, \tilde{\mathbb{P}}) \to (\Omega, \mathbb{P})$$

$$\tilde{\Omega} \ni (\ldots c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots) \xrightarrow{\pi} (c_0, c_1, c_2 \ldots) \in \Omega.$$

This projection transforms the measure $\tilde{\mathbb{P}}$ onto the measure $\mathbb{P}$, that is, $\tilde{\mathbb{P}} \circ \pi^{-1} = \mathbb{P}$.

For each $\tilde{\omega} \in \tilde{\Omega}$ the iterates f_{ω}^{n} are defined as previously, that is, for $\tilde{\omega} = (\ldots, c_{-2}, c_{-1}, c_0, c_1, c_2, \ldots)$,

$$f_{\tilde{\omega}}^{n}(z) = f_{c_{n-1}} \circ \cdots \circ f_{c_1} \circ f_{c_0}(z).$$

The Julia set is defined analogously to (1) and denoted by $J_{\tilde{\omega}}$. Similarly, the Green function $g_{\tilde{\omega}}$ is defined as in (3).

Considering the extended space $\tilde{\Omega}$ in this context may seem artificial, since the iterates $f_{\tilde{\omega}}^{n}$ depend only on the 'future', that is, only non-negative items $(c_j)_{j\geq 0}$ are used to define $f_{\tilde{\omega}}^{n}$ or its Julia set. Nevertheless, the proof is based on the construction of appropriate backward trajectories, which we shall describe below.

Let

$$E_k = \left\{ \tilde{\omega} \in \tilde{\Omega} : g_{\sigma^{-k}\tilde{\omega}}(0) < \frac{1}{2^k} G \right\}, \quad k = 0, 1, 2, \ldots$$

Let us note that the following estimate holds.

PROPOSITION 13. *If the critical point is typically fast escaping, that is, if* (13) *holds, then*

$$\tilde{\mathbb{P}}(E_k) < e^{-\gamma k},$$

where γ *comes from the estimate formulated in* (13). □

Proof. We have the estimates for the measure $\mathbb{P}$ of the set $A_k \subset \Omega$, given by (13). Now, let

$$\tilde{A}_k := \pi^{-1}(A_k) = V^{\mathbb{N}} \times A_k.$$

Then

$$\tilde{\mathbb{P}}(\tilde{A}_k) = \mathbb{P}(A_k).$$

Now note that $E_k = \sigma^k(\tilde{A}_k)$, which implies that

$$\tilde{\mathbb{P}}(E_k) = \tilde{\mathbb{P}}(\tilde{A}_k) = \mathbb{P}(A_k) < e^{-\gamma k}.$$ □

It follows from Proposition 13 and the Borel–Cantelli lemma that almost every $\tilde{\omega} \in \tilde{\Omega}$ belongs to at most finitely many sets E_k. This implies that there exist $K \in \mathbb{N}$ and a set $E \subset \tilde{\Omega}$ such that

$$\tilde{\mathbb{P}}(E) > 0$$

and

$$E \cap \left(\bigcup_{k=K}^{\infty} E_k \right) = \emptyset.$$

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Thus, for every $\tilde{\omega} \in E$ and every $k \geq K$, we have that

$$g_{\sigma^{-k}\tilde{\omega}}(0) \geq \frac{1}{2^k} G.$$

Now, using ergodicity of the left shift σ on $\tilde{\Omega}$, we conclude that $\tilde{\mathbb{P}}$-almost surely a sequence $\tilde{\nu} \in \tilde{\Omega}$ visits E infinitely many times under the iterates of σ.

Let $k \in \mathbb{N}$. For $\nu \in \Omega$ we introduce the following property.

Property (K, k). $\sigma^i \nu \in A_{k-i}$ for more than K indices $i \in \{0, \ldots k-1\}$.

