
On Erdős–Ko–Rado for Random Hypergraphs II[†]

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Denote by $\mathcal{H}_k(n, p)$ the random k -graph in which each k -subset of $\{1, \dots, n\}$ is present with probability p , independent of other choices. More or less answering a question of Balogh, Bohman and Mubayi, we show: there is a fixed $\varepsilon > 0$ such that if $n = 2k + 1$ and $p > 1 - \varepsilon$, then w.h.p. (that is, with probability tending to 1 as $k \rightarrow \infty$), $\mathcal{H}_k(n, p)$ has the ‘Erdős–Ko–Rado property’. We also mention a similar random version of Sperner’s theorem.

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1. Introduction

One of the most interesting combinatorial trends of the last couple decades has been the investigation of ‘sparse random’ versions of some of the classical theorems of the subject – that is, of the extent to which such results hold in a random setting. This issue has been the subject some spectacular successes, particularly those related to the theorems of Ramsey [23], Turán [31] and Szemerédi [30]; see [13, 2, 24, 19] for origins and, for example, [9, 28, 10, 5, 27] (or the survey [25]) for a few of the more recent developments.

Here we are interested in the analogous question for the Erdős–Ko–Rado theorem [11], another cornerstone of extremal combinatorics. This natural problem has already been considered by Balogh, Bohman and Mubayi [4], and we first quickly recall a few of the notions from that paper.

In what follows k and n are always positive integers with $n > 2k$. As usual we write $[n]$ for $\{1, \dots, n\}$ and $\binom{V}{k}$ for the collection of k -subsets of a set V . A k -graph (or k -uniform hypergraph) on V is a multisubset, say \mathcal{H} , of $\binom{V}{k}$. Members of V and \mathcal{H} are called *vertices* and *edges* respectively. We use \mathcal{H}_x for the set of edges containing x ($x \in V$), called the *star* of x in \mathcal{H} or

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the *principal* subhypergraph generated by x . For the present discussion we take $V = [n]$ and write \mathcal{K} for $\binom{V}{k}$.

A collection of sets is *intersecting*, or a *clique*, if no two of its members are disjoint. The Erdős–Ko–Rado theorem says that for any n and k as above, the maximum size of an intersecting k -graph on V is $\binom{n-1}{k-1}$ and, moreover, this bound is achieved only by the stars.

Following [4] we say \mathcal{H} satisfies (*strong*) EKR if every largest clique of \mathcal{H} is a star; thus the EKR Theorem says $\binom{V}{k}$ satisfies EKR. (We also say, again as in [4], that \mathcal{H} satisfies *weak EKR* if *some* largest clique is a star, but this slightly weaker notion will not concern us here.)

For the rest of this introduction we use $\mathcal{H} = \mathcal{H}_k(n, p)$ for the random k -graph on V in which members of $\binom{V}{k}$ are present independently, each with probability p . As suggested above, we are interested in understanding when EKR holds for \mathcal{H} . We now state this a little more formally.

Question 1.1. *For what $p_0 = p_0(n, k)$ is it true that \mathcal{H} satisfies EKR w.h.p. provided $p > p_0$?*

(As usual ‘w.h.p.’ (*with high probability*) means with probability tending to one as $n \rightarrow \infty$.)

The nature of the problem may be said to change around $k = \sqrt{n}$, since for k smaller than this, two random k -sets are typically disjoint, while the opposite is true for larger k . Heuristically we may say that the problem becomes more interesting/challenging as k grows and the potential violations of EKR proliferate (though increasing k does narrow the range of p for which we expect EKR to hold).

In this paper we are interested in what happens when k is as large as possible. The next assertion is our main result.

Theorem 1.2. *There is a fixed $\varepsilon > 0$ such that if $n = 2k + 1$ and $p > 1 - \varepsilon$, then \mathcal{H} satisfies EKR w.h.p.*

This was prompted by Question 1.4 of [4], as follows.

Question 1.3. *Is it true that for $k \in (n/2 - \sqrt{n}, n/2)$ and $p = 0.99$, EKR (or weak EKR) holds w.h.p. for \mathcal{H} ?*

Note that for n, k as in Theorem 1.2, EKR is unlikely unless p is large (so ‘sparse random’ is something of a misnomer), since a simple calculation shows that for p less than about $3/4$ stars are unlikely even to be *maximal* cliques. (This is, of course, reminiscent of the Hilton–Milner Theorem [15], which says that (for any k and $n > 2k$) the largest non-trivial cliques in $\binom{[n]}{k}$ are those of the form $\{A\} \cup \{B \in \binom{[n]}{k} : x \in B, B \cap A \neq \emptyset\}$ (with $A \in \binom{[n]}{k}$ and $x \in [n] \setminus A$)). We expect that, for k, n as in Theorem 1.2, this is in fact the main hurdle – that is, EKR becomes likely as soon as stars are likely to be maximal – but we are far from proving such a statement. On the other hand, as will appear below, the main difficulties in proving the theorem involve cliques that are far from stars.

We have not thought very hard about whether the ε in Theorem 1.2 could be pushed to .01, since this seems somewhat beside the point (and since it does not seem wildly unethical to regard ‘.99’ as really meaning ‘ $1 - \varepsilon$ for some fixed $\varepsilon > 0$ ’). We assume our methods could be adapted to give Theorem 1.2 for smaller k , but confine ourselves to the present statement. This is partly

for simplicity, but also because we do not believe the theorem gives a very satisfactory answer in other cases; for example, even for $n = 2k + 2$ we expect EKR to hold for p down to about $1/k$.

The original paper of Balogh, Bohman and Mubayi dealt mostly with $k < n^{1/2-\varepsilon}$ (for a fixed $\varepsilon > 0$). In a companion paper [14] we precisely settle the question for k up to about $\sqrt{(1/4)n \log n}$ and suggest a possible general answer.

The rest of this paper is organized as follows. Section 2 sets notation and fills in some mostly standard background, and Section 3 reduces Theorem 1.2 to a related, slightly fussier statement. The most interesting part of the argument, given in Section 4, proves the latter statement using, in addition to standard large deviation considerations, asymptotic-enumerative ideas inspired especially by work of A. A. Sapozhenko [26]. A final short section mentions a counterpart of Theorem 1.2 for Sperner’s theorem that follows easily from the method developed in Section 4.

2. Preliminaries

2.1. Usage

Set $M = \binom{2k}{k-1}$ and $N = \binom{2k}{k}$. Unless specified otherwise, we use \mathcal{K} for $\binom{[n]}{k}$. As usual, 2^S is the power set of S and, for a hypergraph \mathcal{H} , $d_{\mathcal{H}}(x)$ is the degree of $x \in V$ in \mathcal{H} (i.e. $|\{A \in \mathcal{H} : x \in A\}|$) and $\Delta_{\mathcal{H}}$ is the maximum of these degrees.

For graphs, xy is an edge joining vertices x and y ; $N(x)$ is, as usual, the neighbourhood of x (and $N(X) = \cup_{x \in X} N(x)$); $\nabla(X, Y)$ is the set of edges joining the disjoint vertex sets X, Y ; and $d_W(x) = |N(x) \cap W|$ (for $W \subseteq V$).

We use $B(m, \alpha)$ for a random variable with the binomial distribution $\text{Bin}(m, \alpha)$ and \log for \ln . We assume throughout that $n = 2k + 1$ is large enough to support our arguments.

2.2. Large deviations

We use Chernoff’s inequality in the following form, which may be found, for example, in [16, Theorem 2.1].

Theorem 2.1. For $\xi = B(m, q)$, $\mu = mq$ and any $\lambda \geq 0$,

$$\mathbb{P}(\xi > \mu + \lambda) < \exp\left[-\frac{\lambda^2}{2(\mu + \lambda/3)}\right],$$

$$\mathbb{P}(\xi < \mu - \lambda) < \exp\left[-\frac{\lambda^2}{2\mu}\right].$$

We will also need the following improvement for larger deviations, for which see [1, Theorem A.1.12], for example.

Theorem 2.2. For $\xi = B(m, q)$ and any K ,

$$\mathbb{P}(\xi > Km) < \exp[-Km \log(K/e)].$$

(Of course this is only meaningful if $K > e$.)

2.3. Isoperimetry and degree

For $A \subseteq \binom{[2k]}{k}$ let $\delta(A) = (|\partial_u A| - |A|)/|A|$, where

$$\partial_u A = \left\{ y \in \binom{[2k]}{k+1} : \exists x \in A, y \supset x \right\}$$

(the *upper shadow* of A). We will use the following consequence of the Kruskal–Katona theorem ([20], [17] or e.g. [7]).

