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Discs area-minimizing in mean convex Riemannian n-manifolds

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In this work, we consider oriented compact manifolds which possess convex mean curvature boundary, positive scalar curvature and admit a map to $\mathbb{D}^2 \times T^n$ with non-zero degree, where $\mathbb{D}^2$ is a disc and T^n is an n-dimensional torus. We prove the validity of an inequality involving a mean of the area and the length of the boundary of immersed discs whose boundaries are homotopically non-trivial curves. We also prove a rigidity result for the equality case when the boundary is strongly totally geodesic. This can be viewed as a partial generalization of a result due to Lucas Ambrózio in (2015, *J. Geom. Anal.*, **25**, 1001–1017) to higher dimensions.

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MSC (2010): Primary 53C24; Secondary 53C42

1. Introduction

An important question in modern differential geometry is about the connection between the curvatures and topology of a manifold. A very significant and historic result on this is the famous Gauss–Bonnet theorem. As a consequence of that theorem, we note that the topological invariant, named Euler characteristic, gives a topological obstruction to the existence of a certain type of Riemannian metrics on surfaces. In higher dimensions, the relationship between curvatures and the topology of a manifold is much more complicated. However, Schoen and Yau, in their celebrated joint work, discovered interesting relations between the scalar curvature of a three-dimensional manifold and the topology of stable minimal surfaces inside it, which emerge when one uses the second variation formula for the area, the Gauss equation and the Gauss–Bonnet theorem.

In a very recent paper Bray, Brendle and Neves [3] proved an elegant rigidity result concerning to an area-minimizing two-sphere embedded in a closed three-dimensional manifold (M^3, g) with positive scalar curvature and $\pi_2(M) \neq 0$. In that work, they showed the following result. Denote by $\mathcal{F}$ the set of all smooth maps

$f : \mathbb{S}^2 \to M$ which represent a non-trivial element in $\pi_2(M)$. Define

$$\mathcal{A}(M, g) = \inf\{Area(\mathbb{S}^2, f^*g) : f \in \mathcal{F}\}.$$

If $R_g \geqslant 2$, the following inequality holds:

$$\mathcal{A}(M, g) \leqslant 4\pi,$$

where R_g denote the scalar curvature of (M, g). Moreover, if the equality holds then the universal cover of (M, g) is isometric to the standard cylinder $\mathbb{S}^2 \times \mathbb{R}$ up to scaling. For more results concerning to rigidity of three-dimensional closed manifolds coming from area-minimizing surfaces, see [**2**, **4**, **16**, **18**, **19**]. In [**21**], Zhou showed a version of Bray, Brendle and Neves [**3**] result for high co-dimension: for $n + 2 \leqslant 7$, let (M^{n+2}, g) be an oriented closed Riemannian manifold with $R_g \geqslant 2$, which admits a non-zero degree map $F : M \to \mathbb{S}^2 \times T^n$. Then $\mathcal{A}(M, g) \leqslant 4\pi$. Furthermore, the equality implies that the universal covering of (M^{n+2}, g) is $\mathbb{S}^2 \times \mathbb{R}^n$.

In the same direction as the results mentioned above for the closed manifolds, let M be a Riemannian manifold with non-empty boundary ∂M. A free boundary minimal surface in M is a minimal surface in M with boundary contained in the boundary ∂M and meeting it orthogonally. Such surfaces arise variationally as critical points of the area among surfaces in M whose boundaries lie on ∂M but are free to vary on ∂M. The simplest examples, considering M as the unit ball with centre at the origin in the Euclidean space, are an equatorial plane disc and the critical catenoid, the unique piece of a suitably scaled catenoid in the unit ball. Fraser and Schoen [**12**] established a connection between free-boundary minimal surfaces and the Steklov eigenvalue problem, and proved existence of an embedded free-boundary minimal surface of genus zero with any number of boundary components. Since then, many works was developed to study free-boundary minimal surfaces. For more results concerning free-boundary minimal surfaces, see the following references and the references therein: [**1**, **5**–**14**].

Consider now a Riemannian n-manifold with non-empty boundary $(M, \partial M, g)$. Let $\mathcal{F}_M$ be the set of all immersed discs in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M. If $\mathcal{F}_M \neq \emptyset$, we define

$$\mathcal{A}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\Sigma|_g \quad \text{and} \quad \mathcal{L}(M, g) = \inf_{\Sigma \in \mathcal{F}_M} |\partial\Sigma|_g$$

In [**1**], Ambrózio proved the following result.

THEOREM 1.1. *Let (M, g) be a compact Riemannian three-manifold with mean convex boundary. Assume that $\mathcal{F}_M \neq \emptyset$. Then*

$$\frac{1}{2} \inf R_g^M \mathcal{A}(M, g) + \inf H_g^{\partial M} \mathcal{L}(M, g) \leqslant 2\pi. \tag{1.1}$$

Moreover, if equality holds, then the universal covering of (M, g) is isometric to $(\mathbb{R} \times \Sigma_0,\, dt^2 + g_0)$, where (Σ_0, g_0) is a disc with constant Gaussian curvature $\frac{1}{2} \inf R_g$ and $\partial\Sigma_0$ has constant geodesic curvature $\inf H_g^{\partial M}$ *in (Σ_0, g_0).*

A question that arises here is the following: *Is it possible to obtain similar result for high co-dimension?* Unfortunately, a general result cannot be true as we can see with the following example. Consider $(M, g) = (\mathbb{S}^2_+(r) \times \mathbb{S}^m(R), h_0 + g_0)$, where $(\mathbb{S}^2_+(r), h_0)$ is the half two-sphere of radius r with the standard metric, and $(\mathbb{S}^m(R), g_0)$ is the m-sphere of radius R with the standard metric, $m \geqslant 2$. This case, we have that

$$\frac{1}{2} \inf R_g^M \mathcal{A}(M, g) + \inf H_g^{\partial M} \mathcal{L}(M, g) > 2\pi.$$

On the other hand, consider $(M, g) = (\mathbb{S}^2_+(r) \times T^m, g_0 + \delta)$, where (T^m, δ) is the flat m-torus, $m \geqslant 2$. Note that the equality holds in (1.1). However, we can see that in this case the universal covering of (M, g) is isometric to $(\mathbb{S}^2_+(r) \times \mathbb{R}^m, g_0 + \delta_0)$, where δ_0 is a standard metric in $\mathbb{R}^m$.

In the first example above, note that there is no map $F : (M, \partial M) \to (\mathbb{D}^2 \times T^n, \partial\mathbb{D}^2 \times T^n)$ with non-zero degree. However, this is a condition that we need in order to obtain a similar result as in [**1**]. However, for the rigidity part, we will assume that the manifold has strongly totally geodesic boundary. We say that (M, g) has strongly totally geodesic boundary if the following two conditions hold simultaneously:

(a) ∂M is a totally geodesic hypersurface of (M, g), i.e. $\nabla_{\partial_n} \partial_i = 0$ on ∂M for $i = 1, \ldots, n-1$;

(b) $\nabla^{2k+1}_{\partial_n} \partial_i = 0$ for all positive integers k and $i = 1, \ldots, n-1$ on ∂M.

Our main result of this work is the following.

Theorem 1.2. *Let $(M, \partial M, g)$ be a Riemannian $(n+2)$-manifold, $3 \leqslant n+2 \leqslant 7$, with positive scalar curvature and mean convex boundary. Assume that there is a map $F : (M, \partial M) \to (\mathbb{D}^2 \times T^n, \partial\mathbb{D}^2 \times T^n)$ with non-zero degree. Then,*

$$\frac{1}{2} \inf R_g^M \mathcal{A}(M, g) + \inf H_g^{\partial M} \mathcal{L}(M, g) \leqslant 2\pi. \tag{1.2}$$

Moreover, if the boundary ∂M is strongly totally geodesic and the equality holds in (1.2), then the universal covering of (M, g) is isometric to $(\mathbb{R}^n \times \Sigma_0, \delta + g_0)$, where δ is the standard metric in $\mathbb{R}^n$ and (Σ_0, g_0) is a disc with constant Gaussian curvature $\frac{1}{2} \inf R_g^M$ and $\partial\Sigma_0$ has null geodesic curvature in (Σ_0, g_0).

