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TATE'S CONJECTURE AND THE TATE–SHAFAREVICH GROUP OVER GLOBAL FUNCTION FIELDS

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Abstract Let $\mathcal{X}$ be a regular variety, flat and proper over a complete regular curve over a finite field such that the generic fiber X is smooth and geometrically connected. We prove that the Brauer group of $\mathcal{X}$ is finite if and only Tate's conjecture for divisors on X holds and the Tate–Shafarevich group of the Albanese variety of X is finite, generalizing a theorem of Artin and Grothendieck for surfaces to arbitrary relative dimension. We also give a formula relating the orders of the group under the assumption that they are finite, generalizing the known formula for a surface.

Keywords: Brauer group; Tate–Shafarevich group; Tate's conjecture

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1. Introduction

Let C be a smooth and proper curve over a finite field of characteristic p with function field K. Let $\mathcal{X}$ be a regular scheme and $\mathcal{X} \to C$ a proper flat map such that $X = \mathcal{X} \times_C K$ is smooth and geometrically connected over K. Let K_v be the quotient field of the henselization of C at a point v, and let $X_v = X \times_K K_v$.

If $\mathcal{X}$ is a surface, then it is a classical result of Artin and Grothendieck [14, § 4] that the Brauer group of $\mathcal{X}$ is finite if and only if the Tate–Shafarevich group of the Jacobian A of X is finite. Moreover, if they are finite, then their orders are related by the formula [11]

$$|\mathrm{Br}(\mathcal{X})|\alpha^2\delta^2 = |\text{Ш}(A)| \prod_{v\in V} \alpha_v\delta_v, \tag{1}$$

where δ and δ_v are the indices of X and X_v, respectively, and α and α_v are the order of the cokernel of the inclusion $\mathrm{Pic}^0(X) \to H^0(K, \mathrm{Pic}^0_X)$ and $\mathrm{Pic}^0(X_v) \to H^0(K_v, \mathrm{Pic}^0_X)$, respectively. The purpose of this paper is to generalize these results to arbitrary relative dimension d.

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Theorem 1.1. *The Brauer group of $\mathcal{X}$ is finite if and only if the Tate–Shafarevich group of the Albanese variety of X is finite and Tate's conjecture for divisors holds for X; i.e., for all $l \neq p$, the cycle map*

$$c_l : \mathrm{Pic}(X) \otimes \mathbb{Z}_l \to H^2_{\mathrm{et}}(X^s, \mathbb{Z}_l(1))^{\mathrm{Gal}(K)}$$

has torsion cokernel, where X^s is the base extension to the separable closure.

Moreover, under the assumptions of the theorem, $\mathrm{coker}\, c_l$ vanishes for almost all l. If $\mathcal{X}$ is a surface, then Tate's conjecture for divisors on X is trivial and we recover the classical result of Artin–Grothendieck.

In order to generalize formula (1), we need to introduce other invariants that vanish for a relative curve. Consider the maps

$$\xi^i_n : H^2(K, H^i_{\mathrm{et}}(X^s, \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}](n))) \longrightarrow \bigoplus_v H^2(K_v, H^i_{\mathrm{et}}(X^s, \mathbb{Q}/\mathbb{Z}[\tfrac{1}{p}](n))),$$
$$l^i_n : H^i_{\mathrm{et}}(X, \mathbb{Z}(n)) \longrightarrow \prod_v H^i_{\mathrm{et}}(X_v, \mathbb{Z}(n)),$$

where $H^i_{\mathrm{et}}(X, \mathbb{Z}(n))$ is the etale hypercohomology of Bloch's cycle complex. The maps ξ^i_n were studied by Jannsen [16], and the argument of *loc. cit.* Theorem 3 shows that they have finite kernel and cokernel if $i \neq 2n-2$.

Let $G = \mathrm{Gal}(K)$, let $G_v = \mathrm{Gal}(K_v)$ be the decomposition group, let α, α_v be defined as above, let β, β_v be the orders of the prime to p-part of the cokernels of

$$H^{2d}_{\mathrm{et}}(X, \mathbb{Z}(d)) \xrightarrow{\rho_d} H^{2d}_{\mathrm{et}}(X^s, \mathbb{Z}(d))^G, \quad H^{2d}_{\mathrm{et}}(X_v, \mathbb{Z}(d)) \longrightarrow H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}_v, \mathbb{Z}(d))^{G_v}$$

and let δ', δ'_v be the order of prime to p-part of the cokernels of the maps

$$CH^d(X^{\mathrm{s}})^G \xrightarrow{\deg} \mathbb{Z}, \quad CH^d(X^{\mathrm{s}}_v)^{G_v} \xrightarrow{\deg} \mathbb{Z},$$

respectively. Then α, δ' and $\prod_v \delta'_v$ are finite, and we use a result of Saito–Sato to show that $\prod_v \beta_v$ is finite. Let c be the product of the maps c_l for all $l \neq p$.

Theorem 1.2. *Let X be smooth and projective over a global field of characteristic p. Assume that X admits a regular proper model, $\text{Ш}(\mathrm{Alb}_X)$ is finite, Tate's conjecture for divisors holds on X and $\ker \xi^{2d-2}_d$ is finite. Then β is finite, and up to a power of p, we have*

$$|\ker l^{2d+1}_d| \cdot |\mathrm{coker} c| \cdot \alpha\beta\delta' = |\text{Ш}(\mathrm{Alb}_X)| \cdot |\ker \xi^{2d-2}_d| \prod_v \beta_v \delta'_v.$$

Note that the terms appearing in the theorem only depend on X. If $\ker \xi^{2d-2}_d$, or equivalently β, is not finite, then the theorem still holds if we replace the corresponding terms by the order of their finite quotients by a common subgroup. If X is a curve, then the formula reduces to (1) because $\ker \xi^0_1 = 0$ and $\ker l^3_1 \cong \mathrm{Br}(\mathcal{X})$.

The same argument gives the following theorem which relates the above-mentioned groups. Let Alb_X be the Albanese variety of X, and for an abelian group A, let A^* and TA be the Pontrjagin dual and Tate module, respectively.

Theorem 1.3. *Modulo the Serre subcategory spanned by finite groups and p-power torsion groups, we have an exact sequence of torsion groups*

$$0 \to \operatorname{coker} \rho_d \to \ker \xi_d^{2d-2} \to \ker l_d^{2d+1} \to Ш(\mathrm{Alb}_X) \\ \to (\operatorname{coker} c)^* \to (T\,\mathrm{Br}(\mathcal{X}))^* \to (T\,Ш(\mathrm{Pic}^0_X))^* \to 0.$$

Theorems 1.2 and 1.3 will be proven in § 4, and in § 5, we show that Theorem 1.3 implies Theorem 1.1. During the proof, we obtain the following result on the maps ξ_d^{2d-2} and l_d^{2d+1}.

Proposition 1.4. *Assume that* $\mathrm{Br}(\mathcal{X})$ *is finite.*

(1) *There is a complex*

$$0 \to \operatorname{coker} \rho_d \to H^2(K, H^{2d-2}_{\mathrm{et}}(X^s, \mathbb{Q}/\mathbb{Z}(d))) \xrightarrow{\xi_d^{2d-2}} \\ \bigoplus_v H^2(K_v, H^{2d-2}_{\mathrm{et}}(X^s, \mathbb{Q}/\mathbb{Z}(d))) \to \mathrm{Hom}(\mathrm{NS}(X), \mathbb{Q}/\mathbb{Z}) \to 0$$

which is exact up to finite groups and p-groups.

(2) *There is a complex*

$$H^{2d+1}_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}(d)) \to H^{2d+1}_{\mathrm{et}}(X, \mathbb{Z}(d)) \xrightarrow{l_d^{2d+1}} \\ \bigoplus_v H^{2d+1}_{\mathrm{et}}(X_v, \mathbb{Z}(d)) \to \mathrm{Hom}(\mathrm{Pic}(X), \mathbb{Q}/\mathbb{Z}) \to 0$$

which is exact up to p-groups.

The idea of the proof of Theorem 1.3 is to work with one-dimensional cycles and to use a theorem of Saito–Sato. For our purposes, it is necessary to give a slight improvement of their main theorem, which we are able to prove using results of Gabber and Kerz–Saito: Let $f : \mathcal{Y} \to \operatorname{Spec} R$ be of finite type over the spectrum of an excellent henselian discrete valuation ring R with residue field k. The following conjecture and theorem on Kato homology were stated and proven by Saito–Sato for $\mathcal{Y}$ projective over R such that the reduced special fiber is a strict normal crossing scheme.

Conjecture 1.5 [29, Conjectures 2.11, 2.12]. *Let $\mathcal{Y}$ be a regular scheme, flat and proper over* $\operatorname{Spec} R$*, and l a prime invertible on R.*

(1) *If k is separably closed, then $KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) = 0$ for all a.*

(2) *If k is finite, then*

$$KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) = \begin{cases} 0 & \text{for } a \neq 1; \\ (\mathbb{Q}_l/\mathbb{Z}_l)^I & \text{for } a = 1, \end{cases}$$

where I is the number of irreducible components of the special fiber.

Theorem 1.6 [29, Theorem 2.13]. *The conjecture holds for $a \leqslant 3$.*

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In the body of this paper, we invert the characteristic p of the base field, i.e., we work with $\mathbb{Z}' = \mathbb{Z}[\frac{1}{p}]$-coefficients. For a compact or discrete abelian group A, we define the Pontrjagin dual to be $A^* = \mathrm{Hom}_{cont}(A, \mathbb{Q}/\mathbb{Z}[\frac{1}{p}])$, the completion to be $A^\wedge = \lim_{p \nmid m} A/m$, the torsion to be $\mathrm{Tor} A = \mathrm{colim}_{p\nmid m\, m} A$ and the Tate module to be $TA = \lim_{p \nmid m\, m} A$. Note that the results for A and $A_{\mathbb{Z}'} := A \otimes \mathbb{Z}'$ agree so that we sometimes omit the subscript $\mathbb{Z}'$.

2. Etale motivic cohomology

Let $f : \mathcal{Y} \to B$ be separated and of finite type over the spectrum B of a Dedekind ring of exponential characteristic p. Consider Bloch's cycle complex with the free abelian group $z_w(\mathcal{Y}, j)$ on closed integral subschemes of dimension $w + j$ on $\mathcal{Y} \times \Delta^j$ which intersect all faces properly in degree j. The higher Chow group $CH_w(\mathcal{Y}, j)$ is defined as the jth homology of this complex. Let $\mathbb{Z}^c(w)$ be the cohomological complex of etale sheaves with $z_w(-, -i - 2w)$ in degree i. For an abelian group A, we define an etale motivic Borel–Moore homology $H^c_i(\mathcal{Y}, A(w))$ to be $H^{-i}_{\mathrm{et}}(\mathcal{Y}, A \otimes \mathbb{Z}^c(w))$. For finite maps $g : \mathcal{Y} \to \mathcal{X}$, the direct image functor is exact so that $g_* = Rg_*$ and the proper push-forward of algebraic cycles induces covariant functoriality. The flat pull-back induces contravariant functoriality with the appropriate shift in weight. Since the homology of Bloch's complex agrees with its etale hypercohomology with rational coefficients [7, Proposition 3.6], the natural map

$$CH_w(\mathcal{Y}, i - 2w) \to H^c_i(\mathcal{Y}, \mathbb{Z}(w))$$

is an isomorphism upon tensoring with $\mathbb{Q}$, and this vanishes if $i < 2w$. In particular, we have $H^c_i(\mathcal{Y}, \mathbb{Z}(w)) \cong H^c_{i+1}(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}(w))$ for $i < 2w - 1$.

