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Global averaging and parametric resonances in damped semilinear wave equations

S. Zelik*
Institut für Analysis, Dynamik und Modelierung,
Universität Stuttgart, Pfaffenwaldring 57,
70569 Stuttgart, Germany

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The long-time behaviour of solutions to a semilinear damped wave equation in a three-dimensional bounded domain with the nonlinearity rapidly oscillating in time ($f = f(\varepsilon, u, t/\varepsilon)$) is studied. It is proved that (under natural assumptions) the behaviour of solutions whose initial energy is not very large can be described in terms of global (uniform) attractors $\mathcal{A}_\varepsilon$ of the corresponding dynamical processes and that, as $\varepsilon \to 0$, these attractors tend to the global attractor $\mathcal{A}_0$ of the corresponding averaged system. We also give the detailed description of these attractors in the case where the limit attractor $\mathcal{A}_0$ is regular.

Moreover, we give explicit examples of semilinear hyperbolic equations where the uniform attractor $\hat{\mathcal{A}}_\varepsilon$ (for the initial data belonging to the whole energy phase space) contains the irregular resonant part, which tends to infinity as $\varepsilon \to 0$, and formulate the additional restrictions on the nonlinearity f which guarantee that this part is absent.

1. Introduction

We consider the following semilinear damped hyperbolic equation in a bounded smooth three-dimensional domain $\Omega \subset \mathbb{R}^3$:

$$\left.\begin{aligned} &\partial_t^2 u + \gamma\partial_t u - \Delta_x u + \lambda_0 u + f\left(\varepsilon, u, \frac{t}{\varepsilon}\right) = g, \\ &u|_{t=\tau} = u_\tau, \qquad \partial_t u|_{t=\tau} = u'_\tau, \qquad \partial_n u|_{\partial\Omega} = 0. \end{aligned}\right\} \tag{1.1}$$

Here γ and λ_0 are fixed positive constants, the external forces $g \in L^2(\Omega)$, $\varepsilon > 0$ is a small parameter, and the nonlinear interaction function $f(\varepsilon, u, z)$ is sufficiently smooth with respect to u and ε and is *almost periodic* with respect to z (see § 2 for the precise conditions).

As usual, we complete the family of problems (1.1) at $\varepsilon = 0$ by the following averaged equation:

$$\partial_t^2 \bar{u} + \gamma\partial_t \bar{u} - \Delta_x \bar{u} + \lambda_0 \bar{u} + \bar{f}(\bar{u}) = g, \tag{1.2}$$

where $\bar{f}(u)$ is the average of the almost-periodic function $f(0, u, z)$. We also impose the standard dissipativity and growth restrictions on the average $\bar{f}(u)$. These guar-

*Present address: Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany (zelik@wias-berlin.de).

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antee the global existence and dissipativity of solutions of equation (1.2) in the energy space

$$E^1(\Omega) := H^2(\Omega) \cap \{\partial_n u|_{\partial\Omega} = 0\} \times H^1(\Omega)$$

(the critical cubic rate of growth of $\bar{f}$ is also allowed; see § 2). It is also worth noting that no growth and dissipativity assumptions on the nonlinearities $f(\varepsilon, u, z)$ for *positive* ε are imposed.

The long-time behaviour of solutions to (1.1) in the autonomous case is usually described in terms of global attractors of the dynamical systems associated with the problem under consideration (see [3, 19, 29] and references therein).

The case of non-autonomous equations is essentially less understood. In fact, until now there have been several natural approaches to extend the attractors theory to the non-autonomous case. One of them is based on a reduction of the non-autonomous dynamical process to the autonomous one, using the skew-product technique. The realization of this approach leads to the so-called uniform attractor $\mathcal{A}_\varepsilon$ of equation (1.1) which is independent of t and is uniform with respect to all the nonlinearities $\phi(\varepsilon, u, t/\varepsilon)$ belonging to the hull $\mathcal{H}(f)$ of the initial nonlinearity f (see [11, 20]). The alternative approach interprets the attractor of the non-autonomous equation (1.1) as a non-autonomous set as well: $\mathcal{A}_f(t)$, $t \in \mathbb{R}$ (see, for example, [12, 22]).

The homogenization problems for individual solutions of evolution equations with rapidly oscillating spatial and temporal terms were investigated in [1, 4, 5, 24, 28, 36] (see also the references therein).

The homogenization of attractors has also been studied by many authors (see, for example, [6, 27] for attractors of reaction–diffusion and hyperbolic equations in non-homogenized spatially periodic media with asymptotic degeneracy). The case of regular spatially almost-periodic media was considered in [13]. The homogenization aspects of the evolution problems for spatially rapidly oscillations in subordinated terms (i.e. for $f = f(x/\varepsilon, u)$ or $g = g(x/\varepsilon)$) were considered in [16, 17]. The temporal averaging of uniform attractors for evolutionary problems was studied in [21] (for the case of the nonlinear wave equation with external forces rapidly oscillating in time) and in [33] (for the case of singularly perturbed reaction–diffusion system with rapidly oscillating external forces). The non-autonomous regular attractors for reaction–diffusion equations with nonlinearities rapidly oscillating in time were investigated in [14]. The homogenization of trajectory attractors associated with ill-posed evolutionary mathematical physics equations (such as three-dimensional Navier–Stokes equations, damped wave equations with supercritical nonlinearities, etc.) were studied in [10, 11].

In this paper, we carry out a detailed analysis of problems related to the local and global averaging of the solutions of semilinear hyperbolic equations (1.1). In particular, we prove (in § 3) that the dissipativity of the averaged system (1.2) in the energy space $E^1(\Omega)$ implies the existence of a global bounded solution for problem (1.1) with the initial data $\xi^\tau := (u_\tau, u'_\tau)$ belonging to a large ball $B_{R_0(\varepsilon)}$ (if $\varepsilon > 0$ is sufficiently small), where the radius $R_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

We also establish (in § 4) that the long-time behaviour of the solutions to equation (1.1) *with the initial data belonging to* $B_{R_0(\varepsilon)}$ can be described in terms of the uniform attractor $\mathcal{A}_\varepsilon$ of the corresponding dynamical process, and that the attractors $\mathcal{A}_\varepsilon$ are uniformly bounded in $E^1(\Omega)$ and tend (as $\varepsilon \to 0$) to the global attractor $\mathcal{A}_0$ of the limit autonomous problem (1.2) (in the sense of the upper semi-continuity in $E^1(\Omega)$).

Moreover, under the additional generic assumption that the limit attractor $\mathcal{A}_0$ is regular, we provide a detailed description (in the spirit of [14]) of the pull-back attractor $\mathcal{A}_f(\tau)$ of equation (1.1) on the ball $B_{R_0(\varepsilon)}$ for small positive ε. In particular, we establish that, in this case, equation (1.1) possesses only a finite number of different almost-periodic solutions (which are close to the equilibria of the averaged system), and that, as $t \to \infty$, any other solution of this equation (belonging to the ball $B_{R_0(\varepsilon)}$) tends to one of these almost-periodic solutions (see § 6). Furthermore, the pull-back attractor $\mathcal{A}_f(\tau)$ has a regular structure (i.e. it is a finite collection of the finite-dimensional unstable manifolds of the almost-periodic solution mentioned above), attracts *exponentially* and uniformly with respect to ε and τ the images of bounded subsets $B \subset B_{R_0(\varepsilon)}$ and tends to the limit global attractor $\mathcal{A}_0$ as $\varepsilon \to 0$ in the sense of upper and lower semi-continuity in $E^1(\Omega)$ (see § 6 for the details).

We however note that, in contrast to the case of parabolic equations or hyperbolic equations with *external forces* rapidly oscillating in time considered in previous papers (see, for example, [11, 14, 34]), in our case the uniform attractor $\hat{\mathcal{A}}_\varepsilon$ of equation (1.1) in the whole phase space $E^1(\Omega)$ (i.e. the initial data outside of the ball $B_{R_0(\varepsilon)}$ is allowed), if it exists, *does not necessarily coincide* with the uniform attractor $\mathcal{A}_\varepsilon$ described above and does not necessarily tend to the averaged attractor $\mathcal{A}_0$ as $\varepsilon \to 0$. Indeed, we give (in § 8) an example of an equation of the form (1.1) whose attractor $\hat{\mathcal{A}}_\varepsilon$ consists of two parts. The first part ($\mathcal{A}_\varepsilon$) is regular, has a large basin of attraction (which contains at least the ball $B_{R_0(\varepsilon)}$) and tends to the limit-averaged attractor $\mathcal{A}_0$ as $\varepsilon \to 0$. The second part is, however, irregular (chaotic) and tends to infinity as $\varepsilon \to 0$ (see examples 8.4 and 8.7). The existence of the irregular part of the attractor $\hat{\mathcal{A}}_\varepsilon$ in our example can be explained in terms of the so-called nonlinear parametric resonance phenomena (see § 8) which are typical for hyperbolic equations and, therefore, we believe that the irregular part of the attractor is non-empty for more-or-less general equations of the form (1.1), where the leading part of the nonlinearity contains rapid oscillations in time.