LEMMA 14. *If Property* (K, k) *holds for* $\nu \in \Omega$ *and if* $\tilde{\nu} \in \pi^{-1}(\nu)$, *then* $\sigma^k \tilde{\nu} \notin E$

Proof. Indeed, let $\tilde{\nu} \in \pi^{-1}(\nu)$. Now, $\sigma^i \nu \in A_{k-i}$ means that

$$g_{\sigma^i \tilde{\nu}}(0) = g_{\sigma^i \nu}(0) < \frac{1}{2^{k-i}} G.$$

Putting $m := k - i$, this can be rewritten as

$$g_{\sigma^{-m}(\sigma^k \tilde{\nu})}(0) < \frac{1}{2^m} G,$$

that is,

$$\sigma^k \tilde{\nu} \in E_m. \tag{16}$$

Since (16) happens for more than K indices m, the definition of the set E implies that $\sigma^k \tilde{\nu} \notin E$. □

Let B be the set of elements $\nu \in \Omega$, for which Property (K, k) happens for all but finitely many integers k. Put $\tilde{B} := \pi^{-1}(B)$. It follows from Lemma 14 that every point $\tilde{\nu} \in \pi^{-1}(B)$ visits E at most finitely many times under the iterates of σ. It thus follows that $\tilde{\mathbb{P}}(\tilde{B}) = 0$ and, consequently, $\mathbb{P}(B) = 0$.

Now let $\nu \notin B$. Then. for infinitely many positive integers k. Property (K, k) does not hold. Pick such k. Then $\sigma^i \nu \notin A_{k-i}$ for all but at most K indices $i \in \{0, \ldots, k-1\}$. Thus, the assumption of Proposition 12(b) is satisfied for all such integers k. Applying this proposition, we see that the assumption of Proposition 10 is satisfied for ν. This allows us to conclude that the Julia set J_ν is totally disconnected for all $\nu \notin B$. This concludes the proof of Theorem 11.

5. *Conclusion. Proof of Theorem A and Theorem B*

In this section we complete the proofs of Theorem A and Theorem B. As shown in Theorem 11, it is enough to check that the estimate (13) holds, that is, the critical point is *typically fast escaping* under the assumptions of both theorems.

Let us note that under the assumptions of Theorem A the estimate (13) was actually proved in [**3**] (see Theorem 2.2 in that paper).

Obviously Theorem B implies Theorem A, thus let us focus on the more general setting presented in Theorem B. We shall conclude the proof of Theorem B with the following proposition.

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PROPOSITION 15. *Let V be a bounded open set such that $D(0, \frac{1}{4}) \subset V$ and $V \neq D(0, \frac{1}{4})$. Take $\Omega = V^{\mathbb{N}}$ to be the product space equipped with the product of uniform distributions on V, denoted by $\mathbb{P}$. There exists a constant $\gamma > 0$ such that*

$$\mathbb{P}\left(\left\{\omega \in \Omega : g_\omega(0) < \frac{G}{2^k}\right\}\right) < e^{-\gamma k},$$

where G is set as in (10).

Proof. To prove Proposition 15 we shall use the estimates (12). We also need a lemma, which follows the general scheme of the proof of [**3**, Theorem 2.2].

LEMMA 16. *Let V be an open and bounded set, such that $\mathbb{D}(0, \frac{1}{4}) \subset V \subset \mathbb{D}(0, R)$ and $V \neq \mathbb{D}(0, \frac{1}{4})$. Consider the space $\Omega = V^{\mathbb{N}}$ with the product of uniform distributions on V. Then there exists $\gamma > 0$ such that for every $z \in \mathbb{C}$,*

$$\mathbb{P}(k(z, \omega) > k) \leqslant e^{-\gamma k},$$

where $k(z, \omega)$ is the escape time of z from the disc $\mathbb{D}_{R_0}$, defined in Definition 2.1.

Proof. Let $c \in V$ be a point such that $|c| > \frac{1}{4}$, say $|c| > \frac{1}{4} + \varepsilon$ for some small $\varepsilon > 0$.

Let us pick a point $c' \in \mathbb{D}(0, \frac{1}{2})$ (not necessarily in V), such that $|c'| = \frac{1}{2} - \varepsilon/2$ and $\arg(c') = \arg(c)/2$. In particular, pick ε small enough so that $\frac{1}{2} - \varepsilon/2 > 0$. Observe that for the parabolic map $f(w) = w^2 + \frac{1}{4}$ we have

$$f^n(w) \xrightarrow[n\to\infty]{} \tfrac{1}{2} \tag{17}$$

for every real w satisfying $|w| < \frac{1}{2}$.