If $\mathcal{Y}$ is regular and of pure dimension $d + 1$, we also write $\mathbb{Z}(n)$ for $\mathbb{Z}^c(d + 1 - n)[-2d - 2]$ and $H^i_{\mathrm{et}}(\mathcal{Y}, A(n))$ for $H^{\mathrm{et}}_{2d+2-i}(\mathcal{Y}, A(d + 1 - n))$. If, in addition, B is a regular curve over a perfect field and $\mathcal{Y}$ is regular, then $\mathbb{Z}^c/m(d + 1 - n)[-2d - 2] \cong \mathbb{Z}/m(n) \cong \mu_m^{\otimes n}$ for $n \geqslant 0$ and m invertible on $\mathcal{Y}$ because this holds for smooth schemes over fields [12].

By [5, XVIII, Theorem 3.1.4], the functor $Rf_!$ has a right adjoint $Rf^!$ on the category of bounded complexes of $\mathbb{Z}/m$-modules for any compactifiable morphism f.

Lemma 2.1. *If B is a regular curve over a perfect field and $\mathcal{Y}$ is regular, then*

$$Rf^! \mu_m^{\otimes -n} \cong \mu_m^{\otimes d-n}[2d] \cong \mathbb{Z}^c/m(n + 1)[-2]$$

for any m invertible on B and $n \leqslant d$.

Taking the limit over etale neighborhoods of a point of B, the lemma remains true for B replaced by the henselization of a regular curve over a perfect field.

Proof. Let $s : B \to k$ be the structure map. Then s and sf are smooth, and $Rs^! \mu_m^{\otimes -n-1}[-2] \cong \mu_m^{\otimes -n}$ on B by [5, XVIII Theorem 3.2.5]. Hence, for the same reason,

$$Rf^! \mu_m^{\otimes -n} \cong R(sf)^! \mu_m^{\otimes -n-1}[-2] \cong \mu_m^{\otimes d-n}[2d] \cong \mathbb{Z}^c/m(n + 1)[-2]. \qquad \square$$

In particular, we obtain an isomorphism

$$H^c_a(\mathcal{Y}, \mathbb{Z}/m(1)) \cong H^{\mathrm{et}}_a(\mathcal{Y}, \mathbb{Z}/m(1)) := H^{2-a}(\mathcal{Y}, Rf^! \mathbb{Z}/m);$$

i.e., for regular schemes, etale motivic Borel–Moore cohomology is isomorphic to the etale motivic homology used in [29].

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Theorem 2.2. *If T is of finite type over a separably closed field k and $w \leqslant 0$, then the natural map*

$$CH_w(T, i-2w)_{\mathbb{Z}'} \to H_i^c(T, \mathbb{Z}'(w))$$

is an isomorphism.

Proof. The proof of [8, Theorem 3.1] works in this situation, but we replicate it for the convenience of the reader. Since $\mathbb{Z}^c(w)$ satisfies the localization property, we can apply Thomason's argument [33, Proposition 2.8] using induction on the dimension of T to reduce to showing that for an artinian local ring R, essentially of finite type over k, the canonical map $\mathbb{Z}^c(w)(\operatorname{Spec} R)_{\mathbb{Z}'} \to R\Gamma_{\text{et}}(\operatorname{Spec} R, \mathbb{Z}^c(w))_{\mathbb{Z}'}$ is a quasi-isomorphism. Since $\mathbb{Z}^c(w)(U) \cong \mathbb{Z}^c(w)(U^{\text{red}})$, we can assume that R is reduced, in which case it is the spectrum of a field F of finite transcendence degree d over k. We have to show that the canonical map $H_i(\mathbb{Z}^c(w)(F))_{\mathbb{Z}'} \to H_i^c(F, \mathbb{Z}^c(w))_{\mathbb{Z}'}$ is an isomorphism for all i. Rationally, Zariski and etale hypercohomologies of the cycle complex agree. With prime to p-coefficients, both sides agree for $i \geqslant d+w$ by the Rost–Voevodsky theorem, and for $i < d$, both sides vanish because $H_i^c(F, \mathbb{Z}/l(w)) \cong H_{\text{et}}^{2d-i}(F, \mathbb{Z}/l(d-w))$ and the l-cohomological dimension of F is d. □

Corollary 2.3. *Let T be a smooth scheme of dimension d over a separably closed field k. Then $H_{\text{et}}^i(T, \mathbb{Z}'(d)) = 0$ for $i > 2d$ and $CH_0(T)_{\mathbb{Z}'} \cong H_{\text{et}}^{2d}(T, \mathbb{Z}'(d))$.*

Taking a K-injective resolution of the motivic complex as in [32], the usual argument gives a Hochschild–Serre spectral sequence

$$E_2^{s,t} = H^s(F, H_{\text{et}}^t(X \times_F F^s, \mathbb{Z}'(d))) \Rightarrow H_{\text{et}}^{s+t}(X, \mathbb{Z}'(d)). \tag{2}$$

The corollary implies that for any field F of characteristic p and a scheme X of dimension d over F, we have $E_2^{s,t} = 0$ for $t > 2d$.

Proposition 2.4. *Let T be a smooth projective scheme over a separably closed field k. Then the Albanese map induces a surjection from the degree zero part*

$$CH_0(T)^0_{\mathbb{Z}'} \to \operatorname{Alb}_T(k)_{\mathbb{Z}'}$$

with uniquely divisible kernel.

Proof. By Rojtman's theorem, the Albanese map induces an isomorphism on prime to p torsion subgroups [2, Theorem 4.2]. Hence, the proposition follows because the Albanese map is surjective, and $CH_0(T)^0$ as well as $\operatorname{Alb}_T(k)$ are divisible by all integers prime to p. □

The following proposition can be proved as in [9, Theorem 1.1].

Proposition 2.5. *Let T be smooth and projective over a separably closed field. Then $H_{\text{et}}^i(T, \mathbb{Z}(n)) \otimes \mathbb{Q}/\mathbb{Z}' = 0$ for $i \neq 2n$.*

2.1. Duality

We recall some facts on duality from [10], [8]. If $g : Y \to \mathbb{F}$ is separated and of finite type over a finite field and m prime to the characteristic of $\mathbb{F}$, then the adjunction $Rg_! \vdash Rg^!$

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gives

$$R\operatorname{Hom}_{Ab}(R\Gamma_c(Y,\mathbb{Z}/m(n)),\mathbb{Z}/m[-1]) \cong R\operatorname{Hom}_{\mathbb{F}}(Rg_!\mathbb{Z}/m(n),\mathbb{Z}/m) \\ \cong R\operatorname{Hom}_Y(\mathbb{Z}/m, Rg^!\mathbb{Z}/m(-n)) \cong R\Gamma(Y, Rg^!\mathbb{Z}/m(-n)),$$

where the shift in degree appears because the left derived functor of coinvariants under $G_{\mathbb{F}}$ is the right derived functor of the invariants functor shifted by one. Using the usual definition $H_i^{\mathrm{et}}(Y,\mathbb{Z}/m(n)) := H_{\mathrm{et}}^{-i}(Y, Rg^!\mathbb{Z}/m(-n))$ for etale homology, we obtain a perfect pairing:

$$H_c^i(Y,\mathbb{Z}/m(n)) \times H_{i-1}^{\mathrm{et}}(Y,\mathbb{Z}/m(n)) \to \mathbb{Z}/m.$$

For finite maps $\iota : Y \to \mathcal{X}$, the adjunction maps $\iota_* R\iota^! Rf^! \to Rf^!$ and $Rf_! \to Rf_!\iota_*\iota^*$ together with the isomorphism $\iota^*\mathbb{Z}/m(n) \cong \mathbb{Z}/m(n)$ induce maps compatible with the pairings

$$\begin{array}{ccc} H_c^i(\mathcal{X},\mathbb{Z}/m(n)) \times H_{i-1}^{\mathrm{et}}(\mathcal{X},\mathbb{Z}/m(n)) & \longrightarrow & \mathbb{Z}/m \\ \iota^*\downarrow \qquad\qquad \iota_*\uparrow & & \| \\ H_c^i(Y,\mathbb{Z}/m(n)) \times H_{i-1}^{\mathrm{et}}(Y,\mathbb{Z}/m(n)) & \longrightarrow & \mathbb{Z}/m. \end{array} \tag{3}$$

Let $\mathcal{X}$ be connected of dimension $d+1$, smooth and proper over a finite field. Then, since $Rf^!\mathbb{Z}/m(-n) \cong \mathbb{Z}/m(d+1-n)[2d+2]$, the pairing above can be rewritten and identified with the cup-product pairing

$$H_{\mathrm{et}}^i(\mathcal{X},\mathbb{Z}/m(n)) \times H_{\mathrm{et}}^{2d+3-i}(\mathcal{X},\mathbb{Z}/m(d+1-n)) \to H_{\mathrm{et}}^{2d+3}(\mathcal{X},\mathbb{Z}/m(d+1)) \cong \mathbb{Z}/m.$$

Moreover, we have a trace map

$$H_{\mathrm{et}}^{2d+4}(\mathcal{X},\mathbb{Z}(d+1)) \overset{\partial}{\cong} H_{\mathrm{et}}^{2d+3}(\mathcal{X},\mathbb{Q}/\mathbb{Z}(d+1)) \cong \mathbb{Q}/\mathbb{Z}$$

and compatible cup-product pairings for any m,

$$\begin{array}{ccc} H_{\mathrm{et}}^i(\mathcal{X},\mathbb{Z}(n)) \times H_{\mathrm{et}}^{2d+4-i}(\mathcal{X},\mathbb{Z}(d+1-n)) & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ \downarrow \qquad\qquad \partial\uparrow & & \partial\uparrow\cong \\ H_{\mathrm{et}}^i(\mathcal{X},\mathbb{Z}/m(n)) \times H_{\mathrm{et}}^{2d+3-i}(\mathcal{X},\mathbb{Z}/m(d+1-n)) & \longrightarrow & \mathbb{Q}/\mathbb{Z}. \end{array} \tag{4}$$

If X_v is connected of dimension d, smooth and proper over a henselian valuation field of residue characteristic p, then the trace isomorphism $H_{\mathrm{et}}^{2d}(X_v^{\mathrm{s}},\mathbb{Q}/\mathbb{Z}'(d)) \cong \mathbb{Q}/\mathbb{Z}'$ together with the Hochschild–Serre spectral sequence induce isomorphisms

$$H_{\mathrm{et}}^{2d+3}(X_v,\mathbb{Z}'(d+1)) \overset{\partial}{\cong} H_{\mathrm{et}}^{2d+2}(X_v,\mathbb{Q}/\mathbb{Z}'(d+1)) \cong \mathbb{Q}/\mathbb{Z}'$$

and compatible cup-product pairings for any m not divisible by p,

$$\begin{array}{ccc} H_{\mathrm{et}}^i(X_v,\mathbb{Z}'(n)) \times H_{\mathrm{et}}^{2d+3-i}(X_v,\mathbb{Z}'(d+1-n)) & \longrightarrow & \mathbb{Q}/\mathbb{Z}' \\ \downarrow \qquad\qquad \partial\uparrow & & \partial\uparrow\cong \\ H_{\mathrm{et}}^i(X_v,\mathbb{Z}/m(n)) \times H_{\mathrm{et}}^{2d+2-i}(X_v,\mathbb{Z}/m(d+1-n)) & \longrightarrow & \mathbb{Q}/\mathbb{Z}. \end{array} \tag{5}$$

Again, the lower pairing is a perfect pairing.