Nevertheless, we introduce a rather wide class of equations of the form (1.1) (the so-called subordinated oscillations), for which we prove that the irregular part of the attractor is empty and

$$\hat{\mathcal{A}}_\varepsilon = \mathcal{A}_\varepsilon \tag{1.3}$$

exactly as in the case of reaction–diffusion equations.

The paper is organized as follows. The precise formulation of our assumptions on the functions $f(\varepsilon, u, z)$ and some auxiliary results, which are of fundamental significance for what follows, are given in § 2.

The local averaging of equation (1.1) (over a finite interval of time) is considered in § 3. Moreover, based on this result, we also establish there the existence of global bounded solutions for problem (1.1), whose initial energy is not very large.

The uniform and pull-back attractors for equation (1.1) (with the initial data belonging to $B_{R_0(\varepsilon)}$) are constructed in § 4 and their convergence to the limit global attractor $\mathcal{A}_0$ is verified there.

We devote § 5 to the study of the behaviour of the solutions to equation (1.1) in a small neighbourhood of the hyperbolic equilibrium $z_0(x)$ of the averaged equation (1.2), which is necessary for the regular attractors theory.

In § 6 we give a detailed description of the pull-back attractors $\mathcal{A}_f(\tau)$ in the case where the limit attractor $\mathcal{A}_0$ is regular. In particular, the upper and lower semi-continuity of attractors $\mathcal{A}_\varepsilon$ at $\varepsilon = 0$ is established.

In § 7, we formulate the additional assumptions on the nonlinearity f which guarantee equality (1.3).

The results obtained are illustrated by several concrete examples of equations of the form (1.1), which are given in § 8.

Finally, several auxiliary estimates for the linear hyperbolic equations are given in the appendix.

2. Main assumptions and preliminary results

In this section we formulate our assumptions on the nonlinear interaction function $f(\varepsilon, u, t/\varepsilon)$, recall some known facts on almost-periodic functions and prepare the technical tools for the next sections. To be precise, we assume from now on that, for every $\varepsilon \geqslant 0$ and $z \in \mathbb{R}$, the functions $f(\varepsilon, \cdot, z)$, $f'_u(\varepsilon, \cdot, z)$, $f''_u(\varepsilon, \cdot, z)$ and $f'''_u(\varepsilon, \cdot, z)$ belong to $C(\mathbb{R})$ and satisfy the following estimate:

$$|f(\varepsilon, u, z)| + |f'_u(\varepsilon, u, z)| + |f''_u(\varepsilon, u, z)| + |f'''_u(\varepsilon, u, z)| \leqslant Q(|u|), \tag{2.1}$$

for some monotonic function Q that is independent of ε and z. We also assume that $f(\varepsilon, u, z) \to f(0, u, z)$ as $\varepsilon \to 0$ in the following sense:

$$\begin{aligned} |f(\varepsilon, u, z) - f(0, u, z)| + |f'_u(\varepsilon, u, z) - f'_u(0, u, z)| \\ + |f''_u(\varepsilon, u, z) - f''_u(0, u, z)| \leqslant C_R\varepsilon, \quad \forall |u| \leqslant R, \end{aligned} \tag{2.2}$$

for every ε and z where the constant C_R is independent of z and ε.

Our next assumption is that, for every $\varepsilon \geqslant 0$, the functions $z \to f(\varepsilon, u, z)$, $f'_u(\varepsilon, u, z)$, $f''_u(\varepsilon, u, z)$ and $f'''_u(\varepsilon, u, z)$ are *almost periodic* as functions with values in the space $C_{\mathrm{loc}}(\mathbb{R})$. We recall that a function $\phi(u, z)$ is almost periodic as a $C_{\mathrm{loc}}(\mathbb{R})$-valued function if and only if $\phi(u, z)$ is an almost-periodic real-valued function (in the Bohr sense; see, for example, [23]) for every fixed $u \in \mathbb{R}$, and it is uniformly continuous with respect to u belonging to bounded subsets and $z \in \mathbb{R}$, i.e. for every $R > 0$ there exists a monotone function $\alpha_R : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{\varepsilon\to 0} \alpha_R(\varepsilon) = 0$ and

$$|\phi(u_1, z) - \phi(u_2, z)| \leqslant \alpha_R(|u_1 - u_2|), \quad \forall u_1, u_2 \in \mathbb{R},\ |u_i| \leqslant R \text{ and every } z \in \mathbb{R} \tag{2.3}$$

(see, for example, [14, 24] for the details). We also recall that every function almost periodic with respect to z, $z \to \phi(u, z)$, possesses the average

$$\mathbb{M}(\phi)(u) := \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} \phi(u, z)\, \mathrm{d}z, \tag{2.4}$$

where the limit is uniform with respect to $t \in \mathbb{R}$. Moreover, if, in addition, ϕ is almost periodic as a $C_{\mathrm{loc}}(\mathbb{R})$-valued function (i.e. (2.3) is satisfied), then the limit (2.4) is uniform with respect to u belonging to bounded subsets as well. Namely,

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for every $R \in \mathbb{R}_+$, there exists a monotone function $\alpha_R : \mathbb{R}_+ \to \mathbb{R}$ such that $\lim_{\varepsilon\to 0} \alpha_R(\varepsilon) = 0$ and

$$\left|\mathbb{M}(\phi)(u) - \frac{1}{T}\int_t^{t+T} \phi(u,z)\,\mathrm{d}z\right| \leqslant \alpha_R\left(\frac{1}{T}\right), \tag{2.5}$$

for every u such that $|u| \leqslant R$ and every $t \in \mathbb{R}$ (see [24] for the details).

We now define, for every $\varepsilon > 0$, the function

$$f_\varepsilon(u,t) := f\left(\varepsilon, u, \frac{t}{\varepsilon}\right) \tag{2.6}$$

and complete this family of functions as $\varepsilon = 0$ by

$$f_0(u) = \bar{f}(u) := \mathbb{M}(f(0,u,z))(u). \tag{2.7}$$

We will consider the function $\bar{f}$ as the average of the functions (2.6).

We do not impose any growth or dissipativity assumptions on the functions f_ε, which guarantee the global solvability of equation (1.1) for $\varepsilon > 0$. In contrast to this, the global solvability of the averaged equation (1.1) (with $\varepsilon = 0$) is crucial for our method, so we need the average $\bar{f}$ to satisfy the following additional assumptions:

$$\bar{f} \in C^3(\mathbb{R}), \qquad |\bar{f}'''(u)| \leqslant C, \quad \forall u \in \mathbb{R}, \qquad \liminf_{|u|\to\infty} \bar{f}'(u) \geqslant 0. \tag{2.8}$$

We start our exposition with the following lemma, which clarifies the sense in which the functions $f_\varepsilon(u,t) \to \bar{f}(u)$ as $\varepsilon \to 0$.

LEMMA 2.1. *Let the above assumptions hold and let φ_ε and $\bar{\varphi}$ be one of the following functions: f_ε, $\partial_u f_\varepsilon$, $\partial_u^2 f_\varepsilon$ and $\bar{f}$, $\partial_u \bar{f}$, $\partial_u^2 \bar{f}$. Then, for every $R > 0$, the following estimate holds:*

$$\left|\int_t^{t+\tau} [\varphi_\varepsilon(u,t) - \bar{\varphi}(u)]\,\mathrm{d}t\right| \leqslant \alpha_R(\varepsilon), \quad \forall |u| \leqslant R, t \in \mathbb{R}, \ \tau \in [0,1], \ \varepsilon \geqslant 0, \tag{2.9}$$

where the monotonic function $\alpha_R(\varepsilon)$ is independent of u, t and τ and satisfies the condition $\lim_{\varepsilon\to 0} \alpha_R(\varepsilon) = 0$.