Remark 1.3. In order to prove the rigidity part of the above result, we consider the double manifold (DM). However, such a double manifold does not inherit a smooth Riemannian metric in general. If the manifold has strongly totally geodesic boundary, we obtain that the double metric is smooth. Hence we apply the theorem 1.1 in [**21**] and obtain the rigidity. If we consider only the totally geodesic condition on the boundary, we think it is also enough to obtain the rigidity. Actually, if the boundary is totally geodesic, the double metric is smooth and it fails to be smooth only across a hypersurface given by the double of the boundary. As in the work of Miao [**17**], related to non-smooth versions of the positive mass theorem, we think is it possible to obtain theorem 1.1 in [**21**] for this type of metrics: smooth metrics which fails to be smooth only across a hypersurface. Hence, applying the conclusion of theorem 1.1 in [**21**], we obtain the rigidity part.

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This work is organized as follows. In § 2, we present some auxiliaries results to be used in the proof of the main results. In § 3, we present the proof of the inequality in our main theorem 1.2. Finally, in § 4, we present the proof of the rigidity part for the case where the equality is achieved and the manifold has strongly totally geodesic boundary.

2. Free-boundary minimal k-slicings

All the manifolds considered here are compact and orientable.

2.1. Definition and examples

Let $(M, \partial M, g)$ be a Riemannian n-manifold and η the outward unit vector field on the boundary ∂M in (M, g). Assume there is a properly embedded free-boundary smooth hypersurface $\Sigma_{n-1} \subset M$ which minimizes volume in (M, g). Choose $u_{n-1} > 0$ a first eigenfunction for the second variation S_{n-1} of the volume of Σ_{n-1} in (M, g) satisfying

$$\frac{\partial u_{n-1}}{\partial \eta_{n-1}} = u_{n-1} B^{\partial M}(\nu_{n-1}, \nu_{n-1}) \text{ on } \partial\Sigma_{n-1}$$

where ν_{n-1} is the unit normal vector field of Σ_{n-1} on (M, g), η_{n-1} is the outward unit normal vector field on the boundary $\partial\Sigma_{n-1}$ in (Σ_{n-1}, g) and $B^{\partial M}$ is the second fundamental form of ∂M in (M, g) with respect to η. Define $\rho_{n-1} = u_{n-1}$ and the weighted volume functional $V_{\rho_{n-1}}$ for hypersurfaces of Σ_{n-1},

$$V_{\rho_{n-1}}(\Sigma) = \int_\Sigma \rho_{n-1}\, \mathrm{d}v_\Sigma,$$

where $\mathrm{d}v_\Sigma$ is the volume form on (Σ, g). Assume that there is a properly embedded free-boundary smooth hypersurface $\Sigma_{n-2} \subset \Sigma_{n-1}$ which minimizes the weighted volume functional $V_{\rho_{n-1}}$. Choose a first eigenfunction $u_{n-2} > 0$ for the second variation S_{n-2} of the weighted volume functional $V_{\rho_{n-1}}$ in Σ_{n-2} satisfying

$$\frac{\partial u_{n-2}}{\partial \eta_{n-2}} = u_{n-2} B^{\partial\Sigma_{n-1}}(\nu_{n-2}, \nu_{n-2}) \text{ on } \partial\Sigma_{n-2},$$

where ν_{n-2} is the unit normal vector field of Σ_{n-2} on (Σ_{n-1}, g), η_{n-2} is the outward unit normal vector field on the boundary $\partial\Sigma_{n-2}$ in (Σ_{n-2}, g) and $B^{\partial\Sigma_{n-1}}$ is the second fundamental form of $\partial\Sigma_{n-1}$ in (Σ_{n-1}, g) with respect to η_{n-1}. Define $\rho_{n-2} = \rho_{n-1} u_{n-2}$. Assume that we can keep doing this, inductively. Hence, we obtain a family of smooth free-boundary minimal submanifolds

$$\Sigma_k \subset \Sigma_{k+1} \subset \cdots \subset \Sigma_{n-1} \subset (\Sigma_n, g) := (M, g),$$

which was constructed by choosing, for each $j \in \{k, \ldots, n-1\}$, a smooth properly embedded free-boundary hypersurface $\Sigma_j \subset \Sigma_{j+1}$ which minimizes the weighted

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volume functional $V_{\rho_{j+1}}$, where $\rho_{j+1} := u_{j+1}u_{j+2}\cdots u_{n-1}$ and

$$\frac{\partial u_j}{\partial \eta_j} = u_j B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) \text{ on } \partial\Sigma_j.$$

We call such family of free-boundary minimal hypersurfaces a *free-boundary minimal k-slicing* in (M, g).

Example 2.1. Let $(N, \partial N, g)$ be a Riemannian k-manifold. Consider the following Riemannian n-manifold $(N \times T^{n-k}, g + \delta)$, where δ is the flat metric on the torus T^{n-k}. The family of smooth hypersurfaces

$$N \subset N \times S^1 \subset N \times T^2 \subset \cdots \subset N \times T^{n-k-1} \subset (N \times T^{n-k}, g + \delta),$$

where $\rho_j \equiv u_j \equiv 1$, for every $j = k, \ldots, n-1$, is a free-boundary minimal k-slicing in $(N \times T^{n-k}, g + \delta)$.

2.2. Geometric formulas for free-boundary minimal *k*-slicing

Let $(M, \partial M, g)$ be a Riemannian n-manifold. Consider a free-boundary k-slicing in M:

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset (\Sigma_n, g) := (M, g).$$

Notation

- $Ric_j :=$ Ricci curvature of (Σ_j, g)
- $R_j :=$ Scalar curvature of (Σ_j, g)
- $\nu_j :=$ Unit normal vector field of Σ_j in (Σ_{j+1}, g)
- $B_j :=$ Second fundamental form of Σ_j in (Σ_{j+1}, g)
- $H_j :=$ Mean curvature of Σ_j in (Σ_{j+1}, g)
- $\eta_j :=$ Outward unit normal vector field on the boundary $\partial\Sigma_j$ in (Σ_j, g)
- $B^{\partial\Sigma_j} :=$ Second fundamental form of $\partial\Sigma_j$ in (Σ_j, g) with respect to η_j
- $H^{\partial\Sigma_j} :=$ Mean curvature of $\partial\Sigma_j$ in (Σ_j, g) with respect to η_j

Remark 2.2. Since Σ_j is a free-boundary hypersurface in (Σ_{j+1}, g), for every $j = k, \ldots, n-1$, we have that

(1) $\eta_j = \eta_p$ in $\partial\Sigma_j$, for every $p \geqslant j$.

(2) $H^{\partial\Sigma_j} = H^{\partial\Sigma_{j+1}} - B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) = H^{\partial M} - \sum_{p=j}^{n-1} B^{\partial\Sigma_{p+1}}(\nu_p, \nu_p)$.

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For each $j \in \{k, \dots, n-1\}$, define on $\Sigma_j \times T^{n-j}$ the Riemannian metric

$$\hat{g}_j = g + \sum_{p=j}^{n-1} u_p^2 \, \mathrm{d}t_p^2.$$

We define

$$\hat{\Sigma}_j = \Sigma_j \times T^{n-j} \quad \text{and} \quad \tilde{\Sigma}_j = \Sigma_j \times T^{n-j-1}.$$

Note that, since Σ_j is free-boundary hypersurface in (Σ_{j+1}, g), we have that $\tilde{\Sigma}_j$ is a free-boundary hypersurface in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$. With the next lemmas and propositions, we will prove that $\Sigma_j \times T^{n-j-1}$ is a stable free boundary minimal hypersurface in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$.

Proposition 2.3. *Let (M, g) be a n-dimensional Riemannian manifold, $\Sigma \subset M$ be a hypersurface and $0 < u \in C^\infty(M)$. Then, the second fundamental form of $\Sigma \times \mathbb{S}^1$ in $(M \times \mathbb{S}^1, \tilde{g} = g + u^2 \, dt^2)$ is given by*

$$\tilde{B} = B - u\nu(u) \, dt^2,$$

where ν is a globally defined unit normal vector field on Σ and B is the second fundamental form of Σ in (M, g).