We end this section with a lemma on topological groups.

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Lemma 2.6. *Let A_i be a system of compact groups. Then the natural map of discrete groups*

$$\operatorname{colim}(A_i^*) \to (\lim A_i)^*$$

is an isomorphism.

The lemma includes the statement $\oplus(A_i^*) \cong (\prod A_i)^*$ and applies, in particular, to finite groups A_i.

Proof. By Pontrjagin duality, it suffices to prove that the map $((\lim A_i)^*)^* \to \lim((A_i^*)^*)$ obtained by dualizing one more time is an isomorphism. But since A_i and $\lim A_i$ are compact, both sides agree with $\lim A_i$. □

3. The local situation

3.1. Motivic cohomology of the model

Let R be an excellent henselian discrete valuation ring with finite or separably closed residue field k and field of fractions K (of arbitrary characteristic). Let $f : \mathcal{Y} \to \operatorname{Spec} R$ be a scheme of finite type and let m be an integer not divisible by $\operatorname{char} k$. As pointed out by Kahn [18], the method of [12] and [22, 12.3] can be applied to construct a cycle map

$$cl : CH_1(\mathcal{Y}, a, \mathbb{Z}/m) \to H^{\mathrm{et}}_{a+2}(\mathcal{Y}, \mathbb{Z}/m(1)) := H^{-a}_{\mathrm{et}}(\mathcal{Y}, Rf^!\mathbb{Z}/m),$$

which is compatible with localization sequences. The difference in twists stems from the fact that the left-hand side is indexed by absolute dimension of cycles, whereas the right-hand side is indexed by relative dimension. The map cl induces a map between the coniveau spectral sequences

$$\tilde{E}^1_{a,b}(\mathcal{Y}, \mathbb{Z}/m) = \bigoplus_{x\in\mathcal{Y}_{(a)}} H^{a-b}_{\mathcal{M}}(k(x), \mathbb{Z}/m(a-1)) \Rightarrow CH_1(\mathcal{Y}, a+b-2, \mathbb{Z}/m)$$

$$E^1_{a,b}(\mathcal{Y}, \mathbb{Z}/m) = \bigoplus_{x\in\mathcal{Y}_{(a)}} H^{a-b}(k(x), \mathbb{Z}/m(a-1)) \Rightarrow H^{\mathrm{et}}_{a+b}(\mathcal{Y}, \mathbb{Z}/m(1)),$$

where the E^1-terms are motivic cohomology and Galois cohomology, respectively. By the Rost–Voevodsky theorem, $\tilde{E}^1_{a,b}(\mathcal{Y}, \mathbb{Z}/m) \cong E^1_{a,b}(\mathcal{Y}, \mathbb{Z}/m)$ for $b \geqslant 1$ and $\tilde{E}^1_{a,b}(\mathcal{Y}, \mathbb{Z}/m) = 0$ for $b \leqslant 0$ for cycle dimension reasons. Taking the colimit over powers of l, the terms $E^1_{a,b}(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l)$ vanish for $b < 0$ and l, a prime different from $\operatorname{char} k$, by [29, Lemma 2.6]. Hence, $H^{\mathrm{et}}_i(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) = 0$ for $i < 0$, and the difference between the two theories is measured by the homology of the $E^1_{*,0}$-row

$$\cdots \to \bigoplus_{x\in\mathcal{Y}_{(1)}} H^1(k(x), \mathbb{Q}_l/\mathbb{Z}_l(0)) \to \bigoplus_{x\in\mathcal{Y}_{(0)}} H^0(k(x), \mathbb{Q}_l/\mathbb{Z}_l(-1)).$$

It is shown in [17, Theorem 2.5.10] that this complex is isomorphic up to sign to the complex [29, Definition 2.1] defining Kato homology $KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l)$ so that we obtain a long exact sequence

$$\to KH_{a+3}(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) \to CH_1(\mathcal{Y}, a, \mathbb{Q}_l/\mathbb{Z}_l) \to H^{\mathrm{et}}_{a+2}(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l(1)) \to KH_{a+2}(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) \to . \tag{6}$$

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Consider the following strengthening of the conjectures [29, Conjectures 2.11, 2.12] of Saito–Sato.

Conjecture 3.1. *Let $\mathcal{Y}$ be a regular scheme, flat and proper over* Spec R.

(1) *If k is separably closed, then $KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) = 0$ for all a.*

(2) *If k is finite, then*

$$KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) = \begin{cases} 0 & \text{for } a \neq 1; \\ (\mathbb{Q}_l/\mathbb{Z}_l)^I & \text{for } a = 1, \end{cases}$$

where I is the number of irreducible components of the special fiber $\mathcal{Y} \times_R k$.

Theorem 3.2. *The conjecture holds for $a \leqslant 3$.*

This is proved in [29, Theorem 2.13] for $\mathcal{Y}$ projective such that the reduced special fiber is a strict normal crossing scheme. We use the theorems of Gabber and Kerz–Saito to reduce to this case. Recall that an l'-alteration is a proper surjective map, generically finite of degree prime to l. We will need the following theorem of Gabber [15, Theorem 3(2)].

Theorem 3.3. *Let S be a noetherian, separated, integral excellent regular scheme of dimension 1 with field of quotients K, and let $\mathcal{Y}$ be a separated scheme, flat and of finite type over S. Then for every prime l invertible on S, there exists a finite extension K'/K of degree prime to l, a projective l'-alteration $h : \tilde{\mathcal{Y}} \to \mathcal{Y}$ over $S' \to S$, where S' is the normalization of S in K', with $\tilde{\mathcal{Y}}$ regular and projective over S'. Moreover, $\tilde{\mathcal{Y}} \to S'$ is smooth outside a finite subset Σ of S', and for every geometric point in a fiber over a point of Σ, $\tilde{\mathcal{Y}}$ is locally, for the etale topology, isomorphic to*

$$S'[t_1, \dots, t_n, u_1^{\pm 1}, \dots u_s^{\pm 1}]/(t_1^{a_1} \cdots t_r^{a_r} u_1^{b_1} \cdots u_s^{b_s} - \pi)$$

at the point $u_i = 1, t_j = 0$, with $1 \leqslant r \leqslant n$, for positive integers $a_1, \dots, a_r, b_1, \dots, b_s$ such that $gcd(p, a_1, \dots, a_r, b_1, \dots, b_s) = 1$, for p the exponential characteristic of η, and π a local uniformizer at s'.

Remark 3.4. This implies that the reduced singular fibers are normal crossing divisors. However, as pointed out in de Jong [4, § 2.4], given a normal crossing divisor $D \subset \tilde{\mathcal{Y}}$, there is a projective birational morphism $\varphi : \mathcal{Y}' \to \tilde{\mathcal{Y}}$ such that $\varphi^{-1}(D)_{\text{red}}$ is a strict normal crossing divisor.

Proof of Theorem 3.2. The theorem was proved by Saito–Sato [29, Theorem 0.8] in case the special fiber has simple normal crossings, i.e., if its irreducible components D_i are regular and if the scheme-theoretical intersection $\cap_{i \in J} D_i$ is empty or regular of codimension $|J|$ in S.

In the general case and $a = 2, 3$, we use Gabber's theorem and the remark to find an l'-alteration $f : \mathcal{Y}' \to \mathcal{Y}$ such that the special fiber is a strict normal crossing divisor. By Kerz–Saito [20, Theorem 4.2, Ex. 4.7], there is a pull-back map $f^* : KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) \to KH_a(\mathcal{Y}', \mathbb{Q}_l/\mathbb{Z}_l)$ such that the composition $f_* f^*$ with the push-forward is multiplication

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by the degree of the alteration; hence, the vanishing of $KH_a(\mathcal{Y}', \mathbb{Q}_l/\mathbb{Z}_l)$ implies the vanishing of $KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l)$.

If $a \leqslant 1$, then by the proper base-change theorem and [29, (1.9)], we obtain for d the relative dimension of f:

$$\begin{aligned} KH_a(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l) &\cong H^{2-a}_{\text{et}}(\mathcal{Y}, Rf^!\mathbb{Q}_l/\mathbb{Z}_l) \\ &\cong H^{2d+2-a}_{\text{et}}(\mathcal{Y}, \mathbb{Q}_l/\mathbb{Z}_l(d)) \cong H^{2d+2-a}_{\text{et}}(\mathcal{Y} \times_B k, \mathbb{Q}_l/\mathbb{Z}_l(d)). \end{aligned}$$

The latter group vanishes for $a \leqslant 1$ and k separably closed and $a \leqslant 0$ and k finite, and it is isomorphic to $(\mathbb{Q}_l/\mathbb{Z}_l)^I$ for $a = 1$ and k finite [29, Lemma 2.15]. □

From the sequence (6), we immediately obtain the following.

Corollary 3.5. *If k is finite, the map $CH_1(\mathcal{Y}, 1, \mathbb{Q}/\mathbb{Z}') \to H^{\text{et}}_3(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}'(1))$ is surjective, the map $CH_1(\mathcal{Y}) \otimes \mathbb{Q}/\mathbb{Z}' \to H^{\text{et}}_2(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}'(1))$ is an isomorphism, $H^{\text{et}}_1(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}'(1)) \cong (\mathbb{Q}/\mathbb{Z}')^I$ and $H^{\text{et}}_0(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}'(1)) = 0$.*

We give an application to integral etale motivic cohomology, which can be viewed as a generalization of the vanishing of Brauer groups of relative curves over B [14, § 3]. We return to cohomological notation $H^i_{\text{et}}(\mathcal{Y}, \mathbb{Z}'(d)) = H^{\text{et}}_{2d+2-i}(\mathcal{Y}, \mathbb{Z}'(1))$ for regular $\mathcal{Y}$ of dimension $d+1$.

Corollary 3.6. *Let $\mathcal{Y}$ be a regular scheme of pure dimension $d+1$, flat and proper over B with finite residue field of characteristic p. Then we have a surjection*

$$CH_1(\mathcal{Y})_{\mathbb{Z}'} \twoheadrightarrow H^{2d}_{\text{et}}(\mathcal{Y}, \mathbb{Z}'(d))$$

and isomorphisms $H^{2d+1}_{\text{et}}(\mathcal{Y}, \mathbb{Z}'(d)) \cong 0$ and $H^{2d+2}_{\text{et}}(\mathcal{Y}, \mathbb{Z}'(d)) \cong (\mathbb{Q}/\mathbb{Z}')^I$.

