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# A spectral decomposition of the attractor of piecewise-contracting maps of the interval

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*Abstract*. We study the asymptotic dynamics of piecewise-contracting maps defined on a compact interval. For maps that are not necessarily injective, but have a finite number of local extrema and discontinuity points, we prove the existence of a decomposition of the support of the asymptotic dynamics into a finite number of minimal components. Each component is either a periodic orbit or a minimal Cantor set and such that the $\omega$-limit set of (almost) every point in the interval is exactly one of these components. Moreover, we show that each component is the $\omega$-limit set, or the closure of the orbit, of a one-sided limit of the map at a discontinuity point or at a local extremum.

Key words: interval map, piecewise contraction, periodic attractor, minimal Cantor set
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## 1. *Introduction*

Let $X \subset \mathbb{R}$ be a compact interval with non-empty interior. A map $f : X \to X$ is a *piecewise-contracting interval map* (PCIM) if there exist $\lambda \in (0, 1)$ and a collection of $N \geqslant 2$ non-empty disjoint open intervals $X_1, X_2, \ldots, X_N$ such that $X = \bigcup_{i=1}^{N} \overline{X_i}$ and

$$|f(x) - f(y)| \leqslant \lambda |x - y| \quad \text{for all } x, y \in X_i \text{ and } i \in \{1, 2, \ldots, N\}. \tag{1}$$

We call the *contracting constant* (or *contracting rate*) of $f$ the real number $\lambda \in (0, 1)$, and the *contraction pieces* the elements of the collection $\{X_i\}_{i=1}^{N}$.

For a PCIM $f : X \to X$, we let $c_0, c_N$ denote the extreme points of $X$ and let $\Delta := \{c_1 < c_2 < \cdots < c_{N-1}\}$ denote the set of the boundaries of the contraction pieces.

That is, $X_1 = [c_0, c_1)$, $X_2 = (c_1, c_2)$, ..., $X_N = (c_{N-1}, c_N]$. For notational convenience we suppose that $X_1$ and $X_N$ are half-closed, but we may also consider the case where one or both pieces are open by adding $c_0$ and/or $c_N$ to the set $\Delta$. In other words, $\Delta$ must contain all the discontinuity points of the map.

From the inequality (1), it follows that the points of $\Delta$ are removable (maybe continuity points) or jump discontinuities. Therefore, for any $i \in \{1, \ldots, N\}$, the map $f|_{X_i}$ admits a unique continuous extension $f_i : \overline{X_i} \to X$, which besides satisfies (1) for any pair of points in $\overline{X_i}$. The one-sided limits of $f$ at the extreme points of its contraction pieces are written as

$$d_0 := f_1(c_0), \quad d_N := f_N(c_N), \quad d_i^- := f_i(c_i) \quad \text{and} \quad d_i^+ := f_{i+1}(c_i)$$

with $i \in \{1, \ldots, N-1\}$. We let $D$ denote the set of these points, that is,

$$D := \{d_0, d_1^-, \ldots, d_{N-1}^-, d_1^+, \ldots, d_{N-1}^+, d_N\}.$$

In this paper, our purpose is to describe the topological structure and dynamical properties of the asymptotic dynamics of PCIMs. To this aim, let $f$ be a PCIM and consider the asymptotic set called the *attractor* of $f$ and which is defined by the following equality:

$$\Lambda := \bigcap_{n\geqslant 1} \Lambda_n, \text{ where } \Lambda_1 := \overline{f(X \setminus \Delta)} \quad \text{and} \quad \Lambda_{n+1} := \overline{f(\Lambda_n \setminus \Delta)} \quad \text{for all } n \geqslant 1. \quad (2)$$

Note that this set does not depend on the particular definition of the map at its discontinuity points. Also, as $\Lambda_n$ is compact, non-empty and $\Lambda_{n+1} \subset \Lambda_n$ for all $n \geqslant 1$, the attractor $\Lambda$ is compact and non-empty. Besides, as shown in [**5**], the attractor contains the $\omega$-limit set of any point of the set

$$\widetilde{X} := \bigcap_{n\geqslant 0} f^{-n}(X \setminus \Delta).$$

A general result, which holds in any compact metric phase space, is that the attractor of a piecewise-contracting map consists of a finite number of periodic orbits whenever it does not intersect the boundary of a contraction piece (see [**5**]). Moreover, for PCIMs defined on a half-closed interval, Nogueira, Pires and Rosales proved that this periodic asymptotic behavior is generic in a metric sense and with a number of periodic orbits which is bounded above by the number of contraction pieces [**10–12**]. This generalizes and refines a previous result obtained by Brémont in [**1**].

Periodic orbits are not the only possible asymptotic sets of PCIMs. In [**7**], Gambaudo and Tresser early studied the attractors of PCIMs with $N = 2$ contraction pieces. Associating a rotation number to the map, they proved that the attractor is either a periodic orbit (rational rotation number) or a Cantor set (irrational rotation number) and that the latter case corresponds to a quasi-periodic asymptotic dynamics with Sturmian complexity. It is in particular the case for the half-closed unit interval map $x \mapsto \lambda x + \mu \bmod 1$, for which the properties of the rotation number as a function of $\lambda$ and $\mu \in [0, 1)$ have been studied in detail [**2**, **3**, **6**, **8**]. For injective PCIMs with $N \geqslant 2$ contraction pieces, it has been proved that the complexity of the itinerary of any orbit is an eventually affine function [**4**, **13**]. The growth rate of the complexity is at most equal to $N - 1$ and there are some examples of PCIMs with such a maximal complexity [**4**]. In these particular examples, the attractor is a minimal Cantor set containing all the boundaries of the contraction pieces. Nevertheless, there is no general description of the topological structure and dynamical properties of the attractor of PCIMs with arbitrary complexity and number of contraction pieces. The aim of this paper is to give such a description.

Before stating the hypothesis and our results, we fix the notation and give some definitions. In the following, $\mathcal{O}(x) := \{f^n(x)\}_{n\geqslant 0}$ denotes the forward orbit of a point $x \in X$ and it is said to be periodic if there exists $p \geqslant 1$ such that $f^p(x) = x$. The $\omega$-limit set of a point $x \in X$ is denoted $\omega(x)$. We recall that $y \in \omega(x)$ if and only if there exists a subsequence of $\mathcal{O}(x)$ which converges to $y$. In practice, we will only study the orbits and the $\omega$-limit sets of the points in $\widetilde{X}$† (nevertheless, the asymptotic sets may contain points of $\Delta$). This allows us to disregard how the map is defined on $\Delta$, the relevant values being actually those of the set $D$.

*Definition 1.1.* (Pseudo-invariant set) We say that $A \subset X$ is *pseudo-invariant* if for any $x \in A$ we have $\lim_{y\to x^-} f(y) \in A$ or $\lim_{y\to x^+} f(y) \in A$.

For a PCIM the $\omega$-limit set of any point is non-empty and compact, but it is not necessarily invariant if it contains a discontinuity point. However, we will see later that the attractor of a PCIM as well as the $\omega$-limit set of any point of $\widetilde{X}$ are pseudo-invariant sets. Note that if $A \subset X$ is pseudo-invariant, then $f(x) \in A$ for any $x \in A \setminus \Delta$ and $A \cap \widetilde{X}$ is invariant.

*Definition 1.2.* We say that $A \subset X$ is $\widetilde{X}$*-minimal* if $\overline{\mathcal{O}(x)} = A$ for any $x \in A \cap \widetilde{X}$.

On some occasions, when a 'property' holds for the intersection of a set $A \subset X$ with $\widetilde{X}$, we will say that the set $A$ is $\widetilde{X}$-'property'. For instance, a set $A \subset X$ is $\widetilde{X}$-invariant if $f(A \cap \widetilde{X}) \subset A \cap \widetilde{X}$. Also, if $A$ and $B \subset X$ satisfy $A \cap B \cap \widetilde{X} = \emptyset$, we say that $A$ and $B$ are $\widetilde{X}$-disjoint.

Now we state Theorem 1.1, which is the main result of this paper.

THEOREM 1.1. *Let $f : X \to X$ be a PCIM which is injective on each of its contraction pieces and such that $D \subset \widetilde{X}$. Then there exist two natural numbers $N_1$ and $N_2$ such that:*

(1) *the attractor $\Lambda$ of $f$ can be decomposed as follows:*

$$\Lambda = \left(\bigcup_{i=1}^{N_1} \mathcal{O}_i\right) \cup \left(\bigcup_{j=1}^{N_2} K_j\right), \tag{3}$$

*where $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{N_1} \subset \widetilde{X}$ are periodic orbits and $K_1, K_2, \ldots, K_{N_2}$ are $\widetilde{X}$-minimal pseudo-invariant Cantor sets of $X$;*

(2) *for any $x \in \widetilde{X}$, either there exists $i \in \{1, \ldots, N_1\}$ such that $\omega(x) = \mathcal{O}_i$ or there exists $j \in \{1, \ldots, N_2\}$ such that $\omega(x) = K_j$;*

(3) *if $N_2 \geqslant 1$, then for any $j \in \{1, \ldots, N_2\}$ there exists $k \in \{1, \ldots, N-1\}$ such that*

$$c_k \in K_j \quad \text{and} \quad K_j = \overline{\mathcal{O}(d_k^+)} = \overline{\mathcal{O}(d_k^-)}; \tag{4}$$

† It is easy to see that the orbit of a point in $X \setminus \widetilde{X}$ eventually falls either in $\widetilde{X}$ or at a point of $\Delta$ which is periodic.

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(4) *if $N_2 \geqslant 1$, then for any $j \in \{1, \ldots, N_2\}$ and $k \in \{1, \ldots, N-1\}$ such that $c_k \in K_j$, we have*

$$K_j = \overline{\mathcal{O}(d_k^+)} \quad or \quad K_j = \overline{\mathcal{O}(d_k^-)}. \tag{5}$$

*Moreover, if $c_k \in K_j$ does not belong to the boundary of a gap of $K_j$, then $\overline{\mathcal{O}(d_k^+)} = \overline{\mathcal{O}(d_k^-)}$;*

(5) *finally, we have $1 \leqslant N_1 + N_2 \leqslant \#D$ and $N_1 + 2N_2 \leqslant 2(N-1)$. Moreover, if $f$ is increasing on each of its contraction pieces, then $N_1$ and $N_2$ also satisfy $N_1 + N_2 \leqslant N$.*

Note that two different Cantor sets $K_i$ and $K_j$ of the decomposition (3) are necessarily $\widetilde{X}$-disjoint. Indeed, if there exits $y \in K_i \cap K_j \cap \widetilde{X}$, then $K_i = \overline{\mathcal{O}(y)} = K_j$, since $K_i$ and $K_j$ are $\widetilde{X}$-minimal. Therefore, Theorem 1.1 ensures a decomposition of the attractor $\Lambda$ into a finite number of topologically transitive, pseudo-invariant and $\widetilde{X}$-disjoint components. So, we may call (3) the 'spectral decomposition' of $\Lambda$ and each of its components a 'basic piece'. Theorem 1.1 also states a dichotomy: a basic piece is either a periodic orbit in $\widetilde{X}$ or a $\widetilde{X}$-minimal Cantor set. This dichotomy does not hold when the phase space is not a subset of $\mathbb{R}$. Indeed, there are examples of PCIMs defined on compact subsets of $\mathbb{R}^n$ ($n \geqslant 2$) for which the attractor is a transitive countable infinite set or an interval; see [**5**].