Proposition 2.3. For $A \subseteq \binom{[2k]}{k}$ with $|A| \leq N/2$,

$$\delta(A) \geq \frac{\log 2}{k} \log_2 \left(\frac{N}{2|A|} \right). \tag{2.1}$$

(Recall $N = \binom{2k}{k}$, and notice that $N/2 = \binom{2k-1}{k}$. The $\log 2$ in (2.1) can probably be replaced by 1, but cannot be replaced by $k/(k-1)$.)

Proof. We use Lovász’s version [21, Problem 13.31] of Kruskal–Katona, which in the present situation says that if $|A| = \binom{x}{k}$ ($:= (x)_k/k!$ for any $x \in \mathbb{R}$), then $|\partial_u(A)| \geq \binom{x}{k-1}$. (This is ordinarily stated for the *lower shadow*, which is equivalent here since our universe is of size $2k$.)

Let $|A| = \binom{2k-t}{k}$, noting that $|A| \leq N/2$ implies $t \geq 1$, and $\psi = k^{-1} \log 2$. Then

$$\frac{N}{2|A|} = \frac{(2k)_k}{2(2k-t)_k}$$

and, from Kruskal–Katona (Lovász),

$$\delta(A) \geq \binom{2k-t}{k-1} / \binom{2k-t}{k} - 1 = \frac{t-1}{k-t+1}.$$

Thus (2.1) will follow from

$$f(t) := \frac{t-1}{k-t+1} - \psi \log_2 \left[\frac{(2k)_k}{2(2k-t)_k} \right] \geq 0 \quad \text{for } t \geq 1,$$

so (since $f(1) = 0$) from $f'(t) \geq 0$. But, recalling the value of ψ , we have

$$f'(t) = \frac{k}{(k-t+1)^2} - \frac{1}{k} \sum_{i=0}^{k-1} \frac{1}{2k-t-i} \geq \frac{k}{(k-t+1)^2} - \frac{1}{k-t+1} \geq 0. \quad \square$$

The following result of P. Frankl [12] will also be helpful in getting things started. (We give the result for general k, n and i , again writing \mathcal{K} for $\binom{[n]}{k}$, but will only use it with $n = 2k + 1$ and $i = 3$.) Given k and $n > 2k$, set, for each $i \in \{3, \dots, k+1\}$,

$$\mathcal{F}_i = \{A \in \mathcal{K} : 1 \in A, A \cap \{2, \dots, i\} \neq \emptyset\} \cup \{A \in \mathcal{K} : A \supseteq \{2, \dots, i\}\}.$$

Theorem 2.4 ([12]). For any k, n and i as above, if $\mathcal{F} \subseteq \mathcal{K}$ is a clique with $|\mathcal{F}| > |\mathcal{F}_i|$, then $\Delta_{\mathcal{F}} > \Delta_{\mathcal{F}_i}$.

2.4. Graphs

Two special graph-theoretic notions will be relevant in what follows. First, for a bigraph Σ with bipartition $\Gamma_1 \cup \Gamma_2$, the *closure* of $X \subseteq \Gamma_i$ is $\{x : N(x) \subseteq N(X)\}$ (and X is *closed* if it is equal to its closure). Second, for a (general) graph Σ and positive integer j , $W \subseteq V(\Sigma)$ is *j-linked* if for all $u, v \in W$ there are $u = u_0, u_1, \dots, u_\ell = v$ with $u_i \in W$ ($\forall i$) and $\rho(u_{i-1}, u_i) \leq j$ for $i \in [\ell]$, where ρ is graph-theoretic distance. We will eventually need the following observation from [26].

Proposition 2.5. *Let Σ be a graph and suppose A and T are subsets of $V(\Sigma)$ with $T \subseteq N(A)$, $A \subseteq N(T)$ and A j -linked. Then T is $(j + 2)$ -linked.*

Proof. Given $u, v \in T$, choose $x, y \in A$ with $x \sim u, y \sim v$, and then $x = x_0, \dots, x_\ell = y$ with $x_i \in A$ and $\rho(x_{i-1}, x_i) \leq j$ ($i \in [\ell]$). If we now let $u_0 = u, u_\ell = v$ and $x_i \sim u_i \in T$ for $i \in [\ell - 1]$, then $\rho(u_{i-1}, u_i) \leq 1 + \rho(x_{i-1}, x_i) + 1 \leq j + 2$ (for $i \in [\ell]$). The proposition follows. \square

We also find some use for the following standard bound.

Proposition 2.6. *In any graph with all degrees at most d , the number of trees of size u rooted at some specified vertex is at most $(ed)^{u-1}$.*

Proof. This follows easily from the fact (see e.g. [18, p. 396, Ex. 11]) that the infinite d -branching rooted tree contains precisely

$$\frac{1}{(d - 1)u + 1} \binom{Du}{u} \leq (ed)^{u-1}$$

rooted subtrees of size u . \square

And so on. We make repeated use of the fact that for positive integers a, b with $a \leq b/2$,

$$\sum_{i \leq a} \binom{b}{i} \leq \exp[a \log(eb/a)]. \tag{2.2}$$

3. Setting up

In what follows, \mathcal{H} denotes a member of \mathcal{M} , the collection of *non-principal* maximal intersecting families in $\binom{[n]}{k}$. We now set $\mathcal{H}_k(n, p) = \mathbf{X}$, where $p = 1 - \varepsilon$, with $\varepsilon > 0$ fixed but small enough to support our arguments. (We make no attempt to optimize.)

The statement we are to prove is

$$\text{w.h.p. } \max_{\mathcal{H} \in \mathcal{M}} |\mathbf{X} \cap \mathcal{H}| < \max_x |\mathbf{X} \cap \mathcal{K}_x|, \tag{3.1}$$

but we will find it better to work with a variant, (3.4) below. This requires a little preparation.

For $x \in [n]$ and $0 \leq \ell \leq n - 1$, let Γ_ℓ^x denote the collection of ℓ -subsets of $[n] \setminus \{x\}$. Let Σ^x be the usual bigraph on $\Gamma_k^x \cup \Gamma_{k+1}^x$ (that is, with adjacency given by set containment), and write N^x for neighbourhood in Σ^x . For $A \subseteq \Gamma_k^x$ set $\delta_x(A) = (|N^x(A)| - |A|)/|A|$ (so $N^x(A)$ is the upper shadow of A in $2^{[n] \setminus \{x\}}$ and our usage here follows that in Proposition 2.3).

For $\mathcal{H} \in \mathcal{M}$ (and $x \in [n]$), let $A^x(\mathcal{H}) = \mathcal{H} \setminus \mathcal{K}_x$, $J^x(\mathcal{H}) = \mathcal{K}_x \setminus \mathcal{H}$ and $G^x(\mathcal{H}) = N^x(A^x(\mathcal{H}))$; thus $A^x(\mathcal{H})$ and $G^x(\mathcal{H})$ are subsets of Γ_k^x and Γ_{k+1}^x respectively. Note that

$$|\mathbf{X} \cap A^x(\mathcal{H})| - |\mathbf{X} \cap J^x(\mathcal{H})| = |\mathbf{X} \cap \mathcal{H}| - |\mathbf{X} \cap \mathcal{K}_x|. \tag{3.2}$$

For $\mathcal{B} \subseteq 2^{[n]}$ set $\mathcal{B}^c = \{[n] \setminus T : T \in \mathcal{B}\}$. It is easy to see that maximality of \mathcal{H} implies that (for any x) $A^x(\mathcal{H})$ is closed in Σ^x and $G^x(\mathcal{H}) = J^x(\mathcal{H})^c$. The converse is also true (and similarly easy): if $x \in [n]$ and A is a non-empty closed subset of Γ_k^x (in Σ^x), then $(\mathcal{K}_x \setminus N^x(A)^c) \cup A \in \mathcal{M}$. (If $A = \emptyset$ then $(\mathcal{K}_x \setminus N^x(A)^c) \cup A = \mathcal{K}_x$ is maximal intersecting but not in \mathcal{M} , whose members are required to be non-principal.)