Proof. In degrees $\leqslant 2d+1$, this follows by combining the corollary, the isomorphism $H^i_{\text{et}}(\mathcal{Y}, \mathbb{Q}(d)) \cong CH_1(\mathcal{Y}, 2d-i)_{\mathbb{Q}}$ (which vanishes for $i > 2d$) and the coefficient sequence

$$\begin{array}{ccccccc} CH_1(\mathcal{Y},1)_{\mathbb{Q}} & \longrightarrow & CH_1(\mathcal{Y},1,\mathbb{Q}/\mathbb{Z}') & \longrightarrow & CH_1(\mathcal{Y})_{\mathbb{Z}'} & \longrightarrow & CH_1(\mathcal{Y})_{\mathbb{Q}} \\ \| & & \downarrow{\scriptstyle surj} & & \downarrow & & \| \\ H^{2d-1}_{\text{et}}(\mathcal{Y},\mathbb{Q}(d)) & \longrightarrow & H^{2d-1}_{\text{et}}(\mathcal{Y},\mathbb{Q}/\mathbb{Z}'(d)) & \longrightarrow & H^{2d}_{\text{et}}(\mathcal{Y},\mathbb{Z}'(d)) & \longrightarrow & H^{2d}_{\text{et}}(\mathcal{Y},\mathbb{Q}(d)) \\ & \longrightarrow & CH_1(\mathcal{Y}) \otimes \mathbb{Q}/\mathbb{Z}' & \longrightarrow & 0 & & \\ & & \| & & \downarrow & & \\ & \longrightarrow & H^{2d}_{\text{et}}(\mathcal{Y},\mathbb{Q}/\mathbb{Z}'(d)) & \longrightarrow & H^{2d+1}_{\text{et}}(\mathcal{Y},\mathbb{Z}'(d)) & \longrightarrow & 0. \end{array}$$

Finally, $H^{2d+2}_{\text{et}}(\mathcal{Y}, \mathbb{Z}'(1)) \cong H^{2d+1}_{\text{et}}(\mathcal{Y}, \mathbb{Q}/\mathbb{Z}'(1)) \cong (\mathbb{Q}/\mathbb{Z}')^I$. □

3.2. Motivic cohomology of the generic fiber

Let K be a global field of exponential characteristic p and X be a connected scheme of dimension d, smooth and projective over K. For a valuation v of K, we let K_v be the henselization of K at v, G_v its Galois group and $X_v = X \times_K K_v$.

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Proposition 3.7. *We have*

$$H^{2d+2}_{\mathrm{et}}(X_v, \mathbb{Z}'(d)) \cong H^2(K_v, \mathbb{Z}')$$

and

$$H^{2d+2}_{\mathrm{et}}(X, \mathbb{Z}'(d)) \cong H^2(K, \mathbb{Z}').$$

The map $l_d^{2d+2} : H^{2d+2}_{\mathrm{et}}(X, \mathbb{Z}(d)) \to \prod_v H^{2d+2}_{\mathrm{et}}(X_v, \mathbb{Z}(d))$ *is injective.*

Proof. Since X_v has dimension d, Corollary 2.3 and the Hochschild–Serre spectral sequence (2) give

$$H^{2d+2}_{\mathrm{et}}(X_v, \mathbb{Z}'(d)) \cong H^2(K_v, H^{2d}_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Z}'(d))) \cong H^2(K_v, CH_0(X_v^{\mathrm{s}})_{\mathbb{Z}'}).$$

By Proposition 2.4, we have $H^2(K_v, CH_0(X_v^{\mathrm{s}})^0_{\mathbb{Z}'}) \cong H^2(K_v, \mathrm{Alb}_{\mathbb{Z}'})$, and the latter group vanishes by comparing to the completion of K_v and using [27, Remark I 3.6, 3.10]. Hence, the degree map induces an isomorphism

$$H^2(K, H^{2d}_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Z}'(d))) \cong H^2(K_v, \mathbb{Z}').$$

In the global case, the same argument works because $H^2(K, \mathrm{Alb}_{\mathbb{Z}'})$ still vanishes by [27, I Corollary 6.24]. The injectivity of l_d^{2d+2} follows from the above and Chebotarev's density theorem. □

Proposition 3.8. *Let X_v be a smooth and proper scheme with good reduction over the henselian valuation field K_v. Then the map*

$$H^i_{\mathrm{et}}(X_v, \mathbb{Q}/\mathbb{Z}'(n)) \xrightarrow{\sigma} H^i_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(n))^{G_v}$$

is surjective. In particular, the map

$$\mathrm{Tor} H^i_{\mathrm{et}}(X_v, \mathbb{Z}'(n)) \xrightarrow{\sigma} (\mathrm{Tor} H^i_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Z}'(n)))^{G_v}$$

is surjective for $i \neq 2n+1$.

Proof. Let Y be the special fiber of a smooth and proper model $\mathcal{Y}$ over the valuation ring R of K_v with strict henselization R^{nr} and residue field k. Consider the specialization diagram

$$\begin{array}{ccccc}
H^i_{\mathrm{et}}(Y, \mathbb{Z}/m(n)) & \xleftarrow{\sim} & H^i_{\mathrm{et}}(\mathcal{Y}, \mathbb{Z}/m(n)) & \longrightarrow & H^i_{\mathrm{et}}(X_v, \mathbb{Z}/m(n)) \\
\downarrow & & \downarrow & & \downarrow \\
H^i_{\mathrm{et}}(Y^{\mathrm{s}}, \mathbb{Z}/m(n))^{\mathrm{Gal}(k)} & \xleftarrow{\sim} & H^i_{\mathrm{et}}(\mathcal{Y} \times_R R^{nr}, \mathbb{Z}/m(n))^{\mathrm{Gal}(k)} & \xrightarrow{\sim} & H^i_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Z}/m(n))^{G_v}.
\end{array}$$

The left horizontal maps are isomorphisms by the proper base-change theorem and the lower right map is an isomorphism by the smooth base-change theorem. The first statement follows because k has cohomological dimension one, and hence the left vertical map is surjective. The second statement follows because $H^{i-1}_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(n)) \cong \mathrm{Tor} H^i_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Z}'(n))$ for $i \neq 2n+1$ by Proposition 2.5. □

The following is a refinement of a theorem of Saito–Sato; see [3, Theorem 3.25].

Theorem 3.9. *Let X be smooth and proper over the fraction field of an excellent henselian discrete valuation ring with finite or separably closed residue field. Then the group of zero-cycles of degree 0 on X is isomorphic to the direct sum of a finite group and a group divisible by all integers prime to the residue characteristic.*

We use this to prove the following.

Theorem 3.10. *Let X be smooth and proper of dimension d over a global field of characteristic p. Then the cokernel B_v of $H^{2d}_{\text{et}}(X_v, \mathbb{Z}'(d)) \to H^{2d}_{\text{et}}(X^{\text{s}}_v, \mathbb{Z}'(d))^{G_v}$ is finite and vanishes for almost all v.*

Proof. We show that the cokernel is finite and vanishes for X_v with good reduction. The maps $CH_0(X_v)_{\mathbb{Z}'} \to H^{2d}_{\text{et}}(X_v, \mathbb{Z}'(d)) \to H^{2d}_{\text{et}}(X^s_v, \mathbb{Z}'(d))^{G_v}$ induced by change of topology and pull-back to the separable closure are compatible with proper push-forward [8, Corollary 3.2] and, hence, induce the map deg in the following map of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^{2d}_{\text{et}}(X_v, \mathbb{Z}'(d))^0 & \longrightarrow & H^{2d}_{\text{et}}(X_v, \mathbb{Z}'(d)) & \xrightarrow{\deg} & \mathbb{Z}' & \longrightarrow & \mathbb{Z}'/\delta_v\mathbb{Z}' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \downarrow & & \\
0 & \longrightarrow & (H^{2d}_{\text{et}}(X^{\text{s}}_v, \mathbb{Z}'(d))^0)^{G_v} & \longrightarrow & H^{2d}_{\text{et}}(X^{\text{s}}_v, \mathbb{Z}'(d))^{G_v} & \xrightarrow{\deg} & \mathbb{Z}'^{G_v} & \longrightarrow & \mathbb{Z}'/\delta'_v\mathbb{Z}' & \longrightarrow & 0.
\end{array}$$

Here, the left-hand groups are defined as the kernel of the degree map and the invariants δ_v and δ'_v are analogs of the (prime to p-part of the) index and period of X_v. The cokernels of the two left vertical maps differ by a finite group, and they agree if X_v has good reduction because in this case a zero-cycle of degree 1 in the special fiber can be lifted to X_v by the henselian property; hence, $\delta_v = \delta'_v = 1$. Now, we consider the following diagram with exact rows (we omit the coefficients $\mathbb{Z}'(d)$):

$$\begin{array}{ccccccc}
0 & \longrightarrow & \text{Tor}\, CH_0(X_v)^0_{\mathbb{Z}'} & \longrightarrow & CH_0(X_v)^0_{\mathbb{Z}'} & \xrightarrow{\tau_1} & CH_0(X_v)^0_{\mathbb{Q}} \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & \text{Tor} H^{2d}_{\text{et}}(X_v)^0 & \longrightarrow & H^{2d}_{\text{et}}(X_v)^0 & \xrightarrow{\tau_2} & H^{2d}_{\text{et}}(X_v)^0_{\mathbb{Q}} \\
 & & \downarrow & & \downarrow & & \| \\
0 & \longrightarrow & (\text{Tor} H^{2d}_{\text{et}}(X^{\text{s}}_v)^0)^{G_v} & \longrightarrow & (H^{2d}_{\text{et}}(X^{\text{s}}_v)^0)^{G_v} & \xrightarrow{\tau_3} & (H^{2d}_{\text{et}}(X^{\text{s}}_v)^0_{\mathbb{Q}})^{G_v}.
\end{array}$$

The lower right equality sign follows by a trace argument. By Theorem 3.9, the cokernel $CH_0(X_v)^0 \otimes \mathbb{Q}/\mathbb{Z}'$ of τ_1 vanishes. This implies that τ_2 and τ_3 are surjective and the cokernels of the two lower left vertical maps are isomorphic. By Rojtman's theorem, we get

$$(\text{Tor} H^{2d}_{\text{et}}(X^{\text{s}}_v, \mathbb{Z}'(d))^0)^{G_v} \cong (\text{Tor}\, \text{Alb}_{X_v}(K^{\text{s}}_v)_{\mathbb{Z}'})^{G_v} \cong \text{Tor}\, \text{Alb}_{X_v}(K_v)_{\mathbb{Z}'}.$$

This group, hence B_v, is finite because an abelian variety over a local field has a subgroup of finite index which is uniquely divisible by all primes different from p [27, I Remark 3.6]. The vanishing of B_v for v with good reduction follows from Proposition 3.8 because the torsions of $H^{2d}_{\text{et}}(X^{\text{s}}_v, \mathbb{Z}'(d))^0$ and of $H^{2d}_{\text{et}}(X^{\text{s}}_v, \mathbb{Z}'(d))$ agree. □

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4. Proof of 1.2, 1.3, and 1.4(2)

Let C be a smooth and proper curve over a finite field $\mathbb{F}$ of characteristic p, with generic point $\operatorname{Spec} K \to C$. Let $\mathcal{X} \to C$ be a flat, projective map, with regular $\mathcal{X}$ of dimension $d+1$. We assume that the generic fiber $X = \mathcal{X} \times_C K$ is smooth and geometrically connected over K. For a closed point v of C, we let $\mathcal{O}_v$ be the henselization of C at v, K_v its quotient field, k_v the residue field and $\mathcal{X}_v = \mathcal{X} \otimes_C \mathcal{O}_v$, with generic fiber $X_v = X \times_K K_v$ and closed fiber $Y_v = \mathcal{X} \times_C k_v$. Let G be the Galois group of K and G_v the Galois group of K_v.