Part (3) states that each Cantor piece must contain a border of a contraction piece. Part (4) states that a Cantor piece is given by the closure of the orbit of a (or both) one-sided limit(s) of the map at any point of $\Delta$ contained in the Cantor piece. An estimation of the number of basic pieces is given by part (5). In particular, we deduce that $N_2 \leqslant N - 1$ and if $N_2 = N - 1$ then $N_1 = 0$. If $N = 2$, then $1 \leqslant N_1 + 2N_2 \leqslant 2$, that is, the attractor consists either of a single $\widetilde{X}$-minimal Cantor set or of one or two periodic orbits. For any of these cases there exist examples of PCIMs with such an attractor [**2**, **3**, **6–8**]. So, the inequality is optimal at least for PCIMs with two contraction pieces. If the map is increasing in each contraction piece, then the number of basic pieces must satisfy the additional inequality $1 \leqslant N_1 + N_2 \leqslant N$. In particular, it complements [**12**, Theorem 1.1] for $\lambda$-piecewise-affine contractions which verify $\lambda \in (0, 1)$ and $D \subset \widetilde{X}$. Finally, it is worth mentioning that for globally injective maps we always have $N_1 \leqslant N$; see [**10**].

In [**4**], it is shown that for injective PCIMs the complexity of the itinerary of any point in $\widetilde{X}$ is an eventually constant or affine function. As a consequence of Theorem 1.1, we obtain that if $D \subset \widetilde{X}$ then the $\omega$-limit sets of the points with affine complexity are $\widetilde{X}$-minimal Cantor sets.

*Remark 1.1.* Note that the hypotheses of Theorem 1.1 require the PCIM being injective only in each contraction piece. Therefore, the theorem can be applied to non-injective PCIMs such as those of Figure 1(a). On the other hand, the collection of the contraction pieces of a PCIM is not unique. The most natural and smallest one is the collection of the continuity pieces (for which $\Delta$ is the set of the discontinuity points of the map). However, Theorem 1.1 applies with any collection of contraction pieces, provided the pieces are chosen in such a way that the map is injective in each of them. For instance, if a PCIM has a finite number of local extrema, the hypotheses of the theorem are satisfied if we choose

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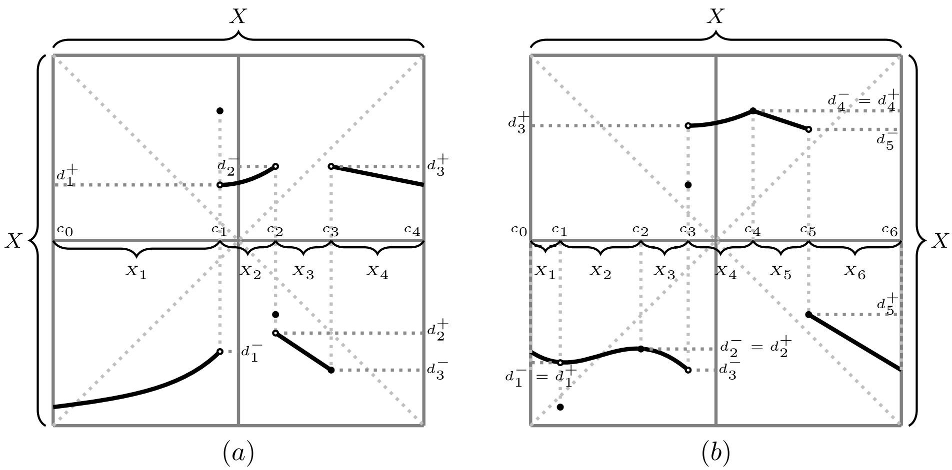


FIGURE 1. Two examples of PCIMs.

the contraction pieces of the map such that the set $\Delta$ contains all the points where the map has a local extremum (in addition to the discontinuity points), as in Figure 1(b).

The paper is organized as follows. In §2, we give the route of the proof of Theorem 1.1. That is, we prove Theorem 1.1, but assuming Theorem 2.3, which is stated without proof. Then, to complete the proof of Theorem 1.1, we give the proof of Theorem 2.3 in §3.

## 2. *Route of the proof of Theorem 1.1*

This section contains three theorems (Theorems 2.1, 2.2 and 2.3), which allow us to prove Theorem 1.1. We will not always assume the hypothesis of Theorem 1.1, which states that $f$ is injective on each of its contraction pieces. We will explicitly mention this hypothesis in the statement of the results whose proof uses it. To prove Theorems 2.1 and 2.2, we will write the attractor $\Lambda$ as the intersection of collections of 'atoms', which are defined as follows.

*Definition 2.1.* (Atoms) Let $\mathcal{P}(X)$ be the power set of $X$ and, for every $i \in \{1, \ldots, N\}$, consider the map $F_i : \mathcal{P}(X) \to \mathcal{P}(X)$ defined by

$$F_i(A) = \overline{f(A \cap X_i)} \quad \text{for all } A \in \mathcal{P}(X).$$

Let $n \geqslant 1$ and $(i_1, i_2, \ldots, i_n) \in \{1, \ldots, N\}^n$. We call the set

$$A_{i_1,\ldots,i_{n-1},i_n} := F_{i_n} \circ F_{i_{n-1}} \circ \cdots \circ F_{i_1}(X)$$

an *atom of generation* $n$ if it is non-empty. We denote by $\mathcal{A}_n$ the family of all the atoms of generation $n$.

The atoms allow us to study the attractor because the sets $\Lambda_n$ that define $\Lambda$ through (2) can also be written as

$$\Lambda_n = \bigcup_{A \in \mathcal{A}_n} A \quad \text{for all } n \geqslant 1.$$

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Also, if $x \in \widetilde{X}$ and $\theta \in \{1, \ldots, N\}^{\mathbb{N}}$ is the itinerary of $x$, i.e. it is the sequence such that $f^n(x) \in X_{\theta_n}$ for all $n \in \mathbb{N}$, then $f^{t+n}(x) \in A_{\theta_t, \theta_{t+1}, \ldots, \theta_{t+n-1}}$ for every $t \geqslant 0$ and $n \geqslant 1$ (see [**4**]).

The basic properties of the atoms are the following ones: any atom of generation $n$ is contained in an atom of generation $n-1$, precisely $A_{i_1, i_2, \ldots, i_n} \subset A_{i_2, i_3, \ldots, i_n} \subset \cdots \subset A_{i_n}$. Moreover, if $f$ is piecewise contracting with contracting constant $\lambda$, then

$$\max_{A \in \mathcal{A}_{n+1}} \operatorname{diam}(A) \leqslant \lambda \max_{A \in \mathcal{A}_n} \operatorname{diam}(A) \quad \text{for all } n \geqslant 1,$$

where $\operatorname{diam}(A)$ denotes the diameter of $A$. It implies that the diameter of any atom of generation $n$ is smaller than $\lambda^n \operatorname{diam}(X)$. Finally, in the case of a PCIM, any atom is a compact interval.

### 2.1. *Decomposition and pseudo-invariance of the attractor.*

LEMMA 2.1. *If $x \in \widetilde{X}$, then $\omega(x)$ is non-empty, compact and pseudo-invariant.*

*Proof.* By compactness of the space $X$, and by definition of $\omega$-limit set, $\omega(x)$ is non-empty and closed and hence compact. To prove that $\omega(x)$ is pseudo-invariant, we show that for any point $x_0 \in \omega(x)$ there exists $i \in \{1, \ldots, N\}$ such that $f_i(x_0) \in \omega(x)$. Let $x_0 \in \omega(x)$ and $\{t_j\}_{j \in \mathbb{N}}$ be a strictly increasing sequence such that $\lim_{j \to \infty} f^{t_j}(x) = x_0$. Then there exist $i \in \{1, \ldots, N\}$ such that $x_0 \in \overline{X}_i$ and a subsequence $\{t_{j_k}\}_{k \in \mathbb{N}}$ of $\{t_j\}_{j \in \mathbb{N}}$ such that $f^{t_{j_k}}(x) \in X_i$ for all $k \in \mathbb{N}$. It follows that $f^{t_{j_k}+1}(x) = f_i(f^{t_{j_k}}(x))$ for any $k \in \mathbb{N}$ and, by continuity of $f_i$ on $\overline{X}_i$, we have $\lim_{k \to \infty} f^{t_{j_k}+1}(x) = f_i(x_0) \in \omega(x)$. □

LEMMA 2.2. *If $f$ has a periodic point $x_0 \in \widetilde{X}$, then there exists $\rho > 0$ such that for any $x$ in the ball $B(x_0, \rho)$ of center $x_0$ and radius $\rho$ we have $\omega(x) = \mathcal{O}(x_0)$.*

*Proof.* Let $\boldsymbol{\nu}$ denote the distance between two subsets of $X$ and let $\rho := \boldsymbol{\nu}(\mathcal{O}(x_0), \Delta)$. As the periodic point $x_0$ belongs to $\widetilde{X}$, we have $\rho > 0$. Therefore, for every $n \in \mathbb{N}$ the ball $B(f^n(x_0), \rho)$ does not contain any point of $\Delta$ and, for each $n \in \mathbb{N}$, it intersects only one of the contraction pieces. It follows that for any point $x \in B(x_0, \rho)$, we have

$$|f^n(x_0) - f^n(x)| < \lambda^n \rho \quad \text{for all } n \in \mathbb{N},$$

where $\lambda \in (0, 1)$ is the contracting rate of $f$. This implies that

$$\boldsymbol{\nu}(\mathcal{O}(x_0), f^n(x)) < \lambda^n \rho \quad \text{for all } n \in \mathbb{N}.$$

Therefore, if for some increasing sequence $\{s_n\}_{n \in \mathbb{N}}$ of natural numbers $\{f^{s_n}(x)\}_{n \in \mathbb{N}}$ converges, then its limit is in $\mathcal{O}(x_0)$. In other words, $\omega(x) \subset \mathcal{O}(x_0)$. On the other hand, by invariance of $\omega(x) \cap \widetilde{X}$, we obtain that $\mathcal{O}(x_0) \subset \omega(x)$. □

The following theorem (Theorem 2.1) is the first key point in the proof of Theorem 1.1. It states that the attractor of a PCIM is completely determined by the $\omega$-limit sets of its one-sided limits at the points of $\Delta$.

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THEOREM 2.1. *Suppose that $f$ is injective on each of its contraction pieces and that $D \subset \widetilde{X}$. Then:*

(1) *the attractor of $f$ can be written as*

$$\Lambda = \bigcup_{d \in D} \omega(d); \tag{6}$$

(2) *for any periodic point $x_0 \in \widetilde{X}$, there exists $d \in D^- \cup D^+$ such that $\mathcal{O}(x_0) = \omega(d)$ with $D^- := \{d_1^-, \ldots, d_{N-1}^-\}$ and $D^+ := \{d_1^+, \ldots, d_{N-1}^+\}$. Moreover, if $f$ is increasing on each of its contraction pieces, then there exist $d^- \in D^- \cup \{d_N\}$ and $d^+ \in D^+ \cup \{d_0\}$ such that $\mathcal{O}(x_0) = \omega(d^-) = \omega(d^+)$.*

*Proof.* Since the $\omega$-limit set of any point of $\widetilde{X}$ is contained in $\Lambda$, we have that $\omega(d) \subset \Lambda$ for all $d \in D$. So, we have to prove that for any point $x_0 \in \Lambda$ there exists $d \in D$ such that $x_0 \in \omega(d)$ and that, besides, $d$ can be chosen in $D^- \cup D^+$ if $x_0$ is periodic.

