

UNBOUNDED DERIVATIONS IN ALGEBRAS ASSOCIATED WITH MONOTHETIC GROUPS

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Abstract

Given an infinite, compact, monothetic group G we study decompositions and structure of unbounded derivations in a crossed product C^* -algebra $C(G) \rtimes \mathbb{Z}$ obtained from a translation on G by a generator of a dense cyclic subgroup. We also study derivations in a Toeplitz extension of the crossed product and the question whether unbounded derivations can be lifted from one algebra to the other.

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1. Introduction

Derivations naturally arise in studying differentiable manifolds, in representation theory of Lie groups and in their noncommutative analogs. They also appear in mathematical aspects of quantum mechanics, in particular in quantum statistical physics. Additionally, derivations are important in analyzing amenability and other structures of operator algebras. Good overviews are in [1] and also in [15].

In this paper we study classification and decompositions of unbounded derivations in C^* -algebras associated to an infinite, compact, monothetic group G , which, by definition, is a Hausdorff topological group with a dense cyclic subgroup. A group translation on G by a generator of a cyclic subgroup is a minimal homeomorphism and one algebra associated with G is the crossed product C^* -algebra $B := C(G) \rtimes \mathbb{Z}$ determined by the translation. This algebra can be naturally represented in the ℓ^2 -Hilbert space of the full orbit. If we consider the analogous algebra on the forward orbit only, we obtain a Toeplitz extension A of the algebra B . When the group is totally disconnected those algebras are precisely Bunce–Deddens and Bunce–Deddens–Toeplitz algebras, which were considered in [10].

The main objects studied in this paper are unbounded derivations $d : \mathcal{A} \rightarrow A$, which are defined on a subalgebra \mathcal{A} of polynomials of the generators of A . Similarly, we study derivations $\delta : \mathcal{B} \rightarrow B$, where \mathcal{B} is the image of \mathcal{A} under the quotient map

$A \rightarrow A/\mathcal{K} = B$. The first of the main results of this paper is that any derivation in those algebras can be uniquely decomposed into a sum of a certain special derivation and an approximately inner derivation. The special derivations are not approximately inner, and can be explicitly described. Analogous classification results were described in [7, 9, 10]. While these papers share similar ideas and contain similar classification results for derivations, the current paper's different settings required numerous modifications. Notable differences include new types of invariant derivations in Lemma 3.9 that were not present in previous papers, and Section 4 on lifting of derivations is substantially more complicated and is using new, different techniques than before.

It turns out that any derivation $d : \mathcal{A} \rightarrow A$ preserves the ideal of compact operators \mathcal{K} and consequently defines a factor derivation $[d] : \mathcal{B} \rightarrow B$ in B . It is an interesting and nontrivial problem to describe properties of the map $d \mapsto [d]$. For any C^* -algebra it is easy to see that bounded derivations preserve closed ideals and so they define derivations on quotients. It was proved in [13] that for bounded derivations and separable C^* -algebras the above map is onto, that is derivations can be lifted from quotients. In nonseparable cases this is not true in general. We prove here that lifting unbounded derivations from B to A is always possible when G is totally disconnected, answering positively a conjecture in [10]. However, we give a simple counterexample of a special derivation in the algebra B for $G = \mathbb{T}^1$ that cannot be lifted to a derivation in the algebra A . Instead, we conjecture that for any compact, infinite, monothetic group approximately inner derivations in B can be lifted to approximately the inner derivation in A .

The paper is organized as follows. In Section 2 we review monothetic groups and discuss their properties. We also describe a crossed product C^* -algebra that is associated to a monothetic group and that algebra Toeplitz extension, as well as discuss a Toeplitz map from one algebra to another. In Section 3 we classify all unbounded derivations on polynomial domains in the C^* -algebras from Section 2. Finally, in Section 4 we consider lifting derivations from a crossed product C^* -algebra to its Toeplitz extension. We prove that all derivations can be lifted for totally disconnected, compact, infinite, monothetic groups and provide an example that shows that not all derivations can be lifted in general.

2. Monothetic groups and associated C^* -algebras

2.1. Monothetic groups. A topological (Hausdorff) group is called *monothetic* if it has a dense cyclic subgroup. Weil observed, Theorem 19 of [12], that if G is a locally compact monothetic group, then $G \cong \mathbb{Z}$ or G is compact. In this paper we only consider the case of compact G . It follows immediately that G is Abelian and separable. We first describe the structure of such groups following [5]. The key tool is the character (dual) group and Pontryagin duality, which translates properties of groups into properties of their duals.

Let S^1 be the unit circle:

$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$

and let \widehat{G} denote the dual group G , the group of continuous homomorphisms from G to S^1 equipped with compact-open topology. It is well known that if G is compact then \widehat{G} is discrete.

We typically use additive notation for an abelian group, however, we use multiplicative notation for the dual group. Given a monothetic group G , let x_1 be a generator of a dense cyclic subgroup, and we set $x_n = nx_1$ for $n \in \mathbb{Z}$, so that $x_0 := 0$ is the neutral element of G . Then we can identify the dual group \widehat{G} of G with a discrete subgroup of S^1 via the map given by

$$\widehat{G} \ni \chi \mapsto \chi(x_1) \in S^1.$$

Conversely, using Pontryagin duality, if H is a discrete subgroup of S^1 , then H is the dual group of a compact monothetic group, namely \widehat{H} , see [5].

To better understand the structure of monothetic groups we look at the torsion subgroup of its dual group. Given a monothetic G , the torsion subgroup of \widehat{G}_{tor} of \widehat{G} is given by

$$\widehat{G}_{\text{tor}} = \{\chi \in \widehat{G} : \chi^n = 1 \text{ for some } n \in \mathbb{N}\}.$$

There are two extreme cases: we say \widehat{G} is of pure torsion if $\widehat{G} = \widehat{G}_{\text{tor}}$. We also say \widehat{G} is torsion free if $\widehat{G}_{\text{tor}} = \{0\}$. The following statements describe the basic properties of monothetic groups. We provide short or outlined proofs with references. A good, concise book on Pontryagin duality is [12].

First we look at the case of torsion free \widehat{G} .

PROPOSITION 2.1. *Let G be a compact monothetic group. G is connected if and only if \widehat{G} is torsion free.*

PROOF. This is [12, Corollary 4 of Theorem 30], which only requires G to be compact and Abelian. \square

We have the following remarkable result proved in [5].

THEOREM 2.2. *Every connected compact separable Abelian topological group is monothetic.*

The n -dimensional torus, $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ is an example of a compact, connected, separable, Abelian group and thus by Theorem 2.2 is monothetic. Consider an element

$$x_1 = (\theta_1, \dots, \theta_n)$$

of \mathbb{T}^n . Then the cyclic subgroup generated by x_1 is dense in \mathbb{T}^n if and only if $\{1, \theta_1, \dots, \theta_n\}$ are linearly independent over \mathbb{Z} , see for example [6].

Next we consider the case when \widehat{G} is of pure torsion.

PROPOSITION 2.3. *Let G be a compact monothetic group. G is totally disconnected if and only if \widehat{G} is of pure torsion.*

PROOF. This result follows from [12, Corollary 1 of Theorem 30], since an element of the discrete group \widehat{G} is compact (that is the smallest closed subgroup containing it is compact) if and only if it has finite order. \square

Before we state the next structural result we need to introduce odometers. Further details on odometers can be found in [3]. The standard definition of an odometer (that inspired the name) uses a sequence of positive integers $b := (b_m)_{m \in \mathbb{N}}$ such that $b_m \geq 2$ for all m , called a multibase. The odometer is then identified (as a set) with the direct product:

$$G(b) := \prod_m \mathbb{Z}/b_m\mathbb{Z},$$

but addition is defined with the carry over rule. Equipped with the product topology $G(b)$ becomes a compact totally disconnected topological group. It is easy to see that the cyclic subgroup generated by $x_1 = (1, 0, 0, 0, \dots)$ is dense and so $G(b)$ is a monothetic group.

An alternative representation of the odometer $G(b)$ uses scales, and this is the description that is used in the proof of Theorem 4.3. Let $s = (s_m)_{m \in \mathbb{N}}$ be a sequence of positive integers such that s_m divides s_{m+1} and $s_m < s_{m+1}$. There are natural homomorphisms between the consecutive finite cyclic groups $\mathbb{Z}/s_{m+1}\mathbb{Z} \rightarrow \mathbb{Z}/s_m\mathbb{Z}$, namely congruence modulo s_m . Thus the inverse limit:

$$G_s = \varprojlim_{m \in \mathbb{N}} \mathbb{Z}/s_m\mathbb{Z}$$

is well defined as the subset of the countable product $\prod_m \mathbb{Z}/s_m\mathbb{Z}$ consisting of sequences (y_1, y_2, y_3, \dots) such that $y_{m+1} \equiv y_m \pmod{s_m}$. Addition in this representation is coordinate-wise, modulo s_m in each coordinate m . G_s becomes a topological group when endowed with the product topology over the discrete topologies in $\mathbb{Z}/s_m\mathbb{Z}$. Obviously, with our assumptions, this group is infinite because s is unbounded.

The relation between the two definitions of an odometer is as follows. Given a multibase $b = (b_m)_{m \in \mathbb{N}}$ define a scale $s = (s_m)_{m \in \mathbb{N}}$ by $s_1 = b_1, s_2 = b_1 b_2, s_3 = b_1 b_2 b_3$ and so on. Equivalently,

$$s_1 = b_1, \quad b_n = \frac{s_n}{s_{n-1}} \quad \text{for } n > 1.$$

Then the map

$$G(b) \ni (k_1, k_2, k_3, \dots) \mapsto (k_1, k_1 + k_2 b_1, k_1 + k_2 b_1 + k_3 b_1 b_2, \dots) \in G_s$$

gives an isomorphism of the groups. In the scales' representation of odometers, the generator x_1 of a cyclic subgroup is given by $x_1 = (1, 1, 1, 1, \dots)$.