Proof of Proposition 1.4(2). Taking the colimit over increasing union of fibers of the localization sequence in etale cohomology, we obtain a map of long exact sequences

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_{\mathrm{et}}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow & H^i_{\mathrm{et}}(X,\mathbb{Z}(n)) & \xrightarrow{\partial} & \bigoplus_v H^{i+1}_{Y_v}(\mathcal{X},\mathbb{Z}(n)) & \longrightarrow \\
& \downarrow & & \downarrow{\scriptstyle l^i_n} & & \downarrow & \\
\longrightarrow & \prod_v H^i_{\mathrm{et}}(\mathcal{X}_v,\mathbb{Z}(n)) & \longrightarrow & \prod_v H^i_{\mathrm{et}}(X_v,\mathbb{Z}(n)) & \xrightarrow{(\partial_v)} & \prod_v H^{i+1}_{Y_v}(\mathcal{X}_v,\mathbb{Z}(n)) & \longrightarrow
\end{array}$$

We claim that identifying the terms $H^i_{\mathrm{et}}(\mathcal{X}_v,\mathbb{Z}(n))$ with Corollary 3.6 gives rise to the following commutative diagram:

$$\begin{array}{ccccc}
H^{2d+1}_{\mathrm{et}}(\mathcal{X},\mathbb{Z}'(d)) & \longrightarrow & H^{2d+1}_{\mathrm{et}}(X,\mathbb{Z}'(d)) & \xrightarrow{\partial} & \bigoplus_v H^{2d+2}_{Y_v}(\mathcal{X},\mathbb{Z}'(d)) \\
\downarrow & & \downarrow{\scriptstyle l^{2d+1}_d} & & \| \\
0 & \longrightarrow & \bigoplus_v H^{2d+1}_{\mathrm{et}}(X_v,\mathbb{Z}'(d)) & \xrightarrow{(\partial_v)} & \bigoplus_v H^{2d+2}_{Y_v}(\mathcal{X}_v,\mathbb{Z}'(d))
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^{2d+1}_{\mathrm{et}}(\mathcal{X},\mathbb{Q}/\mathbb{Z}'(d)) & \longrightarrow & H^{2d+1}_{\mathrm{et}}(X,\mathbb{Q}/\mathbb{Z}'(d)) \\
& \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle l^{2d+2}_d} \\
\xrightarrow{(\tau_v)} & \prod_v \mathbb{Q}/\mathbb{Z}'^{I_v} & \longrightarrow & \prod_v H^{2d+1}_{\mathrm{et}}(X_v,\mathbb{Q}/\mathbb{Z}'(d)).
\end{array} \qquad (7)$$

The shift of degrees appears because we switch from $\mathbb{Z}'$ to $\mathbb{Q}/\mathbb{Z}'$-coefficients in the last four terms. The map l^{2d+1}_d has image in the direct sum because so has ∂. The lower sequence is exact because for almost all v, τ_v is the zero map. Indeed, if $\mathcal{X}_v$ has a good reduction, then by the proper base-change theorem $H^{2d+1}_{\mathrm{et}}(\mathcal{X}_v,\mathbb{Q}/\mathbb{Z}'(d)) \cong H^{2d+1}_{\mathrm{et}}(Y_v,\mathbb{Q}/\mathbb{Z}'(d))$ and by the Hochschild–Serre spectral sequence and the trace map, this is isomorphic to $H^1_{\mathrm{et}}(k_v, H^{2d}_{\mathrm{et}}(\bar{Y}_v,\mathbb{Q}/\mathbb{Z}'(d))) \cong H^1_{\mathrm{et}}(k_v,\mathbb{Q}/\mathbb{Z}')$ so that using Proposition 3.7, the map $H^{2d+1}_{\mathrm{et}}(\mathcal{X}_v,\mathbb{Q}/\mathbb{Z}'(d)) \to H^{2d+1}_{\mathrm{et}}(X_v,\mathbb{Q}/\mathbb{Z}'(d))$ can be identified with the map $H^1_{\mathrm{et}}(k_v,\mathbb{Q}/\mathbb{Z}') \to H^1_{\mathrm{et}}(K_v,\mathbb{Q}/\mathbb{Z}')$, which is the injection dual to the surjection $\mathrm{Gal}(K_v) \to \mathrm{Gal}(k_v)$.

From the diagram and Proposition 3.7, we see that $\ker l^{2d+1}_d = \operatorname{im} H^{2d+1}_{\mathrm{et}}(\mathcal{X},\mathbb{Z}'(d))$ and $\operatorname{coker} l^{2d+1}_d \cong \ker\delta$.

Proposition 4.1. *Up to p-groups, we have an exact sequence*

$$0 \to (T\operatorname{Br}(\mathcal{X}))^* \to \ker\delta \to \operatorname{Pic}(X)^* \to 0.$$

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Proof. The map δ is the colimit of the maps

$$H^{2d+1}_{\text{et}}(\mathcal{X}, \mathbb{Z}/m(d)) \xrightarrow{\delta_m} \prod_v H^{2d+1}_{\text{et}}(Y_v, \mathbb{Z}/m(d)) \xrightarrow{tr} \prod_v (\mathbb{Z}/m)^{I_v}$$

followed by the natural inclusion $\operatorname{colim}_{p \nmid m} \prod_v (\mathbb{Z}/m)^{I_v} \subseteq \prod_v (\mathbb{Q}/\mathbb{Z}')^{I_v}$. Since tr is an isomorphism, it suffices to consider the kernel of δ_m. By Poincaré duality and Lemma 2.6, we have

$$H^{2d+1}_{\text{et}}(\mathcal{X}, \mathbb{Q}/\mathbb{Z}'(d)) \cong \operatorname*{colim}_{p \nmid m} H^2_{\text{et}}(\mathcal{X}, \mathbb{Z}/m(1))^* \cong H^2_{\text{et}}(\mathcal{X}, \hat{\mathbb{Z}}'(1))^*,$$

where, here and in the following, $H^i_{\text{et}}(\mathcal{X}, \hat{\mathbb{Z}}'(n))$ denotes $\lim_{p \nmid m} H^i_{\text{et}}(\mathcal{X}, \mathbb{Z}/m(n))$. If $\iota : Y_v \to \mathcal{X}$ is the inclusion of a closed fiber, then we obtain the commutative diagram (3)

$$\begin{array}{ccccc}
H^{2d+1}_{\text{et}}(\mathcal{X}, \mathbb{Z}/m(d)) & \times H^2_{\text{et}}(\mathcal{X}, \mathbb{Z}/m(1)) & \longrightarrow & \mathbb{Z}/m \\
{\scriptstyle \iota^*}\downarrow & {\scriptstyle \iota_*}\uparrow & & \| \\
H^{2d+1}_{\text{et}}(Y_v, \mathbb{Z}/m(d)) & \times H^{\text{et}}_{2d}(Y_v, \mathbb{Z}/m(d)) & \longrightarrow & \mathbb{Z}/m.
\end{array}$$

Thus, δ_m is dual to the right vertical map in

$$\begin{array}{ccc}
\bigoplus_v \mathrm{CH}_d(Y_v)/m & = & \bigoplus_v H^{\text{et}}_{2d}(Y_v, \mathbb{Z}/m(d)) \\
{\scriptstyle \iota^m_*}\downarrow & & \downarrow{\scriptstyle d_m} \\
\operatorname{Pic}(\mathcal{X})/m & \xrightarrow{\subset} & H^2_{\text{et}}(\mathcal{X}, \mathbb{Z}/m(1)).
\end{array}$$

Since $\operatorname{coker} \iota^m_* = \operatorname{Pic}(X)/m$, the coefficient sequence gives a short exact sequence of cokernels

$$0 \to \operatorname{Pic}(X)/m \to \operatorname{coker} d_m \to {}_m \operatorname{Br} \mathcal{X} \to 0$$

and its dual

$$0 \to ({}_m \operatorname{Br} \mathcal{X})^* \to \ker \delta_m \to (\operatorname{Pic}(X)/m)^* \to 0.$$

In view of the finiteness of ${}_m \operatorname{Br} \mathcal{X}$ and $\operatorname{Pic}(X)/m$, the proposition follows by taking colimits using Lemma 2.6. □

Lemma 4.2. *The composition* $a : \operatorname{coker} l^{2d+1}_d \xrightarrow{\sim} \ker \delta \to \operatorname{Pic}(X)^*$ *is induced by the composition* $a' : H^{2d+1}_{\text{et}}(X_v, \mathbb{Z}'(d)) \to \operatorname{Pic}(X_v)^* \to \operatorname{Pic}(X)^*$, *where the first map is induced by cup product followed by the trace map (5).*

Proof. Consider the following diagram, where the upper horizontal maps are induced by (7) and the lower horizontal maps are induced by functoriality.

$$\begin{array}{ccccc}
H^{2d+1}_{\text{et}}(X_v, \mathbb{Z}'(d)) & \xrightarrow{\partial} & \oplus H^{2d+2}_{Y_v}(\mathcal{X}, \mathbb{Z}'(d)) & \longrightarrow & H^{2d+2}_{\text{et}}(\mathcal{X}, \mathbb{Z}'(d)) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Pic}(X_v)^* & \longrightarrow & \operatorname{Pic}(\mathcal{X}_v)^* & \longrightarrow & \operatorname{Pic}(\mathcal{X})^* \\
& \searrow & & \nearrow & \\
& & \operatorname{Pic}(X)^* & &
\end{array}$$

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The upper vertical maps are the cup products over local fields, over local rings and over finite fields. Commutativity amounts to the compatibility of these pairings. The lower triangle is commutative by functoriality. The upper horizontal composition induces the identification of $\operatorname{coker} l_d^{2d+1}$ with $\ker \delta$ and the right vertical map induces the map $\ker \delta \to \operatorname{Pic}(X)^*$ of the proposition. Commutativity of the diagram shows that a is induced by the lower left composition. □

In particular, we obtain Proposition 1.4(2) because $\operatorname{coker} l_d^{2d+1} \cong \ker \delta \cong \operatorname{Pic}(X)^*$ if $\operatorname{Br}(\mathcal{X})$ is finite.