Define

$$\mathcal{U} := \bigcup_{d \in D} \mathcal{O}(d) \quad \text{and} \quad \mathcal{U}^* := \bigcup_{d \in D^- \cup D^+} \mathcal{O}(d).$$

Since $f$ is injective and continuous on each of its contraction pieces, for each $i \in \{1, \ldots, N\}$ the continuous extension $f_i$ is either strictly increasing or strictly decreasing. This implies that each atom of the first generation is a compact interval the end points of which are different and belong to the set $D$. Moreover, at least one end point of each atom of the first generation belongs to $D^- \cup D^+$. Now, by induction on $n$, we prove that for every $n \geqslant 2$ and every $A \in \mathcal{A}_n$ there exist $a, b \in \mathcal{U}$ such that $A = [a, b]$ with $a \neq b$ and $a$ or $b$ in $\mathcal{U}^*$. Assume that it is true for some $n \geqslant 1$ and let $A := [a, b] \in \mathcal{A}_{n+1}$. Then, by definition of the atoms, there exist $A' := [a', b'] \in \mathcal{A}_n$ and $i \in \{1, \ldots, N\}$ such that $A = \overline{f(A' \cap X_i)} = f_i(\overline{A' \cap X_i})$. If $A' \subset X_i$, then $\{a, b\} = \{f(a'), f(b')\}$. If not, then $\overline{A' \cap X_i}$ is $[c_{i-1}, b']$ or $[a', c_i]$ or $[c_{i-1}, c_i]$ and $\{a, b\}$ is $\{d_{i-1}^+, f(b')\}$ or $\{f(a'), d_i^-\}$ or $\{d_{i-1}^+, d_i^-\}$. In any case, $a \neq b$ belong to $\mathcal{U}$ and $a$ or $b \in \mathcal{U}^*$, because $f_i$ is injective and by the induction hypothesis.

Note that if $f$ is increasing on each of its contraction pieces, then we obtain with a similar induction that for every $n \geqslant 1$ and every $A \in \mathcal{A}_n$ there exist

$$a \in \mathcal{U}^+ := \bigcup_{d \in D^+ \cup \{d_0\}} \mathcal{O}(d) \quad \text{and} \quad b \in \mathcal{U}^- := \bigcup_{d \in D^- \cup \{d_N\}} \mathcal{O}(d)$$

such that $A = [a, b]$ with $a \neq b$ and $a$ or $b$ in $\mathcal{U}^*$.

Now let $x_0 \in \Lambda$ and $\{A_n\}_{n \geqslant 1}$ be a decreasing sequence of atoms such that $A_n \in \mathcal{A}_n$ for all $n \geqslant 1$ and

$$\{x_0\} = \bigcap_{n \geqslant 1} A_n.$$

The existence of $\{A_n\}_{n \geqslant 1}$ is an immediate consequence of the properties of the atoms.

Let $\{a_n\}_{n \geqslant 1}$ and $\{b_n\}_{n \geqslant 1}$ be two sequences of $\mathcal{U}$ such that $A_n = [a_n, b_n]$ for all $n \geqslant 1$. Since the diameter of $A_n$ tends to zero as $n$ goes to infinity, we deduce that $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = x_0$. Besides, as $a_n \neq b_n$ for all $n \geqslant 1$, one of the sequences $\{a_n\}_{n \geqslant 1}$ or $\{b_n\}_{n \geqslant 1}$, let us say $\{a_n\}_{n \geqslant 1}$, is not eventually equal to $x_0$.

(1) As $\{a_n\}_{n\geqslant 1}$ converges to $x_0$ and is not eventually equal to $x_0$, it contains a subsequence $\{a_{n_k}\}_{k\geqslant 1}$ whose terms are all pairwise different. Since $\{a_n\}_{n\geqslant 1} \subset \mathcal{U}$ and $\mathcal{U}$ is a finite union of orbits, we can choose $\{n_k\}_{k\geqslant 1}$ in such a way that for some $d \in D$ the subsequence $\{a_{n_k}\}_{k\geqslant 1}$ satisfies $a_{n_k} \in \mathcal{O}(d)$ for all $k \geqslant 1$. Therefore, there exists a sequence $\{t_k\}_{k\geqslant 1}$ such that

$$a_{n_k} = f^{t_k}(d) \quad \text{for all } k \geqslant 1.$$

Since $a_{n_i} \neq a_{n_j}$ if $i \neq j$, there exists an increasing subsequence $\{t_{k_j}\}_{j\geqslant 1}$ of $\{t_k\}_{k\geqslant 1}$ such that

$$\lim_{j\to\infty} f^{t_{k_j}}(d) = \lim_{k\to\infty} a_{n_k} = x_0$$

and we obtain that $x_0 \in \omega(d)$. This proves that $\Lambda = \bigcup_{d\in D} \omega(d)$.

(2) Now suppose that $x_0 \in \widetilde{X}$ is periodic and let $\rho := \boldsymbol{\nu}(\mathcal{O}(x_0), \Delta)$, as in Lemma 2.2. Let $n_0 \geqslant 1$ be such that the diameter of $A_{n_0} = [a_{n_0}, b_{n_0}]$ is smaller than $\rho$. Then, applying Lemma 2.2, we obtain that $\mathcal{O}(x_0) = \omega(a_{n_0}) = \omega(b_{n_0})$. Since $a_{n_0}$ or $b_{n_0}$ belongs to $\mathcal{U}^*$, we deduce that there exists $d \in D^- \cup D^+$ such that $\omega(d) = \mathcal{O}(x_0)$. Now, if $f$ is increasing on each of its contraction pieces, then $a_{n_0} \in \mathcal{U}^+$ and $b_{n_0} \in \mathcal{U}^-$ and we can conclude that there exist $d^- \in D^- \cup \{d_N\}$ and $d^+ \in D^+ \cup \{d_0\}$ such that $\mathcal{O}(x_0) = \omega(d^-) = \omega(d^+)$. □

Note that Lemma 2.1 and Theorem 2.1 immediately imply that $\Lambda$ is a pseudo-invariant set. Later, we will use the following lemma (Lemma 2.3), which ensures that, besides, the $\omega$-limit set of any point of $\widetilde{X}$ and the attractor contain points of $\widetilde{X}$.

LEMMA 2.3. *If $D \subset \widetilde{X}$ and $\emptyset \neq G \subset X$ is pseudo-invariant, then $G \cap \widetilde{X} \neq \emptyset$.*

*Proof.* Let $y \in G \setminus \widetilde{X}$. Let $t \geqslant 0$ be the smallest integer such that $c_j := f^t(y) \in G \cap \Delta$ for some $j \in \{1, \ldots, N-1\}$. Since $G$ is a pseudo-invariant set, we have that $d_j^+ \in G$ or $d_j^- \in G$. Therefore, $G \cap \widetilde{X} \neq \emptyset$, because by hypothesis $d_j^-$ and $d_j^+$ belong to $\widetilde{X}$. □

2.2. *Periodic and Cantor limit sets.* Here, we relate the asymptotic properties of any orbit in $\widetilde{X}$ to its recurrence properties in a neighborhood of $\Delta$. Precisely, for each point $x \in \widetilde{X}$ we define the (maybe empty) set $\Delta_{lr}(x) \subset \Delta$ consisting of the points in $\Delta$ on which the orbit of $x$ accumulates from both sides (see Definition 2.2). Then we obtain the following dichotomic result: if $\Delta_{lr}(x) = \emptyset$, then the $\omega$-limit set of $x$ is a periodic orbit in $\widetilde{X}$ (Theorem 2.2) and, if $\Delta_{lr}(x) \neq \emptyset$, then the $\omega$-limit set of $x$ is a $\widetilde{X}$-minimal Cantor set (Theorem 2.3).

*Definition 2.2.* (Left–right recurrently visited point) Let $i \in \{1, \ldots, N-1\}$ and $x \in \widetilde{X}$. We say that $c_i \in \Delta$ is *left–right recurrently visited* (in short $lr$-recurrently visited) by the orbit of $x$ if there exist two strictly increasing sequences $\{l_j\}_{j\in\mathbb{N}}$ and $\{r_j\}_{j\in\mathbb{N}}$ of natural numbers such that

$$f^{l_j}(x) \in X_i \quad \text{and} \quad f^{r_j}(x) \in X_{i+1} \quad \text{for all } j \in \mathbb{N} \quad \text{and}$$

$$c_i = \lim_{j\to\infty} f^{l_j}(x) = \lim_{j\to\infty} f^{r_j}(x).$$

We denote by $\Delta_{lr}(x) \subset \Delta$ the set of points in $\Delta$ that are $lr$-recurrently visited by the orbit of $x$, and we denote by $\Delta_{lr}$ the set of points in $\Delta$ which are $lr$-recurrently visited by the orbit of some point in $\widetilde{X}$.

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*Remark 2.1.* Even if not immediate, it is not difficult to check that Definition 2.2 of the set $\Delta_{lr}(x)$ *is equivalent* to the combinatorial definition of the set of $lr$-recurrently visited discontinuities in [**4**, Definition 2.8].

The basic properties of the $lr$-recurrently visited points are given in the following lemma.

LEMMA 2.4. *Let $i \in \{1, \ldots, N-1\}$, $x \in \widetilde{X}$ and suppose that $c_i \in \Delta_{lr}(x)$. Then $c_i$, $d_i^+$ and $d_i^-$ belong to $\omega(x)$. If moreover $D \subset \widetilde{X}$, then $\overline{\mathcal{O}(d_i^-)} \cup \overline{\mathcal{O}(d_i^+)} \subset \omega(x)$.*

*Proof.* By definition of $\omega$-limit set and of $lr$-recurrently visited point, if $c_i \in \Delta_{lr}(x)$, then $c_i \in \omega(x)$. We can show that this implies that $d_i^+$ and $d_i^-$ belong to $\omega(x)$ with a similar proof as that of Lemma 2.1. If we suppose moreover that $D \subset \widetilde{X}$, then $\mathcal{O}(d_i^-)$ and $\mathcal{O}(d_i^+) \subset \omega(x)$, since $\omega(x) \cap \widetilde{X}$ is invariant by pseudo-invariance of $\omega(x)$. The desired inclusion follows from the compactness of $\omega(x)$. □

THEOREM 2.2. (Periodic $\omega$-limit sets) *Suppose that $f$ is such that $D \subset \widetilde{X}$. Let $x \in \widetilde{X}$; then $\omega(x)$ is a periodic orbit contained in $\widetilde{X}$ if and only if $\Delta_{lr}(x) = \emptyset$.*

*Proof.* Let $x \in \widetilde{X}$. Suppose that $\omega(x)$ is contained in $\widetilde{X}$. Then it follows from Lemma 2.4 that $\Delta_{lr}(x) = \emptyset$. Indeed, if $\Delta_{lr}(x) \neq \emptyset$, then there is some point of $\Delta$ in $\omega(x)$ and therefore $\omega(x)$ is not contained in $\widetilde{X}$. Now we suppose that $\Delta_{lr}(x) = \emptyset$ and we prove that $\omega(x)$ is a periodic orbit contained in $\widetilde{X}$.