With the above definitions it is not transparent when two odometers are isomorphic, so we describe yet another way to define odometers that we used in [10]. A *supernatural number* N is defined as the formal product:

$$N = \prod_{p \text{ prime}} p^{\epsilon_p}, \quad \epsilon_p \in \{0, 1, \dots, \infty\}.$$

If $\sum \epsilon_p < \infty$ then N is said to be a finite supernatural number (a regular natural number), otherwise it is said to be infinite. If

$$N' = \prod_{p \text{ prime}} p^{\epsilon'_p}$$

is another supernatural number, then their product is given by

$$NN' = \prod_{p \text{ prime}} p^{\epsilon_p + \epsilon'_p}.$$

A supernatural number N is said to divide M if $M = NN'$ for some supernatural number N' , or equivalently, if $\epsilon_p(N) \leq \epsilon_p(M)$ for every prime p .

Given a supernatural number N let \mathcal{J}_N be the set of finite divisors of N :

$$\mathcal{J}_N = \{j : j|N, j < \infty\}.$$

Then (\mathcal{J}_N, \leq) is a directed set where $j_1 \leq j_2$ if and only if $j_1|j_2|N$. Consider the collection of cyclic groups $\{\mathbb{Z}/j\mathbb{Z}\}_{j \in \mathcal{J}_N}$ and the family of group homomorphisms

$$\begin{aligned} \pi_{ij} : \mathbb{Z}/j\mathbb{Z} &\rightarrow \mathbb{Z}/i\mathbb{Z}, & j \geq i \\ \pi_{ij}(z) &= z \pmod i \end{aligned}$$

satisfying

$$\pi_{ik} = \pi_{ij} \circ \pi_{jk} \quad \text{for all } i \leq j \leq k.$$

Then the inverse limit of this system can be denoted as

$$\mathbb{Z}/N\mathbb{Z} := \lim_{\leftarrow j \in \mathcal{J}_N} \mathbb{Z}/j\mathbb{Z} = \left\{ (z_j) \in \prod_{j \in \mathcal{J}_N} \mathbb{Z}/j\mathbb{Z} : \pi_{ij}(z_j) = z_i \right\}.$$

In particular, if N is finite the above definition coincides with the usual meaning of the symbol $\mathbb{Z}/N\mathbb{Z}$, while if $N = p^\infty$ for a prime p , then the above limit is equal to \mathbb{Z}_p , the ring of p -adic integers, see for example [14].

Given a scale $s = (s_m)_{m \in \mathbb{N}}$ we define the corresponding supernatural number N to be the ‘limit’ of s_m :

$$N = \lim_{m \rightarrow \infty} s_m, \tag{2-1}$$

in the sense that each prime exponent $\epsilon_p(N)$ of N is defined to be the supremum of the prime exponents $\epsilon_p(s_m)$, $m \in \mathbb{N}$. It follows that s_m are divisors of N and for every $j \in \mathcal{J}_N$ there is a natural number $m(j)$ such that $j|s_{m(j)}$. Consequently, a sequence $(z_j) \in \lim_{\leftarrow j \in \mathcal{J}_N} \mathbb{Z}/j\mathbb{Z}$ is completely determined by the subsequence $(z_{s_m}) \in \lim_{\leftarrow m \in \mathbb{N}} \mathbb{Z}/s_m\mathbb{Z}$, which gives an isomorphism $\mathbb{Z}/N\mathbb{Z} \cong G_s$. It turns out that odometers are classified by the supernatural number N , see [3]. As before,

$$x_1 = (1, 1, 1, 1, \dots) \in \mathbb{Z}/N\mathbb{Z}$$

generates a dense cyclic subgroup.

In general, we have the following simple consequence of the Chinese remainder theorem: if $N = \prod_{\substack{p \text{ prime} \\ \epsilon_p \neq 0}} p^{\epsilon_p}$, then

$$\mathbb{Z}/N\mathbb{Z} \cong \prod_{\substack{p \text{ prime} \\ \epsilon_p \neq 0}} \mathbb{Z}/p^{\epsilon_p}\mathbb{Z}.$$

Since the space $\mathbb{Z}/N\mathbb{Z}$ is a compact Abelian topological group, it has a unique normalized Haar measure μ . Also, if N is an infinite supernatural number then $\mathbb{Z}/N\mathbb{Z}$ is a Cantor set [16].

We are now ready to state the next structural result about compact monothetic groups.

PROPOSITION 2.4. *G is a compact, totally disconnected, monothetic group if and only if it is an odometer. In particular, there exists an infinite supernatural number N such that*

$$G \cong \mathbb{Z}/N\mathbb{Z} \cong \prod_i \mathbb{Z}/p_i^{\epsilon_i}\mathbb{Z},$$

where $N = \prod_{i=1}^{\infty} p_i^{\epsilon_i}$.

PROOF. Let G be a compact totally disconnected monothetic group. In [5], between Theorems II' and III on pages 256–257, the authors show that G is isomorphic to a direct product of groups G_{p_i} where p_i runs over all primes and where G_{p_i} isomorphic to the zero group, the cyclic group of order $p_i^{\epsilon_i}$ for some ϵ_i or the group of p -adic integers, the last case corresponds to $\epsilon_i = \infty$. \square

In general, for arbitrary \widehat{G} we have the following structure for compact monothetic groups.

PROPOSITION 2.5. *Let G be a compact monothetic group. If $G_0 \leq G$ is the connected component of the neutral element 0, then G_0 is a connected separable compact Abelian group and G/G_0 is a totally disconnected monothetic group. Moreover, there are natural isomorphisms:*

$$(\widehat{G}/\widehat{G_0}) \cong \widehat{G_{\text{tor}}} \quad \text{and} \quad \widehat{G_0} \cong \widehat{G}/\widehat{G_{\text{tor}}}.$$

PROOF. This proposition is not formally stated but appears as a note in [5], see also [12, Corollary 3 of Theorem 30]. The first part follows from the previous propositions. Recall that the annihilator $A(G_0)$ of G_0 is given by

$$A(G_0) = \{\chi \in \widehat{G} : \chi(g) = 1, \text{ for all } g \in G_0\}.$$

By Pontryagin duality, Theorem 27 of [12],

$$A(G_0) \cong (\widehat{G}/\widehat{G_0}).$$

Notice the right-hand side of the equation is Abelian, discrete and of pure torsion. Thus given $\chi \in A(G_0)$, it defines a character class

$$[\chi] : G/G_0 \rightarrow S^1.$$

Therefore,

$$[\chi] \in (\widehat{G}/\widehat{G_0}) = (\widehat{G}/\widehat{G_0})_{\text{tor}},$$

and hence χ has finite order and thus $A(G_0) \leq \widehat{G_{\text{tor}}}$. Let $\chi \in \widehat{G_{\text{tor}}}$, then

$$\chi|_{G_0} \in (\widehat{G_0})_{\text{tor}} = \{1\}$$

since G_0 is connected and thus $\chi \in A(G_0)$. Therefore, we have $A(G_0) = \widehat{G}_{\text{tor}}$ and hence

$$(\widehat{G}/\widehat{G}_0) \cong \widehat{G}_{\text{tor}}.$$

The second isomorphism relation follows from Pontryagin duality:

$$\widehat{G}_0 \cong \widehat{G}/A(G_0)$$

and the proof is complete. \square

2.2. Minimal systems. By a *topological dynamical system* (X, φ) , we mean a topological space X and a continuous map $\varphi : X \rightarrow X$, see [6]. A topological dynamical system (X, φ) is called *topologically transitive* if there exists a point $x \in X$ such that its orbit $\{\varphi^n(x)\}_{n \in \mathbb{Z}}$ is dense in X . (X, φ) is called *minimal* if every orbit is dense in X . We say and write φ is minimal for brevity.

Other equivalent characterizations of minimal maps is as follows. A set $A \subseteq X$ is called φ -invariant if $\varphi(A) \subseteq A$. Then, φ is minimal if X does not contain any nonempty, proper, closed φ -invariant subset. If in addition X is assumed to be Hausdorff and compact, then a minimal map φ must be surjective. Moreover, if (X, φ) is topologically transitive then there is no φ -invariant nonconstant continuous function on X .

Suppose that G is a compact monothetic group with x_1 the generator of a dense cyclic subgroup. Then we define the map $\varphi : G \rightarrow G$ by the formula:

$$\varphi(x) = x + x_1.$$

It follows that (G, φ) is a minimal system. Let us remark that for metrizable spaces minimal, equicontinuous, dynamical systems coincide with translations by a generator of a dense cyclic subgroup of a compact monothetic group, see [11, Theorem 2.4.2].

We now turn our attention to the algebras that are present in this paper. Let G be a compact infinite monothetic group, $C(G)$ the complex-valued continuous functions on G and μ a normalized Haar measure on G . Recall the notation for the elements of the cyclic subgroup generated by x_1 :

$$x_n = \varphi^n(0) = nx_1,$$

for $n \in \mathbb{Z}$. The set $\{x_n\}_{n \in \mathbb{Z}}$ is the full orbit of 0 under φ and $\{x_n\}_{n \geq 0}$ is the forward orbit. As mentioned above, since φ is a minimal homeomorphism, the forward orbit $\{x_n\}_{n \geq 0}$ is dense in G .

Consider the algebra of trigonometric polynomials on G :

$$\mathcal{F} = \left\{ \sum_n c_n \chi_n : \chi_n \in \widehat{G}, \text{ finite sums} \right\}.$$

We state below two simple but useful properties of \mathcal{F} that we will need later in the paper. First we have the following observation.

