HYPERGEOMETRIC MELLIN TRANSFORMS

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1. *Introduction.* The Mellin transforms of generalized hypergeometric functions are discussed in this paper, and it is shown how some of the most general integrals of the Mellin type can be deduced from them. Four general theorems are considered and a number of special cases are given in detail.

In the usual notation for generalized hypergeometric functions we let

$$ {}_AF_B[(a);\,(b);\,x] = \sum_{n=0}^{\infty} \frac{((a))_n\, x^n}{((b))_n\, n!} = \sum_{n=0}^{\infty} \frac{(a_1)_n\,(a_2)_n \ldots (a_A)_n\, x^n}{(b_1)_n\,(b_2)_n \ldots (b_B)_n\, n!}, \tag{1·1} $$

where $$(a_m)_n = a_m(a_m+1)(a_m+2)\ldots(a_m+n-1), \quad (a_m)_0 = 1,$$

and (a) denotes the sequence of parameters $a_1, a_2, \ldots, a_m, \ldots, a_A$. We assume that there are always A of the a parameters, B of the b parameters, and so on.

Also, for convenience in printing, let

$$ \Gamma[(a);\,(b)] = \Gamma\begin{bmatrix}(a)\\(b)\end{bmatrix} = \Gamma\begin{bmatrix}a_1, a_2, \ldots, a_A\\ b_1, b_2, \ldots, b_B\end{bmatrix} = \frac{\Gamma(a_1)\,\Gamma(a_2)\ldots\Gamma(a_A)}{\Gamma(b_1)\,\Gamma(b_2)\ldots\Gamma(b_B)}. \tag{1·2} $$

A dash will denote the omission of a vanishing factor in a sequence. Thus $(a)'-a_m$ denotes the sequence

$$ a_1-a_m, a_2-a_m, \ldots, a_{m-1}-a_m, a_{m+1}-a_m, a_{m+2}-a_m, \ldots, a_A-a_m. $$

We state the Mellin transform thus: *if*

$$ f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} g(s)\,ds, \tag{1·3} $$

then $$ g(s) = \int_0^\infty x^{s-1} f(x)\,dx, \tag{1·4} $$

provided that $g(s)$ exists in the Lebesgue sense, over the range $0, \infty$. The Mellin transform can be translated into the Laplace transform by putting $\exp(1/x)$ for x, and into the Fourier transform by putting $\exp(1/ix)$ for x (see (3), chap. 2, for fuller details).

2. *The first theorem.* In (2) four theorems are stated on the evaluation of contour integrals in terms of hypergeometric functions. We now apply the Mellin transform

to these theorems and we deduce four further theorems. In the first theorem of (2) the integrand was the product of gamma functions

$$g(s) = \Gamma\begin{bmatrix}(a)+s, (b)-s\\(c)+s, (d)-s\end{bmatrix}, \tag{2·1}$$

and the result proved was: if $\frac{1}{2}\pi(A+B-C-D) > |\arg x|$, then

(i) $f(x) = \Sigma_A(x) \sim \Sigma_B(1/x)$ when $A+D > B+C$ or when $A+D = B+C$ and $|x| < 1$,

(ii) $f(x) = \Sigma_A(x) = \Sigma_B(1/x)$ when $A+D = B+C$, $x = 1$ and $\Re\Sigma(c+d-a-b) > 0$,

and (iii) $f(x) = \Sigma_B(1/x) \sim \Sigma_A(x)$ when $A+D < B+C$ or when $A+D = B+C$ and $|x| > 1$, where

$$\Sigma_A(x) = \sum_{\mu=1}^{A} x^{a_\mu}\Gamma\begin{bmatrix}(a)'-a_\mu, (b)+a_\mu\\(c)-a_\mu, (d)+a_\mu\end{bmatrix}{}_{B+C}F_{A+D-1}\begin{bmatrix}(b)+a_\mu, 1+a_\mu-(c);\ (-1)^{A+C}x\\1+a_\mu-(a)', (d)+a_\mu;\end{bmatrix} \tag{2·2}$$

and

$$\Sigma_B(1/x) = \sum_{\nu=1}^{B} x^{-b_\nu}\Gamma\begin{bmatrix}(a)+b_\nu, (b)'-b_\nu\\(c)+b_\nu, (d)-b_\nu\end{bmatrix}{}_{A+D}F_{B+C-1}\begin{bmatrix}(a)+b_\nu, 1+b_\nu-(d);\ (-1)^{B+D}/x\\(c)+b_\nu, 1+b_\nu-(b)';\end{bmatrix}. \tag{2·3}$$

Let us apply the Mellin transform (1·3) and (1·4) to these results. Then we find that

$$f(x) = \Sigma_A(x) \quad \text{if} \quad A+D > B+C, \quad \text{or} \quad A+D = B+C \quad \text{and} \quad 0 < x < 1,$$

and that

$$f(x) = \Sigma_B(1/x) \quad \text{if} \quad A+D < B+C, \quad \text{or} \quad A+D = B+C \quad \text{and} \quad 1 < x < \infty,$$

where we must have $A+B \geqslant C+D$ for $f(x)$ to exist.

Hence we have

THEOREM 1. (i) $g(s) = \int_0^\infty x^{s-1}\Sigma_A(x)\,dx$ *if* $A+D > B+C$,

(ii) $g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx + \int_1^\infty x^{s-1}\Sigma_B(1/x)\,dx$ *if* $A+D = B+C$,

and (iii) $g(s) = \int_0^\infty x^{s-1}\Sigma_B(1/x)\,dx$ *if* $A+D < B+C$,

provided that $A+B \geqslant C+D$, $-\Re a_\nu < \Re s < \Re b_\nu$, *and* $\Re c_\nu, \Re d_\nu \neq -N$ *for* N *a positive integer or zero, and for all* ν.