First diagram. We consider the long exact sequence of Galois cohomology groups associated with the degree map

$$0 \to CH^d(X^{\mathrm{s}})^0_{\mathbb{Z}'} \to CH^d(X^{\mathrm{s}})_{\mathbb{Z}'} \xrightarrow{\deg} \mathbb{Z}' \to 0.$$

Since higher Galois cohomology of uniquely divisible groups vanish, Proposition 2.4 gives an isomorphism $H^1(K, CH^d(X^{\mathrm{s}})^0_{\mathbb{Z}'}) \cong H^1(K, \mathrm{Alb}_X)_{\mathbb{Z}'}$. Let D and D_v be the cokernels of the global and local degree maps

$$CH^d(X^{\mathrm{s}})^G_{\mathbb{Z}'} \to \mathbb{Z}', \quad CH^d(X^{\mathrm{s}}_v)^{G_v}_{\mathbb{Z}'} \to \mathbb{Z}'$$

and recall that their orders were denoted by δ' and δ'_v. Then we obtain the following commutative diagram with exact middle rows. The map β^1, hence τ, has image in the direct sum by [27, I Lemma 6.3].

$$\begin{array}{ccccccccc}
 & & K_2 & \longrightarrow & Ш(\mathrm{Alb}_X) & \longrightarrow & \Phi & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D & \longrightarrow & H^1(K, \mathrm{Alb}_X)_{\mathbb{Z}'} & \longrightarrow & H^1(K, H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))) & \longrightarrow & 0 \\
 & & {\scriptstyle\theta}\downarrow & & {\scriptstyle\beta^1}\downarrow & & {\scriptstyle\tau}\downarrow & & \\
0 & \longrightarrow & \bigoplus_v D_v & \longrightarrow & \bigoplus_v H^1(K_v, \mathrm{Alb}_X)_{\mathbb{Z}'} & \xrightarrow{u} & \bigoplus_v H^1(K_v, H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & {\scriptstyle\rho}\downarrow & & \\
 & & C_2 & \longrightarrow & (T\,\mathrm{Sel}(\mathrm{Pic}^0_X))^* & \longrightarrow & \Psi. & &
\end{array} \tag{8}$$

The groups K_2 and C_2 are the kernel and cokernel of θ and Φ and Ψ are the kernel and cokernel of τ, respectively. The outer columns are exact by definition, and the middle column is exact because for any abelian variety A with dual A^t, there is a short exact sequence [13]

$$0 \to Ш(A) \to H^1(K, A) \xrightarrow{\beta} \bigoplus_v H^1(K_v, A) \to (T\,\mathrm{Sel}(A^t))^* \to 0$$

with $T\,\mathrm{Sel}(A^t) = \lim_{p \nmid m} \ker H^1(K, {}_mA^t) \to \prod_v H^1(K_v, A^t)$ being the inverse limit of Selmer groups. The middle two rows induce the horizontal maps in the upper and lower rows. An easy diagram chase in (8) gives the exact sequence

$$0 \to K_2 \to Ш(\mathrm{Alb}_X) \to \Phi \to C_2 \to (T\,\mathrm{Sel}(\mathrm{Pic}^0_X))^* \to \Psi \to 0. \tag{9}$$

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Proposition 4.3. (1) *The maps* $Ш(\mathrm{Alb}_X) \to \Phi$ *and* $(T\,\mathrm{Sel}(\mathrm{Pic}^0))^* \to \Psi$ *have finite kernels and cokernels.*

(2) *If* $Ш(\mathrm{Alb}_X)$ *is finite, then* Φ *is finite,* u *induces a surjection* $z : H^0(K, \mathrm{Pic}^0_X)^* \to \Psi$ *and we have an equality*

$$\delta' \cdot |\Phi| \cdot |\ker z| = |Ш(\mathrm{Alb}_X)| \cdot \prod \delta'_v. \tag{10}$$

Proof. (1) It follows because D and $\prod_v D_v$, hence K_2 and C_2, are finite.

(2) If $Ш(A)$ is finite, then $TШ(A) = 0$ and, hence, $A(K)^\wedge \cong T\,\mathrm{Sel}(A)$. Thus, $(T\,\mathrm{Sel}(\mathrm{Pic}^0_X))^* \cong H^0(K, \mathrm{Pic}^0_X)^*$ and we obtain the map z. Now, the equality follows by taking alternating orders in (9) because $|K_2|/|C_2| = \delta'/\prod_v \delta'_v$. □

The second diagram. Let B and B_v be the cokernels of

$$H^{2d}_{\mathrm{et}}(X, \mathbb{Z}'(d)) \xrightarrow{\rho_d} H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))^G, \quad H^{2d}_{\mathrm{et}}(X_v, \mathbb{Z}'(d)) \longrightarrow H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}_v, \mathbb{Z}'(d))^{G_v},$$

respectively, and recall that we denoted their orders by β and β_v. Then $\prod_v \beta_v$ is finite by Theorem 3.10. From the Hochschild–Serre spectral sequence (2), we obtain the sequence

$$\begin{aligned} 0 \to B \to H^2(K, H^{2d-1}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))) \\ \to H^{2d+1}_{\mathrm{et}}(X, \mathbb{Z}'(d)) \to H^1(K, H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))) \to 0. \end{aligned}$$

By Proposition 2.5, the injection

$$H^{2d-2}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(d)) \cong \mathrm{Tor} H^{2d-1}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d)) \hookrightarrow H^{2d-1}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))$$

has uniquely divisible cokernel; hence, we obtain an isomorphism

$$H^2(K, H^{2d-2}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(d))) \cong H^2(K, H^{2d-1}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))).$$

Comparing this with the same exact sequence for the local situation, we obtain the exact middle two rows of the following diagram:

$$\begin{array}{ccccccc}
K_1 & \longrightarrow & \ker \xi_d^{2d-2} & \longrightarrow & \ker l_d^{2d+1} & \longrightarrow & \Phi \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B & \longrightarrow & H^2(K, H^{2d-2}(X^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(d))) & \longrightarrow & H^{2d+1}_{\mathrm{et}}(X, \mathbb{Z}'(d)) & \longrightarrow & H^1(K, H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))) \\
{\scriptstyle \theta'}\downarrow & & {\scriptstyle \xi_d^{2d-2}}\downarrow & & {\scriptstyle l_d^{2d+1}}\downarrow & & {\scriptstyle \tau}\downarrow \\
\bigoplus B_v & \longrightarrow & \bigoplus H^2(K_v, H^{2d-2}(X^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(d))) & \longrightarrow & \bigoplus H^{2d+1}_{\mathrm{et}}(X_v, \mathbb{Z}'(d)) & \longrightarrow & \bigoplus H^1(K_v, H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}_v, \mathbb{Z}'(d))) \\
\downarrow & & {\scriptstyle \xi'}\downarrow & & \downarrow & & {\scriptstyle \rho}\downarrow \\
C_1 & \longrightarrow & (H^2_{\mathrm{et}}(X^{\mathrm{s}}, \hat{\mathbb{Z}}'(1))^G)^* & \xrightarrow{w} & \mathrm{coker}\, l_d^{2d+1} & \longrightarrow & \Psi
\end{array} \tag{11}$$

The maps ξ_d^{2d-2} and τ have image in the direct sum by [27, I Lemma 4.8] and (8), and we have seen above that l_d^{2d+1} has image in the direct sum. The groups K_1 and C_1 are the kernel and cokernel of the canonical map θ', and the map ξ' is the dual of the injection $H^2_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}/m(1))^G \to \prod_v H^2_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}/m(1))^{G_v}$. All columns are short exact by definition except the left middle column, which is exact by Tate–Poitou duality. Hence, the maps in the middle two rows induce the maps in the upper and lower row.

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Proof of Theorem 1.3. In view of the finiteness of $\prod_v \beta_v$ and Proposition 4.3, diagram (11) gives a sequence in the quotient category modulo the Serre subcategory spanned by finite groups and p-groups

$$0 \to \operatorname{coker} \rho_d \to \ker \xi_d^{2d-2} \to \ker l_d^{2d+1} \to Ш(\mathrm{Alb}_X)$$
$$\to (H^2_{\text{et}}(X^{\text{s}}, \hat{\mathbb{Z}}'(1))^G)^* \xrightarrow{w} \operatorname{coker} l_d^{2d+1} \to \Psi \to 0.$$

Lemma 4.4. *The composition of w with the map a of Lemma 4.2 is the dual of the cycle map* $\mathrm{Pic}(X)^\wedge \to H^2_{\text{et}}(X^{\text{s}}, \hat{\mathbb{Z}}'(1))^G$.

Proof. Consider the following commutative diagram. The left middle square is commutative because duality of Galois cohomology of a local field is compatible with duality of etale cohomology over local fields, and the commutativity of the part with a' is (5).

$$\begin{array}{ccccc}
H^2(K_v, H^{2d-1}_{\text{et}}(X^{\text{s}}, \mathbb{Z}'(d))) & \longrightarrow & H^{2d+1}_{\text{et}}(X_v, \mathbb{Z}'(d)) & \xrightarrow{a'} & \\
\cong\uparrow & & \uparrow & & \\
H^2(K_v, H^{2d-2}_{\text{et}}(X^{\text{s}}, \mathbb{Q}/\mathbb{Z}'(d))) & \longrightarrow & H^{2d}_{\text{et}}(X_v, \mathbb{Q}/\mathbb{Z}'(d)) & & \downarrow \\
\cong\downarrow & & \cong\downarrow & & \\
(H^2_{\text{et}}(X^{\text{s}}_v, \hat{\mathbb{Z}}(1))^{G_v})^* & \longrightarrow & H^2_{\text{et}}(X_v, \hat{\mathbb{Z}}(1))^* & \longrightarrow & \mathrm{Pic}(X_v)^* \\
\downarrow & & \downarrow & & \downarrow \\
(H^2_{\text{et}}(X^{\text{s}}, \hat{\mathbb{Z}}(1))^G)^* & \longrightarrow & H^2_{\text{et}}(X, \hat{\mathbb{Z}}(1))^* & \longrightarrow & \mathrm{Pic}(X)^*
\end{array}$$

By Lemma 4.2, the map $a \circ w$ is induced by lifting an element along the left vertical map and taking the upper right composition. Commutativity implies that this is the map in the lower row, i.e., the cycle map. □

Since $\mathrm{Pic}^0(X)^\wedge$ is the kernel of the cycle map, we obtain a commutative diagram with exact rows and surjective map c_1^*.

$$\begin{array}{ccccccccc}
 & & (H^2_{\text{et}}(X^{\text{s}}, \hat{\mathbb{Z}}'(1))^G)^* & \xrightarrow{w} & \operatorname{coker} l_d^{2d+1} & \longrightarrow & \Psi & \longrightarrow & 0 \\
 & & c_1^*\downarrow & & a\downarrow & & b\downarrow & & \\
0 & \longrightarrow & \mathrm{NS}(X)^* & \longrightarrow & \mathrm{Pic}(X)^* & \longrightarrow & \mathrm{Pic}^0(X)^* & \longrightarrow & 0.
\end{array} \tag{12}$$

Taking the kernel of the vertical maps gives the sequence of Theorem 1.3 (still up to finite groups and p-groups). Indeed, $\ker c_1^* = \ker a \circ w \cong (\operatorname{coker} c)^*$ by Lemma 4.4, $\ker a \cong (T\,\mathrm{Br}(\mathcal{X}))^*$ by Proposition 4.1 and $\ker b \cong (T\,Ш(\mathrm{Pic}^0_X))^*$ by Proposition 4.3(1) and the definition of the Selmer group (it is not necessary to check that b is the canonical map as b is surjective so that $\ker b$ and $(T\,Ш(\mathrm{Pic}^0_X))^*$ have the same corank and hence are abstractly isomorphic). Finally, the map $Ш(\mathrm{Alb}_X) \to (H^2_{\text{et}}(X^{\text{s}}, \hat{\mathbb{Z}}'(1))^G)^*$ factors through $\ker c_1^*$ because its composition with w, hence with $a \circ w$, and then with c_1^*, is zero.