PROPOSITION 2.6. *Let $f \in \mathcal{F}$. Then $\int_G f d\mu = 0$ if and only if there is a trigonometric polynomial $g \in \mathcal{F}$ such that*

$$f = g \circ \varphi - g.$$

PROOF. If $f \in \mathcal{F}$ then f has the following decomposition:

$$f = \sum_j c_j \chi_j,$$

where χ_j are characters on G . Notice that

$$\int_G \chi_j d\mu = \begin{cases} 1 & \text{if } \chi_j = 1, \\ 0 & \text{else,} \end{cases}$$

which means that $\int_G f d\mu = 0$ if and only if $\chi_j \neq 1$ for all j . Let χ be a nontrivial character, then the goal is to find a $g \in \mathcal{F}$ such that

$$\chi(x) = g(x + x_1) - g(x). \tag{2-2}$$

Notice that for a nontrivial character we must have $\chi(x_1) \neq 1$. Otherwise, if $\chi(x_1) = 1$, then $\chi(x_n) = \chi(nx_1) = 1$, which in turn implies that $\chi = 1$ on a dense set, and thus $\chi \equiv 1$, which is a contradiction. Therefore, we can choose

$$g(x) = \frac{\chi(x)}{\chi(x_1) - 1},$$

which clearly satisfies equation (2-2). Now that we can find a function $g(x)$ that solves equation (2-2) for a nontrivial character, we just take finite linear combinations of such functions for the general case of a trigonometric polynomial, and thus completing the proof. □

Next we describe another useful property of the space \mathcal{F} .

PROPOSITION 2.7. *For any nonzero $n \in \mathbb{Z}$, there exists a trigonometric polynomial $f \in \mathcal{F}$ such that*

$$f - f \circ \varphi^n \neq 0.$$

PROOF. The key property of the characters is that they separate points of G , see [12, Theorem 14]. Therefore, if $n \neq 0$, we can pick χ such that

$$\chi(x_1) \neq \chi(x_{n+1}) = \chi(\varphi^n(x_1)).$$

As in the previous proposition, the general case is handled by linearity and the proof is complete. □

2.3. C*-algebras. Let G be an infinite, compact, monothetic group. We will now describe two types of C*-algebras that can be naturally associated with such groups. They are defined as concrete C*-algebras of operators in the following Hilbert spaces. The first Hilbert space is the ℓ^2 space of the full orbit:

$$H = \ell^2(\{x_l\}_{l \in \mathbb{Z}}),$$

which is naturally isomorphic with $\ell^2(\mathbb{Z})$. Let $\{E_l\}_{l \in \mathbb{Z}}$ be the canonical basis in H . The second Hilbert space is the ℓ^2 space of the forward orbit:

$$H_+ = \ell^2(\{x_k\}_{k \in \mathbb{Z}_{\geq 0}}),$$

which is naturally isomorphic with the Hilbert space $\ell^2(\mathbb{Z}_{\geq 0})$. We also let $\{E_k^+\}_{k \in \mathbb{Z}_{\geq 0}}$ be the canonical basis on H_+ .

The C^* -algebras associated to G are defined using the following operators. Let $V : H \rightarrow H$ be the shift operator on H :

$$VE_l = E_{l+1}.$$

We also need the unilateral shift operator on H_+ :

$$UE_k^+ = E_{k+1}^+.$$

Notice that V is a unitary while U is an isometry. Then

$$[U^*, U] = P_0,$$

where P_0 is the orthogonal projection onto the one-dimensional subspace spanned by E_0^+ .

For a continuous function $f \in C(G)$ we define two operators $M_f : H \rightarrow H$ and $M_f^+ : H_+ \rightarrow H_+$ via formulas:

$$M_f E_l = f(x_l) E_l \quad \text{and} \quad M_f^+ E_k^+ = f(x_k) E_k^+.$$

They are diagonal multiplication operators on H and H_+ respectively. Due to the density of the orbit $\{x_k\}_{k \in \mathbb{Z}_{\geq 0}}$, immediately

$$\|M_f\| = \|M_f^+\| = \sup_{l \in \mathbb{Z}} |f(x_l)| = \sup_{k \in \mathbb{Z}_{\geq 0}} |f(x_k)| = \sup_{x \in G} |f(x)| = \|f\|_{\infty}.$$

The algebras of operators generated by the M_f or by the M_f^+ are thus isomorphic to $C(G)$ so they carry all the information about the space G , while operators U and V reflect the dynamics φ on G . The relation between those operators is

$$VM_f V^{-1} = M_{f \circ \varphi}.$$

Similarly,

$$UM_f^+ = M_{f \circ \varphi}^+ U.$$

There is also another, less obvious relation between U and the M_f^+ , namely,

$$M_f^+ P_0 = P_0 M_f^+ = f(x_0) P_0. \quad (2-3)$$

We define the algebra B to be the C^* -algebra generated by operators V and M_f :

$$B = C^*\{V, M_f : f \in C(G)\}.$$

We claim that B is isomorphic with the crossed product algebra:

$$B \cong C(G) \rtimes_{\varphi} \mathbb{Z}.$$

Indeed, observe that \mathbb{Z} is amenable, the action of \mathbb{Z} on G given by φ is a free action, and φ is a minimal homeomorphism, and thus the crossed product is simple and equal

to the reduced crossed product, see [4]. Clearly, the operators V and M_f define a representation of $C(G) \rtimes_{\varphi} \mathbb{Z}$, which must be isomorphic to it, by simplicity of the crossed product.

The algebra B has a natural dense $*$ -subalgebra \mathcal{B} of polynomials in V, V^{-1} , and the M_{χ} , where χ is a character of G . Equivalently,

$$\mathcal{B} = \left\{ \sum_n V^n M_{f_n} : f_n \in \mathcal{F}, \text{ finite sums} \right\}.$$

Next we define the other algebra that is of the main interest in this paper, a Toeplitz extension of B . We define the algebra A to be the C^* -algebra generated by operators U and M_f^+ :

$$A = C^*\{U, M_f^+ : f \in C(G)\}.$$

To proceed further we need the following diagonal label operators on H and H_+ , respectively,

$$\mathbb{L}E_l = lE_l \quad \text{and} \quad \mathbb{K}E_k^+ = kE_k^+.$$

If $\{a(k)\}_{k=0}^{\infty}$ is a bounded sequence, then by the functional calculus $a(\mathbb{K})$ is a bounded diagonal operator given by

$$a(\mathbb{K})E_k = a(k)E_k,$$

and all bounded diagonal operators can be expressed this way. There is a similar representation for $a(\mathbb{L})$. Moreover, we have the following commutation relation:

$$a(\mathbb{K})U = Ua(\mathbb{K} + I),$$

and there is also similar relation for V and $a(\mathbb{L})$. Typically, the diagonal operators arise through the following, useful relation:

$$U^n(U^*)^n = P_{\geq n},$$

where $P_{\geq n}$ is the orthogonal projection onto the span of E_k^+ with $k \geq n$.

The algebra A has a natural dense $*$ -subalgebra \mathcal{A} of polynomials in U, U^* , and the M_{χ}^+ , where χ is a character of G , which can be equivalently described as follows, using [9, Proposition 3.1] and also Proposition 2.11 below:

$$\mathcal{A} = \left\{ \sum_{n \geq 0} U^n (a_n^+(\mathbb{K}) + M_{f_n^+}^+) + \sum_{n \geq 1} (a_n^-(\mathbb{K}) + M_{f_n^-}^+) (U^*)^n : f_n^{\pm} \in \mathcal{F}, a_n^{\pm}(k) \in c_{00}(\mathbb{Z}_{\geq 0}) \right\},$$

where the sums above are finite sums and $c_{00}(\mathbb{Z}_{\geq 0})$ is the space of sequences that are eventually zero. Notice that if $a \in \mathcal{A}$ and $x \in c_{00}(\mathbb{Z}_{\geq 0}) \subseteq H_+$, then $ax \in c_{00}(\mathbb{Z}_{\geq 0})$, an observation that is often used below.

Next we establish the key relation between the two algebras A and B . Let $P_+ : \mathcal{H} \rightarrow \mathcal{H}_+$ be the following map from \mathcal{H} onto \mathcal{H}_+ given by

$$P_+ E_k = \begin{cases} E_k^+ & \text{if } k \geq 0, \\ 0 & \text{if } k < 0. \end{cases}$$

We also need another map $s : \mathcal{H}_+ \rightarrow \mathcal{H}$ given by

$$sE_k^+ = E_k.$$

Define the map $T : B(\mathcal{H}) \rightarrow B(\mathcal{H}_+)$, between the spaces of bounded operators on \mathcal{H} and \mathcal{H}_+ , in the following way: given $b \in B(\mathcal{H})$

$$T(b) = P_+bs.$$

T is known as a Toeplitz map. It has the following properties.

PROPOSITION 2.8. *Let T be the Toeplitz map defined above. Then:*

- (1) $T(I_{\mathcal{H}}) = I_{\mathcal{H}_+}$;
- (2) $T(bV^n) = T(b)U^n$ and $T(V^{-n}b) = (U^*)^nT(b)$ for $n \geq 0$ and all $b \in B(\mathcal{H})$;
- (3) $T(bM_f) = T(b)M_f^+$ and $T(M_f b) = M_f^+T(b)$ for all $f \in C(X)$ and all $b \in B(\mathcal{H})$.

Consequently, it follows that T maps B to A and \mathcal{B} to \mathcal{A} .

PROOF. For the first statement, if $h \in \mathcal{H}_+$ then we have the following calculation:

$$T(I_{\mathcal{H}})h = (P_+sI_{\mathcal{H}})h = P_+h = h = I_{\mathcal{H}_+}h.$$

For the second statement we apply $T(bV^n)$ to the basis elements E_k^+ of \mathcal{H}_+ . Then

$$T(bV^n)E_k^+ = (P_+bV^n)s(E_k^+) = (P_+b)E_{k+n} = (P_+bs)U^nE_k^+ = T(b)U^nE_k^+.$$

A similar calculation shows the other equality $T(V^{-n}b) = (U^*)^nT(b)$. Finally, for the last statement, we apply $T(bM_f)$ and $T(M_f b)$ to the basis elements to obtain

$$\begin{aligned} T(bM_f)E_k^+ &= (P_+bM_f)s(E_k^+) = (P_+b)M_fE_k = (P_+b)f(x_k)E_k = (P_+bs)M_f^+E_k^+ \\ &= T(b)M_f^+E_k^+. \end{aligned}$$

This completes the proof. □

The next result describes the main relation between the two algebras A and B .