It should be noted that $g(-s) \to g(s)$ when $f(1/x) \to f(x)$, and so, given any $g(s)$ which falls under (iii) above, it is possible to regard it as equivalent to a function $g(-s)$ which can be treated under (i) by simply substituting b's for a's, d's for c's and *vice versa*. Also, when $A+D = B+C$, $\Sigma_B(1/x)$ is the analytic continuation of $\Sigma_A(x)$ for $1 < x < \infty$. If $B = 0$ and $A = C$, we must also have $D = 0$. In this particular case, the integral over the range 1 to ∞ is zero, and

$$g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx. \tag{2·4}$$

If we leave $f(x)$ expressed as the contour integral $\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g(s)\, x^{-s}\, ds$ without giving its evaluation in terms of power series, we have the Mellin transform of Meijer's G-function ((1), §7·3 (43)).

3. *Some examples on Theorem* 1. In all these examples we assume that

$$-\Re a_\nu < \Re s < \Re b_\nu$$

and that $\Re c_\nu, \Re d_\nu \neq -N$ for all ν. If $g(s)$ contains only one gamma function of s, there are two possibilities, $\Gamma(a+s)$ and $\Gamma(b-s)$, which transform one into the other under $g(-s) \to g(s)$. Both lead to the Euler integral of the second kind,

$$\Gamma(a+s) = \int_0^\infty x^{s-1+a} e^{-x}\, dx. \tag{3·1}$$

The possibilities for $g(s)$ to contain two gamma functions of s reduce in the same way to four distinct cases: two Euler integrals of the first kind,

$$\Gamma\begin{bmatrix} a+s, b-s \\ a+b \end{bmatrix} = \int_0^1 x^{s-1+a}\, {}_1F_0[a+b;\ ;-x]\, dx + \int_1^\infty x^{s-1-b}\, {}_1F_0[a+b;\ ;-1/x]\, dx \quad ((1), §7·3\,(15)) \tag{3·2}$$

and

$$\Gamma\begin{bmatrix} a+s, c-a \\ c+s \end{bmatrix} = \int_0^1 x^{s-1+a}\, {}_1F_0[1+a-c;\ ;x]\, dx \quad ((1), §7·3\,(20) \text{ and } (22)), \tag{3·3}$$

and two integrals of Bessel functions,

$$\Gamma\begin{bmatrix} a+s, a+d \\ d-s \end{bmatrix} = \int_0^\infty x^{s-1+a}\, {}_0F_1[;\ a+d;\ -x]\, dx \quad ((1), §7·3\,(23)) \tag{3·4}$$

and

$$\Gamma[a+s, b+s] = \int_0^\infty x^{s-1+a}\, \Gamma[b-a]\, {}_0F_1[;\ 1+a-b;\ x]\, dx + \int_0^\infty x^{s-1+b}\Gamma[a-b]\, {}_0F_1[;\ 1+b-a;\ x]\, dx \quad ((1), §7·3\,(17)). \tag{3·5}$$

When $g(s)$ contains three gamma functions of s we find five distinct cases:

$$\Gamma\begin{bmatrix} a+s, b-s \\ d-s \end{bmatrix} = \int_0^\infty x^{s-1+a}\, \Gamma\begin{bmatrix} a+b \\ a+d \end{bmatrix} {}_1F_1[a+b;\ a+d;\ -x]\, dx. \tag{3·6}$$

The special case $b = \tfrac{1}{2}$, $d = 1+a$, leads to

$$\sqrt{\pi}\, \Gamma[s+\nu, s-\nu;\ s+\tfrac{1}{2}] = \int_0^\infty x^{s-1}\, e^{-\frac{1}{2}x} K_\nu(\tfrac{1}{2}x)\, dx, \tag{3·7}$$

where $|\Re \nu| < \Re s$ ((1), §7·3 (26)), and the corresponding case for $g(-s)$ leads to

$$\Gamma[\tfrac{1}{2}-s, s+\nu;\ 1+\nu-s] = \sqrt{\pi}\int_0^\infty x^{s-1}\, e^{-\frac{1}{2}x}\, I_\nu(\tfrac{1}{2}x)\, dx, \tag{3·8}$$

where $-\Re\nu < \Re s < \frac{1}{2}$ ((1), §7·3 (25)).

$$\Gamma[a_1+s, a_2+s, b-s] = \int_0^\infty x^{s-1+a_1}\Gamma[a_2-a_1, a_1+b]\,{}_1F_1[a_1+b;\ 1+a_1-a_2;\ x]\,dx$$
$$+\int_0^\infty x^{s-1+a_2}\,\Gamma[a_1-a_2, a_2+b]\,{}_1F_1[a_2+b; 1+a_2-a_1;\ x]\,dx. \quad (3·9)$$

The special case $\mu-\frac{1}{2} = a_1$, $\frac{1}{2}-\mu = a_2$, $-\kappa = b$, leads to

$$\Gamma\begin{bmatrix}\mu-\frac{1}{2}+s,\ -\mu+\frac{1}{2}+s,\ -\kappa-s\\ \mu-\kappa+\frac{1}{2},\ -\mu-\kappa+\frac{1}{2}\end{bmatrix} = \int_0^\infty x^{s-1}\,e^{\frac{1}{2}x}W_{\kappa,\mu}(x)\,dx, \quad (3·10)$$

where $|\Re\mu|-\frac{1}{2} < \Re s < -\Re\kappa$ ((1), §6·9 (7)).