Let \mathcal{Q} be the event that there are $\mathcal{H} \in \mathcal{M}$ and $x \in [n]$ for which $A^x(\mathcal{H})$ is 2-linked (in Σ^x),

$$\delta_x(A^x(\mathcal{H})) > 1/(3k), \tag{3.3}$$

and $|\mathbf{X} \cap \mathcal{H}| \geq |\mathbf{X} \cap \mathcal{K}_x|$. Our main point, the aforementioned variant of (3.1), is

$$\mathbb{P}(\mathcal{Q}) = o(1). \tag{3.4}$$

Before proving this (in Section 4), we show that it implies (3.1), by showing that failure of (3.1) implies \mathcal{Q} . Supposing (3.1) fails, choose $\mathcal{H} \in \mathcal{M}$ with $|\mathbf{X} \cap \mathcal{H}|$ maximum and fix x with $d_{\mathcal{H}}(x) = \Delta(\mathcal{H})$. Let $A = A^x(\mathcal{H})$ and $J = J^x(\mathcal{H})$. By (3.2) (and our assumption that $|\mathbf{X} \cap \mathcal{H}| \geq |\mathbf{X} \cap \mathcal{K}_y| \forall y$) we have

$$|\mathbf{X} \cap A| \geq |\mathbf{X} \cap J|. \tag{3.5}$$

Note also that

$$|A| \leq (k+1)|\mathcal{H}|/n < (k+1)M/n \tag{3.6}$$

(recall $M = \binom{2k}{k-1}$), since $(|\mathcal{H} \setminus A|) = \Delta(\mathcal{H}) \geq k|\mathcal{H}|/n$ (as is true for any $\mathcal{H} \subseteq \mathcal{K}$).

Suppose first that A is 2-linked in Σ^x . In this case we claim that (\mathcal{H}, x) itself satisfies \mathcal{Q} , that is, (3.3) holds. Let $\mathcal{H}^* = \{T \in \mathcal{K} : |T \cap [3]| \geq 2\}$. If $|\mathcal{H}| > |\mathcal{H}^*|$, then Theorem 2.4 gives $\Delta(\mathcal{H}) > \Delta(\mathcal{H}^*) \sim 3M/4$, whence $|A| < (1+o(1))M/4$ and (3.3) (actually a little more) is given by (2.1). If, on the other hand, $|\mathcal{H}| \leq |\mathcal{H}^*|$, then, noting that $M - |\mathcal{H}^*| \sim M/(4k)$, we have, using (3.2) and (3.6),

$$\delta_x(A) = (M - |\mathcal{H}|)/|A| > (2 - o(1))(M - |\mathcal{H}^*|)/M \sim 1/(2k).$$

Now suppose A is not 2-linked. Let A_1, \dots, A_s be the 2-linked components (defined in the obvious way) of A , and $J_i = N^x(A_i)^c$. Then $J_i \cup \dots \cup J_s$ is a partition of J . Moreover, each A_i is closed, so that (see the paragraph following (3.2)) $\mathcal{H}_i := (\mathcal{K}_x \setminus J_i) \cup A_i \in \mathcal{M}$ for each i . Suppose w.l.o.g. that $|A_1| = \max_i |A_i|$. Then for $i \geq 2$ we have $|A_i| \leq |A|/2 < (1/4 + o(1))M$, implying (again using (2.1)) $\delta_x(A_i) > (\log 2 - o(1))/k$. So we have \mathcal{Q} if $|\mathbf{X} \cap A_i| \geq |\mathbf{X} \cap J_i|$ for some $i \geq 2$; but if this is not the case then (again using (3.2))

$$\begin{aligned} |\mathbf{X} \cap \mathcal{H}_1| - |\mathbf{X} \cap \mathcal{K}_x| &= |\mathbf{X} \cap A_1| - |\mathbf{X} \cap J_1| \\ &= |\mathbf{X} \cap A| - |\mathbf{X} \cap J| - \sum_{i \geq 2} (|\mathbf{X} \cap A_i| - |\mathbf{X} \cap J_i|) \\ &> |\mathbf{X} \cap A| - |\mathbf{X} \cap J| = |\mathbf{X} \cap \mathcal{H}| - |\mathbf{X} \cap \mathcal{K}_x|, \end{aligned}$$

contradicting the assumed maximality of $|\mathbf{X} \cap \mathcal{H}|$. □

4. Main point

Here we prove (3.4). For the remainder of our discussion we work with a fixed $x \in [n]$ and drop the super- and subscripts x from our notation; so to begin, we set $\Sigma^x = \Sigma$ and $\Gamma_\ell^x = \Gamma_\ell$. We will use G_A for the neighbourhood of $A \subseteq \Gamma_k$ in Σ and

$$\delta(A) = |G_A|/|A| - 1 \quad (= \delta_x(A)).$$

We extend \mathbf{X} to Γ_{k+1} by declaring that $T \in \mathbf{X}$ if and only if $[n] \setminus T \in \mathbf{X}$ (so here T is a $(k + 1)$ -set off x and $[n] \setminus T$ is a k -set on x); we may then forget about $J(\mathcal{H})$ ($= J^x(\mathcal{H})$) and regard \mathbf{X} as a subset of $\Gamma_k \cup \Gamma_{k+1}$. Note that (cf. (3.2)) ‘ $|\mathbf{X} \cap \mathcal{H}| \geq |\mathbf{X} \cap \mathcal{K}_x|$ ’ in the definition of \mathcal{Q} is then the same as ‘ $|\mathbf{X} \cap G_A| \geq |\mathbf{X} \cap A|$ ’ with $A = \mathcal{H} \setminus \mathcal{K}_x$ and (thus) $G_A = J^x(\mathcal{H})^c$.

For the proof of (3.4) we will bound the probability that \mathcal{Q} occurs at our given x with specified sizes of A and G_A , and then sum over possibilities for these sizes. (Of course we need a bound $o(1/n)$ since we must eventually sum over x .) Thus we assume throughout that we have fixed a, g with

$$\delta := (g - a)/a > \max\{1/(3k), (\log 2/k) \log_2(N/(2a))\} \tag{4.1}$$

(with the second term in the max again given by Proposition 2.3), and write $\mathcal{A} = \mathcal{A}(a, g)$ for the set of A ’s satisfying

$$A \text{ is closed and 2-linked, } |A| = a \text{ and } |G_A| = g. \tag{4.2}$$

Notice that for $A \in \mathcal{A}$ we have

$$\begin{aligned} |\nabla(G_A, \Gamma_k \setminus A)| &= (k + 1)g - ka \\ &= (k + 1)(1 + \delta)a - ka = (1 + (k + 1)\delta)a. \end{aligned} \tag{4.3}$$

Let $\mathcal{Q}(a, g)$ ($= \mathcal{Q}_x(a, g)$) be the event that there is some $A \in \mathcal{A}(a, g)$ with

$$|\mathbf{X} \cap G_A| \leq |\mathbf{X} \cap A|. \tag{4.4}$$

We show

$$\sum_{a,g} \mathbb{P}(\mathcal{Q}(a, g)) = o(1/n), \tag{4.5}$$

which, since the union of the $\mathcal{Q}(a, g)$ ’s is occurrence of \mathcal{Q} at x , gives (3.4).

The bound (4.5) is (of course) the heart of the matter, and the rest of our discussion is devoted to its proof. This turns out to be rather delicate, and a rough indication of where we are headed may be helpful. (The following description refers to the main case, namely $\delta \leq 1$, considered below.)

For $A \in \mathcal{A}$ we have

$$\mathbb{E}|\mathbf{X} \cap G_A| - \mathbb{E}|\mathbf{X} \cap A| = \delta ap, \tag{4.6}$$

so we can rule out (4.4) if we can say that the quantities $|\mathbf{X} \cap G_A|$ and $|\mathbf{X} \cap A|$ are close to their expectations, where ‘close’ means somewhat small relative to δap ($\approx \delta a$). The problem (of course) is that though each of these *individual* events is unlikely, there are too many of them to allow a simple union bound.

Our remedy for this is to exploit similarities among the A 's (and similarly G_A 's, but for this very rough description we stick to A 's) to avoid paying repeatedly for the same unlikely events. To do this we specify each $A \in \mathcal{A}$ via several 'approximations', beginning with a set S_A for which $A \Delta S_A$ is fairly small, and then adding and subtracting lesser pieces. It will then follow that $|\mathbf{X} \cap A|$ is close to its expectation provided this is true of $|\mathbf{X} \cap B|$ for each of the relevant pieces B .

Thus we will want to say that, with B ranging over some to-be-specified collection of subsets of Γ_k , it is likely that all $|\mathbf{X} \cap B|$'s are close to their expectations. Of course the probability that this fails for a particular B grows with $|B|$ (since the benchmark δap does not change), so we would like to arrange that the larger B 's are not too numerous. For example, the above S_A 's will necessarily be large (of size roughly a), but there will be relatively few of them, reflecting the fact that a single S will typically be S_A for many A 's. We may think of \mathcal{A} as consisting of a large number of variations on a relatively small number of themes, though, as we will see, controlling these themes and variations turns out to be not very straightforward.

As mentioned earlier, our approach here has its roots in the beautiful ideas of A. A. Sapozhenko [26], which were originally developed to deal with 'Dedekind's problem' and related questions in asymptotic enumeration.

Proof of (4.5). As our fixed x plays no further role in what follows, we will feel free to recycle and use ' x ' (along with u, v, y, z) to denote a general member of our ground set, which we may now think of as $[2k]$.