PROPOSITION 2.9. *The ideal of compact operators \mathcal{K} in $B(\mathcal{H}_+)$ is an ideal in A . Moreover, B is the factor algebra:*

$$B \cong A/\mathcal{K}$$

and

$$B \ni b \mapsto T(b) + \mathcal{K} \in A/\mathcal{K} \tag{2-4}$$

is an isomorphism.

PROOF. Notice first that

$$P_0 = I - UU^* \in A.$$

It follows that the operators $P_{k,l} := U^kP_0(U^*)^l$ are also in A . Thus, all finite rank operators with respect to the basis $\{E_k^+\}$ belong to A as they are finite linear combinations of $P_{k,l}$. Moreover, since all compact operators in $B(\mathcal{H}_+)$ are norm limits

of these finite rank operators and A is a C^* -algebra, it follows that $\mathcal{K} \subseteq A$. It is clear that \mathcal{K} is an ideal in A . Verifying that the map given by equation (2-4) is an isomorphism, is analogous to the proof of Theorem 2.3 in [8]. We give a short outline. It is clear that T is linear. Checking that the map preserves products requires one to verify that

$$T(b_1)T(b_2) - T(b_1b_2) \in \mathcal{K}$$

for all $b_1, b_2 \in B$. First, by a straightforward calculation, we check this relation on the polynomials in \mathcal{B} . Then, using the fact that \mathcal{B} is dense in B and the continuity of the norm, we conclude the above inclusion is true for all $b_1, b_2 \in B$, making $T(b) + \mathcal{K}$ a homomorphism. By definition of the Toeplitz map T , we have $T(b) + \mathcal{K} = 0$ if and only if $b = 0$, thus the map is injective. Proposition 2.8 gives

$$T(V) = U, \quad T(V^{-1}) = U^*, \quad \text{and} \quad T(M_f) = M_f^+,$$

and hence U, U^* , and M_f^+ are in the range of T . Therefore, by the usual C^* -algebra argument, the map is surjective, completing the proof. \square

It follows from the two previous propositions that we have the following identification.

COROLLARY 2.10. *Under the isomorphism given by equation (2-4), \mathcal{B} is the factor algebra:*

$$\mathcal{B} \cong [\mathcal{A}].$$

For future reference we notice the following formulas:

$$[U] = V, \quad [M_f^+] = M_f, \quad [\mathbb{K}] = \mathbb{L},$$

and also, for every $b \in B$:

$$[T(b)] = b.$$

Useful tools in classifying derivations on A and B are one-parameter groups of automorphisms of A and B , respectively, that are given by the following equations:

$$\begin{aligned} \rho_\theta^{\mathbb{K}}(a) &= e^{i\theta\mathbb{K}} a e^{-i\theta\mathbb{K}} \quad \text{for } a \in A, \\ \rho_\theta^{\mathbb{L}}(b) &= e^{i\theta\mathbb{L}} b e^{-i\theta\mathbb{L}} \quad \text{for } b \in B, \end{aligned}$$

where $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. We have the following formulas:

$$\rho_\theta^{\mathbb{K}}(U) = e^{i\theta} U, \quad \rho_\theta^{\mathbb{K}}(a(\mathbb{K})) = a(\mathbb{K}),$$

and similarly for $\rho_\theta^{\mathbb{L}}$. It immediately follows that $\rho_\theta^{\mathbb{K}} : \mathcal{A} \rightarrow \mathcal{A}$ and that $\rho_\theta^{\mathbb{L}} : \mathcal{B} \rightarrow \mathcal{B}$.

The automorphisms define natural \mathbb{Z} -gradings on A and B given by the spectral subspaces:

$$\begin{aligned} A_n &= \{a \in A : \rho_\theta^{\mathbb{K}}(a) = e^{in\theta} a\}, \\ B_n &= \{b \in B : \rho_\theta^{\mathbb{L}}(a) = e^{in\theta} b\}. \end{aligned}$$

We call the elements of these sets the n -covariant elements of A and B , respectively. When $n = 0$ we call those elements invariant.

Let $c_0(\mathbb{Z}_{\geq 0})$ be the space of sequences that converge to zero. The n -covariant elements of A and B can be described in detail.

PROPOSITION 2.11. *We have the following set equalities:*

$$A_n = \{a \in A : a = U^n(a_n(\mathbb{K}) + M_f^+), a(k) \in c_0(\mathbb{Z}_{\geq 0}), f \in C(G)\}$$

for $n \geq 0$ and

$$A_n = \{a \in A : a = (a(\mathbb{K}) + M_f^+)(U^*)^{-n}, a(k) \in c_0(\mathbb{Z}_{\geq 0}), f \in C(G)\}$$

when $n < 0$. Similarly,

$$B_n = \{b \in B : b = V^n M_f, f \in C(G)\},$$

if $n \geq 0$, and

$$B_n = \{b \in B : b = M_f V^n, f \in C(G)\}$$

for $n < 0$.

PROOF. Consider the invariant elements in A , that is $\rho_\theta^{\mathbb{K}}(a) = a$. It follows from the definition of $\rho_\theta^{\mathbb{K}}$ that these elements are precisely the diagonal operators in A . Moreover, we have the following unique decomposition, which is analogous to [10, Proposition 2.4]:

$$a = a(\mathbb{K}) + M_f^+,$$

where $a(k) \in c_0(\mathbb{Z}_{\geq 0})$ and $f \in C(G)$. Next we consider the n -covariant elements for $n \neq 0$. Without loss of generality we only consider $n > 0$. Since,

$$\begin{aligned} \rho_\theta^{\mathbb{K}}(U^n) &= e^{in\theta} U^n \\ \rho_\theta^{\mathbb{K}}(a(\mathbb{K}) + M_f^+) &= a(\mathbb{K}) + M_f^+ \end{aligned}$$

for $a(k) \in c_0(\mathbb{Z}_{\geq 0})$ and $f \in C(G)$, one containment follows immediately. On the other hand, if $a \in A_n$ then $a(U^*)^n$ is an invariant element and thus by the above has the form

$$a(U^*)^n = a(\mathbb{K}) + M_f^+$$

for some $a(k) \in c_0(\mathbb{Z}_{\geq 0})$ and $f \in C(G)$. The other direction now follows. The same argument also works for B_n , completing the proof. \square

Similarly, we consider n -covariant elements from \mathcal{A} and \mathcal{B} :

$$\mathcal{A}_n = \{a \in \mathcal{A} : \rho_\theta^{\mathbb{K}}(a) = e^{in\theta} a\} \quad \text{and} \quad \mathcal{B}_n = \{b \in \mathcal{B} : \rho_\theta^{\mathbb{L}}(b) = e^{in\theta} b\}.$$

As in Proposition 2.11, $a \in \mathcal{A}_n$ if and only if a has the same element decomposition but with $a(k) \in c_{00}(\mathbb{Z}_{\geq 0})$ and $f \in \mathcal{F}$. Again, there is an analogous result for $b \in \mathcal{B}_n$.

3. Classification of derivations

As in [10], one of the main goals in this paper is to classify unbounded derivations in A and B . We begin with recalling the basic concepts.

Let M be a Banach algebra and let \mathcal{M} be a dense subalgebra of M . A linear map $d : \mathcal{M} \rightarrow M$ is called a *derivation* if the Leibniz rule holds:

$$d(ab) = ad(b) + d(a)b$$

for all $a, b \in \mathcal{M}$. We say a derivation $d : \mathcal{M} \rightarrow M$ is *inner* if there is an element $x \in M$ such that

$$d(a) = [x, a]$$

for $a \in \mathcal{M}$. We say a derivation $d : \mathcal{M} \rightarrow M$ is *approximately inner* if there are $x_n \in M$ such that

$$d(a) = \lim_{n \rightarrow \infty} [x_n, a]$$

for $a \in \mathcal{M}$.

Given $n \in \mathbb{Z}$, a derivation $d : \mathcal{A} \rightarrow A$ is said to be a *n-covariant derivation* if the relation

$$(\rho_\theta^\mathbb{K})^{-1} d(\rho_\theta^\mathbb{K}(a)) = e^{-in\theta} d(a)$$

holds. We have a similar definition for a derivation $\delta : \mathcal{B} \rightarrow B$. Like above, when $n = 0$ we say the derivation is *invariant*.

3.1. Derivations in A . We first classify all invariant derivations $d : \mathcal{A} \rightarrow A$. An example of an invariant derivation is given by

$$d_{\mathbb{K}}(a) = [\mathbb{K}, a],$$

where $a \in \mathcal{A}$. This derivation is well defined because \mathcal{A} is the space of polynomials in U, U^* , and M_f^+ and we have $[\mathbb{K}, U] = U, [\mathbb{K}, U^*] = -U^*$ and $[\mathbb{K}, M_f^+] = 0$.

LEMMA 3.1. *For any $\{\alpha(k)\} \in c_0(\mathbb{Z}_{\geq 0})$ there exists a unique derivation $d_\alpha : \mathcal{A} \rightarrow A$ such that*

$$d_\alpha(U) = U\alpha(\mathbb{K}), \quad d_\alpha(U^*) = -\alpha(\mathbb{K})U^*, \quad d_\alpha(a(\mathbb{K})) = 0$$

for every $a(\mathbb{K}) \in \mathcal{A}_0$. Moreover, this derivation is an approximately inner invariant derivation.