Proof. Thanks to (2.2), it is sufficient to estimate the term

$$\int_t^{t+\tau} \left[f\left(0, u, \frac{t}{\varepsilon}\right) - \bar{f}(u)\right] \mathrm{d}t = \varepsilon \int_{t/\varepsilon}^{t/\varepsilon + \tau/\varepsilon} [f(0,u,z) - \bar{f}(z)]\,\mathrm{d}z \tag{2.10}$$

and its first and second derivatives with respect to u. But, due to our assumptions, the function $z \to f(0,u,z)$ and its first and second derivative with respect to u are almost periodic as $C_{\mathrm{loc}}(\mathbb{R})$-valued functions and, consequently, the required estimate is an immediate corollary of estimates (2.1) and (2.5) (with $\phi = f, f'_u, f''_u$, $t = t/\varepsilon$ and $T = \tau/\varepsilon$). Indeed, if $\tau \leqslant \varepsilon^{1/2}$, estimate (2.9) follows from (2.1) and, if $\varepsilon^{1/2} \leqslant \tau \leqslant 1$, then, due to estimate (2.5), we may estimate the right-hand side of (2.10) in terms of $\tau\alpha_R(\varepsilon/\tau) \leqslant \alpha_R(\varepsilon^{1/2})$ and lemma 2.1 is proven. □

As usual, in order to study the attractors of the *non-autonomous* equation (1.1), it is useful to consider not only the initial nonlinearity f_ε, but also all nonlinearities belonging to the hull of the initial nonlinearity f_ε.

Definition 2.2. Let $f(\varepsilon,t,z)$ satisfy the above assumptions and let the function f_ε be defined by (2.6). The hull $\mathcal{H}(f_\varepsilon)$ of f_ε is then the set

$$\mathcal{H}(f_\varepsilon) := [T_h f_\varepsilon, h\in\mathbb{R}]_{C_b(\mathbb{R},C^3_{\mathrm{loc}}(\Omega))}, \quad (T_h f_\varepsilon)(u,t) := f_\varepsilon(u,t+h), \tag{2.11}$$

where $[\cdot]_V$ denotes the closure in the space V. Then, since the functions $f_\varepsilon(u,t)$, $\partial_u f_\varepsilon(u,t)$, $\partial_u^2 f_\varepsilon(u,t)$ and $\partial_u^3 f_\varepsilon(u,t)$ are assumed to be almost periodic as $C_{\mathrm{loc}}(\mathbb{R})$-valued functions, hull (2.11) is a compact subset of $C_b(\mathbb{R},C_{\mathrm{loc}}(\mathbb{R})^3)$ (due to the Bochner–Amerio criterium [24]).

Lemma 2.3. *Let the above assumptions hold. Then every function*

$$\phi_\varepsilon(u,t) = \phi(\varepsilon,u,z), \quad z = \frac{t}{\varepsilon},$$

belonging to the hull $\mathcal{H}(f_\varepsilon)$ of the initial nonlinearity f_ε satisfies inequalities (2.1), (2.2) and (2.9) with the same constants C_R and monotonic functions Q and α_R as the initial nonlinearity f_ε.

Indeed, this assertion is a standard corollary of the definition of the hull (2.11) (see, for example, [11]).

We now define the scale $E^s(\Omega)$, $s\in\mathbb{R}$, of energy spaces associated with the hyperbolic equation (1.1) with Neumann boundary conditions via

$$E^s(\Omega) := H^{s+1}_N(\Omega)\times H^s_N(\Omega), \quad \text{where } \xi_u(t) := (u(t),\partial_t u(t))\in E^s(\Omega) \tag{2.12}$$

and $H^l_N(\Omega) := D((-\Delta_x+\lambda_0)^{l/2}_N)$ is a scale of Hilbert spaces generated by the Laplace operator in Ω with Neumann boundary conditions. Then, as is well known,

$$H^l_N(\Omega) = \begin{cases} H^l(\Omega), & -\frac{1}{2}<l<\frac{3}{2},\\ H^l(\Omega)\cup\{\partial_n u|_{\partial\Omega}=0\}, & \frac{3}{2}<l<\frac{5}{2}, \end{cases} \tag{2.13}$$

where $H^s(\Omega)$ are the classical Sobolev spaces in Ω (see [31] for the details). To simplify the notation, below we will write $E(\Omega)$ instead of $E^0(\Omega)$.

In the next two lemmas, we obtain the analogues of estimate (2.9) for the case where the parameter u depends on t and x ($u=u(t,x)$).

Lemma 2.4. *Let the above assumptions hold and let $u(t)=u(t,x)$ be a function satisfying*

$$\|\xi_u(t)\|_{E^1(\Omega)} \leqslant R, \quad \forall t\in[0,T]. \tag{2.14}$$

Then, for every $\varepsilon>0$ and $\phi_\varepsilon\in\mathcal{H}(f_\varepsilon)$, the following estimate holds:

$$\left\|\int_t^{t+\tau}[\phi_\varepsilon(u(t),t)-\bar f(u(t))]\,\mathrm{d}t\right\|_{L^2(\Omega)} \leqslant C_R\alpha_{C'_R}(\varepsilon), \tag{2.15}$$

for all $t,t+\tau\in[0,T]$ and $\tau\in[0,1]$, where the monotonic function $\alpha_R(\varepsilon)$ is the same as in (2.9) and the constants C_R and C'_R are independent of ε, t, T, ϕ_ε and u.

Moreover, if $v(t)$ is another function such that $\xi_v\in L^\infty([0,T],E^1(\Omega))$, then

$$\left\|\int_t^{t+\tau}[\partial_u\phi_\varepsilon(u(t),t)-\partial_u\bar f(u(t))]v(t)\,\mathrm{d}t\right\|_{L^2(\Omega)} \leqslant C_R\|\xi_v\|_{L^\infty([0,T],E^1(\Omega))}\alpha_{C'_R}(\varepsilon), \tag{2.16}$$

for all $t,t+\tau\in[0,T]$ and $\tau\in[0,1]$.

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Proof. We consider below only the case $\phi_\varepsilon = f_\varepsilon$ (the general case reduces to this one, due to lemma 2.3). Let us first prove estimate (2.15). To this end, we use the following obvious identity:

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_t^s [f_\varepsilon(u(s),\kappa) - \bar{f}(u(s))]\,\mathrm{d}\kappa \\ = [f_\varepsilon(u(s),s) - \bar{f}(u(s))] + \partial_t u(s)\int_t^s [\partial_u f_\varepsilon(u(s),\kappa) - \partial_u \bar{f}(u(s))]\,\mathrm{d}\kappa. \quad (2.17)$$

Integrating this identity over $s \in [t, t+\tau]$, we derive

$$\int_t^{t+\tau} [f_\varepsilon(u(s),s) - \bar{f}(u(s))]\,\mathrm{d}s \\ = \int_t^{t+\tau} [f_\varepsilon(u(t+\tau),\kappa) - \bar{f}(u(t+\tau))]\,\mathrm{d}\kappa \\ - \int_t^{t+\tau} \partial_t u(s)\biggl(\int_t^s [\partial_u f_\varepsilon(u(s),\kappa) - \partial_u \bar{f}(u(s))]\,\mathrm{d}\kappa\biggr)\,\mathrm{d}s. \quad (2.18)$$

We now note that the estimate (2.14) and the embedding $H^2(\Omega) \subset C(\Omega)$ (we recall that $n = 3$) imply that

$$\|u(s)\|_{C(\Omega)} + \|\partial_t u(s)\|_{L^2(\Omega)} \leqslant C_R \quad (2.19)$$

and, consequently, we may use inequality (2.9) (with $\phi_\varepsilon = f_\varepsilon$ and $\phi_\varepsilon = \partial_u f_\varepsilon$) in order to estimate the first term and the internal integral in the second term on the right-hand side of (2.18) which gives estimate (2.15).

Let us now consider estimate (2.16). Indeed, integrating by parts the left-hand side of (2.16), we have

$$\int_t^{t+\tau} [\partial_u f_\varepsilon(u(s),s) - \partial_u \bar{f}(u(s))]v(s)\,\mathrm{d}s \\ = v(t+\tau)\int_t^{t+\tau} [\partial_u f_\varepsilon(u(s),s) - \partial_u \bar{f}(u(s))]\,\mathrm{d}s \\ - \int_t^{t+\tau} \partial_t v(s)\biggl(\int_t^s [\partial_u f_\varepsilon(u(\kappa),\kappa) - \partial_u \bar{f}(u(\kappa))]\,\mathrm{d}\kappa\biggr)\,\mathrm{d}s. \quad (2.20)$$

Estimating the first term and the internal integral in the second term on the right-hand side of (2.20) using (2.18) (with f_ε replaced by $\partial_u f_\varepsilon$) and lemma 2.1, we derive estimate (2.16) and finish the proof of lemma 2.4. □

We now consider the analogue of estimate (2.15) for the case of less regular functions u. To this end, we need the following additional assumption on the growth of the nonlinearity f:

$$|f_\varepsilon(w,t)| \leqslant C(1+|w|^{3-\delta}), \quad |\partial_u f_\varepsilon(w,t)| \leqslant C(1+|w|), \quad \forall w \in \mathbb{R}, \quad (2.21)$$

where C is independent of $t \in \mathbb{R}$ and δ is some positive number.