We divide the proof of (4.5) into two cases, large and small δ , beginning with the second, which is by far the more interesting. (Our treatment of this case can be adapted to work in general – actually with most of the contortions below becoming unnecessary and/or vacuous – but this seems pointless given how much simpler the proof is for large δ . It also seems worth stressing that, as mentioned earlier, the real challenge is in dealing with quite small δ (and thus, according to (2.1), with quite large a .)

Assume then that $\delta \leq 1$ (say), and note that in this case (2.1) gives

$$a > (4/e)^k (4\sqrt{k})^{-1} =: a_0 \tag{4.7}$$

(which is pretty far from the truth but we have plenty of room here).

Prospectus. Before we continue, some further pointers may be helpful.

This main part of our argument proceeds in two phases. At the end of the first phase we will have associated with each $A \in \mathcal{A}$ several sets (drawn from Γ_k, Γ_{k+1} and $E(\Sigma) = \nabla(\Gamma_k, \Gamma_{k+1})$) from which decent approximations of A and G_A can be built up in a useable way.

The output of this phase, summarized in the paragraphs following the proof of Lemma 4.3, is a collection, \mathcal{R} , of triples encapsulating the relevant information; thus we produce a (typically many-to-one) map, $A \mapsto R(A)$, from \mathcal{A} to \mathcal{R} . We will then, for each $R \in \mathcal{R}$, fix some A_R^* that maps to R , and take this and the associated $G_R^* := G_{A_R^*}$ to be our final approximations to A and G_A for each A with $R(A) = R$.

The second phase of the argument then considers the intersections of \mathbf{X} with our various pieces, as well as with the final bits that are added and subtracted to move from the approximations to our actual A 's and G_A 's. As suggested earlier, we hope to say that (w.h.p.) all these intersections have

sizes close to their expectations, and a central issue will be controlling the numbers of pieces of various sizes: the larger the pieces, the fewer we can afford. This goal is achieved in Lemma 4.4, the workhorse of the second phase, from which the desired application to (4.5) follows fairly immediately: see Corollary 4.5 and the paragraph following its statement.

To get some feel for what’s going on (in both phases) and how the whole thing fits together, the reader might take an early look at the discussion of the second phase through the proof of Corollary 4.5, ignoring the particulars of Lemma 4.4 and reading the proof of the corollary more or less at the level of Venn diagrams (without worrying about the meanings of its many presently undefined ingredients).

First phase. For $X \subseteq V := V(\Sigma)$, let $N^i(X) = \{u \in V : \rho(u, X) \leq i\}$ (where, recall, ρ is graph-theoretic distance). For $A \in \mathcal{A} (= \mathcal{A}(a, g))$, say a path is A -good if it is of the form vx_1yx_2 with $x_1, x_2 \in A$ (so in particular has length 3), and for $v \in \Gamma_{k+1}$, let $\varphi(v, A)$ denote the number of A -good paths beginning with v .

Fix a small $\zeta > 0$ (we just need $\zeta < 1/2$), and set $\vartheta = \zeta/2$ and

$$G_A^0 = \{v \in G_A : \varphi(v, A) \geq (1/4)k^{3-\zeta}\}.$$

For $T \subseteq \Gamma_k$ set $W_T = N^3(T) \cap \Gamma_{k+1}$ and

$$S_T = \{x : d_{W_T}(x) \geq k/2\} \quad (\subseteq \Gamma_k). \tag{4.8}$$

For $T \subseteq A \in \mathcal{A}$, let $F_{A,T} = \nabla(N(T), \Gamma_k \setminus A)$ and $Z_{A,T} = N(N^2(T) \cap A) \subseteq W_T$. Notice that $w \in Z_{A,T}$ if and only if either $w \in N(T)$ or there is a path $xyzw$ with $x \in T$ and $yz \notin F_{A,T}$ (equivalently an A -good path from w to T); in particular $Z_{A,T}$ is determined by T and $F_{A,T}$.

Lemma 4.1. *There is a fixed K such that for each $A \in \mathcal{A}$ there is a $T \subseteq A$ satisfying*

(T1) $|T| \leq K a k^{-3+\zeta} \log k,$

(T2) $|F_{A,T}| \leq K \delta a k^{-1+\zeta} \log k,$

(T3) $|G_A^0 \setminus Z_{A,T}| \leq K a k^{-2},$

(T4) $|W_T \setminus G_A| < K \delta a k^\zeta \log k,$ and

(T5) $|A \setminus S_T| < K \delta a k^{-\vartheta}.$

(The ζ in the definition of G_A^0 is needed for the ϑ in (T5). For the bound in (T5) we could actually get by with $O(\delta a \log^{-1} k)$; see the discussion following (4.28) for more on this relatively delicate point and (4.29) for use of the bound.)

The following auxiliary definitions and lemma will be helpful in the proof of Lemma 4.1 and again later in the proof of Lemma 4.4. Fix $A \in \mathcal{A}$, set $G_A = G$ and $G_A^0 = G^0$, and define

$$H = \{y \in G : d_A(y) < k^{1-\vartheta}\},$$

$$B = \{x \in A : d_H(x) > k/2\},$$

$$I = \{y \in G \setminus H : d_{A \setminus B}(y) < k^{1-\vartheta}/2\},$$

$$C = \{x \in A \setminus B : d_{H \cup I}(x) > k/4\}.$$

Lemma 4.2. *With the above definitions, $|H \cup I| < O(\delta a)$, each of $|B|, |C|$ is $O(\delta a k^{-\vartheta})$, and $G \setminus G^0 \subseteq H \cup I$.*

Proof. We have

$$\begin{aligned} (k + 1 - k^{1-\vartheta})|H| &\leq |\nabla(H, \Gamma_k \setminus A)| \leq |\nabla(G, \Gamma_k \setminus A)| = (1 + (k + 1)\delta)a, \\ (k/2)|B| &< |\nabla(B, H)| < k^{1-\vartheta}|H|, \\ (k^{1-\vartheta}/2)|I| &< |\nabla(I, B)| < k|B|/2, \\ (k/4)|C| &< |\nabla(C, H \cup I)| < |H \cup I|k^{1-\vartheta}, \end{aligned}$$

implying $|H| < (4 + o(1))\delta a$ (using (4.1)), $|B| < (8 + o(1))\delta a k^{-\vartheta}$, $|I| < (8 + o(1))\delta a$ and $|C| < (48 + o(1))\delta a k^{-\vartheta}$. This gives the first two assertions in the lemma. The third is given by the observation that for $y \in G \setminus (H \cup I)$ the number of paths $ywzx$ with $(w, z, x) \in (A \setminus B) \times (G \setminus H) \times A$ is at least $(k^{1-\vartheta}/2)(k/2)k^{1-\vartheta}$. □

Proof of Lemma 4.1. Here we will find it more convenient to use ‘big Oh’ notation; that is, we will prove the lemma with each of the bounds $K \cdot X$ appearing in (T1)–(T5) replaced by $O(X)$. We first show existence of T satisfying (T1)–(T3) and then observe that any such T also satisfies (T4) and (T5).

Let $q = 16k^{-3+\zeta} \log k$ and $\mathbf{T} = A_q$ (the random subset of A in which elements of A appear independently, each with probability q). To show that there is a T satisfying (T1)–(T3), it is enough to show that the stated bounds (again, in their ‘big Oh’ forms) hold for the *expectations* of the set sizes in question, since Markov’s inequality then implies existence of a T for which each of these quantities is at most three times its expectation. This is of course true for $\mathbb{E}|\mathbf{T}| = aq$. For (T2) we have (using (4.3) for the final inequality)

$$\begin{aligned} \mathbb{E}|F_{A,\mathbf{T}}| &= \sum_{x \in G} \mathbb{P}(x \in N(\mathbf{T}))d_{\Gamma_k \setminus A}(x) \\ &\leq q \sum_{x \in G} d_A(x)d_{\Gamma_k \setminus A}(x) \\ &\leq qk|\nabla(G, \Gamma_k \setminus A)| < O(\delta a k^{-1+\zeta} \log k). \end{aligned}$$

To bound the expectation for (T3), notice that for $v \in G^0$, there are at least $(1/8)k^{3-\zeta}$ vertices $x \in A$ for which $x \in T$ implies $v \in Z_{A,T}$. (This is true of any x for which there is an A -good path from v to x and, since two vertices at distance 3 are connected by exactly two paths of length 3 in Σ , the number of such x ’s is at least $\varphi(v, A)/2$.) The probability that such a v does not belong to $Z_{A,\mathbf{T}}$ is thus at most $(1 - q)^{(1/8)k^{3-\zeta}} < k^{-2}$, so that $\mathbb{E}|G^0 \setminus Z_{A,\mathbf{T}}| < gk^{-2}$ (which gives the bound in (T3) since we assume $g = O(a)$; of course the assumption is not really needed here, as we could instead have arranged (for example) $\mathbb{E}|G^0 \setminus Z_{A,\mathbf{T}}| < gk^{-3}$).