Proof. Consider $(x_1, \dots, x_{n-1}, t = x_n)$ a local chart in $\Sigma \times \mathbb{S}^1$ such that $(x_1, \dots, x_{n-1})$ is a local chart in Σ. Denote by $\tilde{\nabla}$ and ∇ the Riemannian connections of $(M \times \mathbb{S}^1, \tilde{g})$ and (M, g), respectively. For $i, j = 1, \dots, n-1$, we have that

$$\tilde{\nabla}_{\partial_i} \partial_j = \nabla_{\partial_i} \partial_j, \quad \tilde{\nabla}_{\partial_i} \partial_t = \frac{\partial_i(u)}{u} \partial_t \quad \text{and} \quad \tilde{\nabla}_{\partial_t} \partial_t = -u \nabla_g u.$$

It follows that

$$\tilde{B}_{ij} = \tilde{g}(\tilde{\nabla}_{\partial_i} \partial_j, \nu) = g(\nabla_{\partial_i} \partial_j, \nu) = B_{ij},$$
$$\tilde{B}_{in} = \tilde{g}(\tilde{\nabla}_{\partial_i} \partial_t, \nu) = \frac{\partial_i(u)}{u} \tilde{g}(\partial_t, \nu) = 0$$

and

$$\tilde{B}_{nn} = \tilde{g}(\tilde{\nabla}_{\partial_t} \partial_t, \nu) = -u g(\nabla_g u, \nu) = -u\nu(u).$$

□

Lemma 2.4. *For every $j = k, \dots, n-1$, the second fundamental form $\tilde{B}_j$ of $\tilde{\Sigma}_j$ in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$ is given by*

$$\tilde{B}_j = B_j - \sum_{p=j+1}^{n-1} u_p \nu_j(u_p) \, dt_p^2. \tag{2.1}$$

In particular,

$$|\tilde{B}_j|^2 = |B_j|^2 + \sum_{p=j+1}^{n-1} (\nu_j(\log u_p))^2.$$

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Proof. To prove (2.1), for $m \in \{j+1, \dots, n-1\}$, define on $\Sigma_{j+1} \times T^{n-m}$ the following Riemannian metric

$$\overline{g}_m = g + \sum_{p=m}^{n-1} u_p^2 \, \mathrm{d}t_p^2.$$

We will prove that the second fundamental form of $\Sigma_j \times T^{n-m}$ in $(\Sigma_{j+1} \times T^{n-m}, \overline{g}_m)$ is

$$\overline{B}_m = B_j - \sum_{p=m}^{n-1} u_p \nu_j(u_p) \, \mathrm{d}t_p^2 \tag{2.2}$$

by a finite reverse induction on m. When $m = n-1$ equality (2.2) follows directly from proposition 2.3. Now suppose that (2.2) is valid for $m+1$. Note that $\overline{g}_m = \overline{g}_{m+1} + u_m^2 \, \mathrm{d}t_m^2$. It follows from proposition 2.3 that $\overline{B}_m = \overline{B}_{m+1} - u_m \nu_j(u_m) \, \mathrm{d}t_m^2$. Equality (2.2) now follows from the inductive assumption. Since $\overline{g}_{j+1} = \hat{g}_{j+1}$, we have showed equality (2.1). □

LEMMA 2.5. *For every $j = k, \dots, n-1$, the second fundamental form $\hat{B}_{j+1}$ of $\partial\hat{\Sigma}_{j+1}$ in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$ with respect to η_j satisfies*

$$\hat{B}_{j+1}(\nu_j, \nu_j) = B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j).$$

Proof. Using a similar argument used to prove (2.1), we can prove that

$$\hat{B}_{j+1} = B^{\partial \Sigma_{j+1}} - \sum_{p=j+1}^{n-1} u_p \eta_{j+1}(u_p) \, \mathrm{d}t_p^2.$$

In particular,

$$\hat{B}_{j+1}(\nu_j, \nu_j) = B^{\partial \Sigma_{j+1}}(\nu_j, \nu_j).$$ □

PROPOSITION 2.6. *Let (M, g) be a n-dimensional Riemannian manifold and $0 < u \in C^\infty(M)$. Then the Ricci curvature of $(M \times \mathbb{S}^1, \tilde{g} = g + u^2 \, dt^2)$ is given by*

$$Ric_{\tilde{g}} = Ric_g - u^{-1}\left(\nabla_g^2 u\right) - u\Delta_g u \, dt^2,$$

where Ric_g is the Ricci curvature of (M, g).

Proof. Consider $(x_1, \dots, x_n, t = x_{n+1})$ a local chart in $M \times \mathbb{S}^1$ such that $(x_1, \dots, x_n)$ is a local chart in M. Denote by ∇ the Riemannian connection of

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(M, g) and $\tilde{R}$, R the curvature tensors of $(M \times \mathbb{S}^1, \tilde{g})$ and (M, g), respectively. Note that

$$(Ric_{\tilde{g}})_{ij} = \sum_{k,l=1}^{n+1} \tilde{g}^{kl} \tilde{R}_{kijl} = \sum_{k,l=1}^{n} g^{kl} \tilde{R}_{kijl} + u^{-2} \tilde{R}_{tijt}.$$

Since, for $i, j, k, l = 1, \ldots, n$, we have that

$$\tilde{R}_{kijl} = R_{kijl}, \quad \tilde{R}_{tijt} = -u \left(\nabla_g^2 u\right)_{ij} \quad \text{and} \quad \tilde{R}_{kitl} = 0$$

then

$$(Ric_{\tilde{g}})_{ij} = \sum_{k,l=1}^{n} g^{kl} R_{kijl} - u^{-1} \left(\nabla_g^2 u\right)_{ij} = (Ric_g)_{ij} - u^{-1} \left(\nabla_g^2 u\right)_{ij},$$

$$(Ric_{\tilde{g}})_{tt} = -u \sum_{k,l=1}^{n} g^{kl} \left(\nabla_g^2 u\right)_{kl} = -u \Delta_g u$$

and

$$(Ric_{\tilde{g}})_{it} = 0$$

for every $i, j = 1, \ldots, n$. □

Lemma 2.7. *For every $j = k, \ldots, n-2$, the Ricci curvature $Ric_{\hat{g}_{j+1}}$ of $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$ satisfies*

$$Ric_{\hat{g}_{j+1}}(\nu_j, \nu_j) = Ric_{j+1}(\nu_j, \nu_j) - \sum_{p=j+1}^{n-1} u_p^{-1} \left(\nabla_{j+1}^2 u_p\right)(\nu_j, \nu_j)$$

where ∇_{j+1}^2 is the Hessian in (Σ_{j+1}, g).

Proof. For $m \in \{j+1, \ldots, n-1\}$, define in $\Sigma_{j+1} \times T^{n-m}$ the Riemannian metric

$$\overline{g}_m = g + \sum_{p=m}^{n-1} u_p^2 \, \mathrm{d}t_p^2.$$

We will prove that the Ricci curvature $\overline{Ric}_m$ of $(\Sigma_{j+1} \times T^{n-m}, \overline{g}_m)$ satisfies

$$\overline{Ric}_m(\nu_j, \nu_j) = Ric_{j+1}(\nu_j, \nu_j) - \sum_{p=m}^{n-1} u_p^{-1} \left(\nabla_{j+1}^2 u_p\right)(\nu_j, \nu_j) \tag{2.3}$$

by a finite reverse induction on m. When $m = n-1$ equality (2.3) follows directly from proposition 2.6. Now suppose (2.3) is valid for $m+1$. Since $\overline{g}_m = \overline{g}_{m+1} +$

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$u_m^2 \mathrm{d}t_m^2$, it follows from proposition 2.6 that

$$\overline{Ric}_m(\nu_j, \nu_j) = \overline{Ric}_{m+1}(\nu_j, \nu_j) - u_m^{-1} \left(\overline{\nabla}^2_{m+1} u_m\right)(\nu_j, \nu_j).$$

where $\overline{\nabla}^2_{m+1}$ denote the Hessian in $(\Sigma_{j+1} \times T^{n-m-1}, \overline{g}_{m+1})$. Note that

$$\left(\overline{\nabla}^2_{m+1} u_m\right)(\nu_j, \nu_j) = \left(\nabla^2_{j+1} u_m\right)(\nu_j, \nu_j).$$

Equality (2.3) now follows from the inductive assumption and the last two equalities. Since $\overline{g}_{j+1} = \hat{g}_{j+1}$, we have proven the lemma. □

Proposition 2.8. *For every $j = k, \ldots, n-1$, $\tilde{\Sigma}_j$ is a free-boundary minimal hypersurfaces in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$.*