The remaining three results in this group do not seem to have been stated before. These are:

$$\Gamma\begin{bmatrix}a+s, b+s;\\ c+s\end{bmatrix} = \int_0^\infty x^{s-1+a}\Gamma[b-a;\ c-a]\,{}_1F_1[1+a-c;\ 1+a-b;\ -x]\,dx$$
$$+\int_0^\infty x^{s-1+b}\Gamma[a-b;\ c-b]\,{}_1F_1[1+b-c;\ 1+b-a;\ -x]\,dx, \quad (3·11)$$

$$\Gamma[a+s, b+s;\ d-s] = \int_0^\infty x^{s-1+a}\Gamma[b-a;\ d+a]\,{}_0F_2[;\ 1+a-b, d+a;\ x]\,dx$$
$$+\int_0^\infty x^{s-1+b}\Gamma[a-b;\ d+b]\,{}_0F_2[;\ 1+b-a, d+b;\ x]\,dx \quad (3·12)$$

and

$$\Gamma[a_1+s, a_2+s, a_3+s] = \sum_{\nu=1}^{3}\int_0^\infty x^{s-1+a_\nu}\,\Gamma[(a)'-a_\nu]\,{}_0F_2[;\ 1+a_\nu-(a)';\ -x]\,dx. \quad (3·13)$$

When $g(s)$ contains four gamma functions in s we find ten distinct cases, each of which can be extended immediately to the generalized hypergeometric series, thus the generalization of (3·13) is

$$\Gamma[(a)+s] = \sum_{\mu=1}^{A}\Gamma[(a)'-a_\mu]\int_0^\infty x^{s-1+a_\mu}\,{}_0F_{A-1}[;\ 1+a_\mu-(a)';\ (-1)^A x]\,dx. \quad (3·14)$$

$$\Gamma[(a)+s;\ (c)+s] = \sum_{\mu=1}^{A}\Gamma[(a)'-a_\mu;\ (c)-a_\mu]$$
$$\times\int_0^\infty x^{s-1+a_\mu}\,{}_CF_{A-1}[1+a_\mu-(c);\ 1+a_\mu-(a)';\ (-1)^{A+C}x]\,dx \quad (3·15)$$

when $A > C$, and

$$\Gamma[(a)+s;\ (c)+s] = \sum_{\mu=1}^{A}\Gamma[(a)'-a_\mu;\ (c)-a_\mu]$$
$$\times\int_0^1 x^{s-1+a_\mu}\,{}_AF_{A-1}[1+a_\mu-(c);\ 1+a_\mu-(a)';\ x]\,dx \quad (3·16)$$

when $A = C$,

$$\Gamma[(a)+s;\ (d)-s] = \sum_{\mu=1}^{A} \Gamma[(a)'-a_\mu;\ (d)+a_\mu] \times \int_0^\infty x^{s-1+a_\mu}\, {}_0F_{A+D-1}[;\ 1+a_\mu-(a)',\ (d)+a_\mu;\ (-1)^A x]\, dx \qquad (3\cdot 17)$$

for $A \geqslant D$,

$$\Gamma[(a)+s;\ (c)+s,\ (d)-s] = \sum_{\mu=1}^{A} \Gamma[(a)'-a_\mu;\ (c)+s,\ (d)-s] \times \int_0^\infty x^{s-1+a_\mu}\, {}_CF_{A+D-1}[1+a_\mu-(c);\ 1+a_\mu-(a)',\ (d)+a_\mu;\ (-1)^{A+C} x]\, dx \qquad (3\cdot 18)$$

for $A \geqslant C+D$,

$$\Gamma[(a)+s,\ (b)-s] = \sum_{\mu=1}^{A} \Gamma[(a)'-a_\mu,\ (b)+a_\mu] \times \int_0^\infty x^{s-1+a_\mu}\, {}_BF_{A-1}[(b)+a_\mu;\ 1+a_\mu-(a)';\ (-1)^A x]\, dx \qquad (3\cdot 19)$$

for $A > B$, and

$$\Gamma[(a)+s,\ (b)-s] = \sum_{\mu=1}^{A} \Bigg\{ \Gamma[(a)'-a_\mu,\ (b)+a_\mu] \times \int_0^1 x^{s-1+a_\mu}\, {}_AF_{A-1}[(b)+a_\mu;\ 1+a_\mu-(a)';\ (-1)^A x]\, dx + \Gamma[(a)+b_\mu,\ (b)'-b_\mu] \int_1^\infty x^{s-1-b_\mu}\, {}_AF_{A-1}[(a)+b_\mu;\ 1+b_\mu-(b)';\ (-1)^A/x]\, dx \Bigg\} \qquad (3\cdot 20)$$

for $A = B$,

$$\Gamma[(a)+s,\ (b)-s;\ (c)+s] = \sum_{\mu=1}^{A} \Gamma[(a)'-a_\mu,\ (b)+a_\mu;\ (c)-a_\mu] \times \int_0^\infty x^{s-1+a_\mu}\, {}_{B+C}F_{A-1}[(b)+a_\mu,\ 1+a_\mu-(c);\ 1+a_\mu-(a)';\ (-1)^{A+C} x]\, dx \qquad (3\cdot 21)$$

for $A > B+C$, and

$$\Gamma[(a)+s,\ (b)-s;\ (c)+s] = \sum_{\mu=1}^{A} \Gamma\begin{bmatrix}(a)'-a_\mu,\ (b)+a_\mu \\ (c)-a_\mu\end{bmatrix} \times \int_0^1 x^{s-1+a_\mu}\, {}_AF_{A-1}\begin{bmatrix}(b)+a_\mu,\ 1+a_\mu-(c);\ (-1)^{A+C} x \\ 1+a_\mu-(a)';\end{bmatrix} dx + \sum_{\nu=1}^{B} \Gamma\begin{bmatrix}(a)+b_\nu,\ (b)'-b_\nu \\ (c)+b_\nu\end{bmatrix} \int_1^\infty x^{s-1-b_\nu}\, {}_AF_{A-1}[(a)+b_\nu;\ (c)+b_\nu,\ 1+b_\nu-(b)';\ (-1)^B/x]\, dx \qquad (3\cdot 22)$$

for $A = B+C$, and finally

$$\Gamma[(a)+s,\ (b)-s;\ (d)-s] = \sum_{\mu=1}^{A} \Gamma\begin{bmatrix}(a)'-a_\mu,\ (b)+a_\mu \\ (d)+a_\mu\end{bmatrix} \times \int_0^\infty x^{s-1+a_\mu}\, {}_BF_{A+D-1}\begin{bmatrix}(b)+a_\mu;\ (-1)^A x \\ 1+a_\mu-(a)',\ (d)+a_\mu;\end{bmatrix} dx \qquad (3\cdot 23)$$

for $B = D$.