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LEMMA 2.5. *Let the nonlinearity f_ε satisfy the additional assumption (2.21) and let $u(t,x)$ be a function satisfying*

$$\|\xi_u\|_{L^\infty([0,T],E(\Omega))} \leqslant R. \tag{2.22}$$

Then, for every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, estimate (2.15) holds (with some new monotonic function α_R that is independent of t, τ, ϕ_ε and u and tends to zero as $\varepsilon \to 0$).

Proof. As in lemma 2.4, we give the proof for $\phi_\varepsilon = f_\varepsilon$ only; the general case is analogous, due to lemma 2.3. We also note, that in contrast to the proof of lemma 2.4, we do not now have the estimate of the C-norm of the function u and, consequently, we cannot directly apply lemma 2.1 to estimate the integrals on the right-hand side of (2.18). In order to overcome this difficulty, we introduce, for every $N > 0$ and $t \in [0,T]$, the sets

$$V_t^N(u) := \{x \in \Omega,\ |u(t,x)| \leqslant N\}, \qquad W_t^N(u) := \Omega \setminus V_t^N(u). \tag{2.23}$$

Then, due to embedding $H^1(\Omega) \subset L^6(\Omega)$, growth restriction (2.21), estimate (2.22) and the Hölder inequality, we have

$$\|f_\varepsilon(u(t),s)\|_{L^2(W_t^N(u))} + \|\bar f(u(s))\|_{L^2(W_t^N(u))} \leqslant C_R|W_t^N(u)|^{\alpha_1} \leqslant C'_R N^{-\alpha_2}, \tag{2.24}$$

where the positive constants C_R, C'_R, α_1 and α_2 are independent of N, ε, u, s and t. Thus, due to estimate (2.24) and lemma 2.1, we have

$$\begin{aligned}
\bigg\| \int_t^{t+\tau} [f_\varepsilon(u(\kappa),s) - \bar f(u(\kappa))]\,\mathrm{d}s \bigg\|_{L^2(\Omega)}
&\leqslant \bigg\| \int_t^{t+\tau} [f_\varepsilon(u(\kappa),s) - \bar f(u(\kappa))]\,\mathrm{d}s \bigg\|_{L^2(V_\kappa^N(u))} + C'_R N^{-\alpha_2} \\
&\leqslant |\Omega|^{1/2}\alpha_N(\varepsilon) + C'_R N^{-\alpha_2},
\end{aligned} \tag{2.25}$$

where the constants C'_R and α_2 are defined in (2.24) and the function $\alpha_N(\varepsilon)$ is the same as in lemma 2.1. Fixing now the parameter $N = N(\varepsilon)$ on the right-hand side of (2.25) in an optimal way (i.e. as a solution of $N^{-\alpha_1} = \alpha_N(\varepsilon)$), we find that the right-hand side of (2.25) tends to zero as $\varepsilon \to 0$ uniformly with respect to u, $t, \kappa \in [0,T]$ and $\tau \in [0,1]$. Moreover, arguing analogously, we can verify that (2.25) remains true with f_ε and $\bar f$ replaced by $\partial_u f_\varepsilon$ and $\partial_u \bar f$, respectively. Now, inserting these estimates into the right-hand side of (2.18) and using $\|\partial_t u(s)\|^2_{L^2(\Omega)} \leqslant R$, we find that

$$\bigg\| \int_t^{t+\tau} [f_\varepsilon(u(s),s) - \bar f(u(s))]\,\mathrm{d}s \bigg\|_{L^1(\Omega)} \leqslant \alpha'_R(\varepsilon), \tag{2.26}$$

where the monotonic function $\alpha'_R(\varepsilon)$ is independent of u, t and $\tau \in [0,1]$ and tends to zero as $\varepsilon \to 0$. In order to deduce the analogue of (2.26) for the L^2-norm, it remains to note that, due to estimate (2.22), embedding $H^1(\Omega)$ and growth restrictions (2.21), we have

$$\bigg\| \int_t^{t+\tau} [f_\varepsilon(u(s),s) - \bar f(u(s))]\,\mathrm{d}s \bigg\|_{L^{2+\delta_1}(\Omega)} \leqslant C''_R, \tag{2.27}$$

for some positive exponent $\delta_1 = \delta_1(\delta)$. Estimates (2.26) and (2.27) together with the interpolation inequality give estimate (2.15) and complete the proof of lemma 2.5. □

To conclude the section, we discuss the rate of convergence to zero of the functions $\alpha_R(\varepsilon)$ in lemma 2.1 as $\varepsilon \to 0$. To this end, we assume, in addition, that the function $f(0,u,z) - \bar{f}(u)$ has a bounded primitive $F(u,z)$, i.e.

$$\partial_z F(u,z) = f(0,u,z) - \bar{f}(u) \tag{2.28}$$

and

$$|F(u,z)| + |\partial_u F(u,z)| + |\partial_u^2 F(u,z)| \leqslant Q(|u|), \tag{2.29}$$

for some monotonic function Q which is independent of z.

Under these assumptions, we have the linear rate of decay of α_R as $\varepsilon \to 0$.

LEMMA 2.6. *Let the additional assumptions (2.28) and (2.29) hold. Then the function $\alpha_R(\varepsilon)$ introduced in lemma 2.1 (and used on the right-hand sides of estimates (2.15) and (2.16)) possesses the upper bound*

$$\alpha_R(\varepsilon) \leqslant C_R\varepsilon, \tag{2.30}$$

where the constant C_R is independent of ε.

Proof. Indeed, due to estimate (2.2), it is sufficient to estimate the term on the left-hand side of (2.10) and its first and second derivatives with respect to u. In order to do so, we transform the right-hand side of (2.10), using (2.28), as follows:

$$\int_t^{t+\tau} \left[f\left(0,u,\frac{t}{\varepsilon}\right) - \bar{f}(u)\right] \mathrm{d}t = \varepsilon\left(F\left(u, \frac{t}{\varepsilon} + \frac{\tau}{\varepsilon}\right) - F\left(u,\frac{t}{\varepsilon}\right)\right). \tag{2.31}$$

Estimate (2.30) is now an immediate corollary of assumption (2.29). Lemma 2.6 is proven. □

REMARK 2.7. By definition (see (2.7)), the function $f_\varepsilon(0,u,z) - \bar{f}(u)$ has zero mean. Therefore, condition (2.29) will be always satisfied if the function $f(0,u,z)$ is *periodic* with respect to z. Consequently, (2.30) is automatically satisfied for periodic nonlinearities f. Unfortunately, for more general quasi-periodic or almost-periodic functions, the sole zero mean assumption is *not sufficient* to obtain the bounded primitive (since the so-called small denominators may appear under the integration [23, 24]). Thus, in this case, some additional assumptions (e.g. some decay assumptions on the Fourier amplitudes of $f(0,u,z)$ or some kind of Diophantine conditions on its Fourier frequencies) are required in order to have estimate (2.30) (see [14, 16, 17, 24]).

3. The local averaging and the global existence of strong solutions

In this section, we prove the existence of a global strong solution of the non-averaged equation (1.1) if ε is sufficiently small and the initial E^1-energy is not very large. We obtain this result by comparison of the solution of equation (1.1) with the

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corresponding solution of the averaged equation, which obviously has the following form:

$$\left.\begin{aligned} \partial_t^2\bar{u} + \gamma\partial_t\bar{u} - \Delta_x\bar{u} + \lambda_0\bar{u} + \bar{f}(\bar{u}) = g, \\ \xi_{\bar{u}}|_{t=\tau} = \xi^\tau, \qquad \partial_n\bar{u}|_{\partial\Omega} = 0. \end{aligned}\right\} \tag{3.1}$$

We start by recalling the classical result on the global solvability of equation (3.1).

Theorem 3.1. *Let the function $\bar{f}$ satisfy assumptions (2.8) and $g \in L^2(\Omega)$. Then, for every $\tau \in \mathbb{R}$ and $\xi^\tau \in E^1(\Omega)$, equation (3.1) has a unique solution $\bar{u}(t)$ that satisfies the following estimate:*

$$\|\xi_{\bar{u}}(t)\|_{E^1(\Omega)} \leqslant Q(\|\xi_{\bar{u}}(\tau)\|_{E^1(\Omega)})\mathrm{e}^{-\alpha(t-\tau)} + Q(\|g\|_{L^2(\Omega)}), \tag{3.2}$$

where $t \geqslant \tau$ and the positive constant α and the monotonic function Q are independent of τ, t, ξ^τ and g.