Proof. Consider $(x_1, \ldots, x_j, t_{j+1}, \ldots, t_{n-1})$ a local chart in $\tilde{\Sigma}_j$ such that $(x_1, \ldots, x_j)$ is a local chart in Σ_j. Denote by $\tilde{H}_j$ the mean curvature of $\tilde{\Sigma}_j$ in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$. It follows from lemma 2.4 that

$$\begin{aligned}
\tilde{H}_j &= \sum_{i,k=1}^{n-1} \hat{g}^{ik}_{j+1} (\tilde{B}_j)_{ik} \\
&= \sum_{i,k=1}^{j} g^{ik} (B_j)_{ik} - \sum_{p=j+1}^{n-1} \frac{\nu_j(u_p)}{u_p} \\
&= H_j - \sum_{p=j+1}^{n-1} \nu_j(\ln u_p) \\
&= H_j - \nu_j(\ln \rho_{j+1}) \\
&= H_j - \langle \nabla_{j+1} \ln \rho_{j+1}, \nu_j \rangle
\end{aligned}$$

where ∇_{j+1} is the gradient in (Σ_{j+1}, g). We have that Σ_j minimizes the weight volume functional $V_{\rho_{j+1}}$, in particular, the $(\ln \rho_{j+1})$-mean curvature of Σ_j in (Σ_{j+1}, g) vanishes everywhere, this is, $H_j = \langle \nabla_{j+1} \ln \rho_{j+1}, \nu_j \rangle$. (See [**15**].) This implies that $\tilde{H}_j = 0$. Therefore, $\tilde{\Sigma}_j$ is a free-boundary minimal hypersurfaces in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$. □

Denote by S_j the second variation for weight volume functional $V_{\rho_{j+1}}$ in Σ_j, $\tilde{S}_j$ the second variation for volume functional of $\tilde{\Sigma}_j$ in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$ and $\tilde{g}_j = \hat{g}_{j+1}|_{\tilde{\Sigma}_j}$.

Proposition 2.9. *For every $j = k, \ldots, n-1$, $\tilde{\Sigma}_j$ is a free-boundary stable minimal hypersurfaces in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$.*

Proof. Let $\varphi \in C^\infty(\Sigma_j)$. We have that

$$S_j(\varphi) = \int_{\Sigma_j} \left[|\nabla_j \varphi|^2 - (|B_j|^2 + Ric_{f_{j+1}}(\nu_j,\nu_j))\varphi^2\right] \rho_{j+1}\, \mathrm{d}v_j \\ - \int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j,\nu_j)\rho_{j+1}\, \mathrm{d}\sigma_j$$

where $Ric_{f_{j+1}}(\nu_j,\nu_j) = Ric_{j+1}(\nu_j,\nu_j) - (\nabla^2_{j+1} f_{j+1})(\nu_j,\nu_j)$, $f_{j+1} = \ln \rho_{j+1}$ (see [**15**]). Here, $\mathrm{d}v_j$ and $\mathrm{d}\sigma_j$ are the volume forms of (Σ_j, g) and $(\partial\Sigma_j, g)$, respectively. Note that

$$\begin{aligned} \nabla_{j+1} f_{j+1} &= \nabla_{j+1} \ln \rho_{j+1} \\ &= \nabla_{j+1}\left(\sum_{p=j+1}^{n-1} \ln u_p\right) \\ &= \sum_{p=j+1}^{n-1} \nabla_{j+1} \ln u_p \\ &= \sum_{p=j+1}^{n-1} \frac{1}{u_p}\nabla_{j+1} u_p. \end{aligned}$$

It follows that

$$\begin{aligned} (\nabla^2_{j+1} f_{j+1})(\nu_j,\nu_j) &= \left\langle \nabla_{\nu_j}\left(\nabla_{j+1} f_{j+1}\right), \nu_j\right\rangle \\ &= \left\langle \nabla_{\nu_j}\left(\sum_{p=j+1}^{n-1} \frac{1}{u_p}\nabla_{j+1} u_p\right), \nu_j\right\rangle \\ &= \sum_{p=j+1}^{n-1}\left\langle \frac{1}{u_p}\nabla_{\nu_j}\left(\nabla_{j+1} u_p\right) - \frac{\nu_j(u_p)}{u_p^2}\nabla_{j+1} u_p, \nu_j\right\rangle \\ &= \sum_{p=j+1}^{n-1} \frac{1}{u_p}\left\langle \nabla_{\nu_j}\left(\nabla_{j+1} u_p\right), \nu_j\right\rangle - \sum_{p=j+1}^{n-1} \frac{1}{u_p^2}[\nu_j(u_p)]^2 \\ &= \sum_{p=j+1}^{n-1} \frac{1}{u_p}\left(\nabla^2_{j+1} u_p\right)(\nu_j,\nu_j) - \sum_{p=j+1}^{n-1} [\nu_j(\ln u_p)]^2 \end{aligned}$$

From lemmas 2.4 and 2.7 we obtain

$$Ric_{f_{j+1}}(\nu_j,\nu_j) + |B_j|^2 = Ric_{\hat{g}_{j+1}}(\nu_j,\nu_j) + |\tilde{B}_j|^2.$$

This implies that

$$S_j(\varphi) = \int_{\Sigma_j} \left(|\nabla_j\varphi|^2 - Q_j\varphi^2\right)\rho_{j+1}\, \mathrm{d}v_j - \int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j,\nu_j)\rho_{j+1}\, \mathrm{d}\sigma_j.$$

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where

$$Q_j = Ric_{\hat{g}_{j+1}}(\nu_j, \nu_j) + |\tilde{B}_j|^2.$$

Consider now $\Psi \in C^\infty(\tilde{\Sigma}_j)$. We have that

$$\tilde{S}_j(\Psi) = \int_{\tilde{\Sigma}_j} \left[|\nabla_{\tilde{g}_j}\Psi|^2 - Q_j\Psi^2\right] \mathrm{d}v_{\tilde{g}_j} - \int_{\partial\tilde{\Sigma}_j} \Psi^2 \hat{B}_{j+1}(\nu_j, \nu_j)\, \mathrm{d}\sigma_{\tilde{g}_j}$$

where $\mathrm{d}v_{\tilde{g}_j}$ and $\mathrm{d}\sigma_{\tilde{g}_j}$ are the volume forms of $(\tilde{\Sigma}_j, \tilde{g}_j)$ and $(\partial\tilde{\Sigma}_j, \tilde{g}_j)$, respectively. From lemma 2.5 we have that

$$\tilde{S}_j(\Psi) = \int_{\tilde{\Sigma}_j} \left(|\nabla_{\tilde{g}_j}\Psi|^2 - Q_j\Psi^2\right) \mathrm{d}v_{\tilde{g}_j} - \int_{\partial\tilde{\Sigma}_j} \Psi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\, \mathrm{d}\sigma_{\tilde{g}_j}$$

Furthermore, since $\mathrm{d}v_{\tilde{g}_j} = \rho_{j+1}\,\mathrm{d}v_j\,\mathrm{d}t$ and $\mathrm{d}\sigma_{\tilde{g}_j} = \rho_{j+1}\,\mathrm{d}\sigma_j\,\mathrm{d}t$, where $\mathrm{d}t = \mathrm{d}t_{j+2}\cdots\mathrm{d}t_{n-1}$, we have that

$$\begin{aligned}\tilde{S}_j(\Psi) = &\int_{T^{n-j-1}} \left(\int_{\Sigma_j} \left(|\nabla_{\tilde{g}_j}\Psi|^2 - Q_j\Psi^2\right)\rho_{j+1}\,\mathrm{d}v_j\right)\mathrm{d}t \\ &- \int_{T^{n-j-1}} \left(\int_{\partial\Sigma_j} \Psi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\rho_{j+1}\,\mathrm{d}\sigma_j\right)\mathrm{d}t\end{aligned}$$

For each $\Psi \in C^\infty(\tilde{\Sigma}_j)$ define $F_\Psi : T^{n-j-1} \to \mathbb{R}$ by $F_\Psi(t) = S_j(\Psi_t)$, where for each $t \in T^{n-j-1}$ the function $\Psi_t \in C^\infty(\Sigma_j)$ is defined by $\Psi_t(x) = \Psi(x, t)$, $x \in \Sigma_j$. Note that

$$\tilde{S}_j(\Psi) \geqslant \int_{T^{n-j-1}} F_\Psi\,\mathrm{d}t. \tag{2.4}$$

Since Σ_j minimizes the weight volume functional $V_{\rho_{j+1}}$ we have that $F_\Psi > 0$ for every $\Psi \in C^\infty(\tilde{\Sigma}_j)$. It follows that $\tilde{S}_j(\Psi) > 0$ for every $\Psi \in C^\infty(\tilde{\Sigma}_j)$. Hence, $\tilde{\Sigma}_j$ is a free-boundary stable minimal hypersurface in $(\hat{\Sigma}_{j+1}, \hat{g}_{j+1})$. □