Most of these results do not seem to have been stated explicitly before, although many special cases of them are well known. In particular the integrals (3·16), (3·20) and (3·22), which involve the Gauss ${}_2F_1(x)$ series when $A = 2$, have often been stated incompletely.

4. *The second theorem.* Let

$$g(s) = \Gamma\begin{bmatrix}(a)+s, (b)-s\\(c)+s, (d)-s\end{bmatrix} {}_{A+B+E}F_{C+D+F}\begin{bmatrix}(a)+s, (b)-s, (e);\ y\\(c)+s, (d)-s, (f);\end{bmatrix}, \tag{4·1}$$

$$\Sigma_A(x) = \sum_{\mu=1}^{A} x^{a_\mu}\Gamma\begin{bmatrix}(a)'-a_\mu, (b)+a_\mu\\(c)-a_\mu, (d)+a_\mu\end{bmatrix} \sum_{m=0}^{\infty} \frac{((b)+a_\mu)_{2m}((e))_m x^m y^m}{((d)+a_\mu)_{2m}((f))_m m!}$$

$$\times {}_{B+C}F_{A+D-1}\begin{bmatrix}(b)+a_\mu+2m, 1+a_\mu-(c);\ (-1)^{A+C}x\\(d)+a_\mu+2m, 1+a_\mu-(a)';\end{bmatrix} \tag{4·2}$$

and $$\Sigma_B(1/x) = \sum_{\nu=1}^{B} x^{-b_\nu}\Gamma\begin{bmatrix}(a)+b_\nu, (b)'-b_\nu\\(c)+b_\nu, (d)-b_\nu\end{bmatrix} \sum_{m=0}^{\infty} \frac{((a)+b_\nu)_{2m}}{((c)+b_\nu)_{2m}}$$

$$\times \frac{((e))_m y^m x^{-m}}{((f))_m m!} {}_{A+D}F_{B+C-1}\begin{bmatrix}(a)+b_\nu+2m, 1+b_\nu-(d);\ (-1)^{B+D}/x\\(c)+b_\nu+2m, 1+b_\nu-(b)';\end{bmatrix}. \tag{4·3}$$

Now let us apply the Mellin transform (1·3) and (1·4) to the results of Theorem 2 of (2). We find

THEOREM 2. *If* $A+B \geqslant C+D$, $A+B+E \leqslant C+D+F+1$, $-\Re a_\mu < \Re s < \Re b_\nu$, *and* $\Re c_\nu$, $\Re d_\nu \neq -N$, *for all* ν, *then*

(i) $g(s) = \int_0^\infty x^{s-1}\Sigma_A(x)\,dx$ *for* $A+D > B+C$,

(ii) $g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx + \int_1^\infty x^{s-1}\Sigma_B(1/x)\,dx$ *for* $A+D = B+C$

and (iii) $g(s) = \int_0^\infty x^{s-1}\,\Sigma_B(1/x)\,dx$ *for* $A+D < B+C$.

Again we note that (iii) and (i) are equivalent under the transform $g(s) \to g(-s)$, and that in (ii) $\Sigma_B(1/x)$ is the analytic continuation of $\Sigma_A(x)$, for $1 < x < \infty$.

In particular, if $B = D = 0$, then $A = C$, the second integral in (ii) is zero and

$$g(s) = \int_0^1 x^{s-1}\,\Sigma_A(x)\,dx. \tag{4·4}$$

To justify the application of the Mellin transform here, we need only note that, under the conditions of this theorem, $g(s)$ is a function of bounded variation, on part of the real axis ((3), Theorem 29).

5. *Special cases of Theorem* 2. In all these examples we assume that the parameters satisfy the restrictions $-\Re a_\mu < \Re s < \Re b_\nu$ and $\Re c_\nu$, $\Re d_\nu \neq -N$ for all ν. The simplest possibility is

$$\Gamma(a+s)\,{}_1F_0[a+s;\ y] = \int_0^\infty x^{s-1+a}\,e^{xy-x}\,dx \tag{5·1}$$

for $|y| < 1$. This is another variation of Euler's integral of the second kind ((1), §6·3 (1)). When an f parameter is added we find

$$\Gamma(a+s)\,{}_1F_1[a+s; f;\ y] = \int_0^\infty x^{s-1+a}\,e^{-x}\,{}_0F_1[; f;\ xy]\,dx \quad ((1), §7·5\,(24)), \tag{5·2}$$

and when e parameters are added also we have

$$\Gamma(a+s)\,{}_{E+1}F_F\,[a+s,(e);\,(f);\,y] = \int_0^\infty x^{s-1+a}\,e^{-x}\,{}_EF_F[(e);\,(f);\,xy]\,dx, \qquad (5\cdot3)$$

where $E < F$ or $E = F$ and $|y| < 1$ ((1), §7·6 (27)).