We first show that, under the hypothesis $\Delta_{lr}(x) = \emptyset$, the itinerary of $x$ is eventually periodic. Let $\eta \in \{1, \ldots, N\}^{\mathbb{N}}$ be the itinerary of $x$ and, for any $n \geqslant 1$, define the set

$$L_n(\eta) := \{(\eta_t, \eta_{t+1}, \ldots, \eta_{t+n-1}) \in \{1, \ldots, N\}^n : t \geqslant 0\}$$

of the words of size $n$ contained in $\eta$. The function $p_\eta$ defined for any $n \geqslant 1$ by $p_\eta(n) := \#L_n(\eta)$ is the complexity function of $\eta$. By the Morse–Hedlund theorem [**9**], if $p_\eta$ is eventually constant, then $\eta$ is eventually periodic. Obviously $\#L_n(\eta) \leqslant \#L_{n+1}(\eta)$. So, we have to show that if $\Delta_{lr}(x) = \emptyset$, then there exists $n_0 \geqslant 1$ such that the converse inequality also holds and therefore

$$\#L_n(\eta) = \#L_{n+1}(\eta) \quad \text{for all } n \geqslant n_0. \tag{7}$$

To that aim, recall that $f^{t+n}(x) \in A_{\eta_t, \eta_{t+1}, \ldots, \eta_{t+n-1}}$ for every $t \geqslant 0$ and $n \geqslant 1$.

First, let us prove that for any $n \geqslant 1$, we have

$$L_{n+1}(\eta) \subset \bigcup_{(i_1, \ldots, i_n) \in L_n(\eta)} \{(i_1, \ldots, i_n, i_{n+1}) : \exists\, t \geqslant 0 \ : \ f^{t+n}(x) \in A_{i_1, \ldots, i_n} \cap X_{i_{n+1}}\}. \tag{8}$$

Indeed, if $(i_1, \ldots, i_{n+1}) \in L_{n+1}(\eta)$, then there exists $t \geqslant 0$ such that

$$(\eta_t, \ldots, \eta_{t+n}) = (i_1, \ldots, i_{n+1})$$

and, by definition of $L_n(\eta)$ and of the itinerary $\eta$, we have that $(i_1, \ldots, i_n) \in L_n(\eta)$ and $f^{t+n}(x) \in X_{i_{n+1}}$. As $f^{t+n}(x) \in A_{i_1, \ldots, i_n}$, we conclude that there exists $t \geqslant 0$ such that

$$(i_1, \ldots, i_n) \in L_n(\eta) \quad \text{and} \quad f^{t+n}(x) \in A_{i_1, \ldots, i_n} \cap X_{i_{n+1}},$$

that is, $(i_1, \ldots, i_{n+1})$ belongs to the set of the right-hand side of the inclusion (8).

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Now, if $\Delta_{lr}(x) = \emptyset$, then there exists $\epsilon > 0$ such that

$$\mathcal{O}(x) \cap (c_i - \epsilon, c_i) = \emptyset \quad \text{or} \quad \mathcal{O}(x) \cap (c_i, c_i + \epsilon) = \emptyset \quad \text{for all } i \in \{1, \ldots, N-1\}.$$

Also, we know that there exists $n_0 \geqslant 1$ such that diam $A < \epsilon$ for all $A \in \mathcal{A}_n$ and $n \geqslant n_0$. Therefore, if $n \geqslant n_0$, then for any $(i_1, \ldots, i_n) \in L_n(\eta)$ fixed, we have that

$$\#\{(i_1, \ldots, i_n, i_{n+1}) : \exists\, t \geqslant 0 \;:\; f^{t+n}(x) \in A_{i_1,\ldots,i_n} \cap X_{i_{n+1}}\} = 1.$$

Thus, from (8), we conclude that $\#L_{n+1}(\eta) \leqslant \#L_n(\eta)$ for all $n \geqslant n_0$, which ends the proof of (7).

Since we have proved that the itinerary $\eta$ of $x$ is eventually periodic, we know that there exist $t \geqslant 0$ and $p \geqslant 1$ such that $\theta := \{\eta_{t+n}\}_{n\in\mathbb{N}}$ is a periodic sequence with period $p$. Let $y := f^t(x)$. As $\omega(x) = \omega(y)$, to finish the proof, we show that $\omega(y)$ is a periodic orbit contained in $\widetilde{X}$.

Since $\theta$ is the itinerary of $y$, we deduce that

$$f^{k+p}(y) \in A_{\theta_k,\ldots,\theta_{k+p-1}} \quad \text{for all } k \in \{0, 1, \ldots, p-1\}.$$

More generally,

$$f^{k+jp}(y) \in A_{\theta_k,\ldots,\theta_{k+jp-1}} \quad \text{for all } j \geqslant 1 \text{ for all } k \in \{0, 1, \ldots, p-1\}. \tag{9}$$

Besides,

$$A_{\theta_k,\ldots,\theta_{k+p-1}} \supset A_{\theta_k,\ldots,\theta_{k+2p-1}} \supset \cdots \supset A_{\theta_k,\ldots,\theta_{k+jp-1}} \supset \cdots$$

is a decreasing sequence of (non-empty compact) atoms whose diameters converge to zero. Then there exists $x_k^* \in X$ such that

$$\bigcap_{j\geqslant 1} A_{\theta_k,\ldots,\theta_{k+jp-1}} = \{x_k^*\}. \tag{10}$$

Considering all the values of $k \in \{0, 1, \ldots, p-1\}$, we conclude that

$$\{x_0^*, x_1^*, \ldots, x_{p-1}^*\} \subset \omega(y). \tag{11}$$

Now let us prove the converse inclusion. If $z \in \omega(y)$, then there exists a strictly increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ such that $f^{m_n}(y)$ converges to $z$ when $n$ goes to infinity. Let $\{q_n\}_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ and $\{r_n\}_{n\in\mathbb{N}} \in \{0, 1, \ldots, p-1\}^{\mathbb{N}}$ be such that

$$m_n = q_n p + r_n \quad \text{for all } n \in \mathbb{N}.$$

Since $\{m_n\}_{n\in\mathbb{N}}$ is strictly increasing and $\{r_n\}_{n\in\mathbb{N}}$ takes only a finite number of values, the sequence of integer quotients $\{q_n\}_{n\in\mathbb{N}}$ is also strictly increasing. Besides, there exist $\{n_j\}_{j\in\mathbb{N}}$ and $k \in \{0, 1, \ldots, p-1\}$ such that $r_{n_j} = k$ for all $j \in \mathbb{N}$. We deduce that

$$z = \lim_{n\to\infty} f^{m_n}(y) = \lim_{j\to\infty} f^{q_{n_j}p+k}(y) \in \bigcap_{j\geqslant 1} A_{\theta_k,\ldots,\theta_{k+q_{n_j}p-1}} = \{x_k^*\}.$$

Therefore, we have proved that $z \in \{x_0^*, x_1^*, \ldots, x_{p-1}^*\}$ for any $z \in \omega(y)$. Together with (11), this implies that

$$\omega(y) = \{x_0^*, x_1^*, \ldots, x_{p-1}^*\}. \tag{12}$$

Finally, let us prove that $\omega(y)$ is a periodic orbit contained in $\widetilde{X}$. By Lemma 2.3, we know that $\omega(y) \cap \widetilde{X} \neq \emptyset$. Thus, there exists $k \in \{0, 1, \ldots, p-1\}$ such that $x_k^* \in \widetilde{X}$. This implies that the distance $\rho$ between $x_k^*$ and any element of $\Delta$ is positive. Since the diameter of the atoms decreases with their generation, there exists $j_0$ such that

$$\operatorname{diam}(A_{\theta_k,\ldots,\theta_{k+jp-1}}) < \rho \quad \text{for all } j \geqslant j_0.$$

From the equality (10), we deduce that for any $j \geqslant j_0$ the atom $A_{\theta_k,\ldots,\theta_{k+jp-1}}$ is contained in the same contraction piece as $x_k^*$. On the other hand, by (9) and the definition of itinerary,

$$f^{k+jp}(y) \in A_{\theta_k,\ldots,\theta_{k+jp-1}} \cap X_{\theta_{k+jp}} \quad \text{for all } j \geqslant 1.$$

This implies that for any $j \geqslant j_0$ the atom $A_{\theta_k,\ldots,\theta_{k+jp-1}}$ is contained in $X_{\theta_{k+jp}}$. Therefore, for every $j \geqslant j_0$, we have

$$f(A_{\theta_k,\ldots,\theta_{k+jp-1}}) = \overline{f(A_{\theta_k,\ldots,\theta_{k+jp-1}} \cap X_{\theta_{k+jp}})} = A_{\theta_k,\ldots,\theta_{k+jp}} \subset A_{\theta_{k+1},\ldots,\theta_{k+jp}}.$$

Now we can conclude from the equality (10) that

$$\{f(x_k^*)\} \subset \bigcap_{j \geqslant j_0} f(A_{\theta_k,\ldots,\theta_{k+jp-1}}) \subset \bigcap_{j \geqslant j_0} A_{\theta_{k+1},\ldots,\theta_{k+jp}} = \{x^*_{k+1 \ (\mathrm{mod}\ p)}\}.$$

Then $f(x_k^*) = x^*_{k+1 \ (\mathrm{mod}\ p)} \in \widetilde{X}$, since $x_k^* \in \widetilde{X}$. So, we can repeat the same argument for all the iterates of $x_k^*$ to obtain $f^l(x_k^*) = x^*_{k+l \ (\mathrm{mod}\ p)} \in \widetilde{X}$ for all $l \geqslant 1$. We conclude that $\omega(y) = \{x_0^*, x_1^*, \ldots, x_{p-1}^*\} = \omega(x)$ is a periodic orbit contained in $\widetilde{X}$, as wanted. □

Now we state the complementary results of Theorem 2.2. Its proof needs a larger development, which is done in §3.

THEOREM 2.3. (Cantor $\omega$-limit sets) *Suppose that $f$ is injective on each of its contraction pieces and that $D \subset \widetilde{X}$. Then, for any $x \in \widetilde{X}$, $\Delta_{lr}(x) \neq \emptyset$ if and only if $\omega(x)$ is a $\widetilde{X}$-minimal Cantor set.*

*Proof.* See §3. □

2.3. *Proof of Theorem 1.1.* Now we prove Theorem 1.1 assuming Theorem 2.3.

(1) For any $d \in D$, either $\Delta_{lr}(d) = \emptyset$ and applying Theorem 2.2 it follows that $\omega(d)$ is a periodic orbit contained in $\widetilde{X}$, or $\Delta_{lr}(d) \neq \emptyset$ and applying Theorem 2.3 we deduce that $\omega(d)$ is a $\widetilde{X}$-minimal Cantor set. So, we can rewrite (6) as follows:

$$\Lambda = \bigcup_{d \in D} \omega(d) = \left(\bigcup_{i=1}^{N_1} \mathcal{O}_i\right) \cup \left(\bigcup_{j=1}^{N_2} K_j\right), \tag{13}$$

where $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{N_1} \subset \widetilde{X}$ are periodic orbits and $K_1, K_2, \ldots, K_{N_2}$ are $\widetilde{X}$-minimal Cantor sets. As $D \subset \widetilde{X}$, Lemma 2.1 ensures that the Cantor sets are pseudo-invariant.