Note that the equality holds in (2.4) if and only if $\Psi \in C^\infty(\Sigma_j)$. So $S_j(\varphi) = \tilde{S}_j(\varphi)$, for every $\varphi \in C^\infty(\Sigma_j)$. It follows that

$$\begin{aligned}S_j(\varphi) &= \int_{\Sigma_j} (|\nabla_j\varphi|^2 - Q_j\varphi^2)\rho_{j+1}\,\mathrm{d}v_j - \int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\rho_{j+1}\,\mathrm{d}\sigma_j \\ &= -\int_{\Sigma_j} \varphi\tilde{L}_j(\varphi)\rho_{j+1}\,\mathrm{d}v_j + \int_{\partial\Sigma_j} \varphi\left(\frac{\partial\varphi}{\partial\eta_j} - \varphi B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j)\right)\rho_{j+1}\,\mathrm{d}\sigma_j\end{aligned}$$

for every $\varphi \in C^\infty(\Sigma_j)$, where $\tilde{L}_j : C^\infty(\Sigma_j) \to C^\infty(\Sigma_j)$ is a differential operator given by $\tilde{L}(\varphi) = \tilde{\Delta}_j\varphi + Q_j\varphi$, where $\tilde{\Delta}_j$ denote the Laplacian operator of $(\tilde{\Sigma}_j, \hat{g}_{j+1})$.

Consider λ_j the first eigenvalue of S_j associated with the first eigenfunction u_j. We have that,

$$\begin{cases} \tilde{L}_j(u_j) = -\lambda_j u_j \text{ on } \Sigma_j \\ \dfrac{\partial u_j}{\partial\eta_j} = u_j B^{\partial\Sigma_{j+1}}(\nu_j, \nu_j) \text{ on } \partial\Sigma_j \end{cases} \tag{2.5}$$

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Lemma 2.10. *For every $j \leqslant p \leqslant n-1$, we have that, in $\partial\Sigma_j$,*

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p) = \langle \nabla_j \log u_p, \eta_j\rangle.$$

Proof. It follows from (2.5) that, in $\partial\Sigma_p$,

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p) = \frac{1}{u_p}\frac{\partial u_p}{\partial \eta_p} = \langle \nabla_p \log u_p, \eta_p\rangle,$$

for every $p = k,\ldots,n-1$. Consider $j \leqslant p \leqslant n-1$. Note that, in $\partial\Sigma_j$,

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p) = \langle \nabla_p \log u_p, \eta_j\rangle,$$

because we have $\eta_p = \eta_j$ in $\partial\Sigma_j$ (see remark 2.2). In Σ_j, we can write

$$\nabla_p \log u_p = \nabla_j \log u_p + \sum_{l=j}^{p-1}\langle \nabla_p \log u_p, \nu_l\rangle \nu_l.$$

Hence, in $\partial\Sigma_j$, we have that

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p) = \langle \nabla_j \log u_p, \eta_j\rangle + \sum_{l=j}^{p-1}\langle \nabla_p \log u_p, \nu_l\rangle\langle \nu_l, \eta_j\rangle.$$

However, we have $\eta_j \perp \nu_l$ in $\partial\Sigma_j$, for every $j \leqslant l \leqslant n-1$. Therefore,

$$B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p) = \langle \nabla_j \log u_p, \eta_j\rangle$$

□

Proposition 2.11. *Let (M,g) be a n-dimensional Riemannian manifold and $0 < u \in C^\infty(M)$. Then the scalar curvature of $(M\times\mathbb{S}^1, \tilde{g} = g + u^2\, dt^2)$ is*

$$R_{\tilde{g}} = R_g - \frac{2}{u}\Delta_g u,$$

where R_g is the scalar curvature of (M,g).

Proof. Consider $(x_1,\ldots,x_n, t = x_{n+1})$ a local chart in $M\times\mathbb{S}^1$ such that $(x_1,\ldots,x_n)$ is a local chart in M. From proposition 2.6, we have that

$$\begin{aligned}
R_{\tilde{g}} &= \sum_{i,j=1}^{n+1} \tilde{g}^{ij}(Ric_{\tilde{g}})_{ij}\\
&= \sum_{i,j=1}^{n} \tilde{g}^{ij}(Ric_{\tilde{g}})_{ij} + \frac{1}{u^2}(Ric_{\tilde{g}})_{tt}\\
&= \sum_{i,j=1}^{n} g^{ij}(Ric_g)_{ij} - \frac{1}{u}\sum_{i,j=1}^{n} g^{ij}\left(\nabla_g^2 u\right)_{ij} - \frac{1}{u}\Delta_g u\\
&= R_g - \frac{2}{u}\Delta_g u.
\end{aligned}$$

□

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LEMMA 2.12. *For $k \leqslant j \leqslant n-1$, the scalar curvature $\tilde{R}_j$ of $(\tilde{\Sigma}_j, \tilde{g}_j)$ is given by*

$$\tilde{R}_j = R_j - 2\sum_{p=j+1}^{n-1} u_p^{-1}\Delta_j u_p - 2\sum_{j+1\leqslant p<q\leqslant n-1} \langle \nabla_j \log u_p, \nabla_j \log u_q\rangle. \tag{2.6}$$

Equivalently,

$$\tilde{R}_j = R_j - 4\rho_{j+1}^{-(1/2)}\Delta_j(\rho_{j+1}^{1/2}) - \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2. \tag{2.7}$$

Proof. To prove (2.6), for $m \in \{j+1, \ldots, n-1\}$, define in $\Sigma_j \times T^{n-m}$ the Riemannian metric

$$\overline{g}_m = g + \sum_{p=m}^{n-1} u_p^2 \, \mathrm{d}t_p^2.$$

We will prove that the scalar curvature of $(\Sigma_j \times T^{n-m}, \overline{g}_m)$ is

$$\overline{R}_m = R_j - 2\sum_{p=m}^{n-1} u_p^{-1}\Delta_j u_p - 2\sum_{m\leqslant p<q\leqslant n-1} \langle \nabla_j \log u_p, \nabla_j \log u_q\rangle \tag{2.8}$$

by a finite reverse induction on m. When $m = n-1$ formula (2.8) follows directly from proposition 2.11. Now suppose formula (2.8) is valid for $m+1$. Note that $\overline{g}_m = \overline{g}_{m+1} + u_m^2 \, \mathrm{d}t_m^2$. It follows from proposition 2.11 that

$$\overline{R}_m = \overline{R}_{m+1} - 2u_m^{-1}\overline{\Delta}_{m+1} u_m$$

where $\overline{\Delta}_{m+1}$ denote the Laplacian operator of $(\Sigma_j \times T^{n-m-1}, \overline{g}_{m+1})$. Note that

$$\overline{\Delta}_{m+1} u_m = \Delta_j u_m + \sum_{p=m+1}^{n-1} g(\nabla_j \log u_p, \nabla_j u_m).$$

Equality (2.8) now follows from the inductive assumption and the last two equalities. Since $\overline{g}_{j+1} = \tilde{g}_j$, we have proven equality (2.6).

To prove (2.7), note that

$$\left|\sum_{p=j+1}^{n-1} \nabla_j \log u_p\right|^2 = \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 + 2\sum_{j+1\leqslant p<q\leqslant n-1} \langle \nabla_j \log u_p, \nabla_j \log u_q\rangle$$

It follows from (2.6) that

$$\tilde{R}_j = R_j - 2\sum_{p=j+1}^{n-1} u_p^{-1}\Delta_j u_p - \left|\sum_{p=j+1}^{n-1} \nabla_j \log u_p\right|^2 + \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2.$$

Since

$$2\Delta_j \log u_p = 2u_p^{-1}\Delta_j u_p - 2|\nabla_j \log u_p|^2 \quad \text{and} \quad \sum_{p=j+1}^{n-1} \log u_p = \log \rho_{j+1}$$

we have that

$$\begin{aligned}\tilde{R}_j &= R_j - \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - 2\Delta_j \left(\sum_{p=j+1}^{n-1} \log u_p\right) - \left|\nabla_j \left(\sum_{p=j+1}^{n-1} \log u_p\right)\right|^2 \\ &= R_j - \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2 - 2\Delta_j \log \rho_{j+1} - |\nabla_j \log \rho_{j+1}|^2 \\ &= R_j - 4\rho_{j+1}^{-(1/2)}\Delta_j(\rho_{j+1}^{1/2}) - \sum_{p=j+1}^{n-1} |\nabla_j \log u_p|^2.\end{aligned}$$