This completes the discussion of (T1)–(T3) and we turn to the last two properties requested of T . We first observe that (T4) follows from (T2), since in fact

$$|W_T \setminus G| \leq k|F_{A,T}|.$$

To see this just notice that if $w \in W_T \setminus G$, then (since $w \in W_T$) there is a path $xyzw$ with $x \in T$ and (therefore) $y \in N(T)$, but $z \notin A$ (since $w \notin G$), so that $yz \in F_{A,T}$ (and each such yz gives rise to at most k such w ’s).

For (T5), note that (according to the definition of S_T in (4.8)) any $x \in A \setminus S_T$ has at least $k/4$ neighbours in one of $G \setminus G^0$, $G^0 \setminus W_T$. By Lemma 4.2, x 's of the first type belong to $B \cup C$ and number at most $O(\delta ak^{-\vartheta})$. On the other hand, by (T3) (and (4.1)), the number of the second type is at most

$$(4/k)|G^0 \setminus W_T|(k+1) < O(ak^{-2}) < o(\delta ak^{-\vartheta}). \quad \square$$

We think of W_T in Lemma 4.1 as a first approximation to G_A , and $Z_{A,T}$ as a second approximation satisfying

$$Z_{A,T} \subseteq W_T \cap G_A \tag{4.9}$$

that discards vertices that got into W_T on spurious grounds. Similarly, the next lemma prunes our first approximation, S_T , of A to get a better second approximation.

Lemma 4.3. *There is a fixed K such that for any $A \in \mathcal{A}$ and $T \subseteq A$ satisfying (T4), there is some $U \subseteq W_T \setminus G_A$ with*

(U1) $|U| \leq K\delta ak^{-1+\zeta} \log^2 k$ and

(U2) $|(S_T \setminus A) \setminus N(U)| \leq K\delta a$.

The second approximation mentioned above is then $S_T \setminus N(U)$, which in particular satisfies

$$S_T \supseteq S_T \setminus N(U) \supseteq S_T \cap A. \tag{4.10}$$

Proof of Lemma 4.3. Here we again (as in the proof of Lemma 4.1) switch to ‘big Oh’ notation. Set $G = G_A$, $W = W_T$ and $S = S_T$. Let $q = 4k^{-1} \log k$ and $\mathbf{U} = (W \setminus G)_q$. By the definition of $S = S_T$, each $x \in S \setminus A$ has at least $k/4$ neighbours in one of $W \setminus G$, G . Let

$$L = \{x \in S \setminus A : d_{W \setminus G}(x) \geq k/4\}.$$

Then $|L| \leq (4/k)|W \setminus G|(k+1) = O(\delta ak^\zeta \log k)$ (by (T4)). On the other hand, for $x \in L$ we have $\mathbb{P}(x \notin N(\mathbf{U})) \leq (1-q)^{k/4} < k^{-1}$, so there is some U with

$$|L \setminus N(U)| \leq \mathbb{E}|L \setminus N(\mathbf{U})| < |L|/k = o(\delta a).$$

Finally, since $x \in (S \setminus A) \setminus L$ implies $d_G(x) > k/4$, we have

$$|(S \setminus A) \setminus L| \leq (4/k)|\nabla(G, \Gamma_k \setminus A)| = 4(1 + (k+1)\delta)a/k = O(\delta a).$$

The lemma follows. □

Now write K for the larger of the constants appearing in Lemmas 4.1 and 4.3. For each $A \in \mathcal{A}$ fix some $T = T_A \subseteq A$ satisfying (T1)–(T5) and then some $U = U_A \subseteq W_T \setminus G_A$ satisfying (U1)–(U2), and set

$$W_A = W_T, \quad S_A = S_T, \quad F_A = F_{A,T}, \quad Z_A = Z_{A,T}, \quad S'_A = S_T \setminus N(U), \quad R_A = R(A) = (T_A, F_A, U_A).$$

(We prefer R_A but will sometimes use $R(A)$ to avoid double subscripts.) We may think of T_A, F_A, U_A as ‘primary’ objects, which we need to specify, and W_A, S_A, Z_A, S'_A as ‘secondary’ objects, which are functions of the primary objects.

Let $\mathcal{R} = \{R_A : A \in \mathcal{A}\}$. If $R = R_A$ then we also set $W_R = W_A$ (which is the same for all A with $R_A = R$), and similarly for the other objects subscripted by A in the preceding paragraph. For each $R \in \mathcal{R}$ fix some $A^* = A_R^* \in \mathcal{A}$ with $R_{A^*} = R$, and let $G_R^* = G_{A^*}$.

Now suppose $A \in \mathcal{A}$, $G = G_A$, $R = R_A$, $A^* = A_R^*$ and $G^* = G_R^*$. Notice that, given A^* and G^* ,

$$A \text{ is determined by } A \setminus A^* \text{ and } G \cap G^*. \tag{4.11}$$

(Actually A is determined by $B, G_B, A \setminus B$ and $G \cap G_B$ whenever $A, B \subseteq \Gamma_k$ are closed with $G = G_A$, since

$$A \cap B = \{x \in B : N(x) \subseteq G \cap G_B\};$$

namely, $x \in A$ if and only if $N(x) \subseteq G$, which for $x \in B$ is the same as $N(x) \subseteq G \cap G_B$.)

Second phase. We now turn to \mathbf{X} . In what follows we assume the constant $\varepsilon (= 1 - p)$ is small enough to support our argument, making no attempt to optimize.

For $\eta > 0$ and $B \subseteq V(\Sigma)$ (we will always have $B \subseteq \Gamma_k$ or $B \subseteq \Gamma_{k+1}$), we will be interested in the event

$$E_{B,\eta} = \{||\mathbf{X} \cap B| - |B|p| > \eta \delta ap\}. \tag{4.12}$$

(The second p on the right-hand side is unnecessary but we keep it as a reminder of where we are: if p were smaller, then this factor *would* be relevant.) Say a collection \mathcal{B} of sets is η -nice if

$$\mathbb{P}(\cup_{B \in \mathcal{B}} E_{B,\eta}) < \exp[-\Omega(ak^{-2})]. \tag{4.13}$$

Fix a smallish η ; for concreteness, say $\eta = 0.08$ (we need $6\eta < 0.5$). The next, regrettably (but as far as we can see unavoidably) elaborate statement is most of the story.

Lemma 4.4. *The following collections are η -nice:*

- (a) $\{W_R : R \in \mathcal{R}\}$,
- (b) $\{S_R : R \in \mathcal{R}\}$,
- (c) $\{W_R \setminus Z_R : R \in \mathcal{R}\}$,
- (d) $\{S_R \setminus S'_R : R \in \mathcal{R}\}$,
- (e) $\{S'_R \setminus A_R^* : R \in \mathcal{R}\}$,
- (f) $\{A_R^* \setminus S'_R : R \in \mathcal{R}\}$,
- (g) $\{G_R^* \setminus Z_R : R \in \mathcal{R}\}$,
- (h) $\{A \setminus A_{R(A)}^* : A \in \mathcal{A}\}$,
- (i) $\{A_{R(A)}^* \setminus A : A \in \mathcal{A}\}$,
- (j) $\{G_A \setminus G_{R(A)}^* : A \in \mathcal{A}\}$,
- (k) $\{G_A \cap (G_{R(A)}^* \setminus Z_{R(A)}) : A \in \mathcal{A}\}$.

Before proving this, we show that it supports (4.5).

Corollary 4.5. *The collections \mathcal{A} and $\{G_A : A \in \mathcal{A}\}$ are (6η) -nice.*

Of course this gives the relevant portion of (4.5), since $\mathcal{Q}(a, g)$ implies that for some $A \in \mathcal{A}$ either $|\mathbf{X} \cap A| \geq |A|p + \delta ap/2$ or $|\mathbf{X} \cap G_A| \leq |G_A|p - \delta ap/2$ (cf. (4.6)), each of which, according to Corollary 4.5, occurs with probability $\exp[-\Omega(ak^{-2})]$ (and – recall (4.7) –

$$\sum_{a>a_0} \sum_{g \leq 2a} \exp[-\Omega(ak^{-2})] = o(1/n).$$

Proof of Corollary 4.5. This is just a matter of building the relevant sets, starting from the collections in Lemma 4.4 and applying the (trivial) observations:

- if $\{K_B : B \in \mathcal{B}\}$ is α -nice, $\{L_B : B \in \mathcal{B}\}$ is β -nice and $K_B \cap L_B = \emptyset \ \forall B \in \mathcal{B}$, then $\{K_B \cup L_B : B \in \mathcal{B}\}$ is $(\alpha + \beta)$ -nice;
- if $\{K_B : B \in \mathcal{B}\}$ is α -nice, $\{L_B : B \in \mathcal{B}\}$ is β -nice and $K_B \supseteq L_B \ \forall B \in \mathcal{B}$, then $\{K_B \setminus L_B : B \in \mathcal{B}\}$ is $(\alpha + \beta)$ -nice.