When $g(s)$ has two gamma functions involving s there are four possibilities. The first case is

$$\Gamma[a+s,b+s]\,{}_2F_1\,[a+s,b+s;f;\,y]$$
$$= \Gamma[b-a]\int_0^\infty x^{s-1+a}\,{}_0F_1\,[;\,1+a-b;\,x]\,{}_0F_1\,[;f;\,xy]\,dx$$
$$+\Gamma[a-b]\int_0^\infty x^{s-1+b}\,{}_0F_1\,[;\,1+b-a;\,x]\,{}_0F_1\,[;f;\,xy]\,dx, \qquad (5\cdot4)$$

from which we can deduce various integrals of products of Bessel functions. When e and f parameters are added, we have

$$\Gamma[a+s,b+s]\,{}_{E+2}F_F\,[a+s,b+s,(e);(f);\,y]$$
$$= \Gamma[b-a]\int_0^\infty x^{s-1+a}\,{}_0F_1\,[;\,1+a-b;\,x]\,{}_EF_F[(e);\,(f);\,xy]\,dx$$
$$+\Gamma[a-b]\int_0^\infty x^{s-1+b}\,{}_0F_1\,[;\,1+b-a;\,x]\,{}_EF_F\,[(e);\,(f);\,xy]\,dx, \qquad (5\cdot5)$$

where $E \leqslant F$. This is a general Hankel transform. The extension to a general number of a parameters is

$$\Gamma[(a)+s]\,{}_{A+E}F_F[(a)+s,(e);\,(f);\,y]$$
$$= \sum_{\mu=1}^{A}\Gamma[(a)'-a_\mu]\int_0^\infty x^{s-1+a_\mu}\,{}_0F_A\,[;\,1+a_\mu-(a)';\,x]\,{}_EF_F[(e);\,(f);\,xy]\,dx \qquad (5\cdot6)$$

for $E \leqslant F$.

The second possibility is

$$\Gamma[a+s,b-s]\,{}_2F_1\,[a+s,b-s;f;\,y]$$
$$= \Gamma[a+b]\int_0^\infty x^{s-1+a}(1+x)^{-a-b}\,{}_2F_1\,[\tfrac{1}{2}(a+b),\tfrac{1}{2}(1+a+b);f;\,4xy/(1+x)^2]\,dx, \qquad (5\cdot7)$$

and when the e and f parameters are added we find that

$$\Gamma[a+s,b-s]\,{}_{E+2}F_F\,[a+s,b-s,(e);\,(f);\,y]$$
$$= \Gamma[a+b]\int_0^\infty x^{s-1+a}(1+x)^{-a-b}$$
$$\times\,{}_{E+2}F_F[\tfrac{1}{2}(a+b),\tfrac{1}{2}(1+a+b),(e);\,(f);\,4xy/(1+x)^2]\,dx \qquad (5\cdot8)$$

for $E < F$. When $A > 1$, or $B > 1$, the double series under the integrand is neither separable nor summable, and so in general it cannot be reduced to a single summation.

Thirdly, we have

$$\Gamma[a+s, c-a;\ c+s]\,{}_1F_1[a+s;\ c+s;\ y] = \int_0^1 x^{s-1+a} e^{xy} (1-x)^{c-a-1} dx \quad ((1), \S 7{\cdot}5\,(25)). \tag{5·9}$$

When we add an e parameter we find

$$\Gamma[a+s, c-a;\ c+s]\,{}_2F_1[a+s, e;\ c+s;\ y] = \int_0^1 x^{s-1+a}(1-x)^{c-a-1}(1-xy)^{-e} dx \quad ((1)\ \S 7{\cdot}5\,(16)) \tag{5·10}$$

for $|y| < 1$, and when we add an f parameter also, we have

$$\Gamma[a+s, c-a;\ c+s]\,{}_2F_2[a+s, e;\ c+s, f;\ y] = \int_0^1 x^{s-1+a}(1-x)^{c-a-1}\,{}_1F_1[e; f;\ xy]\,dx, \tag{5·11}$$

and, for general E and F,

$$\Gamma[a+s, c-a;\ c+s]\,{}_{E+1}F_{F+1}[a+s, (e);\ c+s, (f);\ y] = \int_0^1 x^{s-1+a}(1-x)^{c-a-1}\,{}_EF_F[(e);\ (f);\ xy]\,dx \tag{5·12}$$

for $E < F+1$, or $E = F+1$ and $|y| < 1$ ((1), §6·9 (10)).

For $A = C > 1$, the inner double series is separable but not summable in general, and the result becomes

$$\Gamma[(a)+s;\ (c)+s]\,{}_{A+E}F_{A+F}[(a)+s, (e);\ (c)+s, (f);\ y] = \sum_{\mu=1}^{A} \Gamma[(a)'-a_\mu;\ (c)-a_\mu] \times \int_0^1 x^{s-1+a_\mu}\,{}_AF_{A-1}[1+a_\mu-(c);\ 1+a_\mu-(a)';\ x]\,{}_EF_F[(e);\ (f);\ xy]dx \tag{5·13}$$

for $E < F$ or for $E = F$ and $|y| < 1$. Here the most interesting particular case is $A = 2$, which leads to a ${}_2F_1(x)$ under the integral sign, and in particular when $E = F = 0$, we have

$$\begin{aligned}\Gamma[a+s, b+s;\ c+s, d+s]\,{}_2F_2[a+s, b+s;\ c+s, d+s;\ y] &= \Gamma[b-a;\ c-a, d-a]\int_0^1 x^{s-1+a} e^{-xy}\,{}_2F_1[1+a-c, 1+a-d;\ 1+a-b;\ x]\,dx \\ &+ \Gamma[a-b;\ c-b, d-b]\int_0^1 x^{s-1+b} e^{-xy}\,{}_2F_1[1+b-c, 1+b-d;\ 1+b-a;\ x]\,dx.\end{aligned} \tag{5·14}$$

For $A > C \geqslant 1$, the corresponding general result is

$$\Gamma[(a)+s;\ (c)+s]\,{}_{A+E}F_{C+F}[(a)+s, (e);\ (c)+s, (f);\ y] = \sum_{\mu=1}^{A} \Gamma[(a)'-a_\mu;\ (c)-a_\mu] \times \int_0^\infty x^{s-1+a_\mu}\,{}_CF_{A-1}[1+a_\mu-(c);\ 1+a_\mu-(a)';\ (-1)^{A+C}x]\,{}_EF_F[(e);\ (f);\ xy]\,dx, \tag{5·15}$$

and in particular for $A = 2$, $C = 1$, $E = F = 0$, we find that

$$\Gamma[a+s,b+s;\ c+s]\,{}_2F_1[a+s,b+s;\ c+s;\ y]$$
$$= \Gamma[b-a;\ c-a]\int_0^\infty x^{s-1+a}\,e^{-xy}\,{}_1F_1[1+a-c;\ 1+a-b;\ -x]\,dx$$
$$+\Gamma[a-b;\ c-b]\int_0^\infty x^{s-1+b}\,e^{-xy}\,{}_1F_1[1+b-c;\ 1+b-a;\ -x]\,dx \quad (5\cdot16)$$

for $|y|<1$.