PROOF. Define a sequence $\{\alpha^N(k)\}$ as follows:

$$\alpha_0^N(k) = \begin{cases} \alpha_0(k) & \text{for } k < N, \\ 0 & \text{for } k \geq N. \end{cases}$$

Then $\alpha^N(k) \in c_{00}(\mathbb{Z}_{\geq 0})$ and $\alpha^N(\mathbb{K})$ converges to $\alpha(\mathbb{K})$ as $N \rightarrow \infty$. Next, define a sequence $\{\beta^N(k)\}$ by

$$\beta^N(k) = \sum_{j=0}^{k-1} \alpha^N(j),$$

so that $\beta^N(k)$ is eventually constant. Thus, $d_{\alpha^N} : \mathcal{A} \rightarrow A$ defined by

$$d_{\alpha^N}(a) = [\beta^N(\mathbb{K}), a]$$

is an invariant inner derivation. Then

$$\lim_{N \rightarrow \infty} d_{\alpha^N}(U) = U\alpha(\mathbb{K}), \quad \lim_{N \rightarrow \infty} d_{\alpha^N}(U^*) = -\alpha(\mathbb{K})U^*, \quad \text{and} \quad d_{\alpha^N}(a(\mathbb{K})) = 0$$

for all $a(\mathbb{K}) \in \mathcal{A}_0$. Thus, by the Leibniz rule, the limit

$$\lim_{N \rightarrow \infty} d_{\alpha^N}(a) = d_\alpha(a)$$

exists for all $a \in \mathcal{A}$. Thus, this limit is a derivation from \mathcal{A} to A . It follows that d_α is approximately inner and invariant. \square

LEMMA 3.2. *For any $f \in C(G)$ such that*

$$\int_G f \, d\mu = 0,$$

there exists a unique derivation $d_f : \mathcal{A} \rightarrow A$ such that

$$d_f(U) = UM_f^+, \quad d_f(U^*) = -M_f^+U^*, \quad d_f(a(\mathbb{K})) = 0,$$

where $a(\mathbb{K}) \in \mathcal{A}_0$. Moreover, d_f is an approximately inner invariant derivation.

PROOF. By the density of \mathcal{F} we can pick a sequence $\{f^N\} \subseteq \mathcal{F}$ such that f^N converges to f and

$$\int_G f^N \, d\mu = 0.$$

By Proposition 2.6, there exists a sequence $\{g^N\} \subseteq \mathcal{F}$ such that

$$f^N(x) = g^N(\varphi(x)) - g^N(x).$$

We define

$$d_{f^N}(a) = [M_{g^N}^+, a],$$

and notice that $d_{f^N} : \mathcal{A} \rightarrow A$ is an inner invariant derivation. By direct calculation,

$$\lim_{N \rightarrow \infty} d_{f^N}(U) = UM_f^+, \quad \lim_{N \rightarrow \infty} d_{f^N}(U^*) = -M_f^+U^*, \quad \text{and} \quad d_{f^N}(a(\mathbb{K})) = 0,$$

for every $a(\mathbb{K}) \in \mathcal{A}_0$. Thus, by the Leibniz rule, the limit

$$\lim_{N \rightarrow \infty} d_{f^N}(a) = d_f(a)$$

exists for all $a \in \mathcal{A}$ and is a derivation from \mathcal{A} to A . It follows that d_f is approximately inner and invariant. \square

PROPOSITION 3.3. *Given any invariant derivation $d : \mathcal{A} \rightarrow A$ there exists a number $c_0 \in \mathbb{C}$ such that d is of the unique form*

$$d(a) = c_0 d_{\mathbb{K}}(a) + \tilde{d}(a),$$

where $\tilde{d} : \mathcal{A} \rightarrow A$ is approximately inner.

PROOF. Let $a(\mathbb{K}) \in \mathcal{A}_0$ be a diagonal operator such that $a(k) \in c_{00}(\mathbb{Z}_{\geq 0})$. Then, by invariance of d , we have $d(a(\mathbb{K})) \in A_0$. Notice that since A_0 is precisely the algebra of diagonal operators in A it is therefore a commutative algebra. Let $P^2 = P$ be a projection in \mathcal{A}_0 . Applying d to both sides of the equation and using Leibniz’s rule

$$2Pd(P) = d(P^2) = d(P),$$

which implies that $(1 - 2P)d(P) = 0$ and hence $d(P) = 0$. Since $a(\mathbb{K})$ is a finite linear combination of projections in \mathcal{A}_0 , it follows that $d(a(\mathbb{K})) = 0$.

Let P_k be the one-dimensional orthogonal projection onto the span of E_k . Then $P_k \in \mathcal{A}_0$ and thus $d(P_k) = 0$. We have the following formula:

$$M_f^+ P_k = f(x_k)P_k.$$

Therefore, applying d to both sides,

$$d(M_f^+)P_k + M_f^+ d(P_k) = f(x_k)d(P_k).$$

It follows that $d(M_f^+)P_k = 0$ for all $k \in \mathbb{Z}_{\geq 0}$ and so, $d(M_f^+) = 0$. It follows from Proposition 2.11 that $d(a) = 0$ for all $a \in \mathcal{A}_0$.

Notice that, by the invariance property of d ,

$$\begin{aligned} d(U) &= U(\alpha_0(\mathbb{K}) + M_{f_0}^+) \\ d(U^*) &= -(\alpha_0(\mathbb{K}) + M_{f_0}^+)U^* \end{aligned} \tag{3-1}$$

for some $\alpha_0(k) \in c_0(\mathbb{Z}_{\geq 0})$ and $f_0 \in C(G)$.

Let c_0 be the following integral

$$c_0 = \int_G f_0 d\mu,$$

and set $\tilde{f} = f_0 - c_0$ so that $\int_G \tilde{f} d\mu = 0$. By Lemmas 3.1 and 3.2 and equation (3-1), we have the decomposition

$$d = c_0 d_{\mathbb{K}} + d_{\alpha_0} + d_{\tilde{f}}.$$

Picking $\tilde{d} = d_{\alpha_0} + d_{\tilde{f}}$ completes the proof of existence of the decomposition.

Finally, to verify uniqueness of the decomposition, we only need to check that $d_{\mathbb{K}}$ is not approximately inner. If $d_{\mathbb{K}}$ is approximately inner then we can arrange that it can be approximated by inner invariant derivations of the form $d_j(a) = [\beta_j(\mathbb{K}), a]$ with $\beta_j(k) \in c(\mathbb{Z}_{\geq 0})$. Since $\{\beta_j(k + 1) - \beta_j(k)\} \in c_0(\mathbb{Z}_{\geq 0})$ we would also get $\{(k + 1) - k\} \in c_0(\mathbb{Z})$, which is a contradiction. Full details of an analogous result are given in [10, Theorem 3.10]. □

Next we classify n -covariant derivations in A .

REMARK 3.4. Let $n \neq 0$ and by Proposition 2.7 choose $f \in \mathcal{F}$ such that $f - f \circ \varphi^n \neq 0$. Since $f - f \circ \varphi^n$ is continuous on a compact set, the minimum is achieved and is not equal to zero. Therefore,

$$\inf_{x \in G} |f(x) - f(\varphi^n(x))| \neq 0$$

and hence $M_f^+ - M_{f \circ \varphi^n}^+$ is an invertible operator.

This remark is crucial for the proof of the next proposition.

PROPOSITION 3.5. *Let $d : \mathcal{A} \rightarrow A$ be an n -covariant derivation where $n \neq 0$. There exists an element $\beta(\mathbb{K}) \in A_0$ such that*

$$\begin{aligned} d(a) &= [U^n \beta(\mathbb{K}), a] \quad \text{for } n > 0, \\ d(a) &= [\beta(\mathbb{K})(U^*)^{-n}, a] \quad \text{for } n < 0, \end{aligned}$$

and hence d is an inner derivation.

PROOF. We only discuss the case of $n > 0$ as the case of $n < 0$ is completely analogous. By definition of n -covariance there exists an $\alpha(\mathbb{K}) \in A_0$ such that

$$d(U) = U^{n+1} \alpha(\mathbb{K} + I) \quad \text{and} \quad d(U^*) = -U^{n-1} \alpha(\mathbb{K}).$$

We define a ‘twisted’ derivation $\tilde{d} : \mathcal{A}_0 \rightarrow A_0$ by $d(a(\mathbb{K})) = U^n \tilde{d}(a(\mathbb{K}))$ for $a(\mathbb{K}) \in \mathcal{A}_0$. A direct computation yields

$$\begin{aligned} \tilde{d}(a(\mathbb{K})b(\mathbb{K})) &= \tilde{d}(a(\mathbb{K}))b(\mathbb{K}) + a(\mathbb{K} + nI)\tilde{d}(b(\mathbb{K})) \\ \tilde{d}(b(\mathbb{K})a(\mathbb{K})) &= \tilde{d}(b(\mathbb{K}))a(\mathbb{K}) + b(\mathbb{K} + nI)\tilde{d}(a(\mathbb{K})) \end{aligned}$$

for $a(\mathbb{K}), b(\mathbb{K}) \in \mathcal{A}_0$. Since \mathcal{A}_0 and A_0 are commutative algebras,

$$\tilde{d}(a(\mathbb{K}))(b(\mathbb{K}) - b(\mathbb{K} + nI)) = \tilde{d}(b(\mathbb{K}))(a(\mathbb{K}) - a(\mathbb{K} + nI)).$$

Similarly to the proof in [10, Theorem 3.4], there must exist a $\beta(\mathbb{K})$ such that

$$\tilde{d}(a(\mathbb{K})) = \beta(\mathbb{K})(a(\mathbb{K}) - a(\mathbb{K} + nI)).$$

Consequently, we have the following formula:

$$d(a(\mathbb{K})) = U^n \beta(\mathbb{K})(a(\mathbb{K}) - a(\mathbb{K} + nI)).$$

Next we apply d to the commutation relation $U^* a(\mathbb{K}) = a(\mathbb{K} + I)U^*$ for a diagonal operator $a(\mathbb{K}) \in \mathcal{A}_0$, and obtain

$$\begin{aligned} U^{n-1}(-\alpha(\mathbb{K})a(\mathbb{K}) + \beta(\mathbb{K})a(\mathbb{K}) - \beta(\mathbb{K})a(\mathbb{K} + nI)) &= \\ = U^{n-1}(\beta(\mathbb{K} - I)a(\mathbb{K}) - a(\mathbb{K} + nI)\beta(\mathbb{K} - I) - a(\mathbb{K} + nI)\alpha(\mathbb{K})), \end{aligned}$$

where we define $\beta(-1) := 0$. Rearranging these terms gives

$$\alpha(\mathbb{K})(a(\mathbb{K} + nI) - a(\mathbb{K})) = (\beta(\mathbb{K}) - \beta(\mathbb{K} - I))(a(\mathbb{K} + nI) - a(\mathbb{K}))$$

for all $a(\mathbb{K}) \in \mathcal{A}_0$. It therefore follows that $\beta(\mathbb{K}) - \beta(\mathbb{K} - I) = \alpha(\mathbb{K})$. Thus $\beta(k)$ is uniquely determined by

$$\beta(k) = \sum_{j=0}^k \alpha(j),$$

and it follows that

$$d(a) = [U^n \beta(\mathbb{K}), a]$$

for any $a \in \mathcal{A}$, since both sides of the above equation are derivations, and they agree on the generators of the polynomial algebra \mathcal{A} . By the remark preceding the statement of the proposition, if $f \in \mathcal{F}$ is such that $M_f^+ - M_{f \circ \varphi^n}^+$ is invertible, we can apply d to M_f^+ to get

$$A_0 \ni d(M_f^+) = U^n \beta(\mathbb{K})(M_f^+ - M_{f \circ \varphi^n}^+).$$

Therefore, it follows that we must have $\beta(\mathbb{K}) \in A_0$, and the proof is complete. \square

To classify all derivations $d : \mathcal{A} \rightarrow A$ we need to define the Fourier coefficients of d following the ideas of [2].