Using these (in combination with Lemma 4.4), we find that

$$\begin{aligned} \{Z_R = W_R \setminus (W_R \setminus Z_R) : R \in \mathcal{R}\} &\text{ is } (2\eta)\text{-nice,} \\ \{S'_R = S_R \setminus (S_R \setminus S'_R) : R \in \mathcal{R}\} &\text{ is } (2\eta)\text{-nice,} \\ \{A^*_R = (S'_R \setminus (S'_R \setminus A^*_R)) \cup (A^*_R \setminus S'_R) : R \in \mathcal{R}\} &\text{ is } (4\eta)\text{-nice,} \\ \{A = (A \setminus A^*_{R(A)}) \cup (A^*_{R(A)} \setminus (A^*_{R(A)} \setminus A)) : A \in \mathcal{A}\} &= \mathcal{A} \text{ is } (6\eta)\text{-nice,} \\ \{G_A = (G_A \setminus G^*_{R(A)}) \cup (G_A \cap (G^*_{R(A)} \setminus Z_{R(A)})) \cup Z_{R(A)} : A \in \mathcal{A}\} &= \{G_A : A \in \mathcal{A}\} \text{ is } (4\eta)\text{-nice.} \end{aligned}$$

(Note that $Z_{R(A)}$ is the same as Z_A but seems slightly more natural here.) □

Proof of Lemma 4.4. For the rest of this discussion we write E_B for $E_{B,\eta}$. We want to show that (4.13) holds for each of the collections – say \mathcal{B} – appearing in (a)–(k). This is all based on the union bound: in each case we bound the size of the \mathcal{B} in question and show, using what we know about the sizes of members of \mathcal{B} , that $\mathbb{P}(E_B)$ is much smaller than $|\mathcal{B}|^{-1}$ for each $B \in \mathcal{B}$.

We are interested in bounding probabilities of the type

$$\mathbb{P}(|\mathbf{X} \cap B| - |B|p > \eta \delta ap)$$

using Theorems 2.1 and 2.2; but, since $p = 1 - \varepsilon \approx 1$, we can do a little better by applying these theorems with $\xi = |B \setminus \mathbf{X}|$ (which has the distribution $\text{Bin}(|B|, \varepsilon)$), using the trivial observation that, for any $\lambda > 0$ (always equal to $\eta \delta ap$ in what follows),

$$\mathbb{P}(|\mathbf{X} \cap B| - |B|p > \lambda) = \mathbb{P}(|B \setminus \mathbf{X}| - |B|\varepsilon > \lambda). \tag{4.14}$$

(For most of the argument this change will make little difference, but it will be crucial when we come to items (h)–(k).)

Items (a) and (b). To make things easier to read, set $b = Kak^{-3+\zeta} \log k$ (the bound in (T1) of Lemma 4.1). The number of possibilities for each of W_R, S_R is bounded by the number of possible T_R 's, which (by (2.2)) is at most

$$\exp[b \log(eN/b)] < \exp[b \log(Nk^3/a)] \tag{4.15}$$

(recall $N = \binom{2k}{k}$). On the other hand, (T4) and the fact that $|S_T| \leq 2(k+1)|W_T|/k$ (see the definition of S_T in (4.8)) imply that, for any T ,

$$|W_T|, |S_T| < O(\delta a k^\zeta \log k + g) = O(\delta a k^\zeta \log k + a),$$

so that Theorem 2.1 gives (for any T)

$$\max\{\mathbb{P}(E_{W_T}), \mathbb{P}(E_{S_T})\} < \exp[-\Omega(\delta^2 a / (\delta k^\zeta \log k + 1))]. \tag{4.16}$$

(In a little more detail: we apply Theorem 2.1 (using (4.14), though, as noted above, this is not really needed here), with $m = O(\delta a k^\zeta \log k + a)$, $q = \varepsilon$ and $\lambda = \eta \delta a p$, to bound the left side of (4.16) by $\exp[-\Omega(\lambda^2 / \max\{m\varepsilon, \lambda\})]$, and observe that $\max\{m\varepsilon, \lambda\} = O(\delta a k^\zeta \log k + a)$.)

That the collections in (a) and (b) are η -nice now follows upon multiplying the bounds in (4.15) and (4.16) and checking that (4.1) implies (with room to spare)

$$b \log(Nk^3/a) = o(\delta^2 a / (\delta k^\zeta \log k + 1)).$$

Item (c). Since each of Z_R, W_R is determined by T_R and F_R , the number of possibilities for $W_R \setminus Z_R$ is at most the product of the bound in (4.15) (which will be negligible here) and the number of possibilities for F_R given $T := T_R$. The latter is at most the number of subsets of $\nabla(N(T), \Gamma_k \setminus T)$ of size less than $c := K\delta a k^{-1+\zeta} \log k$ (the bound in (T2)), which, since

$$|\nabla(N(T), \Gamma_k \setminus T)| \leq k^2 |T| < K a k^{-1+\zeta} \log k =: d \tag{4.17}$$

(see (T1)), is less than

$$\begin{aligned} \exp[c \log(ed/c)] &= \exp[O(\delta a k^{-1+\zeta} \log k \log(e/\delta))] \\ &= \exp[O(\delta a k^{-1+\zeta} \log^2 k)]. \end{aligned} \tag{4.18}$$

(Here we again use (2.2) (for the initial bound) and (4.1) (for the second line). Strictly speaking, the application of (2.2) is only justified when $\delta \leq 1/2$; but for larger δ we can bound the number of possibilities for F_R by the trivial 2^d , which (for such δ) is smaller than the left side of (4.18).)

On the other hand, again using (T2), we have

$$|W_R \setminus Z_R| \leq k |F_R| = O(\delta a k^\zeta \log k).$$

(For the first inequality, fix A with $R(A) = R$ and note that for any $w \in W_R \setminus Z_R (= W_A \setminus Z_A)$ there is a path $xyzw$ with $x \in T (= T_A)$ – such a path exists since $w \in W_A$ – and $yz \in F_A$ (since otherwise $y \in A$ and $w \in Z_A$).)

Thus Theorem 2.1 gives (for any R)

$$\begin{aligned} \mathbb{P}(E_{W_R \setminus Z_R}) &< \exp[-\Omega(\eta^2 \delta^2 a^2 / (\delta a k^\zeta \log k))] \\ &= \exp[-\Omega(\eta^2 \delta a / (k^\zeta \log k))], \end{aligned} \tag{4.19}$$

which, combined with the (here insignificant) bounds in (4.15) and (4.18), gives

$$\sum_R \mathbb{P}(E_{W_R \setminus Z_R}) = \exp[-\Omega(\eta^2 \delta a / (k^\zeta \log k))].$$

Items (d)–(g). For each of these the number of sets in question is $|\mathcal{R}|$, the number of possibilities for (T_R, F_R, U_R) . As already observed, the number of (T_R, F_R) 's is at most the product of the

bounds in (4.15) and (4.18). On the other hand, with $c = K\delta ak^{-1+\zeta} \log^2 k$ (the bound on $|U|$ in (U1)) and $d = K\delta ak^\zeta \log k$ (the bound on $|W_T \setminus G_A|$ in (T4)) – so c and d have changed from what they were above – the number of possibilities for U_R given T_R is at most

$$\exp[c \log(ed/c)] = \exp[O(\delta ak^{-1+\zeta} \log^3 k)] \tag{4.20}$$

(which dominates the bounds from (4.15) and (4.18)).

We next need to bound the sizes of the various sets under discussion. We have

$$|S_R \setminus S'_R| \leq (k+1)c = O(\delta ak^\zeta \log^2 k) \tag{4.21}$$

(again, since (U1) bounds $|U_R|$ by c (and $S_R \setminus S'_R \subseteq N(U_R)$));

$$|S'_R \setminus A_R^*| = O(\delta a) \tag{4.22}$$

(given by (U2), once we recall that $S'_R = S_R \setminus N(U_R)$);

$$|A_R^* \setminus S'_R| = O(\delta ak^{-\vartheta}) \tag{4.23}$$

(using (T5) and the fact – see (4.10) – that $A_R^* \setminus S'_R = A_R^* \setminus S_R$); and, with $A_R^* = A$ (so $G_R^* = G_A$),

$$|G_R^* \setminus Z_R| \leq |G_A^0 \setminus Z_R| + |G_R^* \setminus G_A^0| = O(ak^{-2} + \delta a) = O(\delta a) \tag{4.24}$$

(using (T3), Lemma 4.2 and (4.1)). Note, for use below, that for *any* A with $R(A) = R$, (4.24) remains true if we replace G_R^* by G_A . (Similarly (4.23) holds with any such A in place of A_R^* , but we do not need this.)