The fourth case is

$$\Gamma[a+s,a+d;\ d-s]\,{}_1F_1[a+s;\ d-s;\ x]$$
$$=\int_0^\infty x^{s-1+a}\sum_{m=0}^{\infty}\frac{x^m y^m}{(d+a)_{2m}\,m!}\,{}_0F_1[;\ a+d+2m;\ -x]\,dx. \quad (5\cdot17)$$

Here the inner series are neither separable nor summable, and the introduction of further parameters, a's, d's, e's or f's does not alter this fact.

When $g(s)$ contains three different parameters involving s three further groups of cases arise. Firstly we have

$$g(s) = \Gamma[a+s,b-s;\ d-s]\,{}_2F_1[a+s,b-s;\ d-s;\ y],$$

where $|y|<1$, and its extensions

$$g(s) = \Gamma[a+s,b-s;\ d-s]\,{}_{E+2}F_{F+1}[a+s,b-s,(e);\ d-s,(f);\ y], \quad (5\cdot18)$$

where $E \leqslant F$, and

$$g(s) = \Gamma[(a)+s,(b)-s;\ (d)-s]\,{}_{A+B+E}F_{D+F}[(a)+s,(b)-s,(e);\ (d)-s,(f);\ y], \quad (5\cdot19)$$

where $A+B+E \leqslant D+F+1$ and $A+B \geqslant D$.

Secondly, we have

$$g(s) = \Gamma[a+s,b-s;\ c+s]\,{}_2F_1[a+s,b-s;\ c+s;\ y], \quad (5\cdot20)$$

where $|y|<1$, and its extensions,

$$g(s) = \Gamma[a+s,b-s;\ c+s]\,{}_{E+2}F_{F+1}[a+s,b-s,(e);\ c+s,(f);\ y], \quad (5\cdot21)$$

where $E \leqslant F$, and

$$g(s) = \Gamma[(a)+s,(b)-s;\ (c)+s]\,{}_{A+B+E}F_{C+F}[(a)+s,(b)-s,(e);\ (c)+s,(f);\ y], \quad (5\cdot22)$$

where $A+B+E \leqslant C+F+1$ and $A+B \geqslant C$.

Thirdly, we have

$$g(s) = \Gamma[a_1+s,a_2+s,a_3+s;\ c+s,d-s]\,{}_3F_2[a_1+s,a_2+s,a_3+s;\ c+s,d-s;\ y], \quad (5\cdot23)$$

and its extensions,

$$g(s) = \Gamma[a_1+s,a_2+s,a_3+s;\ c+s,d-s]$$
$${}_{E+3}F_{F+2}[a_1+s,a_2+s,a_3+s,(e);\ c+s,d-s,(f);\ y], \quad (5\cdot24)$$

where $E \leqslant F+1$, and

$$g(s) = \Gamma[(a)+s;\ (c)+s, (d)-s]$$
$$_{A+E}F_{C+D+F}[(a)+s, (e);\ (c)+s, (d)-s, (f);\ y], \qquad (5\cdot25)$$

where $A \geqslant C+D$ and $A+E \leqslant C+D+F+1$.

In these three groups, however, the inner double sum is neither separable nor summable, owing to the presence of an a and a d parameter or a b and a c parameter at one and the same time.

6. *The third theorem.* Let

$$g(s) = \Gamma\begin{bmatrix}(a)+s, (b)-s, (g)+s, (h)-s\\(c)+s, (d)-s, (j)+s, (k)-s\end{bmatrix}{}_{A+B+E}F_{C+D+F}\begin{bmatrix}(a)+s, (b)-s, (e);\ x\\(c)+s, (d)-s, (f);\end{bmatrix}, \qquad (6\cdot1)$$

$$\Sigma_A(x) = \sum_{\mu=1}^{A}\Gamma\begin{bmatrix}(a)'-a_\mu, (b)+a_\mu, (g)-a_\mu, (h)+a_\mu\\(c)-a_\mu, (d)+a_\mu, (j)-a_\mu, (k)+a_\mu\end{bmatrix}x^{a_\mu}$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((b)+a_\mu)_{2m+n}((h)+a_\mu)_{m+n}((e))_m(1+a_\mu-(c))_n}{(1+a_\mu-(a)')_n\,(1+a_\mu-(g))_{m+n}((f))_m\,((d)+a_\mu)_{2m+n}}$$
$$\times\frac{(1+a_\mu-(j))_{m+n}x^{m+n}y^m(-1)^{n(A+G-C-J)}}{((k)+a_\mu)_{m+n}\,m!\,n!}$$
$$+\sum_{\mu=1}^{G}\Gamma\begin{bmatrix}(a)-g_\mu, (b)+g_\mu, (g)'-g_\mu, (h)+g_\mu\\(c)-g_\mu, (d)+g_\mu, (j)-g_\mu, (k)+g_\mu\end{bmatrix}x^{g_\mu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a)+g_\mu)_{m-n}}{((c)-g_\mu)_{m-n}}$$
$$\times\frac{((b)+g_\mu)_{m+n}\,(1+g_\mu-(j))_n\,((h)+g_\mu)_n\,((e))_m\,y^m x^n(-1)^{n(G-J)}}{((d)+g_\mu)_{m+n}\,(1+g_\mu-(g)')_n\,((k)+g_\mu)_n\,((f))_m\,m!\,n!}, \qquad (6\cdot2)$$