The proof of this result can be found, for example, in [3].

Let us now consider the non-averaged equation (1.1). As usual, in order to study the long-time behaviour of solutions of this equation, it is useful to consider a family of equations of the form (1.1) with all nonlinearities ϕ_ε, belonging to the hull $\mathcal{H}(f_\varepsilon)$ of the initial nonlinearity. To be more precise, for every $\varepsilon > 0$, $\tau \in \mathbb{R}$ and every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, we consider the following problem:

$$\left.\begin{aligned} \partial_t^2 u + \gamma\partial_t u - \Delta_x u + \lambda_0 u + \phi_\varepsilon(u,t) = g, \\ \xi_u|_{t=\tau} = \xi^\tau, \qquad \partial_n u|_{\partial\Omega} = 0. \end{aligned}\right\} \tag{3.3}$$

We first establish the existence of a solution $u(t)$ on a finite interval $[0, T]$ if ε is sufficiently small.

Theorem 3.2. *Let the function $f(\varepsilon, u, z)$ satisfy the assumptions of lemma 2.1. Then, for every $T > 0$ and $R > 0$, there exists $\varepsilon_0 = \varepsilon_0(T, R)$ such that, for every $\varepsilon < \varepsilon_0$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, $\tau \in \mathbb{R}$ and $\xi^\tau \in E^1(\Omega)$ that satisfies $\|\xi^\tau\|_{E^1(\Omega)} \leqslant R$, equation (3.3) has a unique strong solution $u_\varepsilon(t)$ on the interval $t \in [\tau, \tau + T]$ and the following estimate holds:*

$$\|\xi_{u_\varepsilon}(t) - \xi_{\bar{u}}(t)\|_{E^1(\Omega)} \leqslant C_{T,R}\alpha_{C_R}(\varepsilon)^\beta, \quad \forall t \in [\tau, \tau + T], \tag{3.4}$$

where the function $\alpha_R(\varepsilon)$ is the same as in lemma 2.1 and positive constants $C_{T,R}$, C_R and β are independent of τ, ξ^τ, $\varepsilon \leqslant \varepsilon_0$ and $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$.

Proof. We will construct the desired solution $u_\varepsilon(t)$ as a small perturbation of the corresponding solution $\bar{u}(t)$ of the averaged equation (3.1) (with the same initial conditions) based on estimates of lemma 2.4, estimate (3.2) and the implicit function theorem. For simplicity, we consider below only the case $\phi_\varepsilon = f_\varepsilon$ and $\tau = 0$ (the general case is analogous to this, due to lemma 2.3). We introduce a new unknown function $w_\varepsilon(t) := u_\varepsilon(t) - \bar{u}(t)$, which should satisfy

$$\begin{aligned} \partial_t^2 w_\varepsilon + \gamma\partial_t w_\varepsilon - \Delta_x w_\varepsilon + \lambda_0 w_\varepsilon + [\bar{f}(w_\varepsilon + \bar{u}(t)) - \bar{f}(\bar{u}(t))] \\ = [f_\varepsilon(w_\varepsilon + \bar{u}, t) - \bar{f}(w_\varepsilon + \bar{u})], \quad \xi_{w_\varepsilon}|_{t=0} = 0. \end{aligned} \tag{3.5}$$

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We will apply the implicit function theorem to equation (3.5). To this end, we define the space

$$\mathcal{L}_T := \{\xi_u \in C([0,T], E^1(\Omega)),\ \xi_u(0) = 0\} \tag{3.6}$$

and invert the linear part of equation (3.5). We then obtain the relation

$$\xi_{w_\varepsilon} + \mathbb{H}_T(\bar f(w_\varepsilon + \bar u) - \bar f(\bar u)) = \mathbb{H}_T(f_\varepsilon(w_\varepsilon + \bar u, t) - \bar f(w_\varepsilon + \bar u)), \tag{3.7}$$

where $\mathbb{H}_T : h \to \xi_\theta$ is the solution operator of the linear hyperbolic problem

$$\partial_t^2\theta + \gamma\partial_t\theta - \Delta_x\theta + \lambda_0\theta = h, \quad t \in [0,T],\ \xi_\theta(0) = 0. \tag{3.8}$$

Let us introduce an operator $\Phi : \mathcal{L}_T \times E^1(\Omega) \times \mathbb{R}_+ \to \mathcal{L}_T$ as follows:

$$\Phi(\xi_w, \xi^0, \varepsilon) := \xi_w - \mathbb{T}_0(\xi_w, \xi^0) + \mathbb{T}_1(\xi_w, \xi^0, \varepsilon), \tag{3.9}$$

where

$$\mathbb{T}_0(\cdot) := \mathbb{H}_T(\bar f(w + \bar u(\xi^0)) - \bar f(\bar u(\xi^0))), \mathbb{T}_1(\cdot) = \mathbb{H}_T(f_\varepsilon(w + \bar u(\xi^0), t) - \bar f(w + \bar u(\xi^0)))$$

and $\bar u(\xi^0)$ is a solution of (3.1) on $[0,T]$ with $\xi_{\bar u}(0) = \xi^0$. In order to study function (3.9), we need the following lemma.

LEMMA 3.3. *The operator $\mathbb{T}_1$ satisfies the estimate*

$$\|\mathbb{T}_1(\xi_w, \xi^0, \varepsilon)\|_{\mathcal{L}_T} + \|D_{\xi_w}\mathbb{T}_1(\xi_w, \xi^0, \varepsilon)\|_{\mathcal{L}(\mathcal{L}_T, \mathcal{L}_T)} \leqslant C_{T,R}\alpha_{C_R}(\varepsilon)^\beta \tag{3.10}$$

if $\|\xi_w\|_{\mathcal{L}_T} + \|\xi^0\|_{E^1(\Omega)} \leqslant 2R$, where the function $\alpha_R(\varepsilon)$ is the same as in lemma 2.1 and the positive constants $C_{T,R}$, C_R and β are independent of ε, ξ^0 and ξ_w.

Proof. Indeed, due to estimates (2.15), (3.2) and (A 4) (with $s = 0$), we have

$$\|\mathbb{T}_1(\xi_w, \xi^0, \varepsilon)\|_{C([0,T], E^{-1}(\Omega))} \leqslant C'_{T,R}\alpha_{C'_R}(\varepsilon). \tag{3.11}$$

On the other hand, it follows from assumption (2.1) and the dissipative estimate (3.2) that $\|f_\varepsilon(w(t) + \bar u(t), t) - \bar f(w(t) + \bar u(t))\|_{H^2(\Omega)} \leqslant C''_R$ and, consequently, due to (A 2), we have

$$\|\mathbb{T}_1(\xi_w, \xi^0, \varepsilon)\|_{C([0,T], E^{1+\delta}(\Omega))} \leqslant C'_R, \tag{3.12}$$

for some $0 < \delta < \frac{1}{2}$. Inequalities (3.11) and (3.12), together with the interpolation inequality, give (3.10). Estimates of the derivative can be proven analogously, except that estimate (2.16) should be used instead of (2.15). Lemma 3.3 is proven. □

Estimate (3.10) implies, in particular, that

$$\Phi(0, \xi^0, 0) \equiv 0. \tag{3.13}$$

Moreover, since the differentiability of the operator $\mathbb{T}_0$ with respect to ξ_w is obvious,

$$D_{\xi_w}\Phi(0, \xi^0, 0)\xi_\theta = \xi_\theta + \mathbb{H}_T(\bar f'(\bar u(\xi))\theta) \tag{3.14}$$

and $\Phi(\xi_w, \xi^0, \varepsilon)$ and $D_{\xi_w}\Phi(\xi_w, \xi^0, \varepsilon)$ tend to $\Phi(\xi_w, \xi^0, 0)$ and $D_{\xi_w}\Phi(\xi_w, \xi^0, 0)$, respectively, as $\varepsilon \to 0$ (and this convergence is uniform with respect to ξ^0). Thus, in order to deduce estimate (3.4) from (3.7) and the implicit function theorem, it remains only to verify that the operator (3.14) is (uniformly with respect to ξ^0) invertible in $\mathcal{L}_T$.

Lemma 3.4. *The equation*

$$D_{\xi_w}\Phi(0,\xi^0,0)\xi_\theta = \xi_v \tag{3.15}$$

has a unique solution ξ_θ*, for every* $\xi_v \in \mathcal{L}_T$*, and the estimate*

$$\|\xi_\theta\|_{\mathcal{L}_T} \leqslant C_{R,T}\|\xi_v\|_{\mathcal{L}_T}, \tag{3.16}$$

holds for all $\xi^0 \in E^1(\Omega)$ *such that* $\|\xi^0\|_{E^1(\Omega)} \leqslant R$.