(2) Now let us prove that the $\omega$-limit set of any point $x \in \widetilde{X}$ coincides either with one periodic orbit $\mathcal{O}_i$ or with one Cantor set $K_j$. First, recall that the $\omega$-limit set $\omega(x)$ of any point $x \in \widetilde{X}$ satisfies $\omega(x) \cap \widetilde{X} \neq \emptyset$ (see Lemma 2.3). Then there exists $y \in \omega(x) \cap \widetilde{X}$. Since $\omega(x) \subset \Lambda$, from Theorem 2.1 we deduce that there exists $d \in D$ such that $y \in \omega(d)$,

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so $y \in \omega(x) \cap \omega(d) \cap \widetilde{X}$. Besides, $x, d \in \widetilde{X}$, so we can apply Theorems 2.2 and 2.3 to deduce that both $\omega(x)$ and $\omega(d)$ are $\widetilde{X}$-minimal sets. Therefore,

$$\omega(x) = \overline{\mathcal{O}(y)} = \omega(d).$$

This proves that $\omega(x)$ coincides with some set of the decomposition (13) and it also proves that the sets of the decomposition (13) are all pairwise $\widetilde{X}$-disjoint. We conclude that, for any $x \in \widetilde{X}$, either there exists $i \in \{1, \ldots, N_1\}$ such that $\omega(x) = \mathcal{O}_i$ or there exists $j \in \{1, \ldots, N_2\}$ such that $\omega(x) = K_j$.

(3) Suppose that $N_2 \geqslant 1$. Let $j \in \{1, \ldots, N_2\}$ and let $d \in D$ be such that $\omega(d) = K_j$. Since $\omega(d) = K_j$, according to Theorem 2.3 there exists $k \in \{1, \ldots, N-1\}$ such that $c_k \in \Delta_{lr}(d)$. From Lemma 2.4, it follows that $c_k, d_k^-$ and $d_k^+ \in \omega(d) = K_j$. As $D \subset \widetilde{X}$ and $K_j$ is $\widetilde{X}$-minimal, we have that $\overline{\mathcal{O}(d_k^-)} = K_j = \overline{\mathcal{O}(d_k^+)}$.

(4) Let $j \in \{1, \ldots, N_2\}$ and $k \in \{1, \ldots, N-1\}$ be such that $c_k \in K_j$. Since $K_j$ is pseudo-invariant, we deduce that $d_k^-$ or $d_k^+ \in K_j$. As $D \subset \widetilde{X}$ and $K_j$ is $\widetilde{X}$-minimal, we have that $K_j = \overline{\mathcal{O}(d_k^+)}$ or $K_j = \overline{\mathcal{O}(d_k^-)}$. Suppose moreover that $c_k \in K_j$ does not belong to the boundary of a gap of $K_j$. If $K_j = \overline{\mathcal{O}(d_k^+)}$, then $c_k \in \Delta_{lr}(d_k^+)$ and from Lemma 2.4 it follows that $d_k^- \in K_j$. Since $K_j$ is $\widetilde{X}$-minimal, we obtain that $\overline{\mathcal{O}(d_k^-)} = K_j$. An analogous proof allows us to show that $K_j = \overline{\mathcal{O}(d_k^+)}$ in the case where $K_j = \overline{\mathcal{O}(d_k^-)}$.

(5) From (13), it follows immediately that $1 \leqslant N_1 + N_2 \leqslant \# D$. Now we show that

$$N_1 + 2N_2 \leqslant 2(N-1).$$

Let $d'_1, d'_2, \ldots, d'_{2(N-1)}$ be such that

$$d'_{2k-1} := d_k^- \quad \text{and} \quad d'_{2k} := d_k^+ \quad \text{for all } k \in \{1, \ldots, N-1\}.$$

Consider the sets $C_1 := \{l \in \{1, \ldots, 2(N-1)\} : \Delta_{lr}(d'_l) = \emptyset\}$ and $C_2 := \{l \in \{1, \ldots, 2(N-1)\} : \Delta_{lr}(d'_l) \neq \emptyset\}$. Let $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_{N_1} \subset \widetilde{X}$ and $K_1, K_2, \ldots, K_{N_2}$ be the periodic orbits and the $\widetilde{X}$-minimal Cantor sets of the decomposition (13), respectively.

From part (2) of Theorem 2.1, we know that for every $i \in \{1, \ldots, N_1\}$ there exists $l(i) \in C_1$ such that

$$\mathcal{O}_i = \omega(d'_{l(i)}).$$

The function $l : \{1, \ldots, N_1\} \to C_1$ defined by $i \mapsto l(i)$ being injective, we have that $N_1 \leqslant \# C_1$.

From part (3), we know that for every $j \in \{1, \ldots, N_2\}$ there exists an odd number $\ell(j) \in C_2$ such that

$$K_j = \overline{\mathcal{O}(d'_{\ell(j)})} = \overline{\mathcal{O}(d'_{\ell(j)+1})}.$$

The function $(j, s) \mapsto \ell(j) + s$ from the set $\{1, \ldots, N_2\} \times \{0, 1\}$ to the set $C_2$ being injective, we obtain that $2N_2 \leqslant \# C_2$, which together with $N_1 \leqslant \# C_1$ gives

$$N_1 + 2N_2 \leqslant \# C_1 + \# C_2 = \#(C_1 \cup C_2) = 2(N-1).$$

Finally, suppose that $f$ is increasing on each of its contraction pieces. Let $d'_0 := d_0$, $d'_{2N-1} := d_N$ and

$$C := \{l \in \{0, 1, \ldots, 2N-1\} \; : \; \Delta_{lr}(d'_l) = \emptyset\}.$$

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Then, from part (2) of Theorem 2.1, we know that for every $i \in \{1, \ldots, N_1\}$ there exist an odd number $l_1(i) \in C$ and an even number $l_2(i) \in C$ such that

$$\mathcal{O}_i = \omega(d'_{l_1(i)}) = \omega(d'_{l_2(i)}).$$

The function $(i, s) \mapsto l_s(i)$ from the set $\{1, \ldots, N_1\} \times \{1, 2\}$ to the set $C$ being injective, we obtain that $2N_1 \leqslant \#C$, which together with $2N_2 \leqslant \#C_2$ gives

$$2N_1 + 2N_2 \leqslant \#C + \#C_2 = \#(C \cup C_2) = 2N.$$

This ends the proof of Theorem 1.1 assuming Theorem 2.3.

## 3. *Proof of Theorem 2.3*

All through this section we assume that $f$ is such that $D \subset \widetilde{X}$ and $\Delta_{lr} \neq \emptyset$. In other words, we suppose that $f$ has at least one point $c \in \Delta$ which is $lr$-recurrently visited by the orbit of some point $x \in \widetilde{X}$. We already know by Theorem 2.2 that this implies that $\omega(x)$ is not a periodic orbit in $\widetilde{X}$. In §3.1, we will first show a stronger preliminary result: $\omega(x)$ cannot contain a periodic point belonging to $\widetilde{X}$. It will imply that the orbits of the one-sided limits of $f$ at the points of $\Delta_{lr}(x)$ do not accumulate at periodic points contained in $\widetilde{X}$. These preliminary results will be used in §3.4 to prove that the $\omega$-limit set of some particular points of $D$ is $\widetilde{X}$-minimal.

In §3.2, we construct a partial order in a quotient set of $\Delta_{lr}$. This allows us to define minimal classes of points of $\Delta$, which are the minimal nodes in the Hasse graph of such a partial order (Definition 3.3). The study of the asymptotic dynamics of a point $x$ satisfying $\Delta_{lr}(x) \neq \emptyset$ can be done by analyzing the minimal classes. Indeed, in §3.3, we show that if $\Delta_{lr}(x) \neq \emptyset$, then $\omega(x)$ is equal to $\omega(d)$, where $d \in D$ is a one-sided limit of $f$ at a point of $\Delta_{lr}(x)$ belonging to a minimal class (Theorem 3.1). In §3.4, we study the $\omega$-limit sets of the elements of $D$ associated to a minimal class and show that they are $\widetilde{X}$-minimal Cantor sets (Theorem 3.2). These two results allow us to complete the proof of Theorem 2.3.

### 3.1. *Preliminary results.*

LEMMA 3.1. *Let $x \in \widetilde{X}$ and suppose that $f$ has a periodic point $p \in \widetilde{X}$. If $p \in \omega(x)$, then $\omega(x) = \mathcal{O}(p)$.*

*Proof.* It is a direct consequence of Lemma 2.2. □

COROLLARY 3.1. *Let $x \in \widetilde{X}$ and $i \in \{1, \ldots, N-1\}$. If $c_i \in \Delta_{lr}(x)$, then $\omega(x) \cap \widetilde{X}$, $\omega(d_i^+) \cap \widetilde{X}$ and $\omega(d_i^-) \cap \widetilde{X}$ do not contain any periodic point.*

*Proof.* Suppose that $c_i \in \Delta_{lr}(x)$; then from Theorem 2.2 we deduce that $\omega(x)$ is not a periodic orbit of $\widetilde{X}$. Therefore, by Lemma 3.1, it does not contain any periodic point belonging to $\widetilde{X}$. On the other hand, since $D \subset \widetilde{X}$, by Lemma 2.4, we have that $\omega(d_i^+) \cup \omega(d_i^-) \subset \omega(x)$. It follows that neither $\omega(d_i^+)$ nor $\omega(d_i^-)$ contains a periodic point in $\widetilde{X}$. □

COROLLARY 3.2. *Let $i \in \{1, \ldots, N-1\}$ and $c_i \in \Delta_{lr}$. Then $\Delta_{lr}(d_i^-) \neq \emptyset$ and $\Delta_{lr}(d_i^+) \neq \emptyset$.*

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*Proof.* Suppose that $c_i \in \Delta_{lr}$; then, by Definition 2.2, there exists $x \in \widetilde{X}$ such that $c_i \in \Delta_{lr}(x)$. From Corollary 3.1, we deduce that neither $\omega(d_i^+)$ nor $\omega(d_i^-)$ is a periodic orbit contained in $\widetilde{X}$. Applying Theorem 2.2, we deduce that $\Delta_{lr}(d_i^-) \neq \emptyset$ and $\Delta_{lr}(d_i^+) \neq \emptyset$. □

3.2. *Equivalence classes in $\Delta_{lr}$ and their partial order.* Here, we introduce an equivalence relation in $\Delta_{lr}$ and a partial order in the resulting quotient space. This allows us to identify some classes of points of $\Delta_{lr}$ which are minimal elements with respect to the partial order. These minimal classes will be of special importance to study the non-periodic asymptotic dynamics.

Before defining our equivalence relation, let us prove the following lemma.