and
$$\Sigma_B(1/x) = \sum_{\nu=1}^{B}\Gamma\begin{bmatrix}(a)+b_\nu, (b)'-b_\nu, (g)+b_\nu, (h)-b_\nu\\(c)+b_\nu, (d)-b_\nu, (j)+b_\nu, (k)-b_\nu\end{bmatrix}x^{-b_\nu}$$
$$\times\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a)+b_\nu)_{2m+n}\,((g)+b_\nu)_{m+n}(1+b_\nu-(d))_n\,(1+b_\nu-(k))_{m+n}}{(1+b_\nu-(b)')_n\,(1+b_\nu-(k))_{m+n}\,((c)+b_\nu)_{2m+n}\,((j)+b_\nu)_{m+n}}$$
$$\times\frac{((e))_m\,y^m x^{-m-n}(-1)^{n(B+H-D-K)}}{((f))_m\,m!\,n!}$$
$$+\sum_{\nu=1}^{H}\Gamma\begin{bmatrix}(a)+h_\nu, (b)-h_\nu, (g)+h_\nu, (h)'-h_\nu\\(c)+h_\nu, (d)-h_\nu, (j)+h_\nu, (k)-h_\nu\end{bmatrix}x^{-h_\nu}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a)+h_\nu)_{m+n}}{(1+h_\nu-(h)')_n}$$
$$\times\frac{((b)-h_\nu)_{m-n}((g)+h_\nu)_n((e))_m(1+h_\nu-(k))_n\,y^m x^{-n}(-1)^{n(H-K)}}{((c)+h_\nu)_{m+n}((d)-h_\nu)_{m-n}((f))_m((j)+h_\nu)_n\,m!\,n!}. \qquad (6\cdot3)$$

Again we see that $g(s)$ is a function of bounded variation over part of the real axis, so that we can apply the Mellin transform to Theorem 3 of (2). Then

THEOREM 3. (i) $g(s) = \int_0^\infty x^{s-1}\Sigma_A(x)\,dx$ *if* $A+G+D+K > B+H+C+J$,

(ii) $g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx + \int_1^\infty x^{s-1}\Sigma_B(1/x)\,dx$

if $A+G+D+K = B+H+C+J$,

and (iii) $g(s) = \int_0^\infty x^{s-1}\Sigma_B(1/x)\,dx$ *if* $A+G+D+K < B+H+C+J$,

provided that

$$-\Re a_\nu,\ -\Re g_\nu < \Re s < \Re b_\nu,\ \Re h_\nu,$$

$\Re c_\nu$, $\Re d_\nu$, $\Re j_\nu$ *and* $\Re k_\nu \neq -N$ *for all* ν,

$$A+B+G+H \geqslant C+D+J+K$$

and $A+B+E < C+D+F+1$, *or* $A+B+E = C+D+F+1$ *and* $|x| < 1$.

As before, (iii) and (i) are equivalent under the transform $g(s) \to g(-s)$, and, in (ii) $\Sigma_B(1/x)$ is the analytic continuation of $\Sigma_A(x)$, for $1 < x < \infty$. In particular, if $B = D = H = K = 0$, then $A+G = C+J$, the second integral in (ii) is zero, and

$$g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx. \qquad (6\cdot4)$$

7. *Special cases of Theorem* 3. There are only three cases in which the series under the integral can be reduced to a simpler form. In these examples we assume that the conditions of Theorem 3 are satisfied.

First, $\Gamma[a+s, h-s]\,{}_1F_0[a+s;\,y]$ leads to

$$\Gamma\begin{bmatrix} a+s, h-s \\ h+a \end{bmatrix}(1-y)^{-a-s} = \int_0^\infty x^{s-1+a}(1+x+xy)^{-h-a}\,dx \quad ((\mathbf{1}), \S 6\cdot2\,(19)). \qquad (7\cdot1)$$

This can be extended to

$$\Gamma[a+s, h-s]\,{}_2F_1[a+s, e; f;\, y] = \Gamma[h+a] \times \int_0^\infty x^{s-1+a}(1+x)^{-h-a}\,{}_2F_1\left[h+a, e; f;\, \frac{xy}{1+x}\right]dx \qquad (7\cdot2)$$

and $\Gamma[a+s, h-s;\, h+a]\,{}_{E+1}F_F[a+s, (e);\, (f);\, y]$

$$= \int_0^\infty x^{s-1+a}(1+x)^{-h-a}\,{}_{E+1}F_F\left[a+h, (e);\, (f);\, \frac{xy}{1+x}\right]dx \qquad (7\cdot3)$$

for $E \leqslant F$.

A similar result from $\Gamma[a+s;\, j+s]\,{}_1F_0[a+s;\ ;\, y]$ is

$$\Gamma[a+s, j-a;\, j+s]\,(1-y)^{-a-s} = \int_0^1 x^{s-1+a}(1-xy-x)^{j-a-1}\,dx. \qquad (7\cdot4)$$

This can be extended to

$$\Gamma[a+s, j-a;\, j+s]\,{}_2F_1[a+s, e; f;\, y] = \int_0^1 x^{s-1+a}(1-x)^{j-a-1}\,{}_2F_1\left[1+a-j, e; f;\, \frac{xy}{1-x}\right]dx \qquad (7\cdot5)$$

and $\Gamma[a+s, j-a;\, j+s]\,{}_{E+1}F_F[a+s, (e);\, (f);\, y]$

$$= \int_0^1 x^{s-1+a}(1-x)^{j-a-1}\,{}_{E+1}F_F\left[1+a-j, (e);\, (f);\, \frac{xy}{1-x}\right]dx \qquad (7\cdot6)$$

for $E \leqslant F$.