Proof. Indeed, let $w(t) := \theta(t) - v(t)$, where ξ_θ is a solution of (3.15). This function then obviously satisfies

$$\partial_t^2 w + \gamma\partial_t w - \Delta_x w + \lambda_0 w + f'(\bar{u}(t))(w(t)+v(t)) = 0, \quad \xi_w(0) = 0. \tag{3.17}$$

We recall that, due to estimate (3.2), we have

$$\|f'(\bar{u}(t))\|_{C(\Omega)} + \|f'(\bar{u}(t))\|_{H^2(\Omega)} \leqslant C_R, \quad \forall t \geqslant 0. \tag{3.18}$$

The existence of a solution of (3.17) and estimate (3.16) is now an immediate corollary of this estimate and the classical E^1-energy estimates for the solutions of linear hyperbolic equations (see, for example, [3, 29]). Lemma 3.4 is proven. □

Thus, we have verified that operator (3.9) (in which ξ^0 is interpreted as a parameter) satisfies all of the assumptions of the implicit function theorem and, consequently, the desired solution ξ_{w_ε} can be found in a unique way from

$$\Phi(\xi_{w_\varepsilon},\xi^0,\varepsilon) \equiv 0 \tag{3.19}$$

if ε is sufficiently small. Moreover, estimate (3.4) is now a standard corollary of estimates (3.10). Theorem 3.2 is proven. □

Remark 3.5. We note that the right-hand side of (3.4) tends to zero as $\varepsilon \to 0$ and, consequently, the solution $u_\varepsilon(t)$ of equation (3.3) with rapidly oscillating-in-time coefficients tends to the corresponding solution $\bar{u}(t)$ as $\varepsilon \to 0$ on every *finite* interval $[\tau, \tau+T]$. Thus, theorem 3.2 can be interpreted as the analogue of Bogolubov's first theorem for the hyperbolic equation of the form (3.3) (see [5, 26]).

We now recall that only the averaged nonlinearity $\bar{f}$ is assumed to satisfy the dissipativity and growth assumptions that guarantee the existence of the global solutions. Thus, in general, we do not have the existence of a global solution for equation (3.3) if $\varepsilon > 0$. Nevertheless, estimates (3.2) and (3.4) allow us to prove the global existence if $\varepsilon > 0$ is sufficiently small and the E^1-energy of the initial data ξ^τ is not very large. To be more precise, the following result holds.

Corollary 3.6. *Let the assumptions of theorem 3.2 hold. There then exists* $\varepsilon_0 > 0$ *and a monotone decreasing function* $R_0 : (0, \varepsilon_0] \to \mathbb{R}_+$ *such that* $\lim_{\varepsilon\to 0} R_0(\varepsilon) = \infty$ *and, for every* $\varepsilon \leqslant \varepsilon_0$, $\tau \in \mathbb{R}$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ *and* $\xi^\tau \in E^1(\Omega)$ *that satisfies the condition*

$$\|\xi^\tau\|_{E^1(\Omega)} \leqslant R_0(\varepsilon), \tag{3.20}$$

equation (3.3) possesses a unique global solution $u \in L^\infty(\mathbb{R}_+, E^1(\Omega))$ *and the estimate*

$$\|\xi_u(t)\|_{E^1(\Omega)} \leqslant \tilde{Q}(\|\xi_u(\tau)\|_{E^1(\Omega)})\mathrm{e}^{-\alpha(t-\tau)} + \tilde{Q}(\|g\|_{L^2(\Omega)}) \tag{3.21}$$

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is valid, where the positive constant α and monotonic function $\tilde{Q}$ are independent of ε, τ, ϕ_ε and ξ^τ.

Proof. Instead of constructing the function $R_0(\varepsilon)$, it is more convenient to construct the inverse function $\varepsilon_0(R)$. Indeed, let R be an arbitrary sufficiently large number ($R \geqslant 4Q(\|g\|_{L^2(\Omega)})$, where Q is the same as in estimate (3.2)). Then, we fix $T = T(R)$ as a solution of

$$Q(R)e^{-\alpha T} = Q(\|g\|_{L^2(\Omega)}). \tag{3.22}$$

Finally, we fix $\varepsilon_0 = \varepsilon_0(R)$ such that

$$C_{T,R}\alpha_{C_R}(\varepsilon_0)^\beta = Q(\|g\|_{L^2(\Omega)}), \tag{3.23}$$

where the constants C_R, β and $C_{T,R}$ and the function $\alpha_R(\varepsilon)$ are the same as in theorem 3.2. We claim that, if $\varepsilon \leqslant \varepsilon_0$ and $\|\xi^\tau\|_{E^1(\Omega)} \leqslant R$, then equation (3.3) has a solution $u(t)$ that satisfies estimate (3.21). Indeed, due to theorem 3.2, equation (3.3) has a solution $u(t)$ that satisfies the estimate

$$\|\xi_u(t) - \xi_{\bar{u}}(t)\|_{E^1(\Omega)} \leqslant Q(\|g\|_{L^2(\Omega)}), \quad \forall t \in [\tau, \tau + T], \tag{3.24}$$

where $\bar{u}(t)$ is the corresponding solution of the averaged equation (3.1). On the other hand, thanks to estimate (3.2) and equation (3.22), we have the estimate $\|\xi_{\bar{u}}(\tau + T)\|_{E^1(\Omega)} \leqslant 2Q(\|g\|_{L^2(\Omega)})$. Combining this estimate with (3.24), we can derive

$$\|\xi_u(\tau + T)\|_{E^1(\Omega)} \leqslant 3Q(\|g\|_{L^2(\Omega)}) < R. \tag{3.25}$$

Thus, we may again apply theorem 3.2 in order to construct the solution of equation (3.3) on the interval $[\tau + T, \tau + 2T]$ with $\xi_u|_{t=\tau+T} = \xi_u(\tau + T)$. By iterating this procedure, we obtain the global solution $u(t)$ of equation (3.3) defined for every $t \in [\tau, +\infty)$ such that

$$\|\xi_u(\tau + nT)\|_{E^1(\Omega)} \leqslant 3Q(\|g\|_{L^2(\Omega)}), \quad \forall n \in \mathbb{N}. \tag{3.26}$$

Estimate (3.21) is now a corollary of (3.2), (3.26) and (3.24). Since the strong solution $\xi_u \in L^\infty([\tau, +\infty], E^1(\Omega))$ is unique, corollary 3.6 is proven. □

Thus, for sufficiently small ε and every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, equation (3.3) defines a family of solution operators

$$\left.\begin{aligned} U_{\phi_\varepsilon}(t,\tau) : B_{R_0(\varepsilon)} \to E^1(\Omega), \quad t, \tau \in \mathbb{R},\ t \geqslant \tau \text{ on the ball,} \\ B_{R_0(\varepsilon)} := \{\xi \in E^1(\Omega),\ \|\xi\|_{E^1(\Omega)} \leqslant R_0(\varepsilon)\} \end{aligned}\right\} \tag{3.27}$$

via $U_{\phi_\varepsilon}(t,\tau)\xi^\tau = \xi_u(t)$, where $u(t)$ is a solution of (3.3) which exists due to corollary 3.6. Moreover, these families, obviously, satisfy the following translation identity:

$$U_{\phi_\varepsilon}(t + s, \tau + s) = U_{T_s\phi_\varepsilon}(t, \tau), \tag{3.28}$$

for every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, $t, \tau, s \in \mathbb{R}$ and $t \geqslant \tau$, where the shift operator is defined in (2.11). We also note that the limit case $\varepsilon = 0$ corresponds to the autonomous equation (3.1), whose solutions exist globally for every $\xi^\tau \in E^1(\Omega)$. Consequently, this equation generates a semigroup $\{S_t, t \geqslant 0\}$ in the whole phase space $E^1(\Omega)$:

$$S_t\xi := U_{\bar{f}}(t, 0)\xi, \quad \forall \xi \in E^1(\Omega) \quad \text{and} \quad S_t \circ S_h = S_{t+h}, \quad \forall t, h \in \mathbb{R}_+. \tag{3.29}$$

The rest of this section is devoted to the study of the analytic properties of operators (3.27). We start with the standard result on the differentiability with respect to the initial data ξ^τ.