DEFINITION 3.6. If d is a derivation in A , the n th Fourier component of d is defined as

$$d_n(a) = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} (\rho_\theta^\mathbb{K})^{-1} d\rho_\theta^\mathbb{K}(a) d\theta.$$

A direct calculation shows that if $d : \mathcal{A} \rightarrow A$ is a derivation then $d_n : \mathcal{A} \rightarrow A$ is an n -covariant derivation.

We have the following key Cesàro mean convergence result for Fourier components of d , which is more generally valid for unbounded derivations in any Banach algebra with the continuous circle action preserving the domain of the derivation: if d is a derivation in A then

$$d(a) = \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{j=0}^M \left(\sum_{n=-j}^j d_n(a) \right), \tag{3-2}$$

for every $a \in \mathcal{A}$, see [10, Lemma 4.2] for more details.

The following theorem classifies all derivations $d : \mathcal{A} \rightarrow A$.

THEOREM 3.7. Let $d : \mathcal{A} \rightarrow A$ be any derivation. Then there exists $c_0 \in \mathbb{C}$ such that d has the following decomposition:

$$d(a) = c_0 d_\mathbb{K}(a) + \tilde{d}(a),$$

where \tilde{d} is an approximately inner derivation.

PROOF. Let d_0 be the 0th Fourier component of d . It is an invariant derivation, so by Proposition 3.3 we have the unique decomposition:

$$d_0(a) = c_0 d_\mathbb{K}(a) + \tilde{d}_0(a) = c_0 [\mathbb{K}, a] + \tilde{d}_0(a),$$

for every $a \in \mathcal{A}$, where \tilde{d}_0 is an approximately inner derivation. From Proposition 3.5 we have that the Fourier components d_n , $n \neq 0$ are inner derivations. It follows from equation (3-2), by extracting d_0 , that

$$d(a) = d_0(a) + \lim_{M \rightarrow \infty} \frac{1}{M+1} \sum_{j=1}^M \left(\sum_{|n| \leq j, n \neq 0} d_n(a) \right).$$

The terms under the limit sign are all finite linear combinations of n -covariant derivations and so they are inner derivations themselves, meaning that the limit is approximately inner, which ends the proof. \square

We also have the following useful but weaker convergence result for the Fourier components of derivations.

PROPOSITION 3.8. *Let $d : \mathcal{A} \rightarrow A$ be any derivation. Then for every $x \in c_{00}(\mathbb{Z}_{\geq 0}) \subseteq \ell^2(\mathbb{Z}_{\geq 0})$ and $a \in \mathcal{A}$,*

$$\sum_{n \in \mathbb{Z}} d_n(a)x = d(a)x.$$

We say that $\sum_{n \in \mathbb{Z}} d_n(a)$ converges densely pointwise on the set $c_{00}(\mathbb{Z}_{\geq 0})$.

PROOF. By Leibniz rule we only need to verify the above formula on generators of \mathcal{A} . Moreover, it is enough to consider only $x = E_k^+$, since $c_{00}(\mathbb{Z}_{\geq 0})$ consists of finite linear combinations of such x . Below we show the details for $a = M_f^+$, as the calculations for $a = U$ and $a = U^*$ are very similar. We have the following basis decomposition:

$$d(a)E_k^+ = \sum_{j=0}^{\infty} \langle E_j^+, d_n(a)E_k^+ \rangle E_j^+.$$

Using the definition of the n th Fourier components d_n of d and the fact that d_n are n -covariant, a direct calculation gives

$$\langle E_j^+, d_n(a)E_k^+ \rangle = \begin{cases} \langle E_j^+, d(a)E_k^+ \rangle & \text{if } n + k = j, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\left\langle E_j^+, \sum_{n \in \mathbb{Z}} d_n(a)E_k^+ \right\rangle = \langle E_j^+, d(a)E_k^+ \rangle,$$

completing the proof. □

3.2. Derivations in B . Next we classify derivations in B starting with the invariant derivations. It turns out that there are new types of invariant derivations in B that were not present in A . We describe these in the following lemma.

LEMMA 3.9. *Let $\partial : \mathcal{F} \rightarrow C(G)$ be any derivation such that*

$$\partial f \circ \varphi = \partial(f \circ \varphi)$$

for all $f \in \mathcal{F}$, which we call a φ invariant derivation in $C(G)$. Then there exists a unique invariant derivation $\delta_{\partial} : \mathcal{B} \rightarrow \mathcal{B}$ such that

$$\delta_{\partial}(V) = 0 \quad \text{and} \quad \delta_{\partial}(M_f) = M_{\partial f}.$$

PROOF. Since (V, M_f) is a defining representation for $C(G) \rtimes_{\varphi} \mathbb{Z}$, the only relation in the polynomial algebra \mathcal{B} is

$$VM_fV^{-1} = M_{f \circ \varphi}.$$

Define the δ_{∂} on the generators as above by $\delta_{\partial}(V) = 0$ and $\delta_{\partial}(M_f) = M_{\partial f}$. Using the Leibniz rule we try to extend this definition to all \mathcal{B} . To verify that δ_{∂} is a well-defined derivation from $\mathcal{B} \rightarrow \mathcal{B}$, we thus need to check that it preserves the relation. Applying δ_{∂} to both sides of the relation yields $M_{\partial f \circ \varphi} = M_{\partial(f \circ \varphi)}$, completing the proof. □

As with algebra A there is a simple example of an invariant derivation, which is given by

$$\delta_{\mathbb{L}}(b) = [\mathbb{L}, a],$$

where $b \in \mathcal{B}$. This derivation is well defined because \mathcal{B} is the space of polynomials in V, V^{-1} , and M_f , and we have $[\mathbb{L}, V] = V, [\mathbb{L}, V^{-1}] = -V^{-1}$ and $[\mathbb{L}, M_f] = 0$.

LEMMA 3.10. *For any $f \in C(G)$ such that*

$$\int_G f \, d\mu = 0,$$

there exists a unique derivation $\delta_f : \mathcal{B} \rightarrow B$ such that

$$\delta_f(V) = VM_f, \quad \delta_f(V^{-1}) = -M_fV^{-1}, \quad \delta_f(M_g) = 0,$$

where $M_g \in \mathcal{B}_0$. Moreover, δ_f is an approximately inner invariant derivation.

The proof is identical to that of Lemma 3.2.

PROPOSITION 3.11. *Let $\delta : \mathcal{B} \rightarrow B$ be any invariant derivation, then there exist $c_0 \in \mathbb{C}$ and a φ invariant derivation in $C(G)$, $\partial : \mathcal{F} \rightarrow C(G)$, such that δ is of the unique form*

$$\delta(b) = c_0\delta_{\mathbb{L}}(b) + \delta_{\partial}(b) + \tilde{\delta}(b),$$

where $\delta_{\partial} : \mathcal{B} \rightarrow B$ is the derivation defined in Lemma 3.9 and $\tilde{\delta}$ is approximately inner.

PROOF. Since δ is invariant and $V \in \mathcal{B}_1$, we must have $\delta(V) \in B_1$. Hence, there exists $f_0 \in C(G)$ such that

$$\delta(V) = VM_{f_0}.$$

Similarly, since $M_f \in \mathcal{B}_0$, we have $\delta(M_f) \in B_0$ and thus there exists a linear map $\partial : \mathcal{F} \rightarrow C(G)$ such that

$$\delta(M_f) = M_{\partial f}.$$

Applying δ to the relation $M_{fg} = M_fM_g$ gives

$$M_{\partial(fg)} = M_{\partial f}M_g + M_fM_{\partial g}.$$

Hence ∂ satisfies the Leibniz rule and thus is a derivation. Applying δ to both sides of the relation $VM_fV^{-1} = M_{f \circ \varphi}$ yields

$$\partial f \circ \varphi = \partial(f \circ \varphi),$$

that is, ∂ is φ invariant.

Now write $f_0 = c_0 + \tilde{f}$ with $c_0 \in \mathbb{C}$ and

$$\int_G \tilde{f} \, d\mu = 0.$$

It follows that

$$\delta(b) = c_0\delta_{\mathbb{L}}(b) + \delta_{\partial}(b) + \delta_{\tilde{f}}(b),$$

where $\delta_\theta : \mathcal{B} \rightarrow B$ is the derivation defined in Lemma 3.9 and $\delta_{\tilde{f}}$ is defined in Lemma 3.10. Arguing as in the proof of Proposition 3.3 we obtain that $\delta_{\mathbb{L}}$ is not approximately inner. To complete the proof we notice that a nonzero derivation δ_θ cannot be approximately inner since \mathcal{F} is commutative and hence has no nonzero inner and approximately inner derivations. This proves the uniqueness of the decomposition and finishes the proof of the proposition. \square

Because the proof of classifying all n -covariant derivations in B is essentially the same as in the case of A , we only state the result.