Consider z such that $|z| < \frac{1}{2}$. We claim that one can choose $N \in \mathbb{N}$ and the parameters $c_1, c_2, \ldots, c_N \in \mathbb{D}(0, \frac{1}{4})$ in a way that $f_\omega^N(z) = c'$. Indeed, note that since $|z| < \frac{1}{2}$ the set

$$\left\{z^2 + c : c \in \mathbb{D}\left(0, \tfrac{1}{4}\right)\right\}$$

contains the disc $\{w : |w| < \rho\}$, where $\rho = \frac{1}{4} - |z|^2 > 0$. So, we can choose c_0 such that, putting $w = z^2 + c_0$, we have $|w| < \rho$, and, adjusting c_0, we can additionally achieve that the argument of w is as we wish.

Using (17), we find $N > 0$ and real parameters $\tilde{c}_1, \ldots, \tilde{c}_{N-1} \in (0, \frac{1}{4})$ such that

$$f^{N-1}_{\tilde{c}_{N-1}, \ldots, \tilde{c}_1}(|w|) = |c'|.$$

Now, choosing appropriate c_0, we adjust the argument of w in such a way that

$$[2^{N-1} \arg(w) = 2^{N-1} \arg(f_{c_0}(z)]_{\mathrm{mod}\, 2\pi} = \arg(c'). \tag{18}$$

Next, for $n = 1, \ldots, N-1$, we choose c_n in such a way that $|c_n| = \tilde{c}_n$ and

$$\arg(c_n) = \arg((f^n_{c_{n-1}, \ldots c_1, c_0}(z))^2),$$

so that

$$\begin{aligned} |f^{n+1}_{c_n, c_{n-1}, \ldots, c_1, c_0}(z)| &= |(f^n_{c_{n-1}, \ldots c_1, c_0}(z))^2 + c_n| = |f^n_{c_{n-1}, \ldots c_1, c_0}(z))|^2 + |c_n| \\ &= |f^n_{c_{n-1}, \ldots c_1, c_0}(z))|^2 + \tilde{c}_n, \end{aligned}$$

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and, in consequence, $|f^N_{c_{N-1},\dots c_0}(z)| = |c'|$ and $\arg(f^N_{c_{N-1},\dots c_0}(z)) = \arg(c')$, thus $f^N_{c_{N-1},\dots c_0}(z) = c'$. Now since $f^N_{c_{N-1},\dots c_0}(z) = c'$ and $\arg[(c')^2] = \arg(c)$, putting $c_N := c$, we obtain

$$|f^{N+1}_{c_{N+1},\dots c_0}(z)| = |c'|^2 + c > \left(\frac{1}{2} - \frac{\varepsilon}{2}\right)^2 + \frac{1}{4} + \varepsilon > \frac{1}{2}.$$

Recall that for v real, $v > \frac{1}{2}$, we have $f^n(v) \xrightarrow[n\to\infty]{} \infty$. This means we can pick parameters $c_{N+2}, c_{N+3}, \dots, c_{N+N_1-1} \in \mathbb{D}(0, \frac{1}{4})$ for some N_1 (again, adjusting the argument appropriately) in such a way that $|f^{N+N_1}_{c_{N+N_1-1},\dots,c_0}(z)| > R_0 + 1$. For $|z| > \frac{1}{2}$ we obtain the same statement even in a easier way; one only has to repeat the second part of the reasoning above. The case of z with $|z| = \frac{1}{2}$ needs a small modification: choosing an appropriate c_0 in $\mathbb{D}(0, \frac{1}{4})$, we obtain $|z^2 + c_0| < \frac{1}{2}$ and the previously described procedure applies.

So, finally, this means we have the following. For every $z \in \mathbb{C}$, there exist $M = M_z$ and a sequence $c_0, c_1, \dots, c_M$, $c_i \in V$, such that

$$|f^M_{c_{M-1},\dots,c_0}(z)| > R_0 + 1.$$

Clearly, the same is true with c_i slightly perturbed, so, if we take $\delta > 0$ sufficiently small and put

$$A_z = \mathbb{D}(c_0, \delta) \times \cdots \times \mathbb{D}(c_{M-1}, \delta) \times \mathbb{D}(0, R)^{\mathbb{N}},$$