LEMMA 3.2. *Let $x \in \widetilde{X}$. If there exist $i$ and $k \in \{1, \ldots, N-1\}$ such that $c_i \in \Delta_{lr}(d_k^+)$ and $c_k \in \Delta_{lr}(x)$, then $c_i \in \Delta_{lr}(x)$.*

*Proof.* If $c_k \in \Delta_{lr}(x)$, then $\mathcal{O}(d_k^+) \subset \omega(x)$; see Lemma 2.4. This implies that the orbit of $x$ accumulates at any point of the orbit of $d_k^+$. On the other hand, we have $c_i \in \Delta_{lr}(d_k^+)$. This means that the orbit of $d_k^+$ accumulates at $c_i$ from the left and from the right. Joining the two latter assertions, we conclude that the orbit of $x$ also accumulates at $c_i$ from the left and from the right. In other words, $c_i \in \Delta_{lr}(x)$. □

*Definition 3.1.* Let $i$ and $j \in \{1, \ldots, N-1\}$ be such that $c_i$ and $c_j \in \Delta_{lr}$. We write $c_i \sim^+ c_j$ and we say that $c_i$ and $c_j$ are related if and only if

$$c_i = c_j \quad \text{or} \quad c_i \in \Delta_{lr}(d_j^+) \quad \text{and} \quad c_j \in \Delta_{lr}(d_i^+).$$

LEMMA 3.3. *The relation $\sim^+$ is an equivalence relation on $\Delta_{lr}$.*

*Proof.* The reflexive and the symmetric properties follow immediately from the definition of the relation $\sim^+$. So, it remains to prove the transitive property. Let $i$, $j$ and $k \in \{1, \ldots, N-1\}$ be such that $c_i$, $c_j$ and $c_k \in \Delta_{lr}$. Let us suppose that $c_i \sim^+ c_j$ and $c_j \sim^+ c_k$ and let us show that $c_i \sim^+ c_k$. This assertion holds trivially if $c_i = c_j$ or $c_j = c_k$. If $c_i \neq c_j$ and $c_j \neq c_k$, by definition of the relation $\sim^+$, we have

$$c_i \in \Delta_{lr}(d_j^+), \quad c_j \in \Delta_{lr}(d_k^+), \quad c_k \in \Delta_{lr}(d_j^+) \quad \text{and} \quad c_j \in \Delta_{lr}(d_i^+).$$

Applying Lemma 3.2, we conclude that $c_i \in \Delta_{lr}(d_k^+)$ and $c_k \in \Delta_{lr}(d_i^+)$, which implies that $c_i \sim^+ c_k$. □

For any point $c \in \Delta_{lr}$, we let $[c]$ denote the equivalence class of $c$. In order to define an order relation on the (non-empty) set $\Delta_{lr}/\sim^+$ of the equivalence classes of $\Delta_{lr}$, we first prove the following lemma.

LEMMA 3.4. *Let $i$ and $j \in \{1, \ldots, N-1\}$ be such that $c_i$ and $c_j \in \Delta_{lr}$. If $c_i \in \Delta_{lr}(d_j^+)$, then $c_{i'} \in \Delta_{lr}(d_{j'}^+)$ for all $i'$ and $j' \in \{1, \ldots, N-1\}$ such that $c_{i'} \in [c_i]$ and $c_{j'} \in [c_j]$.*

*Proof.* Suppose that $c_{i'} \sim^+ c_i$ and $c_{j'} \sim^+ c_j$. First, assume that $c_{i'} \neq c_i$ and $c_{j'} \neq c_j$. In this case, the definition of $\sim^+$ implies that

$$c_{i'} \in \Delta_{lr}(d_i^+) \quad \text{and} \quad c_j \in \Delta_{lr}(d_{j'}^+).$$

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Applying Lemma 3.2 for $c_{i'} \in \Delta_{lr}(d_i^+)$ and $c_i \in \Delta_{lr}(d_j^+)$, we obtain that $c_{i'} \in \Delta_{lr}(d_j^+)$. Applying once again the same lemma but for $c_{i'} \in \Delta_{lr}(d_j^+)$ and $c_j \in \Delta_{lr}(d_{j'}^+)$, we conclude that $c_{i'} \in \Delta_{lr}(d_{j'}^+)$, as wanted. To obtain the same result in the complementary case $c_{i'} = c_i$ or $c_{j'} = c_j$, we can use similar arguments. □

*Definition 3.2.* Let $i$ and $j \in \{1, \ldots, N-1\}$ be such that $c_i$ and $c_j \in \Delta_{lr}$. We define the relation $\preccurlyeq^+$ between the equivalence classes $[c_i]$ and $[c_j]$ in $\Delta_{lr}/\sim^+$ by

$$[c_i] \preccurlyeq^+ [c_j] \quad \text{if and only if } [c_i] = [c_j] \quad \text{or} \quad c_i \in \Delta_{lr}(d_j^+).$$

Note that Lemma 3.4 proves that the above definition is well posed, since it is independent of the choice of the elements $c_i$, $c_j$ in the equivalence classes $[c_i]$ and $[c_j]$.

LEMMA 3.5. *$(\Delta_{lr}/\sim^+, \preccurlyeq^+)$ is a partially ordered set.*

*Proof.* Take $[c]$, $[c']$ and $[c''] \in \Delta_{lr}/\sim^+$. Let $i$, $j$ and $k \in \{1, \ldots, N-1\}$ be such that $[c_i] = [c]$, $[c_j] = [c']$ and $[c_k] = [c'']$.

*Reflexive property*: It follows trivially from Definition 3.2.

*Antisymmetric property*. Suppose that $[c_i] \preccurlyeq^+ [c_j]$ and $[c_j] \preccurlyeq^+ [c_i]$. Then, from Definition 3.2, it follows that either $[c_i] = [c_j]$, and we are done, or $c_i \in \Delta_{lr}(d_j^+)$ and $c_j \in \Delta_{lr}(d_i^+)$. In this last case, we deduce from Definition 3.1 that $c_i \sim^+ c_j$, which implies that $[c_i] = [c_j]$.

*Transitive property*: Suppose that $[c_i] \preccurlyeq^+ [c_j]$ and $[c_j] \preccurlyeq^+ [c_k]$. If $[c_i] = [c_j]$ or $[c_j] = [c_k]$, then $[c_i] \preccurlyeq^+ [c_k]$. Otherwise, we have $c_i \in \Delta_{lr}(d_j^+)$ and $c_j \in \Delta_{lr}(d_k^+)$. Applying Lemma 3.2, we obtain $c_i \in \Delta_{lr}(d_k^+)$ and we conclude that $[c_i] \preccurlyeq^+ [c_k]$. □

*Definition 3.3.* (Minimal classes) Let $[c] \in \Delta_{lr}/\sim^+$. We say that $[c]$ is a *minimal class* if it is a minimal element of the partially ordered set $(\Delta_{lr}/\sim^+, \preccurlyeq^+)$. In other words, $[c]$ is a minimal class if for every $[c'] \in \Delta_{lr}/\sim^+$ such that $[c'] \preccurlyeq^+ [c]$, we have $[c'] = [c]$.

It is well known that any finite partially ordered set has at least one minimal element. Since our partially ordered set $(\Delta_{lr}/\sim^+, \preccurlyeq^+)$ is finite, it always has minimal classes.

PROPOSITION 3.1.

(a) *Let $j \in \{1, \ldots, N-1\}$ be such that $c_j \in \Delta_{lr}$. Then there exists $i \in \{1, \ldots, N-1\}$ such that $[c_i]$ is a minimal class and $[c_i] \preccurlyeq^+ [c_j]$.*

(b) *Let $[c] \in \Delta_{lr}/\sim^+$ and $i \in \{1, \ldots, N-1\}$ be such that $c_i \in [c]$. Then $[c]$ is a minimal class if and only if $c_i \in \Delta_{lr}(d_j^+)$ for every $j \in \{1, \ldots, N-1\}$ such that $c_j \in \Delta_{lr}(d_i^+)$.*

*Proof.* (a) For any Hasse graph of a partial order on a finite non-empty set, and for any of its nodes, say $j$, there exists at least one minimal node, say $i$, smaller than or equal to $j$. Applying this assertion to the partially ordered set $(\Delta_{lr}/\sim^+, \preccurlyeq^+)$, we deduce that for all $[c_j] \in \Delta_{lr}/\sim^+$, there exists at least one minimal class $[c_i]$ such that $[c_i] \preccurlyeq^+ [c_j]$.

(b) Let $[c] \in \Delta_{lr}/\sim^+$ and let $i \in \{1, \ldots, N-1\}$ be such that $c_i \in [c]$.

Suppose that $[c]$ is a minimal class. If $c_j \in \Delta_{lr}(d_i^+)$ for some $j \in \{1, \ldots, N-1\}$, then $[c_j] \preccurlyeq^+ [c_i]$. This implies that $[c_j] = [c_i]$, because $[c_i] = [c]$ and $[c]$ is a minimal class. It follows that $c_i \sim^+ c_j$ and therefore we have that $c_i \in \Delta_{lr}(d_j^+)$.

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Now suppose that $c_i \in \Delta_{lr}(d_j^+)$ for all $j \in \{1, \ldots, N-1\}$ such that $c_j \in \Delta_{lr}(d_i^+)$. Let $j \in \{1, \ldots, N-1\}$ be such that $[c_j] \preccurlyeq^+ [c]$. Since $[c] = [c_i]$, to prove that $[c]$ is a minimal class, we have to show that $[c_j] = [c_i]$. By definition of $\preccurlyeq^+$, either $[c_j] = [c_i]$, and we are done, or $c_j \in \Delta_{lr}(d_i^+)$. By hypothesis, the second case implies that $c_i \in \Delta_{lr}(d_j^+)$. It follows that $c_i \sim^+ c_j$ and therefore $[c_j] = [c_i]$. □

3.3. *Asymptotic dynamics and minimal classes.* In this section, we show that the non-periodic asymptotic dynamics is supported on the closure of the orbits of the one-sided limits of the map at its minimal class points. Precisely, we will prove the following theorem.

THEOREM 3.1. *If $x \in \widetilde{X}$ and $\Delta_{lr}(x) \neq \emptyset$, then there exists $i \in \{1, \ldots, N-1\}$ such that $c_i \in \Delta_{lr}(x)$ and $[c_i]$ is a minimal class. Moreover, if $f$ is injective on each of its contraction pieces, then $\omega(x) = \omega(d_i^+) = \overline{\mathcal{O}(d_i^+)}$.*

Note that we can define equivalence classes and a partial order $\preccurlyeq^-$ based on the left-sided limits of the map $f$ at the points of $\Delta_{lr}$, just exchanging the superscripts $+$ and $-$ in our definitions and proofs. Therefore, the same Theorem 3.1 is also true for the left-sided limits of the map. Actually, in the next subsection, Theorem 3.2 will make precise and (re)prove this assertion.

To prove Theorem 3.1, we need the following two lemmas.