PROPOSITION 3.12. *Let $\delta : \mathcal{B} \rightarrow B$ be an n -covariant derivation where $n \neq 0$. There exists an $\eta(\mathbb{L}) \in B_0$ such that*

$$\delta(b) = [U^n \eta(\mathbb{L}), b] \quad \text{for } n \neq 0.$$

Moreover, δ is an inner derivation.

Finally, putting Propositions 3.11 and 3.12 together along with the Cesàro mean convergence result for Fourier components of d we have the following result.

THEOREM 3.13. *Let $\delta : \mathcal{B} \rightarrow B$ be any derivation. Then there exists $c_0 \in \mathbb{C}$ and a φ invariant derivation $\partial : \mathcal{F} \rightarrow C(G)$ such that δ has the following unique decomposition:*

$$\delta(b) = c_0 \delta_{\mathbb{L}}(b) + \delta_\theta(b) + \tilde{\delta}(b),$$

where δ_θ is the derivation defined in Lemma 3.9 and $\tilde{\delta}$ is an approximately inner derivation.

We also state here a dense pointwise convergence result for Fourier components δ_n of a derivation $\delta : \mathcal{B} \rightarrow B$, which is similar to Proposition 3.8 and has completely analogous proof.

PROPOSITION 3.14. *Let $\delta : \mathcal{B} \rightarrow B$ be any derivation. Then for every $x \in c_{00}(\mathbb{Z}) \subseteq \ell^2(\mathbb{Z})$ and $b \in \mathcal{B}$,*

$$\sum_{n \in \mathbb{Z}} \delta_n(b)x = \delta(b)x,$$

and we say that $\sum_{n \in \mathbb{Z}} \delta_n(b)$ converges densely pointwise on the set $c_{00}(\mathbb{Z})$.

4. Lifting derivations

The first important observation is that any derivation in algebra A preserves compact operators.

PROPOSITION 4.1. *If $d : \mathcal{A} \rightarrow A$ is a derivation, then $d : \mathcal{A} \cap \mathcal{K} \rightarrow \mathcal{K}$.*

PROOF. It is enough to prove that $d(P_0)$ is compact, where P_0 is the orthogonal projection onto the one-dimensional subspace spanned by E_0^+ , because $\mathcal{A} \cap \mathcal{K}$

is comprised of linear combinations of expressions of the form $U^l P_0 (U^*)^j$ and compactness would follow immediately from the Leibniz property. To see that $d(P_0)$ is compact, apply d to both sides of the relation $P_0 = P_0^2$ to obtain

$$d(P_0) = d(P_0)P_0 + P_0d(P_0) \in \mathcal{K},$$

which completes the proof. □

As a consequence of Proposition 4.1, if $d : \mathcal{A} \rightarrow A$ is a derivation in A , then $[d] : \mathcal{B} \rightarrow B$ defined by

$$[d](a + \mathcal{K}) := [d(a)]$$

gives a derivation in B , which, by Corollary 2.10, is defined on $\mathcal{B} \cong [\mathcal{A}]$.

As a consequence to Proposition 4.1,

$$[d_{\mathbb{K}}] = \delta_{\mathbb{L}}.$$

Clearly, if d is an approximately inner derivation, then so is $[d]$. In general, given a derivation $\delta : \mathcal{B} \rightarrow B$, if there exists a derivation $d : \mathcal{A} \rightarrow A$ such that $[d] = \delta$ we call such a d a *lift* of δ .

A natural question is as follows: which derivations $\delta : \mathcal{B} \rightarrow B$ can be lifted to a derivation $d : \mathcal{A} \rightarrow A$? It follows from Theorems 3.7 and 3.13 that if there is a nonzero φ invariant derivation in $C(G)$, $\partial : \mathcal{F} \rightarrow C(G)$, then there is no $d : \mathcal{A} \rightarrow A$ such that $[d] = \delta_{\partial}$, because δ_{∂} is not approximately inner. A natural example of this is $G = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ with $x_k = \theta k \pmod{\mathbb{Z}}$, $k \in \mathbb{Z}$ and θ irrational, giving a dense subgroup of \mathbb{T}^1 . In this case, \mathcal{F} is the actual space of trigonometric polynomials. Any derivation $\partial : \mathcal{F} \rightarrow C(\mathbb{T}^1)$ invariant with respect to $\varphi(x) = x + \theta \pmod{\mathbb{Z}}$ is of the form:

$$\partial(f) = \text{const} \frac{d}{dx} f(x).$$

In this case, the algebra B is generated by V and $W = M_{e^{2\pi i x}}$ satisfying the relation

$$VW = e^{2\pi i \theta} WV.$$

Consequently, B is isomorphic with the irrational rotation algebra. \mathcal{B} is the algebra of polynomials in V and W and the derivation $\delta_{d/dx} : \mathcal{B} \rightarrow B$ is given on generators by

$$\delta_{d/dx}(V) = 0 \quad \text{and} \quad \delta_{d/dx}(W) = 2\pi i W$$

and it cannot be lifted to a derivation in A . The key reason is that there is an additional relation on A given by equation (2-3), which prevents existence of such a lift. We conjecture however, that for any compact infinite monothetic group, any approximately inner derivation $\delta : \mathcal{B} \rightarrow B$ can be lifted to a derivation $d : \mathcal{A} \rightarrow A$.

For the remainder of the section we let G be totally disconnected, in other words G is an odometer, and thus by Proposition 2.4, there exists an infinite supernatural number N such that $G \cong \mathbb{Z}/N\mathbb{Z}$. It was proved in [10] that for such G , the algebras A and B are precisely the Bunce–Deddens–Toeplitz, $A(N)$, and Bunce–Deddens algebras, $B(N)$, respectively. It follows from [10, Theorem 4.4] that there are no nontrivial φ invariant

derivations $\partial : \mathcal{F} \rightarrow C(G)$. Below we prove one of the main results of this paper that for odometers, any unbounded derivation in $B(N)$ can be lifted to an unbounded derivation in $A(N)$.

We will need the following useful result for computing Hilbert–Schmidt norms of operators in $\ell^2(\mathbb{Z})$ and $\ell^2(\mathbb{Z}_{\geq 0})$. Since below we work mostly with algebra A , we only state the corresponding version for brevity.

PROPOSITION 4.2. *Let $a : \ell^2(\mathbb{Z}_{\geq 0}) \rightarrow \ell^2(\mathbb{Z}_{\geq 0})$ be defined by*

$$a = \sum_{n=0}^{\infty} U^n a_n(\mathbb{K}) + \sum_{n=-\infty}^{-1} a_n(\mathbb{K})(U^*)^{-n},$$

where $\{a_n(k)\}_{n \in \mathbb{Z}, k \in \mathbb{Z}_{\geq 0}} \in \ell^2(\mathbb{Z} \times \mathbb{Z}_{\geq 0})$. Then a is an integral operator with the Hilbert–Schmidt norm given by

$$\|a\|_{HS}^2 = \sum_{n \in \mathbb{Z}} \sum_{k=0}^{\infty} |a_n(k)|^2.$$

PROOF. Write $f \in \ell^2(\mathbb{Z}_{\geq 0})$ in the canonical basis:

$$f = \sum_{k=0}^{\infty} f(k)E_k^+.$$

Applying the formula for a to f yields

$$af = \sum_{n \geq 0} \sum_{k \geq 0} a_n(k)f(k)E_{k+n}^+ + \sum_{0 \leq k+n} \sum_{n < 0} a_n(k+n)f(k)E_{k+n}^+.$$

Resumming both terms gives

$$\begin{aligned} af &= \sum_{n \geq 0} \sum_{n \geq k} a_{n-k}(k)f(k)E_n^+ + \sum_{n \geq 0} \sum_{n < k} a_{n-k}(n)f(k)E_n^+ \\ &= \sum_{n \geq 0} \left(\sum_{k=0}^{\infty} a_{n-k}(\min\{n, k\})f(k) \right) E_n^+. \end{aligned}$$

This shows that a is an integral operator with integral kernel

$$\kappa(k, n) = a_{n-k}(\min\{n, k\}) \in \ell^2(\mathbb{Z} \times \mathbb{Z}_{\geq 0}).$$

Therefore, by writing a in the following way

$$(af)(n) = \sum_{k=0}^{\infty} \kappa(k, n)f(k) = \sum_{k=0}^n a_{n-k}(k)f(k) + \sum_{k > n} a_{n-k}(n)f(k),$$

the Hilbert–Schmidt norm formula now follows, completing the proof. □

The following is the main result of this section.

THEOREM 4.3. *Let $\delta : \mathcal{B}(N) \rightarrow B(N)$ be any derivation. There exists a derivation $d : \mathcal{A}(N) \rightarrow A(N)$ such that $[d] = \delta$.*

PROOF. Let $\delta : \mathcal{B}(N) \rightarrow B(N)$ be an approximately inner derivation in $B(N)$, then by Theorem 3.13 and by Propositions 3.11 and 3.12,

$$\delta(b) = \left[\sum_{n \geq 0} V^n M_{f_n} + \sum_{n < 0} M_{f_n} V^n, b \right]$$

for $b \in \mathcal{B}(N)$, where the convergence of infinite sums is understood as being densely pointwise on $c_{00}(\mathbb{Z})$. In order to construct a lift of δ we need to consider derivations $d : \mathcal{A}(N) \rightarrow A(N)$ given by the following expression, densely pointwise convergent on $c_{00}(\mathbb{Z}_{\geq 0})$:

$$d(a) = \left[\sum_{n \geq 0} U^n \beta_n(\mathbb{K}) + \sum_{n < 0} \beta_n(\mathbb{K})(U^*)^{-n}, a \right]$$

for $a \in \mathcal{A}(N)$, where $\beta_n(k)$ has the following decomposition:

$$\beta_n(\mathbb{K}) = \beta_{n,0}(\mathbb{K}) + M_{f_n}^+$$

where $\beta_{n,0}(k) \in c_0(\mathbb{Z}_{\geq 0})$. We need to find conditions on $\beta_{n,0}(k)$ so that d is a well-defined derivation in $A(N)$ such that

$$[d](a) = \delta([a])$$

for all $a \in \mathcal{A}(N)$. But, by the properties of the Toeplitz map, this is equivalent to checking that the difference

$$d(a) - T(\delta([a]))$$

is compact for all $a \in \mathcal{A}(N)$. By the Leibniz rule we only need to check this property on the generators U, U^* , and M_{χ}^+ , where χ is a character on $\mathbb{Z}/N\mathbb{Z}$.