Proof of Theorem 1.2. If $\mathrm{Br}(\mathcal{X})$, hence $Ш(\mathrm{Alb}_X)$, $Ш(\mathrm{Pic}^0_X)$ and $\operatorname{coker} c$ are finite, then the maps a and b in (12) are isomorphisms. This implies that $\ker w = \ker c_1^* = \ker c^*$, hence counting alternating orders of the first and fourth rows of (11), we get

$$|\ker \xi_d^{2d-2}/K_1| \cdot |\Phi| \cdot |C_1| = |\ker l_d^{2d+1}| \cdot |\operatorname{coker} c|. \tag{13}$$

We now compare (10) and (13). We have the identities $|C_1| = \prod \beta_v / |B/K_1|$, and by the following lemma, $|\ker z| = |\ker bz|$ is equal to the order α of the cokernel of the canonical map $\mathrm{Pic}^0(X) \to H^0(K, \mathrm{Pic}^0_X)$ so that

$$|\ker l_d^{2d+1}| \cdot |\operatorname{coker} c| \cdot |B/K_1| \cdot \alpha\delta' = |Ш(\mathrm{Alb}_X)| \cdot |\ker \xi_d^{2d-2}/K_1| \prod_v \beta_v \delta'_v.$$

If $B = \operatorname{coker} \rho_d$ is finite, we can remove the common factor $|K_1|$ and obtain the formula of Theorem 1.2. It remains to show the following.

Lemma 4.5. *If $Ш(\mathrm{Pic}^0_X)$ is finite, then the composition $H^0(K, \mathrm{Pic}^0_X)^* \xrightarrow{z} \Psi \xrightarrow{b} \mathrm{Pic}^0(X)^*$ of the maps from Proposition 4.3 and diagram (12) is the canonical map.*

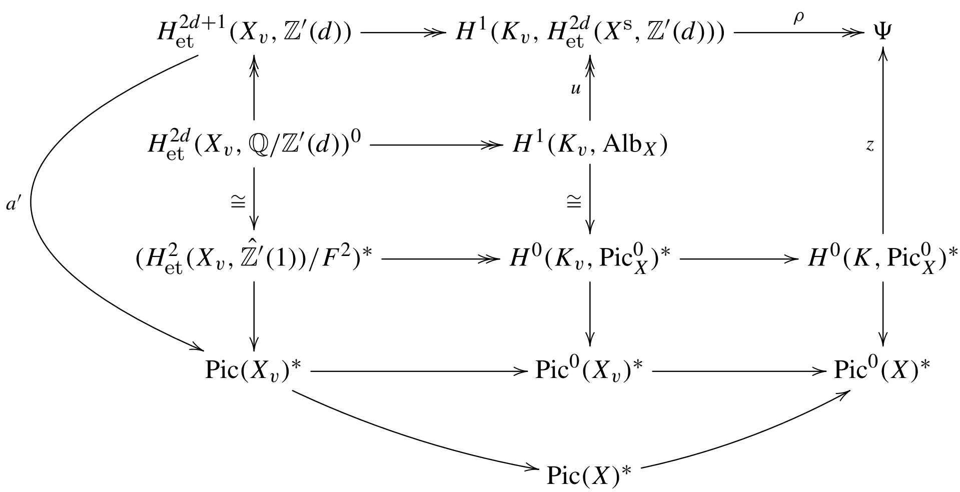
Proof. Consider the filtration induced by the Hochschild–Serre spectral sequence. By functoriality, we have a duality between

$$H^2_{\mathrm{et}}(X_v, \hat{\mathbb{Z}}'(1))/F^2 = \operatorname{coker}(\hat{\mathbb{Z}}' \cong H^2(K_v, \hat{\mathbb{Z}}'(1)) \to H^2_{\mathrm{et}}(X_v, \hat{\mathbb{Z}}'(1)))$$

and

$$H^{2d}_{\mathrm{et}}(X_v, \mathbb{Q}/\mathbb{Z}')^0 = \ker(H^{2d}_{\mathrm{et}}(X_v, \mathbb{Q}/\mathbb{Z}') \to H^{2d}_{\mathrm{et}}(X_v^{\mathrm{s}}, \mathbb{Q}/\mathbb{Z}'(d))^{G_v} \cong \mathbb{Q}/\mathbb{Z}').$$

The latter still surjects onto $H^{2d+1}_{\mathrm{et}}(X_v, \mathbb{Z}'(d))$ because $H^{2d}_{\mathrm{et}}(X_v, \mathbb{Z}(d)) \otimes \mathbb{Q}/\mathbb{Z}'$ surjects onto the copy of $\mathbb{Q}/\mathbb{Z}'$ and maps to zero in $H^{2d+1}_{\mathrm{et}}(X_v, \mathbb{Z}'(d))$. We obtain a diagram



(14)

The map u is the map of diagram (8), and the upper right square defines the map z. The map b is induced by lifting elements of Ψ to $H^{2d+1}_{\mathrm{et}}(X_v, \mathbb{Z}'(d))$ and then following the curved arrows. The middle left square is the following composition:

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$$
\begin{array}{ccccc}
H^{2d}_{\mathrm{et}}(X_v, \mathbb{Q}/\mathbb{Z}'(d))^0 & \longrightarrow & H^1(K_v, H^{2d-1}_{\mathrm{et}}(X^{\mathrm{s}}_v, \mathbb{Q}/\mathbb{Z}'(d))) & \xrightarrow{\sim} & \\
\cong\downarrow & & \cong\downarrow & & \\
(H^2_{\mathrm{et}}(X_v, \hat{\mathbb{Z}}'(1))/F^2)^* & \longrightarrow & H^1(K_v, H^1_{\mathrm{et}}(X^{\mathrm{s}}_v, \hat{\mathbb{Z}}'(1)))^* & \xrightarrow{\sim} &
\end{array}
$$

$$
\begin{array}{ccccc}
\xrightarrow{\sim} & H^1(K_v, \mathrm{Tor\,Alb}_X) & \xrightarrow{\sim} & H^1(K_v, \mathrm{Alb}_X) \\
& \cong\downarrow & & \cong\downarrow \\
\xleftarrow{\sim} & H^1(K_v, T\,\mathrm{Pic}^0_X)^* & \xrightarrow{\sim} & H^0(K_v, \mathrm{Pic}^0_X)^*.
\end{array}
$$

The left square commutes by compatibility of duality for schemes over local fields with Galois cohomology of the field, and the middle square is the compatibility of the e_m-pairing with the cup-product pairing:

$$
\begin{array}{ccc}
H^{2d-1}_{\mathrm{et}}(X^{\mathrm{s}}_v, \mathbb{Q}/\mathbb{Z}'(d)) & \xrightarrow{\sim} & \mathrm{Tor\,Alb}(X^{\mathrm{s}}_v) \\
\cong\downarrow & & \cong\downarrow e_m \\
H^1_{\mathrm{et}}(X^{\mathrm{s}}_v, \hat{\mathbb{Z}}'(1))^* & \xleftarrow{\sim} & (T\,\mathrm{Pic}^0(X^{\mathrm{s}}_v))^*.
\end{array}
$$

It is easy to see that all other squares in (14) commute, and this implies the claim of the lemma. □

If $\mathcal{X}$ is a surface, i.e., $d = 1$, then $\ker l^{2d+1}_d = \mathrm{Br}(\mathcal{X})$, $\ker \xi^0_1$ vanishes and δ' and δ'_v are the periods of X and X_v. By Lichtenbaum [23], β_v is equal to the index δ_v of X_v, whereas c is the (completed) degree map whose cokernel has order equal to the index δ. Hence, the formula reduces to the formula in [11] in view of $\delta\alpha = \beta\delta'$, which can be seen from the following diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}^0(X) & \longrightarrow & \mathrm{Pic}\,X & \longrightarrow & \delta\mathbb{Z} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(K, \mathrm{Pic}^0_X) & \longrightarrow & H^0(K, \mathrm{Pic}_X) & \longrightarrow & \delta'\mathbb{Z} & \longrightarrow & 0
\end{array}
$$

5. Other consequences

We recall the following well-known results:

(1) The finiteness of the Brauer group and of the Tate–Shafarevich group is implied by the finiteness of its l-primary part for any prime l; see [26, Remark 8.5] for the Brauer group and [19] for the Tate–Shafarevich group. In particular, Tate's conjecture for divisors on $\mathcal{X}$ holds if and only if $\mathrm{Br}(\mathcal{X})$ is finite.

(2) The Tate–Shafarevich group $Ш(\mathrm{Pic}^0_X)$ is finite if and only if $Ш(\mathrm{Alb}_X)$ is finite [27, I Remark 6.14 (c)].

Proof of Theorem 1.1. If Tate's conjecture holds for X and the Tate–Shafarevich group of Alb_X is finite, then $(T\,\mathrm{Br}(\mathcal{X}))^*$ vanishes by Theorem 1.3. But the Brauer group is torsion with ${}_m\mathrm{Br}(\mathcal{X})$ finite for every m; hence, the vanishing of the Tate module implies that the l-primary part $\mathrm{Br}(\mathcal{X})\{l\}$ is finite for every prime $l \neq p$.

Conversely, assume that the Brauer group $\mathrm{Br}(\mathcal{X})$ is finite. Then $T\,\mathrm{Br}(\mathcal{X}) = 0$; hence, $(T\text{Ш}(\mathrm{Pic}^0_X))^* = 0$ by Theorem 1.3. But the Tate–Shafarevich group is torsion with finite m-torsion for every m [27, I Remark 6.7]. This implies that the l-primary part $\text{Ш}(\mathrm{Pic}^0_X)\{l\}$ is finite for every prime $p \neq l$ and, hence, that $\text{Ш}(\mathrm{Pic}^0_X)$ is finite and then $\text{Ш}(\mathrm{Alb}_X)$ is finite. Consequently, $(\mathrm{coker}\, c)^*$ is finite by Theorem 1.3.

Proof of Proposition 1.4(1). First, we consider $\ker \xi_d^{2d-2}$. If $\mathrm{Br}(\mathcal{X})$ is finite, then Tate's conjecture for divisors holds on $\mathcal{X}$, and, hence, Tate's conjecture for dimension-one cycles holds on $\mathcal{X}$ [26, Proposition 8.4] or, equivalently, $T_l H^{2d+1}_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}(d))$ vanishes for all $l \neq p$. This implies that $H^{2d+1}_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}(d))\{l\}$ is the cotorsion of $H^{2d}_{\mathrm{et}}(\mathcal{X}, \mathbb{Q}_l/\mathbb{Z}_l(d))$, and this is finite and vanishes for almost all l by Gabber's theorem [6] and semi-simplicity at the eigenvalue 1. We conclude that $H^{2d+1}_{\mathrm{et}}(\mathcal{X}, \mathbb{Z}'(d))$ is finite and surjects onto $\ker l_d^{2d+1}$ in (7). By (11), we see that $\mathrm{coker}\,\rho_d \to \ker \xi_d^{2d-2}$ has finite kernel and cokernel.

By the Tate–Poitou duality, $\mathrm{coker}\, \xi_d^{2d-2}$ is dual to $H^2_{\mathrm{et}}(X^s, \hat{\mathbb{Z}}(1))^{G_K}$. Finiteness of $\mathrm{Br}(\mathcal{X})$ implies Tate's conjecture for X in codimension one; hence, this agrees with the dual of $\mathrm{NS}(X) \otimes \hat{\mathbb{Z}}$, which is isomorphic to the dual of $\mathrm{NS}(X)$ as $\mathrm{NS}(X)$ is finitely generated.

In characteristic 0, our method gives the following weaker result (to obtain the full result, one would need to prove Theorem 3.10 in this situation).