LEMMA 3.6. *Let $x \in \widetilde{X}$. There exists $\epsilon(x) > 0$ such that if for some $l, r \in \mathbb{N}$ and $c \in \Delta$ we have $f^l(x) \in (c - \epsilon(x), c)$ and $f^r(x) \in (c, c + \epsilon(x))$, then $c \in \Delta_{lr}(x)$.*

*Proof.* If $\Delta_{lr}(x) = \Delta$, then the lemma is true for any $\epsilon(x) > 0$. Now suppose that $\Delta \setminus \Delta_{lr}(x) \neq \emptyset$. By Definition 2.2, we have that for any $c \in \Delta \setminus \Delta_{lr}(x)$ there exists $\epsilon_c > 0$ such that $f^t(x) \notin (c - \epsilon_c, c)$ for all $t \in \mathbb{N}$ or $f^t(x) \notin (c, c + \epsilon_c)$ for all $t \in \mathbb{N}$. Now we define

$$\epsilon(x) := \min_{c \in \Delta \setminus \Delta_{lr}(x)} \epsilon_c > 0.$$

Suppose that there exist $l, r \in \mathbb{N}$ and $c \in \Delta$ such that

$$f^l(x) \in (c - \epsilon(x), c) \quad \text{and} \quad f^r(x) \in (c, c + \epsilon(x)).$$

Then, by definition of $\epsilon(x)$, we must have that $c \notin \Delta \setminus \Delta_{lr}(x)$. Therefore, $c \in \Delta_{lr}(x)$. □

LEMMA 3.7. *Suppose that $f$ is injective on each of its contraction pieces and let $x \in \widetilde{X}$ be such that $\Delta_{lr}(x) \neq \emptyset$. If there exist $i, j \in \{1, \ldots, N-1\}$ such that*

$$c_i \in \Delta_{lr}(d_j^+) \cap \Delta_{lr}(x) \quad \textit{and} \quad c_j \in \Delta_{lr}(d_i^+) \cap \Delta_{lr}(x), \tag{14}$$

*then there exist $\epsilon_0 > 0$, $m_0 \geqslant 0$ and two sequences $\{\alpha_k\}_{k \in \mathbb{N}}$ and $\{\beta_k\}_{k \in \mathbb{N}}$ such that:*

(1) *$\{\alpha_k\}_{k \geqslant 1}$ is a subsequence of $\mathcal{O}(d_i^+)$ and $\{\beta_k\}_{k \geqslant 1}$ is a subsequence of $\mathcal{O}(d_j^+)$;*

(2) *the closed interval $I_k$ whose end points are $\alpha_k$ and $\beta_k$ satisfies*

$$|\beta_k - \alpha_k| < \lambda^k \epsilon_0 \quad \textit{and} \quad f^{m_0 + k}(x) \in I_k \quad \textit{for all } k \in \mathbb{N}. \tag{15}$$

*Proof.* First, we construct $\epsilon_0$, $m_0$, $\alpha_0$ and $\beta_0$. Let $\epsilon(d_i^+)$ and $\epsilon(d_j^+)$ be as in Lemma 3.6 and

$$0 < \epsilon_1 := \min\{|c - c'| : c, c' \in \Delta, c \neq c'\}. \tag{16}$$

We define $\epsilon_0$ as $\epsilon_0 := \min\{\epsilon(d_i^+), \epsilon(d_j^+), \epsilon_1\}$.

As $c_i \in \Delta_{lr}(d_j^+) \cap \Delta_{lr}(x)$, from Definition 2.2, we deduce that there are $n_0 \geqslant 0$ and $m_0 \geqslant 0$ such that

$$f^{m_0}(x) \in (c_i, f^{n_0}(d_j^+)) \subset (c_i, c_i + \epsilon_0) \subset X_{i+1}.$$

Denote $\alpha_0 := c_i$ and $\beta_0 := f^{n_0}(d_j^+)$. Since $d_j^+ \in \widetilde{X}$, we have that $\alpha_0 \neq \beta_0$ and the relation above implies that

$$0 < |\beta_0 - \alpha_0| < \epsilon_0 \quad \text{and} \quad f^{m_0}(x) \in (\alpha_0, \beta_0) \subset X_{i+1}, \tag{17}$$

which shows that (15) holds for $k = 0$.

Now we show by induction that for any $k \geqslant 1$ there exist two points $\alpha_k$ and $\beta_k \in X$ that satisfy the following properties:

$$\alpha_k \in \mathcal{O}(d_i^+), \quad \beta_k \in \mathcal{O}(d_j^+), \quad |\beta_k - \alpha_k| < \lambda^k \epsilon_0 \quad \text{and} \quad f^{m_0+k}(x) \in I_k, \tag{18}$$

where $I_k$ is the compact interval whose end points are $\alpha_k$ and $\beta_k$.

Let us show (18) for $k = 1$. Let $I_0 := [\alpha_0, \beta_0]$. According to (17), we have that $I_0 \subset \overline{X_{i+1}}$ and, as $f_{i+1}$ is $\lambda$-Lipschitz, we deduce that $I_1 := f_{i+1}(I_0)$ is a compact interval of size smaller than $\lambda\epsilon_0$ such that $f^{m_0+1}(x) \in I_1$. As $f_{i+1}$ is a strictly monotonic function, the end points of $I_1$ are

$$\alpha_1 := d_i^+ \quad \text{and} \quad \beta_1 := f(\beta_0) \tag{19}$$

and belong to $\mathcal{O}(d_i^+)$ and $\mathcal{O}(d_j^+)$, respectively. It follows that (18) holds for $k = 1$.

Assume that (18) holds for some $k \geqslant 1$. We discuss two cases.

*Case 1:* There is no point of $\Delta$ in the interval $I_k$. Then $f|_{I_k}$ is a $\lambda$-Lipschitz strictly monotonic function and, using the induction hypothesis (18), we obtain that

$$\alpha_{k+1} := f(\alpha_k) \quad \text{and} \quad \beta_{k+1} := f(\beta_k) \tag{20}$$

satisfy (18) replacing $k$ by $k + 1$.

*Case 2:* There exists a point $c_\ell \in I_k \cap \Delta$. First, note that such a point $c_\ell$ is unique, because of (16) and

$$\text{length}(I_k) = |\alpha_k - \beta_k| < \lambda^k \epsilon_0 \leqslant \lambda^k \epsilon_1.$$

Second, note that

$$c_\ell \in \text{int}(I_k),$$

because the end points $\alpha_k$ and $\beta_k$ of $I_k$ belong to $\widetilde{X}$. Indeed, by the induction hypothesis $\alpha_k \in \mathcal{O}(d_i^+) \subset \widetilde{X}$ and $\beta_k \in \mathcal{O}(d_j^+) \subset \widetilde{X}$ (recall that $D \subset \widetilde{X}$). Therefore,

$$\alpha_k, \beta_k \in (c_\ell - \lambda^k \epsilon_0, c_\ell + \lambda^k \epsilon_0)$$

and one of the two points $\alpha_k$, $\beta_k$ is at the left-hand side of $c_\ell$ while the other one is at the right-hand side of $c_\ell$. Without loss of generality, we will suppose that

$$\alpha_k \in (c_\ell - \lambda^k \epsilon_0, c_\ell) \quad \text{and} \quad \beta_k \in (c_\ell, c_\ell + \lambda^k \epsilon_0). \tag{21}$$

Now we show that $c_\ell \in \Delta_{lr}(\alpha_k) \cap \Delta_{lr}(\beta_k)$. Recall that by (14) we have $c_j \in \Delta_{lr}(d_i^+)$ and that by Lemma 2.4 this implies that $\mathcal{O}(d_j^+) \subset \omega(d_i^+)$. As $\alpha_k \in \mathcal{O}(d_i^+)$, we have $\omega(\alpha_k) = \omega(d_i^+)$ and, as $\beta_k \in \mathcal{O}(d_j^+)$, we deduce from the right-hand relation of (21) that there exists $n > 0$ such that

$$f^n(\alpha_k) \in (c_\ell, c_\ell + \lambda^k \epsilon_0).$$

Then, from the left-hand relation of (21), the definition of $\epsilon_0$ and Lemma 3.6, it follows that $c_\ell \in \Delta_{lr}(\alpha_k)$. Analogously, using that $c_i \in \Delta_{lr}(d_j^+)$, we obtain $c_\ell \in \Delta_{lr}(\beta_k)$. This ends the proof of $c_\ell \in \Delta_{lr}(\alpha_k) \cap \Delta_{lr}(\beta_k)$.

Now let us construct $\alpha_{k+1}$ and $\beta_{k+1}$. By (18), we have $f^{m_0+k}(x) \in [\alpha_k, \beta_k]$. Suppose that $f^{m_0+k}(x) \in (c_\ell, \beta_k]$. Since $c_\ell \in \Delta_{lr}(\alpha_k)$, there exists $r > 0$ such that

$$f^r(\alpha_k) \in (c_\ell, f^{m_0+k}(x)).$$

Therefore, the interval $[f^r(\alpha_k), \beta_k]$ satisfies the same properties (18) as the interval $I_k$ and moreover does not intersect $\Delta$. So, we can use the same proof as in Case 1, to show that

$$\alpha_{k+1} := f^{r+1}(\alpha_k) \quad \text{and} \quad \beta_{k+1} := f(\beta_k) \tag{22}$$

satisfy (18) replacing $k$ by $k+1$. Now, if we suppose that $f^{m_0+k}(x) \in [\alpha_k, c_\ell)$, then using this time that $c_\ell \in \Delta_{lr}(\beta_k)$ we obtain that there exists $l > 0$ such that

$$f^l(\beta_k) \in (f^{m_0+k}(x), c_\ell).$$

Therefore, for the same reason as for the case where $f^{m_0+k}(x) \in (c_\ell, \beta_k]$, we conclude that

$$\alpha_{k+1} := f(\alpha_k) \quad \text{and} \quad \beta_{k+1} := f^{l+1}(\beta_k) \tag{23}$$

satisfy (18) replacing $k$ by $k+1$.

We have constructed by induction two sequences $\{\alpha_k\}_{k\geqslant 1}$ and $\{\beta_k\}_{k\geqslant 1}$ satisfying (18) for all $k \geqslant 1$, which are moreover subsequences of $\mathcal{O}(d_i^+)$ and $\mathcal{O}(d_j^+)$, respectively (see (19), (20), (22) and (23)). □

Note that in Lemma 3.7, as well as in its following Corollary 3.3, the integers $i$ and $j$ are not necessarily different. As a consequence, their results can be applied even if $\Delta_{lr}(x)$ contains only one point.

COROLLARY 3.3. *Suppose that $f$ is injective on each of its contraction pieces and let $x \in \widetilde{X}$ be such that $\Delta_{lr}(x) \neq \emptyset$. If $i, j \in \{1, \ldots, N-1\}$ are such that*

$$c_i \in \Delta_{lr}(d_j^+) \cap \Delta_{lr}(x) \quad \textit{and} \quad c_j \in \Delta_{lr}(d_i^+) \cap \Delta_{lr}(x),$$

*then $\omega(x) = \omega(d_i^+) = \omega(d_j^+)$.*

*Proof.* Applying Lemma 2.4, we immediately obtain that $\omega(d_i^+) \subset \omega(x)$ and $\omega(d_j^+) \subset \omega(x)$. Now, according to Lemma 3.7, there exist $m_0 \geqslant 0$, $\epsilon_0 > 0$, a subsequence $\{\alpha_k\}_{k\geqslant 1}$ of $\mathcal{O}(d_i^+)$ and a subsequence $\{\beta_k\}_{k\geqslant 1}$ of $\mathcal{O}(d_j^+)$ such that

$$|f^{m_0+k}(x) - \alpha_k| \leqslant \lambda^k \epsilon_0 \quad \text{and} \quad |f^{m_0+k}(x) - \beta_k| \leqslant \lambda^k \epsilon_0 \quad \text{for all } k \geqslant 1. \tag{24}$$

Let $y \in \omega(x)$ and $\{k_n\}_{n\in\mathbb{N}}$ be an increasing sequence such that $\lim_{n\to\infty} f^{k_n}(f^{m_0}(x)) = y$. Then (24) implies that $\lim_{n\to\infty} \alpha_{k_n} = y = \lim_{n\to\infty} \beta_{k_n}$ and therefore $y \in \omega(d_i^+) \cap \omega(d_j^+)$. So, we have proved that $\omega(x) \subset \omega(d_i^+)$ and $\omega(x) \subset \omega(d_j^+)$. □

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*Proof of Theorem 3.1.* Let $x \in \widetilde{X}$ and suppose that $\Delta_{lr}(x) \neq \emptyset$. Then there exists $k \in \{1, \ldots, N-1\}$ such that $c_k \in \Delta_{lr}(x)$. Applying part (a) of Proposition 3.1, we know that there exists $i \in \{1, \ldots, N-1\}$ such that $[c_i] \in \Delta_{lr}/\sim^+$ is a minimal class and $[c_i] \preccurlyeq^+ [c_k]$. From Definition 3.2, it follows that either $c_i \in \Delta_{lr}(d_k^+)$ and Lemma 3.2 ensures that $c_i \in \Delta_{lr}(x)$, or $[c_i] = [c_k]$ and we also conclude that $c_i \in \Delta_{lr}(x)$. We have proved that there exists a point

$$c_i \in \Delta_{lr}(x)$$

whose equivalence class $[c_i]$ is minimal.