Using the above formula for δ ,

$$\delta(V) = \sum_{n \geq 0} V^{n+1}(M_{f_n \circ \varphi} - M_{f_n}) + \sum_{n < 0} (M_{f_n} - M_{f_n \circ \varphi^{-1}})V^{n-1}.$$

Then, from the properties of T we have the following:

$$T(\delta(V)) = \sum_{n \geq 0} U^{n+1}(M_{f_n \circ \varphi}^+ - M_{f_n}^+) + \sum_{n < 0} (M_{f_n}^+ - M_{f_n \circ \varphi^{-1}}^+)(U^*)^{-n-1}.$$

Finally, a direct computation yields the following formula:

$$\begin{aligned} & d(U) - T(\delta(V)) \\ &= \sum_{n \geq 0} U^{n+1}(\beta_{n,0}(\mathbb{K} + I) - \beta_{n,0}(\mathbb{K})) \\ & \quad + \sum_{n < 0} (\beta_{n,0}(\mathbb{K}) - \beta_{n,0}(\mathbb{K} - I))(U^*)^{-n-1} + P_0 \left(\sum_{n < 0} M_{f_n \circ \varphi^{-1}}^+ (U^*)^{-n-1} \right). \end{aligned}$$

Similarly, on U^* ,

$$\begin{aligned} & d(U^*) - T(\delta(V^{-1})) \\ &= \sum_{n > 0} U^{n-1}(\beta_{n,0}(\mathbb{K} - I) - \beta_{n,0}(\mathbb{K})) \\ & \quad + \sum_{n \leq 0} (\beta_{n,0}(\mathbb{K}) - \beta_{n,0}(\mathbb{K} + 1))(U^*)^{-n+1} + \left(\sum_{n > 0} U^{n-1} M_{f_n \circ \varphi^{-1}}^+ \right) P_0. \end{aligned}$$

Finally, we get the following expression for diagonal operators M_χ^+ :

$$\begin{aligned} d(M_\chi^+) - T(\delta(M_\chi)) &= \sum_{n \geq 0} U^n \beta_{n,0}(\mathbb{K})(M_\chi^+ - M_{\chi \circ \varphi^n}^+) \\ &\quad + \sum_{n < 0} \beta_{n,0}(\mathbb{K})(M_{\chi \circ \varphi^n}^+ - M_\chi^+)(U^*)^{-n}. \end{aligned}$$

The result follows provided we can choose $\beta_{n,0}(k)$ so that the right-hand sides of the above equations are compact operators. We compute the Hilbert–Schmidt norm of the above operators to show the compactness. A direct calculation using Proposition 4.2 yields the following formulas:

$$I := \|d(M_\chi^+) - T(\delta(M_\chi))\|_{\text{HS}}^2 = \sum_{n \in \mathbb{Z}} \sum_{k \geq 0} |\beta_{n,0}(k)|^2 |\chi(x_k) - \chi(x_{k+n})|^2$$

$$\begin{aligned} II &:= \|d(U) - T(\delta(V))\|_{\text{HS}}^2 = \sum_{n \in \mathbb{Z}} \sum_{k \geq 1} |\beta_{n,0}(k) - \beta_{n,0}(k-1)|^2 + \sum_{n \in \mathbb{Z}} |\beta_{n,0}(0) - f_n(x_{-1})|^2 \\ &= \|d(U^*) - T(\delta(V^{-1}))\|_{\text{HS}}^2. \end{aligned}$$

We define $\beta_{n,0}(k)$ to have the following form:

$$\beta_{n,0}(k) = \begin{cases} -f_n(x_{-1}) \binom{N_n - k}{N_n} & 0 \leq k < N_n, \\ 0 & k \geq N_n, \end{cases}$$

where the numbers N_n will be chosen later.

Notice that any character on $\mathbb{Z}/N\mathbb{Z}$ is of the form:

$$\chi(x_k) = \exp\left(\frac{2\pi i j k}{M}\right),$$

where $M \mid N$ and $j \in \mathbb{Z}$. Therefore, I and II become

$$\begin{aligned} I &= \sum_{n \in \mathbb{Z}} \sum_{k=0}^{N_n} |f_n(x_{-1})|^2 \left(\frac{N_n - k}{N_n}\right)^2 |1 - e^{2\pi i j n / M}|^2 \\ II &= \sum_{n \in \mathbb{Z}} \sum_{k=0}^{N_n-1} \frac{|f_n(x_{-1})|^2}{N_n^2} = \sum_{n \in \mathbb{Z}} \frac{|f_n(x_{-1})|^2}{N_n}. \end{aligned}$$

To estimate I we need the following key bound on the Fourier coefficients f_n of δ .

LEMMA 4.4. *The coefficients f_n of the Fourier decomposition of the derivation $\delta : \mathcal{B} \rightarrow \mathcal{B}$ satisfy the following condition: for all $M \mid N$:*

$$\sum_{M \nmid n} |f_n(x_{-1})|^2 < \infty. \tag{4-1}$$

PROOF. To obtain the estimate we compute norms of the following two Hilbert–Schmidt operators:

$$P_{\geq 0} \delta(\exp(2\pi i j \mathbb{L} / M)) P_{-1} \quad \text{and} \quad P_{-1} \delta(\exp(2\pi i j \mathbb{L} / M)) P_{\geq 0}.$$

Here P_{-1} is the orthogonal projection in $\ell^2(\mathbb{Z})$ onto the one-dimensional subspace spanned by E_{-1} , while $P_{\geq 0}$ is the orthogonal projection onto the subspace spanned by $\{E_l\}_{l \geq 0}$. A direct calculation of the Hilbert–Schmidt norms yields

$$\sum_{n \geq 0} |f_n(x_{-1})|^2 |1 - e^{2\pi i j n / M}|^2 = \|P_{\geq 0} \delta(\exp(2\pi i j \mathbb{L} / M)) P_{-1}\|_{\text{HS}}^2 < \infty,$$

and a similar formula for $n < 0$:

$$\sum_{n < 0} |f_n(x_{-1})|^2 |1 - e^{2\pi i j n / M}|^2 = \|P_{-1} \delta(\exp(2\pi i j \mathbb{L} / M)) P_{\geq 0}\|_{\text{HS}}^2 < \infty.$$

The above equation imply that

$$\begin{aligned} \infty &> \sum_{n \in \mathbb{Z}} |f_n(x_{-1})|^2 |1 - e^{2\pi i j n / M}|^2 = \sum_{M \nmid n} |f_n(x_{-1})|^2 |1 - e^{2\pi i j n / M}|^2 \\ &> \text{const} \sum_{M \nmid n} |f_n(x_{-1})|^2, \end{aligned}$$

since, for fixed M , the factor $1 - e^{2\pi i j n / M}$ has only finitely many values. This concludes the proof. □

The above lemma implies the following estimate on I :

$$\begin{aligned} I &= \sum_{n \in \mathbb{Z}} \sum_{k=0}^{N_n} |f_n(x_{-1})|^2 \left(\frac{N_n - k}{N_n}\right)^2 |1 - e^{2\pi i j n / M}|^2 \\ &\leq 4 \sum_{M \nmid n} \sum_{k=0}^{N_n} |f_n(x_{-1})|^2 \left(\frac{N_n - k}{N_n}\right)^2 \sim \text{const} \sum_{M \nmid n} N_n |f_n(x_{-1})|^2. \end{aligned}$$

To proceed further we choose a scale $s = (s_m)_{m \in \mathbb{N}}$ for the supernatural number N , which is a sequence of positive integers such that s_m divides s_{m+1} , $s_m < s_{m+1}$, and such that $N = \lim_m s_m$, see equation (2-1). For every $n \in \mathbb{Z}$ there is an index m such that $s_m \mid n$ but $s_{m+1} \nmid n$. We then write

$$n = s_m n',$$

where n' is such that $s_{m+1} / s_m \nmid n'$. Using this decomposition we choose $N_n = C_m$ to be a constant depending on m only, to be determined later. Also, without loss of generality, we can choose M , in the formula for the character χ , to be equal to one of the elements of the scale: $M = s_q$. It is then important to notice that $s_q \nmid n = s_m n'$ if and only if $m < q$. Consequently, we have the following expressions:

$$\begin{aligned} I &\leq \text{const} \sum_{m=1}^{q-1} C_m \sum_{s_{m+1} / s_m \nmid n'} |f_{s_m n'}(x_{-1})|^2 = \text{const} \sum_{m=1}^{q-1} C_m \sum_{s_m \mid n, s_{m+1} \nmid n} |f_n(x_{-1})|^2 \\ &\leq \text{const} \sum_{m=1}^{q-1} C_m \sum_{s_{m+1} \nmid n} |f_n(x_{-1})|^2 < \infty \end{aligned}$$

for any choice of C_m because the sum over $s_{m+1} \nmid n$ is finite by equation (4-1).

Next, for II we have an estimate:

$$II = \sum_{m=1}^{\infty} \frac{1}{C_m} \sum_{s_m|n, s_{m+1} \nmid n} |f_n(x_{-1})|^2 \leq \sum_{m=1}^{\infty} \frac{1}{C_m} \sum_{s_{m+1} \nmid n} |f_n(x_{-1})|^2.$$

By equation (4-1) the interior sum is finite. Finally, we can always choose C_m large enough so that $II < \infty$. This completes the proof. \square

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