The largest of the bounds in (4.21)–(4.24) is the $O(\delta ak^\zeta \log^2 k)$ in (4.21); so for each of the sets B appearing in (d)–(g) (*i.e.* $B = S_R \setminus S'_R$ in (d) and so on), we have

$$\begin{aligned} \mathbb{P}(E_B) &< \exp[-\Omega(\eta^2 \delta^2 a^2 / (\delta ak^\zeta \log^2 k))] \\ &= \exp[-\Omega(\eta^2 \delta a / (k^\zeta \log^2 k))]; \end{aligned} \tag{4.25}$$

and, since $\eta^2 \delta a / (k^\zeta \log^2 k)$ in (4.25) is much larger than the exponent in (4.20), it follows that the collections in (d)–(g) are η -nice.

Items (h)–(k). Here we first dispose of the sizes of the individual sets, before turning to the more interesting problem of bounding the sizes of the collections in question.

For (h) and (i), notice that for any $A, A' \in \mathcal{A}$ with $R(A) = R(A')$ we have

$$|A \setminus A'| \leq |A \cap (S'_R \setminus A')| + |A \setminus S'_R| = O(\delta a + \delta ak^{-\vartheta}) = O(\delta a)$$

(using (U2) and (T5), as earlier in (4.22) and (4.23)); in particular this bounds the sizes of the sets in (h), (i) (namely $|A \setminus A_R^*|$ and $|A_R^* \setminus A|$, where $R = R(A)$) by $O(\delta a)$. For (j) and (k), a similar bound – that is,

$$\max\{|G_A \setminus G_R^*|, |G_A \cap (G_R^* \setminus Z_R)|\} = O(\delta a)$$

for A with $R(A) = R$ – follows from $G_R^* \supseteq Z_R$ (see (4.9)), $|G_A \setminus Z_R| < O(\delta a)$ (noted following (4.24)) and (4.24) itself.

We now turn to the sizes of the collections in (h)–(k), each of which is at most $|\mathcal{A}|$. We will show

$$|\mathcal{A}| < \exp[O(\delta a)]. \tag{4.26}$$

Before doing so we observe that this is enough to show that the collections in (h)–(k) are η -nice; namely, for B belonging to any of these collections (so $|B| = O(\delta a)$) and small enough ε , Theorem 2.2 (applied with $m = O(\delta a)$ and $q = \varepsilon$ – and now really using (4.14) – gives

$$\mathbb{P}(E_B) < \exp[-\eta \delta a(\log(1/\varepsilon) - O(1))]. \tag{4.27}$$

Here $|B \setminus \mathbf{X}| < |B|\varepsilon - \eta \delta a p$ is impossible, so we are just using

$$\mathbb{P}(|B \setminus \mathbf{X}| > |B|\varepsilon + \eta \delta a p) < \mathbb{P}(|B \setminus \mathbf{X}| > \eta \delta a p) < \exp\left[-\eta \delta a p \log\left(\frac{\eta \delta a p}{\varepsilon m \varepsilon}\right)\right].$$

Assertions (h)–(k) then follow on multiplying the bounds in (4.26) and (4.27).

Proof of (4.26). According to (4.11), we may bound $|\mathcal{A}|$ by the number of possibilities for the pair $(A \setminus A_R^*, G_A \cap G_R^*)$ (with $R_A = R$), so by our earlier bound on $|\mathcal{R}|$ – essentially that in (4.20); see the discussion of items (d)–(g) – multiplied by the number of possibilities for $(A \setminus A_R^*, G_A \cap G_R^*)$ given R . So it is enough to show that, once we know R – and therefore A_R^* and G_R^* – the number of choices for each of $A \setminus A_R^*, G_A \cap G_R^*$ is less than $\exp[O(\delta a)]$.

The second of these is easy: since (by (4.9)) each of G_A, G_R^* contains Z_R (which is determined by R), the number of possibilities for $G \cap G_R^*$ given R (and therefore G_R^*) is at most $\exp_2[|G_R^* \setminus Z_R|]$, and we have already seen in (4.24) that $|G_R^* \setminus Z_R| = O(\delta a)$.

The case of $A \setminus A_R^*$ is more interesting. Here we decompose

$$A \setminus A_R^* = (A \cap (S'_R \setminus A_R^*)) \cup (A \setminus (S'_R \cup A_R^*))$$

and consider the two terms on the right-hand side separately. The number of possibilities for the first term is at most $\exp_2[|S'_R \setminus A_R^*|]$ (again, given R , which determines S'_R and A_R^*), while (U2) (or (4.22)) gives $|S'_R \setminus A_R^*| = O(\delta a)$.

So it is enough to show that the number of possibilities for $A \setminus (S'_R \cup A_R^*)$ is $\exp[O(\delta a)]$ (it will actually be much smaller). In fact, it is enough to prove such a bound on the number of possibilities for $A \setminus S'_R$, which determines $A \setminus (S'_R \cup A_R^*)$ since we know A_R^* . Here we recall that (4.10) gives $A \setminus S'_R = A \setminus S_R$ (so we may use these interchangeably, and similarly for $A \cap S'_R = A \cap S_R$), and that – crucially – (T5) gives

$$|A \setminus S_R| = O(\delta a k^{-\vartheta}). \tag{4.28}$$

Note that this final point differs from its earlier counterparts in that we now have less control over the size of the universe from which the set in question (*i.e.* $A \setminus S_R$) is being drawn (in contrast to, for example, F_R in (c), which was drawn from $\nabla(N(T), \Gamma_k \setminus T)$, whose size was bounded in (4.17), or, in the present case, $A \cap (S'_R \setminus A_R^*)$, which is drawn from the quite small $S'_R \setminus A_R^*$). Thus, for example, if we try to apply (2.2) with a the bound in (4.28) and $\mathbf{b} = N (= \binom{2k}{k})$, then we can only say that the number of possibilities for $A \setminus S_R$ is less than $\exp[O(\delta a k^{-\vartheta}) \log(eN/\delta a k^{-\vartheta})]$, which for somewhat small a may be far larger than the desired $\exp[O(\delta a)]$. This little difficulty will be handled by Proposition 2.6.

Write $t (= O(\delta ak^{-\vartheta}))$ for the bound on $|A \setminus S_R|$ given in (4.28). Denote by Λ the (‘Johnson’) graph on Γ_k in which two vertices (a.k.a. k -sets) are adjacent if they are at distance 2 in Σ , and set $d = k^2$ (so Λ is d -regular). Since our A ’s induce connected subgraphs of Λ (another way of saying they are 2-linked), there is, for each A under discussion, a rooted forest with roots in $S_R \cap A = S'_R \cap A$, set of (at most t) non-roots equal to $A \setminus S_R$, and at least one non-root in each component; thus we just need to bound the number of such forests.

(Note that existence of said forest requires $S_R \cap A \neq \emptyset$, which, since we assume δ is not too large, holds because the bound in (T5) is less than a . If $S_R \cap A = \emptyset$, as is possible for large δ , we may bound the number of choices for $A \setminus S_R = A$ by the number of *trees* of size up to t , but this count should include a factor N (in place of the bound for (ii) below) for the choice of a root – a change that can cause trouble in the present regime, but not for large δ , where, as will appear below, our probability bounds improve.)

For the desired bound we may think of specifying a forest as above by specifying:

- (i) the number, say $q \leq t$, of roots;
- (ii) the set of roots, $\{x_1, \dots, x_q\} \subseteq S'_R \cap A$;
- (iii) for each $i \in [q]$, the size, say α_i , of the component (tree) rooted at x_i ; and
- (iv) the components themselves.