Proposition 3.7. *Let the assumptions of corollary 3.6 hold. Then, for every* $\xi^\tau \in B_{R_0(\varepsilon)}$*, the function* $\xi^\tau \to U_{\phi_\varepsilon}(t,\tau)\xi^\tau$ *is Frechet differentiable and its derivative* $D_{\xi^\tau}U_{\phi_\varepsilon}(t,\tau)$ *can be computed as follows:* $[D_{\xi^\tau}U_{\phi_\varepsilon}(t,\tau)\xi^\tau]\theta := \xi_{w_\theta}(t)$*, where* $\theta \in E^1(\Omega)$ *is an arbitrary vector and* $w_\theta(t)$ *is a solution of the following equation of variations:*

$$\left.\begin{aligned}\partial_t^2 w_\theta + \gamma\partial_t w_\theta - \Delta_x w_\theta + \lambda_0 w_\theta + \partial_u\phi_\varepsilon(u(t),t)w_\theta = 0,\\ \xi_{w_\theta}|_{t=\tau} = \theta, \quad u(t) := U_{\phi_\varepsilon}(t,\tau)\xi^\tau.\end{aligned}\right\} \tag{3.30}$$

Moreover, this derivative satisfies the following estimates:

$$\|D_{\xi^\tau}U_{\phi_\varepsilon}(t,\tau)\|_{\mathcal{L}(E^1,E^1)} \leqslant C\mathrm{e}^{K(t-\tau)}, \tag{3.31}$$

where the constants C *and* K *depend on* $\|\xi^\tau\|_{E^1}$*, but are independent of* ε*,* t*,* τ *and* ϕ_ε *and, for every* ξ_1^τ *and* ξ_2^τ *belonging to* $B_{R_0(\varepsilon)}$*, we have*

$$\|D_{\xi^\tau}U_{\phi_\varepsilon}(t,\tau)(\xi_1^\tau) - D_{\xi^\tau}U_{\phi_\varepsilon}(t,\tau)(\xi_2^\tau)\|_{\mathcal{L}(E^1,E^1)} \leqslant C\mathrm{e}^{K(t-\tau)}\|\xi_1^\tau - \xi_2^\tau\|_{E^1(\Omega)}, \tag{3.32}$$

where the constants C *and* K *are also independent of* t*,* τ*,* ε *and* ϕ_ε*.*

Indeed, proposition 3.7 is a standard corollary of estimate (A 2) with $s = 1$, estimate (3.21) and assumption (2.1) on the nonlinearity f (since all of these estimates are uniform with respect to ε, (3.31) and (3.32) will also be uniform with respect to ε), so we leave its rigorous proof to the reader.

We now establish the convergence of operators (3.27) to the limit semigroup S_t as $\varepsilon \to 0$.

Proposition 3.8. *Let the assumptions of corollary 3.6 hold. Then the operators* $U_{\phi_\varepsilon}(t,\tau)$ *tend to* $S_{t-\tau}$ *as* $\varepsilon \to 0$ *in the following sense:*

$$\begin{aligned}\|U_{\phi_\varepsilon}(t,\tau)(\xi^\tau) - S_{t-\tau}(\xi^\tau)\|_{E^1(\Omega)} + \|D_{\xi^\tau}U_{\phi_\varepsilon}(t,\tau)(\xi^\tau)\\ - D_{\xi^\tau}S_{t-\tau}(\xi^\tau)\|_{\mathcal{L}(E^1,E^1)} \leqslant C_R\mathrm{e}^{K(t-\tau)}\alpha_{C_R}(\varepsilon)^\beta,\end{aligned} \tag{3.33}$$

where the constants C_R *and* K *depend on* R *(we recall that* $\|\xi^\tau\|_{E^1(\Omega)} \leqslant R \leqslant R_0(\varepsilon)$*) but are independent of* t*,* τ*,* ε *and* ϕ_ε*, and the positive constant* β *and the monotonic function* $\alpha_R(\varepsilon)$ *are the same as in theorem 3.2.*

Proof. Estimate (3.33) can be easily deduced from the implicit function theorem (to this end, we need only to verify that function (3.9) is differentiable with respect to ξ^0), but we prefer to give an independent proof of this fact. We restrict ourselves to considering only the case $\phi_\varepsilon = f_\varepsilon$ and $\tau = 0$ (the general case is analogous, due to lemma 2.3). Let us first verify estimate (3.33) for the first term on the left-hand side. Indeed, let $u_\varepsilon(t) := U_{f_\varepsilon}(t,0)\xi^0$ and $\bar{u}(t) := S_t\xi^0$ be solutions of equations (3.3) and (3.1), respectively, and let $w_\varepsilon(t) := u_\varepsilon(t) - \bar{u}(t)$. Then, this function satisfies equation (3.5). Let us now introduce a new function $\theta(t)$ as a solution of the following auxiliary equation:

$$\partial_t^2\theta + \gamma\partial_t\theta - \Delta_x\theta + \lambda_0\theta = [f_\varepsilon(u_\varepsilon(t),t) - \bar{f}(u_\varepsilon(t))] := h_{u_\varepsilon}(t), \quad \xi_\theta(0) = 0. \tag{3.34}$$

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Estimates (2.1), (2.15) and (3.21) imply that

$$\left\| \int_t^{t+\tau} h_{u_\varepsilon}(s)\,\mathrm{d}s \right\|_{L^2(\Omega)} \leqslant C_R \alpha_{C_R}(\varepsilon) \quad \text{and} \quad \|h_{u_\varepsilon}(t)\|_{H^2(\Omega)} \leqslant C'_R, \tag{3.35}$$

for every $t \in \mathbb{R}_+$, $\tau \in [0,1]$, where the constant C_R depends only on R, and the function α_R is the same as in (2.15). Thus, due to estimates (A 2), (A 4) and the interpolation inequality, we have (see the proof of lemma 3.3)

$$\|\theta(t)\|_{E^1(\Omega)} \leqslant C''_R \alpha_{C_R}(\varepsilon)^\beta, \quad t \in \mathbb{R}_+, \tag{3.36}$$

where the constants C''_R and C_R depend only on R, and the positive constant β and the function α_R are the same as in theorem 3.1.

Now let $v_\varepsilon(t) := w_\varepsilon(t) - \theta(t)$. This function then satisfies

$$\partial_t^2 v_\varepsilon + \gamma\partial_t v_\varepsilon - \Delta_x v_\varepsilon + \lambda_0 v_\varepsilon = [\bar{f}(\bar{u}(t)) - \bar{f}(\bar{u}(t) + v_\varepsilon(t) + \theta(t))] := h_\theta(t), \quad \xi_{v_\varepsilon}(0) = 0. \tag{3.37}$$

Now using estimates (2.1), (3.2), (3.21) and the fact that the space $H^2(\Omega)$ is an algebra (we recall that $n = 3$), we deduce in a standard way that

$$\|h_\theta(t)\|_{H^1(\Omega)} \leqslant C'''_R(\|v_\varepsilon(t)\|_{H^2(\Omega)} + \|\theta(t)\|_{H^2(\Omega)}), \tag{3.38}$$

where C'''_R depends only on R. Applying estimate (A 2) to equation (3.37) and using estimates (3.36), (3.38) and Gronwall inequality, we finally obtain

$$\|v_\varepsilon(t)\|_{E^1(\Omega)} \leqslant C_R \mathrm{e}^{Kt} \alpha_{C_R}(\varepsilon)^\beta. \tag{3.39}$$

Thus, the first term on the right-hand side of (3.33) is estimated. The second term can be estimated analogously: we should consider the difference between the non-averaged (see (3.30)) and averaged equation of variations and use estimate (3.16) instead of (3.15). Proposition 3.8 is proven. □

Remark 3.9. We now discuss some generalizations of the results obtained. We first note that the almost-periodicity of functions f_ε with respect to t is necessary *only* for the proof of estimate (2.9) in lemma 2.1. Thus, all the results of this section remain true if, instead of the almost-periodicity, we postulate the existence of a function $\bar{f}(u)$ that satisfies estimates (2.9).

We also note that, although we consider only the spatially homogeneous nonlinearities $f_\varepsilon(u,t)$, this assumption is not crucial for our method and the results remain true for more general nonlinearities $f_\varepsilon(u,t,x)$ (under some smoothness assumptions on f_ε with respect to x).