Theorem 5.1. *Let $\mathcal{X}$ be regular, proper and flat over the rings of integers of a number field. If $\mathcal{X}$ has good reduction at all places above p and if the p-primary component $\mathrm{Br}(\mathcal{X})\{p\}$ of the Brauer group is finite, then the p-primary component $\text{Ш}(\mathrm{Alb}_X)\{p\}$ of the Tate–Shafarevich group of the Albanese of the generic fiber is finite.*

Proof. If $\mathcal{X}$ has good reduction at p, then motivic cohomology agrees with Sato's p-adic Tate twists, $\mathbb{Z}/p^r(n) \cong \mathcal{T}_r(n)$ by [7] and [34, Theorem 4.8]; see [31, § 1.4]. Hence, the analog of Proposition 4.1 can be proved by using [31, Theorem 1.2.2]. On the other hand, the vanishing of $H^{2d+1}_{\mathrm{et}}(\mathcal{X}_v, \mathbb{Z}_{(p)}(d))$ can be proved as in Corollary 3.6 by using [30, Theorem 1.3.1]. Then the diagram chase in diagram (11) gives the result. □

We show that the cokernel of ρ_d is finite under the following conjecture, which was stated by Beilinson [1, Conjecture 5.2] for number fields.

Conjecture 5.2. *If X^s is smooth and projective over the separable closure K^s of a global field, then the Albanese map*

$$\mathrm{CH}_0(X^s)^0 \to \mathrm{Alb}_{X^s}(K^s)$$

is an isomorphism.

Proposition 5.3. *Assuming Conjecture 5.2, the cokernel of*

$$\rho_d : H^{2d}_{\mathrm{et}}(X, \mathbb{Z}'(d)) \to H^{2d}_{\mathrm{et}}(X^{\mathrm{s}}, \mathbb{Z}'(d))^{G_K}$$

is finite up to a p-group.

Proof. A norm argument shows that the cokernel of ρ_d is torsion; hence, it suffices to show that the target of ρ_d is finitely generated. But by the conjecture and Proposition 2.4, it is an extension of $\mathbb{Z}'$ by $(\mathrm{CH}_0(X^{\mathrm{s}})^0_{\mathbb{Z}'})^{G_K} \cong \mathrm{Alb}_X(K^{\mathrm{s}})^{G_K}_{\mathbb{Z}'} \cong \mathrm{Alb}_X(K)_{\mathbb{Z}'}$, a finitely generated $\mathbb{Z}'$-module. □

Finally, we mention that our result implies the following corollary, of which the equivalence of (1)–(4) is folklore.

Corollary 5.4. *The first four statements are equivalent and imply the fifth statement:*

(1) *Tate's conjecture holds for smooth and proper surfaces over finite fields.*

(2) *Tate's conjecture holds for divisors on smooth and projective varieties over finite fields.*

(3) *Tate's conjecture holds for one-dimensional cycles on smooth and projective varieties over finite fields.*

(4) *The Tate–Shafarevich group of any abelian variety A over a global function field is finite.*

(5) *Tate's conjecture holds for divisors and for one-dimensional cycles on smooth and proper varieties over global function fields for all $l \neq p$.*

Proof. (1) $\Leftrightarrow$ (3) follows by a Lefschetz theorem argument [26, Remark 8.7], and (1) $\Leftrightarrow$ (2) is proven in [28, Theorem 4.3].

The equivalence of (1) and (4) is known to the experts, but we repeat the argument for the convenience of the reader. Given a smooth and projective surface $\mathcal{X}$ over a finite field, it is explained in [24, Proof of Theorem 1] how to obtain a surface satisfying the hypothesis of Theorem 1.1. Then the finiteness of the Tate–Shafarevich group of the Jacobian of the generic fiber implies the finiteness of $\mathrm{Br}(\mathcal{X})$.

Conversely, given an abelian variety A over a function field K, we can find a Jacobian variety J of a smooth and proper curve X over K surjecting onto A [25, Theorem 10.1]. Choosing a polarization on A and dualizing, we obtain a morphism $A \to A^t \to J^t \cong J$ with finite kernel. Applying [21, Theorem 6] to the image of A in J, we obtain another abelian subvariety A' of J and an isogeny $A \times A' \to J$. Since $Ш(J)$ and $Ш(A) \oplus Ш(A')$ differ by a finite group, it suffices by [14] to observe that the Brauer group of a smooth and projective model $\mathcal{X}$ of X is finite by (1).

It remains to show that (2) implies (5). Let X be a smooth and projective variety over a function field K, and fix a prime l different from $\mathrm{char}\, K$. By Gabber's Theorem 3.3 applied to some proper model of X over C, we can find a finite extension K' of degree prime to l such that $X' = X \times_K K'$ has a regular projective model $\mathcal{X}'$ over the smooth and proper curve C' with function field K'. Hence, $\mathcal{X}'$ is smooth and projective over a finite field, and its Brauer group is finite by (2). From Theorem 1.1, we can then conclude that Tate's conjecture for divisors holds for X', which implies Tate's conjecture for X. □

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References

1. A. A. Beilinson, Height pairings between algebraic cycles, in *K-theory, Arithmetic and Geometry (Moscow, 1984–1986)*, Lecture Notes in Mathematics, Volume 1289, pp. 1–25 (Springer, Berlin, 1987).
2. J. L. Colliot-Thélène, Cycles algébriques de torsion et K-théorie algébrique, in *Arithmetic Algebraic Geometry (Trento, 1991)*, Lecture Notes in Mathematics, Volume 1553, pp. 1–49 (Springer, Berlin, 1993).
3. J. L. Colliot-Thélène, Groupe de Chow des zéro-cycles sur les variétés p-adiques (d'après S. Saito, K. Sato et al.). Séminaire Bourbaki. Vol. 2009/2010. Exposés 1012–1026. Astérisque No. 339 (2011), Exp. No. 1012, vii, 1–30.
4. J. A. de Jong, Smoothness, semi-stability and alterations, *Inst. Hautes Études Sci. Publ. Math.* **83** (1996), 51–93.
5. P. Deligne, La formule de dualite globale, in *Théorie des topos et cohomologie étale des schémas. Tome 3. Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964* (ed. Dirigé par M. Artin, A. Grothendieck and J. L. Verdier), Lecture Notes in Mathematics, Volume 305, p. vi+640 (Springer, Berlin–New York, 1973).
6. O. Gabber, Sur la torsion dans la cohomologie l-adique d'une variétè, *C. R. Acad. Sci. Paris Sér. I Math.* **297**(3) (1983), 179–182.
7. T. Geisser, Motivic cohomology over Dedekind rings, *Math. Z.* **248**(4) (2004), 773–794.
8. T. Geisser, Duality via cycle complexes, *Ann. of Math. (2)* **172**(2) (2010), 1095–1126.
9. T. Geisser, On the structure of etale motivic cohomology, *J. Pure Appl. Algebra* **221**(7) (2017), 1614–1628.
10. T. Geisser, Duality of integral étale motivic cohomology, in *K-Theory-Proceedings of the International Colloquium, Mumbai, 2016*, pp. 195–209 (Hindustan Book Agency, New Delhi, 2018).
11. T. Geisser, Comparing the Brauer group and the Tate–Shafarevich group, *J. Inst. Math. Jussieu.* to appear, doi:10.1017/S1474748018000294.
12. T. Geisser and M. Levine, The Bloch–Kato conjecture and a theorem of Suslin-Voevodsky, *J. Reine Angew. Math.* **530** (2001), 55–103.
13. C. D. González-Avilés and K.-S. Tan, A generalization of the Cassels-Tate dual exact sequence, *Math. Res. Lett.* **14**(2) (2007), 295–302.
14. A. Grothendieck, Le groupe de Brauer. III, in *Exemples et compléments. Dix exposés sur la cohomologie des schémas, 88–188*, Advanced Studies in Pure Mathematics, Volume 3 (North-Holland, Amsterdam, 1968).
15. L. Illusie, Y. Laszlo and F. Orgogozo, Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents, in *Séminaire à l'École Polytechnique 2006–2008*, Astérisque No. 363–364 (Société Mathématique de France, Paris, 2014).
16. U. Jannsen, On the l-adic cohomology of varieties over number fields and its Galois cohomology, in *Galois Groups Over $\mathbb{Q}$ (Berkeley, CA, 1987)*, Mathematical Sciences Research Institute Publications, Volume 16, pp. 315–360 (Springer, New York, 1989).
17. U. Jannsen, S. Saito and K. Sato, Étale duality for constructible sheaves on arithmetic schemes, *J. Reine Angew. Math.* **688** (2014), 1–65.
18. B. Kahn, Algebraic K-theory, algebraic cycles and arithmetic geometry, in *Handbook of K-theory*, Volume 1, 2, pp. 351–428 (Springer, Berlin, 2005).
19. K. Kato and F. Trihan, On the conjectures of Birch and Swinnerton–Dyer in characteristic $p > 0$, *Invent. Math.* **153**(3) (2003), 537–592.
20. M. Kerz and S. Saito, Cohomological Hasse principle and motivic cohomology for arithmetic schemes, *Publ. Math. Inst. Hautes Études Sci.* **115** (2012), 123–183.

21. S. Lang, Abelian varieties, in *Interscience Tracts in Pure and Applied Mathematics* p. xii+256 (No. 7 Interscience Publishers, Inc., New York, 1959). Interscience Publishers Ltd., London.
22. M. Levine, K-theory and motivic cohomology of schemes, https://faculty.math.illinois.edu/K-theory/0336/mot.pdf.
23. S. Lichtenbaum, Duality theorems for curves over p-adic fields, *Invent. Math.* **7** (1969), 120–136.
24. Q. Liu, D. Lorenzini and M. Raynaud, On the Brauer group of a surface, *Invent. Math.* **159**(3) (2005), 673–676.
25. J. S. Milne, Jacobian varieties, in *Arithmetic Geometry (Storrs, Conn., 1984)*, pp. 167–212 (Springer, New York, 1986).
26. J. Milne, Values of zeta functions of varieties over finite fields, *Amer. J. Math.* **108**(2) (1986), 297–360.
27. J. S. Milne, *Arithmetic Duality Theorems*, Second Edition p. viii+339 (BookSurge, LLC, Charleston, SC, 2006). ISBN: 1-4196-4274-X.
28. M. Morrow, A Variational Tate Conjecture in crystalline cohomology, *J. Eur. Math. Soc.*, to appear, https://arxiv.org/pdf/1408.6783.pdf.
29. S. Saito and K. Sato, A finiteness theorem for zero-cycles over p-adic fields. With an appendix by Uwe Jannsen, *Ann. of Math. (2)* **172**(3) (2010), 1593–1639.
30. S. Saito and K. Sato, Zero-cycles on varieties over p-adic fields and Brauer groups, *Ann. Sci. Éc. Norm. Supér. (4)* **47**(3) (2014), 505–537.
31. K. Sato, p-adic étale Tate twists and arithmetic duality, *Ann. Sci. Éc. Norm. Supér. (4)* **40**(4) (2007), 519–588. With an appendix by Kei Hagihara.
32. N. Spaltenstein, Resolutions of unbounded complexes, *Compos. Math.* **65**(2) (1988), 121–154.
33. R. W. Thomason, Algebraic K-theory and étale cohomology, *Ann. Sci. Éc. Norm. Supér. (4)* **18**(3) (1985), 437–552.
34. C. Zhong, Comparison of dualizing complexes, *J. Reine Angew. Math.* **695** (2014), 1–39.