We may bound the numbers of possibilities in (ii), (iii) and (iv) by $\binom{a+O(\delta a)}{q}$, $\binom{t}{q}$ and $(ed)^t$ respectively. The first of these derives from (U2), according to which we have $|S'_R| < a + O(\delta a)$; the second is the number of sequences $(\alpha_1, \dots, \alpha_q)$ of positive integers summing to at most t ; and the third is given by Proposition 2.6. Thus (recalling from (4.7) that a is not very small), we find that the number of forests as above is at most

$$\sum_{q \leq t} \binom{a+O(\delta a)}{q} \binom{t}{q} (ed)^t = \exp[\Theta(t \log k)] \quad (= \exp[O(\delta a)]). \tag{4.29}$$

□

Finally we turn to the case of large δ ($\delta > 1$), showing (for any a, g , with $\delta = (g - a)/a > 1$)

$$\mathbb{P}(\mathcal{Q}(a, g)) < \varepsilon^{g/3}, \tag{4.30}$$

which, with the trivial $g \geq k$, bounds the contribution to (4.5) of the terms under discussion by

$$\sum_{g \geq ka < g} \varepsilon^{g/3} = o(1/n).$$

For (4.30), first notice that in the present situation Theorem 2.2 bounds the probability of (4.4) (for a given $A \in \mathcal{A}(a, g)$) by

$$\mathbb{P}(|G_A \setminus \mathbf{X}| > g/2) < (2e\varepsilon)^{g/2}. \tag{4.31}$$

On the other hand, to bound the number of possibilities for A (i.e. the size of $\mathcal{A}(a, g)$), we may think of specifying A via the following steps.

- (i) Choose $T \subseteq G := G_A$ of size $C(g/k) \log k$ such that, with

$$S = S_T = \{x \in \Gamma_k : d_T(x) > (C/2) \log k\},$$

we have

$$|A \setminus S| < k^{-2}a \tag{4.32}$$

and, with $Z = Z(G) = \{x \in \Gamma_k : d_G(x) \geq k/4\}$,

$$|S \setminus Z| < k^{-1}g. \tag{4.33}$$

(The proof of the existence of such a T is similar to – but easier than – the proof of Lemma 4.1, and we omit the details, just noting that, since $S \subseteq N(G)$, fewer than gk vertices are candidates for $S \setminus Z$.)

Notice that by (4.33) (and the definition of Z), we have

$$|S| \leq (4/k)g(k+1) + g/k = O(g). \tag{4.34}$$

- (ii) For each $x \in A \setminus N(T) (\subseteq A \setminus S)$, choose some neighbour of x (necessarily in G) and let T' be the collection of these neighbours; thus $T' \cap T = \emptyset$ and $|T'| \leq |A \setminus S| < k^{-2}a$ (by (4.32)). Notice also that $T \cup T'$ is 4-linked (by Proposition 2.5 and the fact that A is 2-linked).
- (iii) Finally, choose A from $S \cup N(T')$.

We should then bound the number of ways in which these steps can be carried out:

- (i) Since $T \cup T'$ is 4-linked, Proposition 2.6 (applied to the graph on Γ_k in which adjacency is Σ -distance at most 4, so a d -regular graph for some $d < k^4$) bounds the number of choices for $T \cup T'$ by

$$N \exp[O((g/k) \log k) \log d] < N \exp[O(g/k) \log^2 k]$$

(where the $N (= |\Gamma_k|)$ corresponds to choosing a root in $T \cup T'$.)

- (ii) The number of choices for T' given $T \cup T'$ is $\exp[O(k^{-2}a \log(gk/a))]$. Note that once we know $T \cup T'$ and T' , we also know T and thus S .
- (iii) Given T' and S , there are at most $\exp[a(\log(g/a) + O(1))]$ choices for $A \subseteq S \cup N(T')$ (since $|S \cup N(T')| < O(g)$; see (4.34) and the specification of $|T|$ in (i)).

Of course for sufficiently (not very) small ε , all of these bounds are dominated by the one in (4.31), so we have (4.30). □

5. Sperner

As one might expect, there has also been some consideration of Sperner’s theorem [29] – usually considered the first result in extremal set theory – from the sparse random viewpoint. Here \mathbf{X} is the random subset of $2^{[n]}$ in which each $A \subseteq [n]$ is present with probability p (independent of other choices), and one is interested in the size of a largest antichain (collection of pairwise incomparable sets) in \mathbf{X} (called the *width* of \mathbf{X} and denoted $w(\mathbf{X})$; see [7] for general background and [6] for a review of work related to the present question). In particular, proving a conjecture of Osthus [22], it is shown in [6] and [8] (both using the ‘container’ technology of [5, 27]) that $w(\mathbf{X}) \sim \binom{n}{\lfloor n/2 \rfloor} p$ w.h.p. provided $p > C/n$ for a suitable fixed C .

Here again it is natural to ask for a more literal counterpart of Sperner’s theorem, namely, for the property

$$w(\mathbf{X}) = \max\{|\mathbf{X} \cap \Gamma_{\lfloor n/2 \rfloor}|, |\mathbf{X} \cap \Gamma_{\lceil n/2 \rceil}|\} \tag{5.1}$$

(where we again take $\Gamma_\ell = \binom{[n]}{\ell}$). As for the Erdős–Ko–Rado question considered above (and for similar reasons), it is easy to see that (5.1) is unlikely for p less than about $3/4$ (and easy to guess that it is likely above this). Here we just observe that the method of Section 4 at least gives the weaker statement analogous to our Theorem 1.2.

Theorem 5.1. *There is a fixed $\varepsilon > 0$ such that (5.1) holds w.h.p. provided $p > 1 - \varepsilon$.*

(Though not in print as far as we know, this seems to have been of some interest; the present authors first heard the question in a lecture of J. Balogh [3].)

We just indicate how this goes. The main point is that the argument of Section 4 is easily adapted to show that (for ε, p as in Theorem 5.1) w.h.p.

$$|\mathbf{X} \cap \partial_u(A)| > |\mathbf{X} \cap A| \tag{5.2}$$

whenever $A \subseteq \Gamma_i$ is closed and non-empty and either $i < \lfloor n/2 \rfloor$ or $n = 2k + 1, i = k$ and $|A| \leq \frac{1}{2} \binom{n}{k}$, and (with ∂_l denoting lower shadow)

$$|\mathbf{X} \cap \partial_l(B)| > |\mathbf{X} \cap B| \tag{5.3}$$

whenever $B \subseteq \Gamma_i$ is closed and non-empty and either $i > \lceil n/2 \rceil$ or $n = 2k + 1, i = k + 1$ and $|B| \leq \frac{1}{2} \binom{n}{k}$. (Note (5.2) and (5.3) hold in the specified regimes provided they do so when A and B are 2-linked.)

(We have preferred not to extend the material of Section 4 to cover the present situation, feeling that the extra generality would make the argument even harder to follow than it already is. It should at least be intuitively clear that (5.2) and (5.3) are in fact less delicate than what’s gone before; for example, (5.2) gets easier as i shrinks (with n fixed, so the hardest case would be $i = n/2$, which corresponds to what we did earlier and does not even appear here).

For the warier reader we may also argue as follows (for (5.2) say). Given $i < n/2$, identify $2^{[n]}$ in the natural way with $\{B \subseteq [n] \cup J : B \supseteq J\}$, where J is some $(n - 2i)$ -set disjoint from $[n]$. Our Γ_i then becomes a subset of $\binom{[n] \cup J}{k}$, where $k = n - i = \lfloor [n] \cup J \rfloor / 2$, and the results of Section 4 apply directly. We will not elaborate, apart from noting that (i) in this case the lower bound on δ in (4.1) is automatic, and (ii) the need to sum failure probabilities over possible values of i causes no trouble since the bounds on these probabilities (essentially those in (4.16),(4.19),(4.25),(4.27),(4.31)) are so small.)

It remains to observe that (5.2) and (5.3) (for the stated ranges) imply (5.1). (They actually imply that $\mathbf{X} \cap \Gamma_{\lfloor n/2 \rfloor}$ and $\mathbf{X} \cap \Gamma_{\lceil n/2 \rceil}$ are the only possible antichains of size $w(\mathbf{X})$; so if n is odd, then w.h.p. one of these is the *unique* largest antichain, since their sizes differ w.h.p.)

For n even the implication is immediate: if $\cup A_i$ is an antichain of \mathbf{X} with $A_i \subseteq \Gamma_i$ and $i = \min\{j : A_j \neq \emptyset\} < n/2$, then replacing A_i by $\mathbf{X} \cap \partial_u(A_i)$ gives a larger antichain, and similarly if $A_i \neq \emptyset$ for some $i > n/2$.

When $n = 2k + 1$ the same argument shows that any largest antichain of \mathbf{X} is $C \cup D$ with $C \subseteq \Gamma_k$ and $D \subseteq \Gamma_{k+1}$. But then the union of the closures, say A and B , of C and D is an antichain of $2^{[n]}$, so $\min\{|A|, |B|\} \leq \frac{1}{2} \binom{n}{k}$; and if (for example) this minimum is $|A| > 0$, then, according to (5.2), replacing C by $\mathbf{X} \cap \partial_u(A)$ ($= \mathbf{X} \cap \partial_u(C)$) increases the size of our antichain, a contradiction. \square

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