The third result, which involves Bessel functions, is

$$\Gamma[g+s, c-g; c+s]\,{}_0F_1[; c+s; y]$$
$$= \int_0^1 x^{s-1+g}(1-x)^{c-g-1}\,{}_0F_1[; c-g; y(1-x)]\,dx \quad ((1), \S 7{\cdot}4\,(5)), \qquad (7{\cdot}7)$$

with the generalization

$$\Gamma[g+s, c-g; c+s]\,{}_EF_{F+1}[(e); c+s, (f); y]$$
$$= \int_0^1 x^{s-1+g}(1-x)^{c-g-1}\,{}_EF_{F+1}[(e); c-g, (f); y(1-x)]\,dx \qquad (7{\cdot}8)$$

for $E \leqslant F+2$.

The other special cases of Theorem 3 lead to double series which are neither separable nor summable.

8. *The fourth theorem.* Let

$$g(s) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \Gamma\begin{bmatrix}(a)+u_a m+v_a n+s, (b)+u_b m+v_b n-s\\ (c)+u_c m+v_c n+s, (d)+u_d m+v_d n-s\end{bmatrix} \alpha(m,n)\,y^m z^n, \qquad (8{\cdot}1)$$

where each u or v is unity or zero according as each parameter a, b, c or d occurs within the m or n series or not. $\alpha(m, n)$ is a function independent of s such that the double series in m and n is absolutely and uniformly convergent in both y and z. Also let

$$\Sigma_A(x) = \sum_{\mu=1}^{A}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \Gamma\begin{bmatrix}(a)'+u_a m+v_a n-a_\mu-u_\mu m-v_\mu n-p,\\ (c)+u_c m+v_c n-a_\mu-u_\mu m-v_\mu n-p,\end{bmatrix.$$
$$\left.\begin{matrix}(b)+u_b m+v_b n+a_\mu+u_\mu m+v_\mu n+p\\ (d)+u_d m+v_d n+a_\mu+u_\mu m+v_\mu n+p\end{matrix}\right] \alpha(m,n)\,y^m z^n x^{a_\mu+u_\mu m+v_\mu n+p}, \qquad (8{\cdot}2)$$

and

$$\Sigma_B(1/x) = \sum_{\nu=1}^{B}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \Gamma\begin{bmatrix}(a)+u_a m+v_a n+b_\nu+u_\nu m+v_\nu n+p,\\ (c)+u_c m+v_c n+b_\nu+u_\nu m+v_\nu n+p,\end{bmatrix.$$
$$\left.\begin{matrix}(b)'+u_b m+v_b n-b_\nu-u_\nu m-v_\nu n-p\\ (d)+u_d m+v_d n-b_\nu-u_\nu m-v_\nu n-p\end{matrix}\right] \alpha(m,n)\,y^m z^n x^{-b_\nu-u_\nu m-v_\nu n-p}. \qquad (8{\cdot}3)$$

Then, since $g(s)$ is again a function of bounded variation we can apply the Mellin transform to Theorem 4 of (2), and we find

THEOREM 4. (i) $g(s) = \int_0^\infty x^{s-1}\Sigma_A(x)\,dx$ *if* $A+D > B+C$,

(ii) $g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx + \int_1^\infty x^{s-1}\Sigma_B(1/x)\,dx$ *if* $A+D = B+C$,

and (iii) $g(s) = \int_0^\infty x^{s-1}\Sigma_B(1/x)\,dx$ *if* $A+D < B+C$,

provided that $A+B \geqslant C+D$, *and that* $g(s)$ *is an absolutely and uniformly convergent series in* y *and* z.

Again (iii) and (i) are equivalent under the transform $g(s) \to g(-s)$, and in (ii) the sum $\Sigma_B(1/x)$ is the analytic continuation of $\Sigma_A(x)$ for $1 < x < \infty$. In particular, if $B = D = 0$, then $A = C$, the second integral in (ii) is zero, and

$$g(s) = \int_0^1 x^{s-1}\Sigma_A(x)\,dx. \tag{8·4}$$

This theorem contains as special cases Theorems 1, 2 and 3, and other transforms of Meijer's G-function, for example (1), §7·5 (30), (31) and (32).

9. *Examples on Theorem* 4. The double series in $g(s)$ can assume many forms. For example, it may be one of the Appell series, ((1), p. 384), though none of the integrals which arise assumes a simple form. Alternatively, the double series in $g(s)$ may be separable into two simple series in m and n respectively. In a few cases the resulting triple series under the integral is separable into double series, and occasionally this integrand will reduce even further. Thus $\Gamma[a+s, b-s]\,{}_1F_0\,[a+s;\, y]\,{}_1F_0\,[b-s;\, z]$ leads to

$$\Gamma[a+s, b-s;\, a+b]\,(1-y)^{-a-s}\,(1-z)^{s-b} = \int_0^\infty x^{s-1+a}(1+x-y-xz)^{-a-b}\,dx \tag{9·1}$$

and $\Gamma[a+s, b-s;\, a+b]\,{}_1F_1\,[a+s;\, e;\, y]\,{}_1F_1\,[b-s;\, f;\, z]$

$$= \int_0^\infty x^{s-1+a}(1+x)^{-a-b} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{x^m y^m z^n (a+b)_{m+n}}{(1+x)^{m+n}\, m!\, n!\, (e)_m\, (f)_n}\,dx. \tag{9·2}$$

This double series under the integral is one of Horn's hypergeometric series ((1), p. 385).

Another related result is

$$\Gamma[a+s, c-a;\, c+s]\,{}_1F_1\,[a+s;\, f;\, y]\,{}_1F_1\,[e;\, c+s;\, z]$$

$$= \int_0^\infty x^{s-1+a}(1+x)^{c-a-1} \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(e)_n\, y^m z^n x^m (-1)^n\, (1+a-c)_{m-n}}{(f)_m\, m!\, n!\, (1+x)^{m-n}}\,dx, \tag{9·3}$$

and there are many other similar results involving Bessel functions, confluent hypergeometric functions and other more general hypergeometric functions.

Results similar to Theorem 4 will hold for triple and multiple series in general, in $g(s)$, but these results are so complicated that it is unlikely that they can have any practical application.

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