Applying Corollary 3.2, we deduce that there exists $j \in \{1, \ldots, N-1\}$ such that $c_j \in \Delta_{lr}(d_i^+)$. Using once more Lemma 3.2, we obtain that

$$c_j \in \Delta_{lr}(d_i^+) \cap \Delta_{lr}(x).$$

On the other hand, as the class of $c_i$ is a minimal class, $c_j \in \Delta_{lr}(d_i^+)$ also implies that $c_i \in \Delta_{lr}(d_j^+)$; see part (b) of Proposition 3.1. It follows that

$$c_i \in \Delta_{lr}(d_j^+) \cap \Delta_{lr}(x).$$

Therefore, the hypotheses of Corollary 3.3 are verified and $\omega(x) = \omega(d_i^+)$. Besides, as $c_i \in \Delta_{lr}(x)$, by Lemma 2.4, we have

$$\overline{\mathcal{O}(d_i^+)} \subset \omega(x) = \omega(d_i^+) \subset \overline{\mathcal{O}(d_i^+)},$$

which ends the proof of Theorem 3.1. □

3.4. *End of proof of Theorem 2.3.* In this section, we study the orbits of the points of $D$ corresponding to the minimal classes of $\Delta_{lr}/\sim^+$. By Theorem 3.1, we know that these orbits determine all the non-periodic asymptotic dynamics. Among other results, we show that the closure of such an orbit is a $\widetilde{X}$-minimal Cantor set, which together with Theorem 3.1 will achieve the proof of Theorem 2.3.

LEMMA 3.8. *Let $i \in \{1, \ldots, N-1\}$ and suppose that $[c_i] \in \Delta_{lr}/\sim^+$ is a minimal class. Then, for any $x \in \omega(d_i^+) \cap \widetilde{X}$, we have $c_i \in \Delta_{lr}(x)$ and*

$$\omega(x) = \overline{\mathcal{O}(x)} = \omega(d_i^+) = \overline{\mathcal{O}(d_i^+)}.$$

*Proof.* Let $x \in \omega(d_i^+) \cap \widetilde{X}$. Since $\omega(d_i^+) \cap \widetilde{X}$ is invariant, we have that

$$\omega(x) \subset \overline{\mathcal{O}(x)} \subset \omega(d_i^+). \tag{25}$$

As $c_i \in \Delta_{lr}$, from Corollary 3.1, we know that $\omega(d_i^+) \cap \widetilde{X}$ does not contain any periodic point and, therefore, by (25), $\omega(x) \cap \widetilde{X}$ does not either. It follows by Theorem 2.2 that there exists $j \in \{1, \ldots, N-1\}$ such $c_j \in \Delta_{lr}(x)$.

Moreover, still by (25), we have that $\mathcal{O}(x) \subset \overline{\mathcal{O}(d_i^+)}$, which allows us to deduce that $c_j \in \Delta_{lr}(d_i^+)$. Since $c_i$ is of minimal class, we must have that $c_i \in \Delta_{lr}(d_j^+)$, which together with $c_j \in \Delta_{lr}(x)$ implies by Lemma 3.2 that $c_i \in \Delta_{lr}(x)$.

Once we know that $c_i \in \Delta_{lr}(x)$, we deduce from Lemma 2.4 that $\overline{\mathcal{O}(d_i^+)} \subset \omega(x)$ and using (25) we obtain that

$$\overline{\mathcal{O}(d_i^+)} \subset \omega(x) \subset \overline{\mathcal{O}(x)} \subset \omega(d_i^+) \subset \overline{\mathcal{O}(d_i^+)}.$$ □

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THEOREM 3.2. *Let $i \in \{1, \ldots, N-1\}$ and suppose that $[c_i] \in \Delta_{lr}/\sim^+$ is a minimal class. Then $K_i := \omega(d_i^+)$ is a $\widetilde{X}$-minimal Cantor set. Moreover, if $f$ is injective on each of its contraction pieces, then for any $k \in \{1, \ldots, N-1\}$ such that $[c_i] \preccurlyeq^+ [c_k]$, we have*

$$c_k \in K_i \quad \textit{and} \quad K_i = \overline{\mathcal{O}(d_k^+)} = \overline{\mathcal{O}(d_k^-)}. \tag{26}$$

*Proof.* Let $i \in \{1, \ldots, N-1\}$, $K_i := \omega(d_i^+)$ and suppose that $[c_i] \in \Delta_{lr}/\sim^+$ is a minimal class.

*$K_i$ is $\widetilde{X}$-minimal:* It is a direct consequence of Lemma 3.8. It also proves that $K_i$ is a compact set.

*$K_i$ is a perfect set:* Let $y \in K_i$. As $K_i$ is pseudo-invariant (see Lemma 2.1), there exists $x \in K_i \cap \widetilde{X}$ (see Lemma 2.3) and $\overline{\mathcal{O}(x)} = K_i$. As $c_i \in \Delta_{lr}$ and $D \subset \widetilde{X}$, from Corollary 3.1, we deduce that $K_i \cap \widetilde{X}$ does not contain periodic points. Therefore, $\mathcal{O}(x) \subset \widetilde{X}$ does not contain periodic points and there exists $n_0 \in \mathbb{N}$ such that $y \notin \mathcal{O}(f^{n_0}(x))$. As $\mathcal{O}(f^{n_0}(x))$ is dense in $K_i$, there exists $\{y_n\}_{n\in\mathbb{N}} \subset \mathcal{O}(f^{n_0}(x)) \subset K_i \setminus \{y\}$ which converges to $y$.

*$K_i$ is totally disconnected:* In [**5**, Theorem 5.2], it is proved that, if $f$ is a piecewise-contracting map on a one-dimensional compact space $X$, then its attractor $\Lambda$ is totally disconnected. As any $\omega$-limit set is contained in $\Lambda$, we conclude that $K_i$ is also totally disconnected.

Now let $k \in \{1, \ldots, N-1\}$ be such that $[c_i] \preccurlyeq^+ [c_k]$. As $c_k \in \Delta_{lr}$, there exists $x \in \widetilde{X}$ such that

$$c_k \in \Delta_{lr}(x). \tag{27}$$

According to Theorem 3.1, this implies that there exists $i' \in \{1, \ldots, N-1\}$ such that $[c_{i'}]$ is a minimal class and $\omega(x) = \omega(d_{i'}^+)$. We have proved above that if $[c_{i'}]$ is a minimal class, then $K_{i'} := \omega(d_{i'}^+)$ is a $\widetilde{X}$-minimal Cantor set. Therefore, Lemma 2.4 and (27) imply that

$$c_k, d_k^+, d_k^- \in K_{i'} \quad \text{and} \quad K_{i'} = \overline{\mathcal{O}(d_k^+)} = \overline{\mathcal{O}(d_k^-)}.$$

To finish the proof of the theorem, we only have to show that $K_{i'} = K_i$. To this end, note that

$$c_i \in \Delta_{lr}(x). \tag{28}$$

Indeed, (28) follows from $[c_i] \preccurlyeq^+ [c_k]$, (27) and Lemma 3.2. We deduce from (28) and Lemma 2.4 that $\omega(d_i^+) \subset \omega(x)$, that is,

$$K_i \subset K_{i'}.$$

Since $K_i$ and $K_{i'}$ are both $\widetilde{X}$-minimal, and $K_i \cap \widetilde{X} \neq \emptyset$, we conclude that $K_{i'} = K_i$. □

Now we can prove Theorem 2.3, which, as said in §2.3, will also complete the proof of Theorem 1.1.

*Proof of Theorem 2.3.* Suppose that $f$ is injective on each of its contracting pieces and that $D \subset \widetilde{X}$. Let $x \in \widetilde{X}$. If $\Delta_{lr}(x) \neq \emptyset$, then, according to Theorem 3.1, there exists $i \in \{1, \ldots, N-1\}$ such that $[c_i]$ is a minimal class and $\omega(x) = \omega(d_i^+)$. Using Theorem 3.2, we deduce that $\omega(x)$ is a $\widetilde{X}$-minimal Cantor set. Reciprocally, if $\omega(x)$ is a $\widetilde{X}$-minimal Cantor set, then $\omega(x)$ is not a periodic orbit and we obtain from Theorem 2.2 that $\Delta_{lr}(x) \neq \emptyset$. □

Note that Theorem 3.2 allows us to prove Theorem 2.3, but also states in addition, through (26), that all the points in $\Delta$ belonging to a same minimal class, as well as those belonging to a class comparable with it, generate the same Cantor set (through the orbits of both lateral limits) and belong to it.

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## REFERENCES

[**1**] J. Brémont. Dynamics of injective quasi-contractions. *Ergod. Th. & Dynam. Sys.* **26** (2006), 19–44.
[**2**] Y. Bugeaud. Dynamique de certaines applications contractantes, linéaires par morceaux, sur [0, 1[. *C. R. Acad. Sci. Sér. 1 Math.* **317** (1993), 575–578.
[**3**] Y. Bugeaud and J.-P. Conze. Calcul de la dynamique de transformations linéaires contractantes mod 1 et arbre de Farey. *Acta Arith.* **88** (1999), 201–218.
[**4**] E. Catsigeras, P. Guiraud and A. Meyroneinc. Complexity of injective piecewise contracting interval maps. *Ergod. Th. & Dynam. Sys.* **40** (2020), 64–88.
[**5**] E. Catsigeras, P. Guiraud, A. Meyroneinc and E. Ugalde. On the asymptotic properties of piecewise contracting maps. *Dyn. Syst.* **31** (2016), 107–135.
[**6**] R. Coutinho. Dinámica simbólica linear. *PhD Thesis*, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1999.
[**7**] J. M. Gambaudo and C. Tresser. On the dynamics of quasi-contractions. *Bull. Braz. Math. Soc. (N.S.)* **19** (1988), 61–114.
[**8**] M. Laurent and A. Nogueira. Rotation number of contracted rotations. *J. Mod. Dyn.* **12** (2018), 175–191.
[**9**] M. Morse and G. A. Hedlund. Symbolic dynamics II. Sturmian trajectories. *Amer. J. Math.* **62** (1940), 1–42.
[**10**] A. Nogueira and B. Pires. Dynamics of piecewise contractions of the interval. *Ergod. Th. & Dynam. Sys.* **35** (2015), 2198–2215.
[**11**] A. Nogueira, B. Pires and R. A. Rosales. Asymptotically periodic piecewise contractions of the interval. *Nonlinearity* **27** (2014), 1603–1610.
[**12**] A. Nogueira, B. Pires and R. A. Rosales. Topological dynamics of piecewise $\lambda$-affine maps. *Ergod. Th. & Dynam. Sys.* **38** (2018), 1876–1893.
[**13**] B. Pires. Symbolic dynamics of piecewise contractions. *Nonlinearity* **32** (2019), 4871–4889.