□

LEMMA 2.13. *For $k \leqslant j \leqslant n-1$, the scalar curvature $\hat{R}_j$ of $(\hat{\Sigma}_j, \hat{g}_j)$ is given by*

$$\hat{R}_j = R_j - 2\sum_{p=j}^{n-1} u_p^{-1}\Delta_j u_p - 2\sum_{j\leqslant p<q\leqslant n-1} \langle \nabla_j \log u_p, \nabla_j \log u_q\rangle \tag{2.9}$$

$$= \hat{R}_{j+1} + |\tilde{B}_j|^2 + 2\lambda_j \tag{2.10}$$

$$= R^M + \sum_{p=j}^{n-1} |\tilde{B}_p|^2 + 2\sum_{p=j}^{n-1} \lambda_p. \tag{2.11}$$

Proof. We can prove (2.9) using a similar argument used to prove (2.6). To prove (2.10), note that $(\hat{\Sigma}_j, \hat{g}_j) = (\tilde{\Sigma}_j \times \mathbb{S}^1, \tilde{g}_j + u_j^2\, dt_j^2)$. It follows from proposition 2.11 that

$$\hat{R}_j = \tilde{R}_j - 2u_j^{-1}\tilde{\Delta}_j u_j$$

where $\tilde{\Delta}_j$ denote the Laplacian operator of $(\tilde{\Sigma}_j, \tilde{g}_j)$. So, from (2.5) we have that

$$2\lambda_j = -2u_j^{-1}\tilde{\Delta}_j u_j - \hat{R}_{j+1} + \tilde{R}_j - |\tilde{B}_j|^2 = \hat{R}_j - \hat{R}_{j+1} - |\tilde{B}_j|^2$$

Hence, it follows equality (2.10). To get (2.11) we iterate (2.10) $n-j$ times. □

PROPOSITION 2.14. *If $R^M > 0$ and $H^{\partial M} \geqslant 0$ then*

$$4\int_{\Sigma_j} |\nabla_j \varphi|^2\, dv_j > -2\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j}\, d\sigma_j - \int_{\Sigma_j} \varphi^2 R_j\, dv_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$ and $j = k, \ldots, n-1$.

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Proof. Since Σ_j minimizes the weighted volume functional $V_{\rho_{j+1}}$, we have that $S_j(\varphi) \geqslant 0$, for every $\varphi \in C^\infty(\Sigma_j)$. It follows that,

$$4\int_{\Sigma_j} |\nabla_j\varphi|^2\rho_{j+1}\,\mathrm{d}v_j \geqslant 2\int_{\Sigma_j} c_j\varphi^2\rho_{j+1}\,\mathrm{d}v_j + 2\int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j,\nu_j)\rho_{j+1}\,\mathrm{d}\sigma_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$. From Gauss equation we have that

$$Q_j = \frac{1}{2}(\hat{R}_{j+1} - \tilde{R}_j + |\tilde{B}_j|^2).$$

Since $R^M > 0$, from lemma 2.13, we have that $\hat{R}_i > 0$, for every $k \leqslant i \leqslant n-1$. It follows from lemma 2.12 that

$$2Q_j > -R_j + 4\rho_{j+1}^{-(1/2)}\Delta_j(\rho_{j+1}^{1/2})$$

Thus,

$$4\int_{\Sigma_j} |\nabla_j\varphi|^2\rho_{j+1}\,\mathrm{d}v_j > -\int_{\Sigma_j} R_j\varphi^2\rho_{j+1}\,\mathrm{d}v_j + 4\int_{\Sigma_j} \rho_{j+1}^{1/2}\Delta_j(\rho_{j+1}^{1/2})\varphi^2\,\mathrm{d}v_j$$
$$+ 2\int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j,\nu_j)\rho_{j+1}\,\mathrm{d}\sigma_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$. Replacing φ by $\varphi\rho_{j+1}^{-(1/2)}$ at the last inequality, we obtain that

$$4\int_{\Sigma_j} |\nabla_j(\varphi\rho_{j+1}^{-(1/2)})|^2\rho_{j+1}\,\mathrm{d}v_j > -\int_{\Sigma_j} R_j\varphi^2\,\mathrm{d}v_j + 4\int_{\Sigma_j} \rho_{j+1}^{-(1/2)}\Delta_j(\rho_{j+1}^{1/2})\varphi^2\,\mathrm{d}v_j$$
$$+ 2\int_{\partial\Sigma_j} \varphi^2 B^{\partial\Sigma_{j+1}}(\nu_j,\nu_j)\,\mathrm{d}\sigma_j.$$

Observe that

$$\nabla_j(\varphi\rho_{j+1}^{-(1/2)}) = \varphi\nabla_j\rho_{j+1}^{-(1/2)} + \rho_{j+1}^{-(1/2)}\nabla_j\varphi$$

This implies that

$$|\nabla_j(\varphi\rho_{j+1}^{-(1/2)})|^2 = \rho_{j+1}^{-1}|\nabla_j\varphi|^2 + \varphi^2|\nabla_j\rho_{j+1}^{-(1/2)}|^2 + 2\varphi\rho_{j+1}^{-(1/2)}\langle\nabla_j\rho_{j+1}^{-(1/2)}, \nabla_j\varphi\rangle$$

Thus,

$$\rho_{j+1}|\nabla_j(\varphi\rho_{j+1}^{-(1/2)})|^2 = |\nabla_j\varphi|^2 + \varphi^2\rho_{j+1}|\nabla_j\rho_{j+1}^{-(1/2)}|^2 + \langle\nabla_j\log\rho_{j+1}^{-(1/2)}, \nabla_j(\varphi^2)\rangle$$

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Using integration by parts, we have that

$$\begin{aligned}
&\int_{\Sigma_j} \langle \nabla_j \log \rho_{j+1}^{-(1/2)}, \nabla_j(\varphi^2)\rangle \,\mathrm{d}v_j \\
&\quad = -\int_{\Sigma_j} \varphi^2 \Delta_j \log \rho_{j+1}^{-(1/2)} \,\mathrm{d}v_j + \int_{\partial\Sigma_j} \varphi^2 \frac{\partial(\log \rho_{j+1}^{-(1/2)})}{\partial \eta_j} \,\mathrm{d}\sigma_j \\
&\quad = +\int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-(1/2)} \Delta_j \rho_{j+1}^{1/2} \,\mathrm{d}v_j - \int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^{1/2}|^2) \,\mathrm{d}v_j \\
&\qquad - \frac{1}{2}\int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j\rangle \,\mathrm{d}\sigma_j \\
&\quad = -\int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^{1/2}|^2 \,\mathrm{d}v_j + \int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-(1/2)} \Delta_j \rho_{j+1}^{1/2} \,\mathrm{d}v_j \\
&\qquad - \frac{1}{2}\int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j\rangle \,\mathrm{d}\sigma_j
\end{aligned}$$

Then,

$$\begin{aligned}
&4\int_{\Sigma_j} \rho_{j+1} |\nabla_j(\varphi \rho_{j+1}^{-(1/2)})|^2 \,\mathrm{d}v_j \\
&\quad = 4\int_{\Sigma_j} |\nabla_j \varphi|^2 \,\mathrm{d}v_j + 4\int_{\Sigma_j} \varphi^2 \rho_{j+1} |\nabla_j \rho_{j+1}^{-(1/2)}|^2 \,\mathrm{d}v_j \\
&\qquad - 4\int_{\Sigma_j} \varphi^2 |\nabla_j \log \rho_{j+1}^{1/2}|^2 \,\mathrm{d}v_j + 4\int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-(1/2)} \Delta_j \rho_{j+1}^{1/2} \,\mathrm{d}v_j \\
&\qquad - 2\int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j\rangle \,\mathrm{d}\sigma_j
\end{aligned}$$

Since,

$$\nabla_j \rho_{j+1}^{-(1/2)} = -\rho_{j+1}^{-1} \nabla_j \rho_{j+1}^{1/2},$$

we obtain that

$$\rho_{j+1} |\nabla_j \rho_{j+1}^{-(1/2)}|^2 = |\nabla_j \log \rho_{j+1}^{1/2}|^2.$$

This implies that

$$\begin{aligned}
4\int_{\Sigma_j} \rho_{j+1} |\nabla_j(\varphi \rho_{j+1}^{-(1/2)})|^2 \,\mathrm{d}v_j &= 4\int_{\Sigma_j} |\nabla_j \varphi|^2 \,\mathrm{d}v_j + 4\int_{\Sigma_j} \varphi^2 \rho_{j+1}^{-(1/2)} \Delta_j \rho_{j+1}^{1/2} \,\mathrm{d}v_j \\
&\quad - 2\int_{\partial\Sigma_j} \varphi^2 \langle \nabla_j \log \rho_{j+1}, \eta_j\rangle \,\mathrm{d}\sigma_j
\end{aligned}$$