then $\mathbb{P}(A_z) > 0$ and, for all $\omega \in A_z$,

$$|f^M_\omega(z)| > R_0 + \tfrac{1}{2}.$$

Since the family

$$\{f^M_{\omega\,|\mathbb{D}_{R_0}}, \omega \in \Omega\},$$

with fixed M, is equicontinuous, we conclude that there exists $U_z \ni z$, an open neighbourhood of z such that, for all $v \in U_z$, $\omega \in A_z$, we have $|f^M_\omega(v)| > R_0$. Actually because of (5), for all $N \geq M$, we have

$$|f^N_\omega(v)| > R_0.$$

By compactness of $\overline{\mathbb{D}}_{R_0}$, there exists a finite cover of $\overline{\mathbb{D}}_{R_0}$ by a finite collection of the sets U_{z_i}. Taking $\alpha := \min_{z_i} \mathbb{P}(A_{z_i})$ and $M = \max_{z_i} M_{z_i}$, we can write:

$$\text{there exists } M \in \mathbb{N},\ \text{there exists } \alpha > 0,\ \text{for all } z \in \mathbb{C} \quad \mathbb{P}(\{|f^M_\omega(z)| > R_0\}) > \alpha.$$

Consequently, putting $S^k(z) = \{\omega \in \Omega : |f^k_\omega(z)| < R_0\}$, we know that for any z we have $\mathbb{P}(S^M(z)) < 1 - \alpha$.

We proceed to estimate $\mathbb{P}(S^k(z))$ exactly as in [3], using the fact that $\mathbb{P}$ is the product measure:

$$\mathbb{P}(S^{k+M}(z)) = \int_{S^k(z)} \mathbb{P}(S^M(f^k_\omega(z)))\, d\mathbb{P}(\omega) \leqslant (1-\alpha)\mathbb{P}(S^k(z)),$$

which, applied repeatedly, yields the existence of a constant $\gamma > 0$ such that

$$\mathbb{P}(k(z, \omega) > k) = \mathbb{P}(S^k(z)) \leqslant e^{-k\gamma}. \qquad \square$$

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Applying the above result for $z = 0$, together with the previously established (12), yields the claim, with possibly modified constant γ. This ends the proof of Proposition 15. □

It is important to point out that Lemma 16 is the only part of the proof of the main result that uses the assumption on the parameter space, that is, that it contains points from outside of the disc $D(0, \frac{1}{4})$. As mentioned before, if $R \leqslant \frac{1}{4}$ then the resulting Julia set is always connected, thus the proof above illustrates exactly the role this assumption fulfils.

Taking V to be the main cardioid yields the following interesting corollary of Theorem B.

THEOREM 17. *Let $\Omega = B^{\mathbb{N}}$, where B is the main cardioid of the Mandelbrot set, and let Ω be equipped with the product of uniform distributions on V. Then, for almost every sequence $\omega \in \Omega$, the Julia set J_ω is totally disconnected.*

6. *Further generalizations*

A number of other generalizations can be made by simple adaptations of the proof. For instance, it can be seen by inspecting the proof of Lemma 16 that the uniform distribution does not play any important role.

THEOREM 18. *Let $R > \frac{1}{4}$, and let μ be a Borel probability distribution on $\mathbb{D}(0, R)$ such that* $\operatorname{supp}(\mu) \supset \mathbb{D}(0, \frac{1}{4})$ *and* $\mu(\mathbb{D}(0, R) \setminus (\overline{\mathbb{D}(0, \frac{1}{4})})) > 0$. *Now consider the product measure of μ on $\mathbb{D}(0, R)^{\mathbb{N}}$. The Julia set for a sequence $\{c_n\} \subset \mathbb{D}(0, R)^{\mathbb{N}}$ is almost always totally disconnected, with respect to this product measure.*

The following result comes from [**8**, Theorem 2.2], but can also be inferred easily from our proof.

Remark. For every $c \notin \mathcal{M}$, there exists a neighbourhood $U(c)$ such that $J(c_n)$ is totally disconnected if all $c_n \in U(c)$.

Indeed, in this case it is easy to see that

$$\inf_{\omega=(c_n), c_n \in U} g_\omega(0) > a > 0$$

for some constant a, depending on U. So, with K sufficiently large, the set E defined in §4 is just the whole space $\tilde{\Omega}$. By Lemma 14 we conclude that, for every $\nu \in \Omega = U(c)^{\mathbb{N}}$ and for all k,

$$\sigma^i \nu \notin A_{k-i}$$

happens for all but at most K indices $i \in \{0, \ldots, k-1\}$, which, by Proposition 12 and Proposition 10, immediately implies that every Julia set J_ω is totally disconnected.