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Consequently,

$$4\int_{\Sigma_j} |\nabla_j\varphi|^2 \, \mathrm{d}v_j > 2\int_{\partial\Sigma_j} \varphi^2 \left(B^{\partial\Sigma_{j+1}}(\nu_j,\nu_j) + \langle \nabla_j \log\rho_{j+1}, \eta_j\rangle\right) \mathrm{d}\sigma_j - \int_{\Sigma_j} R_j\varphi^2 \, \mathrm{d}v_j$$

Since $H_g^{\partial M} \geqslant 0$, from remark 2.2 and lemma 2.10 that

$$\begin{aligned} 4\int_{\Sigma_j} |\nabla_j\varphi|^2 \, \mathrm{d}v_j &> 2\int_{\partial\Sigma_j} \varphi^2 \left(\sum_{p=j}^{n-1} B^{\partial\Sigma_{p+1}}(\nu_p,\nu_p)\right) \mathrm{d}\sigma_j - \int_{\Sigma_j} R_j\varphi^2 \, \mathrm{d}v_j \\ &= 2\int_{\partial\Sigma_j} \varphi^2 \left(H_g^{\partial M} - H^{\partial\Sigma_j}\right) \mathrm{d}\sigma_j - \int_{\Sigma_j} R_j\varphi^2 \, \mathrm{d}v_j \\ &\geqslant -2\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} \, \mathrm{d}\sigma_j - \int_{\Sigma_j} R_j\varphi^2 \, \mathrm{d}v_j \end{aligned}$$

Therefore,

$$4\int_{\Sigma_j} |\nabla_j\varphi|^2 \, \mathrm{d}v_j > -2\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} \, \mathrm{d}\sigma_j - \int_{\Sigma_j} \varphi^2 R_j \, \mathrm{d}v_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$. □

THEOREM 2.15. *Let $(M, \partial M, g)$ be a Riemannian n-manifold such that $R^M > 0$ and $H^{\partial M} \geqslant 0$. Consider the free-boundary minimal k-slicing in (M, g)*

$$\Sigma_k \subset \cdots \subset \Sigma_{n-1} \subset \Sigma_n = M.$$

Then:

(1) *The manifold Σ_j has a metric with positive scalar curvature and minimal boundary, for every $3 \leqslant k \leqslant j \leqslant n-1$.*

(2) *If $k = 2$, then the connected components of Σ_2 are discs.*

Proof. (1) Consider $j \in \{k, \ldots, n-1\}$, here $k \geqslant 3$. It follows from proposition 2.14 that

$$-4k_j\int_{\Sigma_j} |\nabla_j\varphi|^2 \, \mathrm{d}v_j < 2k_j\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} \, \mathrm{d}\sigma_j + k_j\int_{\Sigma_j} \varphi^2 R_j \, \mathrm{d}v_j,$$

for every $\varphi \in C^\infty(\Sigma_j)$ such that $\varphi \not\equiv 0$ and $k_j = (j-2)/(4(j-1)) > 0$. This implies that

$$\int_{\Sigma_j} |\nabla_j\varphi|^2 \, \mathrm{d}v_j + 2k_j\int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j} \, \mathrm{d}\sigma_j + k_j\int_{\Sigma_j} \varphi^2 R_j \, \mathrm{d}v_j > (1-4k_j)\int_{\Sigma_j} |\nabla_j\varphi|^2 \, \mathrm{d}v_j,$$

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for every $\varphi \in H^1(\Sigma_j)$ such that $\varphi \not\equiv 0$. It follows that

$$\lambda_j = \inf_{0\not\equiv\varphi\in H^1(\Sigma_j)} \frac{\displaystyle\int_{\Sigma_j} |\nabla_j\varphi|^2\,\mathrm{d}v_j + 2k_j \int_{\partial\Sigma_j} \varphi^2 H^{\partial\Sigma_j}\,\mathrm{d}\sigma_j + k_j \int_{\Sigma_j} \varphi^2 R_j\,\mathrm{d}v_j}{\displaystyle\int_{\Sigma_j} \varphi^2\,\mathrm{d}v_j} > 0.$$

Therefore, there exists a metric in Σ_j with positive scalar curvature and minimal boundary.

(2) From proposition 2.14 we have that

$$4\int_{\Sigma_2} |\nabla_2\varphi|^2\,\mathrm{d}v_2 > -2\int_{\partial\Sigma_2} \varphi^2 H^{\partial\Sigma_2}\,\mathrm{d}\sigma_2 - 2\int_{\Sigma_2} \varphi^2 K\,\mathrm{d}v_2,$$

for every $\varphi \in C^\infty(\Sigma_2)$ such that $\varphi \not\equiv 0$, because $R_2 = 2K_2$, where K_2 is the Gaussian curvature of (Σ_2, g). In particular, for $\varphi \equiv 1$ we have that

$$\int_{\partial\Sigma_2} H^{\partial\Sigma_2}\,\mathrm{d}\sigma_2 + \int_{\Sigma_2} K\,\mathrm{d}v_2 > 0. \tag{2.12}$$

Let S be a connected component of Σ_2. From inequality (2.12) and from Gauss–Bonnet theorem, we have that $\chi(S) > 0$. Therefore S is a disc. □

3. Proof of inequality

PROPOSITION 3.1. *There is a free-boundary minimal two-slicing*

$$\Sigma_2 \subset \Sigma_3 \subset \cdots \subset \Sigma_{n+1} \subset (M, g),$$

such that Σ_k is connected and the map $F_k := F|_{\Sigma_k} : (\Sigma_k, \partial\Sigma_k) \to (\mathbb{D}^2 \times T^{k-2}, \partial\mathbb{D}^2 \times T^{k-2})$ has non-zero degree, for every $k = 2, \ldots, n+1$.

Proof. Without loss of generality, we assume that F is a smooth function. Consider the projection $p_j : \mathbb{D}^2 \times T^j \to S^1$ given by

$$p_j(x, (t_1, \ldots, t_j)) = t_j,$$

for every $x \in \Sigma$ and $(t_1, \ldots, t_j) \in T^j = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$.

We will start constructing the manifold Σ_{n+1}. For this, define $f_n = p_n \circ F$. It follows from the Sard's theorem that there is $\theta_n \in S^1$ which is a regular value of f_n and ∂f_n. Define

$$S_{n+1} := f_n^{-1}(\theta_n) = F^{-1}(\mathbb{D}^2 \times T^{n-1} \times \{\theta_n\}).$$

Note that $S_{n+1} \subset M$ is a properly embedded hypersurface which represents a non-trivial class in $H_{n+1}(M, \partial M)$ and

$$F|_{S_{n+1}} : (S_{n+1}, \partial S_{n+1}) \to (\mathbb{D}^2 \times T^{n-1}, \partial\mathbb{D}^2 \times T^{n-1})$$

is a non-zero degree map. It follows from geometric measure theory that there is a properly embedded free-boundary smooth hypersufrace $\Sigma'_{n+1} \subset M$ which minimizes

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volume in (M, g) and represents the class $[S_{n+1}] \in H_{n+1}(M, \partial M)$. Since Σ'_{n+1} and S_{n+1} represent the same homology class in $H_{n+1}(M, \partial M)$, we have that

$$F|_{\Sigma'_{n+1}} : (\Sigma'_{n+1}, \partial\Sigma'_{n+1}) \to (\mathbb{D}^2 \times T^{n-1}, \partial\mathbb{D}^2 \times T^{n-1})$$

has non-zero degree. Consider Σ_{n+1} a connected component of Σ'_{n+1} such that $F_{n+1} := F|_{\Sigma_{n+1}} : (\Sigma_{n+1}, \partial\Sigma_{n+1}) \to (\mathbb{D}^2 \times T^{n-1}, \partial\mathbb{D}^2 \times T^{n-1})$ has non-zero degree. It follows from lemma 33.4 in [**20**] that Σ_{n+1} is still a properly embedded free-boundary hypersurface which minimizes volume in (M, g). Consider $u_{n+1} \in C^\infty(\Sigma_{n+1})$ a positive first eigenfunction for the second variation S_{n+1} of the volume of Σ_{n+1} in (M, g). Define $\rho_{n+1} = u_{n+1}$.