Another easy adaptation of the proof yields an answer to a question from [**3**] (see [**3**, Remark 2.5]), whether we can choose the parameters randomly, according to the uniform distribution, from a circle of radius $\delta > 1/4$.

Actually, the authors ask in [**3**, Remark 2.5] whether the set Julia set is almost surely disconnected. Our approach gives much more.

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PROPOSITION 19. *Let $\Omega = \partial\mathbb{D}(0, R)^{\mathbb{N}}$, where $R > \frac{1}{4}$, be equipped with the product of uniform distributions on the circle $\partial\mathbb{D}(0, R)$. Then, for almost every $\omega \in \Omega$, the Julia set J_ω is totally disconnected.*

To repeat our proof in the above case, we need the following version of Lemma 16.

LEMMA 20. *Let $K = \partial\mathbb{D}(0, R)$ where $R > \frac{1}{4}$. Consider the space $\Omega = K^{\mathbb{N}}$ with the product of uniform distributions on K. Then there exists $\gamma > 0$ such that, for all $z \in \mathbb{C}$,*

$$\mathbb{P}(k(z, \omega) > k) \leqslant e^{-\gamma k},$$

where $k(z, \omega)$ is the value defined in (11).

Proof. Take an arbitrary point $z \in \mathbb{C}$, and let $c_1, c_2, c_3, \ldots, c_N \in K$ be a sequence of N parameters such that, for all $n \leq N$,

$$|f_\omega^n(z)| = |f_\omega^{n-1}(z)^2 + c_n| = |f_\omega^{n-1}(z)|^2 + |c_n|.$$

Recall that, for iterations on the real line, with $f(x) = x^2 + R$ and $R > \frac{1}{4}$, we have, for all x,

$$\lim_{n\to\infty} f^n(x) = \infty.$$

Since $|c_n| = R > \frac{1}{4}$ by our choice of the numbers $c_1, \ldots, c_N$, for a large enough N, we will have $|f_\omega^N(z)| > R_0$. By continuity and compactness arguments, used exactly as in the proof of Lemma 16, we see that one can show something more, that is,

$$\text{there exists } N \in \mathbb{N}, \text{ there exists } \delta > 0, \text{ for all } z \in \mathbb{C} \quad \mathbb{P}(\{\omega \in \Omega : |f_\omega^N(z)| > R_0\}) > \delta.$$

We finish the proof in exactly the same way as the proof of Lemma 16. □

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REFERENCES

[1] B. Branner and J. H. Hubbard. The iteration of cubic polynomials part II: Patterns and parapatterns. *Acta Math.* **169**(3–4) (1992), 229–325.
[2] R. Brück. Connectedness and stability of Julia sets of the composition of polynomials of the form $z^2 + c_n$. *J. Lond. Math. Soc.* **61** (2000), 462–470.
[3] R. Brück, M. Büger and S. Reitz. Random iteration of polynomials of the form $z^2 + c$: connectedness of Julia sets. *Ergod. Th. & Dynam. Sys.* **19** (1999), 1221–1231.
[4] M. Büger. On the composition of polynomials of the form $z^2 + c_n$. *Math. Ann.* **310** (1998), 661–683.
[5] M. Comerford. Hyperbolic non-autonomous Julia sets. *Ergod. Th. & Dynam. Sys.* **26** (2006), 353–377.
[6] M. Comerford. Non-autonomous Julia sets with escaping critical points. *J. Difference Equ. Appl.* **17** (2011), 1813–1826.
[7] J. E. Fornæss and N. Sibony. Random iterations of rational functions. *Ergod. Th. & Dynam. Sys.* **11** (1991), 687–708.
[8] Z. Gong, W. Qiu and Y. Li. Connectedness of Julia sets for a quadratic random dynamical system. *Ergod. Th. & Dynam. Sys.* **23** (2003), 1807–1815.
[9] V. Mayer, B. Skorulski and M. Urbański. *Distance Expanding Random Mappings, Thermodynamical Formalism, Gibbs Measures and Fractal Geometry* (*Lecture Notes in Mathematics, 2036*) (Springer, Berlin, 2011).
[10] C. T. McMullen. *Complex Dynamics and Renormalization* (Princeton University Press, Princeton, NJ, 1994).