By a similar reasoning used to construct Σ_{n+1}, we obtain a properly embedded free-boundary connected smooth hypersurface $\Sigma_n \subset \Sigma_{n+1}$ which minimizes the weighted volume functional $V_{\rho_{n+1}}$ and

$$F_n := F|_{\Sigma_n} : (\Sigma_n, \partial\Sigma_n) \to (\mathbb{D}^2 \times T^{n-2}, \partial\mathbb{D}^2 \times T^{n-2})$$

has non-zero degree. Consider $u_n \in C^\infty(\Sigma_{n+1})$ a positive first eigenfunction for the second variation S_n of $V_{\rho_{n+1}}$ on Σ_n. We then define $\rho_n = u_n\rho_{n+1}$ and we continue this process. □

Lemma 3.2. *We have that* $\Sigma_2 \in \mathcal{F}_M$.

Proof. From theorem 2.15 that Σ_2 is a disc. Since there is a non-zero degree map $F_2 : (\Sigma_2, \partial\Sigma_2) \to (\mathbb{D}^2, \partial\mathbb{D}^2)$, we have that $\partial\Sigma_2$ is a curve homotopically non-trivial in ∂M. Therefore, $\Sigma_2 \in \mathcal{F}_M$. □

Lemma 3.3. *We have that,*

$$\frac{1}{2} \inf R^M |\Sigma_2|_g + \inf H^{\partial M} |\Sigma_2|_g \leqslant 2\pi.$$

Moreover, if equality holds then $R_2 = \inf R^M$, $H^{\partial\Sigma_2} = \inf H^{\partial M}$ *and* $u_k|_{\Sigma_2}$ *are positive constants for every* $k = 2, \ldots, n+1$.

Proof. From remark 2.2 and lemma 2.10

$$\inf H^{\partial M} \leqslant \sum_{p=2}^{n+1} \langle \nabla_2 \log u_p, \eta_2 \rangle + H^{\partial\Sigma_2}.$$

This implies that

$$\inf H^{\partial M} |\partial\Sigma_2|_g \leqslant \sum_{p=2}^{n+1} \int_{\partial\Sigma_2} \langle \nabla_2 \log u_p \,\mathrm{d}\sigma_2, \eta_2 \rangle + \int_{\partial\Sigma_2} H^{\partial\Sigma_2} \,\mathrm{d}\sigma_2. \tag{3.1}$$

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From lemma 2.13, we have that

$$\begin{aligned}\hat{R}_2 &= R_2 - 2\sum_{p=2}^{n+1} u_p^{-1}\Delta_2 u_p - 2\sum_{2\leqslant p<q\leqslant n+1}\langle \nabla_2 \log u_p, \nabla_2 \log u_q\rangle \\ &= R_2 - 2\sum_{p=2}^{n+1} u_p^{-1}\Delta_2 u_p - \left|\sum_{p=2}^{n+1} X_p\right|^2 + \sum_{p=2}^{n+1}|X_p|^2,\end{aligned}$$

where $X_p := \nabla_2 \log u_p$. Since

$$u_p^{-1}\Delta_2 u_p = \Delta_2 \log u_p + |X_p|^2,$$

we have that

$$\hat{R}_2 = R_2 - 2\sum_{p=2}^{n+1}\Delta_2 \log u_p - \left|\sum_{p=2}^{n+1} X_p\right|^2 - \sum_{p=2}^{n+1}|X_p|^2.$$

Since $\hat{R}_2 \geqslant \inf R^M$, we obtain

$$\begin{aligned}\frac{1}{2}\inf R^M|\Sigma_2|_g &\leqslant \frac{1}{2}\int_{\Sigma_2}\hat{R}_2\,\mathrm{d}v_2 \\ &= \frac{1}{2}\int_{\Sigma_2} R_2\,\mathrm{d}v_2 - \sum_{p=2}^{n+1}\int_{\Sigma_2}\Delta_2 \log u_p\,\mathrm{d}v_2 \\ &\quad - \frac{1}{2}\int_{\Sigma_2}\left|\sum_{p=2}^{n+1} X_p\right|^2 \mathrm{d}v_2 - \frac{1}{2}\sum_{p=2}^{n+1}\int_{\Sigma_2}|X_p|^2\,\mathrm{d}v_2 \\ &\leqslant \frac{1}{2}\int_{\Sigma_2} R_2\,\mathrm{d}v_2 - \sum_{p=2}^{n+1}\int_{\Sigma_2}\Delta_2 \log u_p\,\mathrm{d}v_2.\end{aligned}$$

It follows from divergence theorem that

$$\frac{1}{2}\inf R^M|\Sigma_2|_g \leqslant \frac{1}{2}\int_{\Sigma_2} R_2\,\mathrm{d}v_2 - \sum_{p=2}^{n+1}\int_{\partial\Sigma_2}\langle \nabla_2 \log u_p, \eta_2\rangle\,\mathrm{d}\sigma_2. \tag{3.2}$$

By inequalities (3.1) and (3.2), we have that

$$\frac{1}{2}\inf R^M|\Sigma_2|_g + \inf H^{\partial M}|\partial\Sigma_2|_g \leqslant \frac{1}{2}\int_{\Sigma_2} R_2\,\mathrm{d}v_2 + \int_{\partial\Sigma_2} H^{\partial\Sigma_2}\,\mathrm{d}\sigma_2.$$

Therefore, from Gauss–Bonnet theorem, we obtain

$$\frac{1}{2}\inf R^M|\Sigma_2|_g + \inf H^{\partial M}|\partial\Sigma_2|_g \leqslant 2\pi\mathcal{X}(\Sigma_2) = 2\pi.$$

However, note that if holds equality then the field $X_p = 0$ for every $p = 2,\ldots,n+1$. It follows that $u_p|_{\Sigma_2}$ are positive constants for every $p = 2,\ldots,n+1$. Consequently,

$R_2 = \hat{R}_2 \geqslant \inf R^M$ and $H^{\partial\Sigma_2} \geqslant \inf H^{\partial M}$. Therefore, from Gauss–Bonnet theorem, we have that $R_2 = \inf R^M$ and $H^{\partial\Sigma_2} = \inf H^{\partial M}$. □

COROLLARY 3.4. *We have that,*

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leqslant 2\pi.$$

Moreover, if equality holds then $R_2 = \inf R^M$, $H^{\partial\Sigma_2} = \inf H^{\partial M}$ *and* $u_k|_{\Sigma_2}$ *are positive constants for every* $k = 2, \ldots, n+1$.

Proof. We have that

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leqslant \frac{1}{2}\inf R^M |\Sigma|_g + \inf H^{\partial M}|\partial\Sigma|_g$$

for every $\Sigma \in \mathcal{F}_M$. From proposition 3.1 and lemmas 3.2 and 3.3 we have that there is $\Sigma_2 \in \mathcal{F}_M$ such that

$$\frac{1}{2}\inf R^M |\Sigma_2|_g + \inf H^{\partial M}|\Sigma_2|_g \leqslant 2\pi. \tag{3.3}$$

It follows that

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leqslant 2\pi. \tag{3.4}$$

If the equality holds in (3.4) then the equality holds in (3.3). Therefore, from lemma 3.3 we have that $R_2 = \inf R^M$, $H^{\partial\Sigma_2} = \inf H^{\partial M}$ and $u_k|_{\Sigma_2}$ are positive constants for every $k = 2, \ldots, n+1$. □

4. Proof of the rigidity

Proof. Without loss of generality, we can assume that $R_g \geqslant 2$. Using an idea in the Gromov–Lawsons paper on positive scalar curvature and mean-convex manifolds, we obtain that the doubling DM of M has a metric g with $R_g \geqslant 2$. Moreover, if $F : (M, \partial M) \to (\mathbb{D}^2 \times T^n, \partial\mathbb{D}^2 \times T^n)$ is a non-zero degree map, then the induced map $DF : DM \to D\mathbb{D}^2 \times T^n$ has the same non-zero degree, simply by looking at the preimage of a non-singular point. Hence, DM admits a map to $\mathbb{S}^2 \times T^n$ with non-zero degree, since $D\mathbb{D}^2 = \mathbb{S}^2$. Note that such a double manifold does not inherit a smooth Riemannian metric in general. However, since the boundary ∂M is strongly totally geodesic, we obtain that the double metric is a smooth metric. Now, we obtain that equality in (1.2) implies that the equality is achieved in the main inequality of theorem 1.1 in [**21**] for our doubling manifold DM. Therefore, the rigidity part can be obtained from theorem 1.1 in [**21**]. □

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