To conclude, we observe that the Neumann boundary condition is also not essential for our technique. The only difference (e.g. with the case of Dirichlet boundary conditions) is that the analogue of estimate (A 2) with $s > \frac{1}{2}$ requires the boundary condition $h(t)|_{\partial\Omega} = 0$ (in fact, we do not know whether or not estimate (A 2) holds without this assumption). Thus, in the case of Dirichlet boundary conditions, we need to assume, in addition, that

$$f_\varepsilon(0,t) \equiv 0, \quad \forall t \in \mathbb{R}. \tag{3.40}$$

4. The attractors and their averaging

In this section, we start to study the long-time behaviour of solutions of (3.3) in the case where $\varepsilon > 0$ is sufficiently small. We recall that, in contrast to the case $\varepsilon = 0$, for positive ε, we have the global existence of a solution for the initial data belonging to the ball $B_{R_0(\varepsilon)}$ of radius $R_0(\varepsilon)$ in $E^1(\Omega)$ only (see §§ 6 and 7 for a discussion of the case where the initial data do not belong to this ball). Therefore, it is natural to consider this ball as the phase space for problem (3.3) and, thus, to construct the attractors for the solutions whose initial data belong to $B_{R_0(\varepsilon)}$ only. We also note that the main estimate (3.21) is not strong enough for us to conclude that the solution operators (3.27) map $B_{R_0(\varepsilon)}$ to itself, for every $t \geqslant \tau$. Nevertheless, it follows from (3.21) that there exists $T_0 = T_0(\varepsilon)$, which is independent of t, τ and ϕ_ε such that

$$U_{\phi_\varepsilon}(t,\tau) : B_{R_0(\varepsilon)} \to B_{R_0(\varepsilon)}, \quad \text{for all } t,\tau \in \mathbb{R},\ t-\tau \geqslant T_0,\ \phi_\varepsilon \in \mathcal{H}(f_\varepsilon), \tag{4.1}$$

which is quite enough for the attractors theory.

It is worth recalling here that, in contrast to the limit equation (3.1), equations (3.3) are *non-autonomous*. Thus, operators (4.1) do not generate a semigroup in the phase space $B_{R_0(\varepsilon)}$ and the standard concept of a global attractor is not directly applicable here. Until now, two major possibilities of generalizing the concept of a global attractor to non-autonomous equations have been known. The first is to reduce the non-autonomous dynamical system to the autonomous system defined on the properly extended phase space. This approach naturally leads to the concept of the so-called *uniform* attractor (see [11, 19] and the explanations below). The alternative approach is the so-called *pull-back* attractor, which treats the attractor of the non-autonomous equation as a non-autonomous set as well and, therefore, does not require the reduction to the autonomous system (see [12, 22]).

We start our exposition by the uniform attractor (and the pull-back attractor will be discussed at the end of the section). To this end (following the standard scheme; see, for example, [11]), we define the extended phase space for dynamical system (4.1) as

$$\Phi_\varepsilon := B_{R_0(\varepsilon)} \times \mathcal{H}(f_\varepsilon) \tag{4.2}$$

(where the hull $\mathcal{H}(f_\varepsilon)$ is endowed by the topology of $C_b(\mathbb{R}_+, C^3_{\mathrm{loc}}(\mathbb{R}))$) and define the extended semigroup $\mathbb{S}^\varepsilon_t$ associated with problems (3.3) on Φ_ε via

$$\mathbb{S}^\varepsilon_t(\xi^0,\phi_\varepsilon) := (U_{\phi_\varepsilon}(t,0)\xi^0, T_t\phi_\varepsilon), \quad \xi^0 \in B_{R_0(\varepsilon)},\ \phi_\varepsilon \in \mathcal{H}(f_\varepsilon). \tag{4.3}$$

It is well known that (4.3) indeed generates a semigroup in Φ_ε. Therefore, we may consider its global attractor.

Definition 4.1. The set $\mathbb{A}_\varepsilon \subset \Phi_\varepsilon$ is a (global) attractor of semigroup (4.3) if

(i) the set $\mathbb{A}_\varepsilon$ is compact in Φ_ε;

(ii) this set is strictly invariant, i.e. $\mathbb{S}^\varepsilon_t \mathbb{A}_\varepsilon = \mathbb{A}_\varepsilon$;

(iii) this set attracts all (bounded) subsets of Φ_ε, i.e. for every $B \subset \Phi_\varepsilon$ and every neighbourhood $\mathcal{O}(\mathbb{A}_\varepsilon)$ of the attractor $\mathbb{A}_\varepsilon$ in Φ_ε, there exists $T = T(B,\mathcal{O})$ such that

$$\mathbb{S}^\varepsilon_t B \subset \mathcal{O}(\mathbb{A}_\varepsilon), \quad \text{for every } t \geqslant T. \tag{4.4}$$

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If the set $\mathbb{A}_\varepsilon$ is the global attractor of the extended semigroup (4.3), then, by definition, the *uniform* attractor $\mathcal{A}_\varepsilon$ of the family (4.1) is a projection of $\mathbb{A}_\varepsilon$ to the first component of the Cartesian product $\Phi_\varepsilon = B_{R_0(\varepsilon)} \times \mathcal{H}(f_\varepsilon)$:

$$\mathcal{A}_\varepsilon = \Pi_1 \mathbb{A}_\varepsilon \tag{4.5}$$

(see [11] for details).

The following theorem establishes the existence of the uniform attractor $\mathcal{A}_\varepsilon$ for the family (4.1) associated with non-autonomous hyperbolic equations (3.3).

THEOREM 4.2. *Let the assumptions of corollary 3.6 hold. Then, semigroup (4.3) possesses a global attractor $\mathbb{A}_\varepsilon$ and, consequently, each dynamical processes (4.1) possesses a uniform attractor $\mathcal{A}_\varepsilon$. Moreover, these attractors are uniformly (with respect to ε) bounded in the space $E^1(\Omega)$, i.e.*

$$\|\mathcal{A}_\varepsilon\|_{E^1(\Omega)} \leqslant \bar{R}, \quad \text{for every } \varepsilon \in [0, \varepsilon_0] \tag{4.6}$$

and possess the following description:

$$\mathcal{A}_\varepsilon = \bigcup_{\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)} \Pi_{t=0} \mathcal{K}_{\phi_\varepsilon}, \tag{4.7}$$

where $\mathcal{K}_{\phi_\varepsilon}$ is a union of all solutions $u(t)$ of equation (3.3) (with the fixed nonlinearity $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$) that are defined for every $t \in \mathbb{R}$ and satisfy $\|\xi_u(t)\|_{E^1(\Omega)} \leqslant \bar{R}$, for every $t \in \mathbb{R}$ ($\mathcal{K}_{\phi_\varepsilon}$ is a kernel of equation (3.3) in the terminology of [11]).

Proof. According to the existence theorem for the global and uniform attractors (see [11, 29]), we need to verify the following conditions:

(i) operators (4.3) are continuous in Φ_ε for every fixed t;

(ii) there exists a compact attracting set $\mathcal{B}_\varepsilon \subset B_{\bar{R}} \times \mathcal{H}(f_\varepsilon)$ of this semigroup.

Moreover, the first condition of this theorem is obvious (see proposition 3.7), so it remains only to construct the compact attracting set $\mathcal{B}_\varepsilon$. To this end, we first note that, due to estimate (3.21), the set

$$\tilde{\mathcal{B}}_\varepsilon := B_{\bar{R}} \times \mathcal{H}(f_\varepsilon), \tag{4.8}$$

where $B_{\bar{R}}$ is the $\bar{R}$ ball of the space $E^1(\Omega)$, will be an absorbing set for semigroup (4.3) if $\bar{R}$ is sufficiently large (but which is, however, not compact in Φ_ε). In order to construct the compact analogue of (4.8), we split an arbitrary solution u of (3.3) with the initial data belonging to $B_{\bar{R}}$ as a sum of three functions: $u(t) = G + v(t) + w(t)$, where G solves the linear elliptic problem

$$-\Delta_x G + \lambda_0 G = g, \qquad \partial_n G|_{\partial\Omega} = 0, \tag{4.9}$$

the function $v(t)$ solves the linear homogeneous hyperbolic problem

$$\partial_t^2 v + \gamma \partial_t v - \Delta_x v + \lambda_0 v = 0, \quad \xi_v|_{t=\tau} = \xi_u|_{t=\tau} \tag{4.10}$$

and the remainder $w(t)$ is a solution of

$$\partial_t^2 w + \gamma \partial_t w - \Delta_x w + \lambda_0 w = h_u(t) := -\phi_\varepsilon(u(t), t), \quad \xi_w|_{t=\tau} = 0. \tag{4.11}$$

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Then, obviously $G \in H^2(\Omega)$ and, consequently, $\xi_G := (G, 0) \in E^1(\Omega)$ and, due to proposition A.1, we have

$$\|\xi_v(t)\|_{E^1(\Omega)} \leqslant C\mathrm{e}^{-\alpha(t-\tau)}\|\xi_u(0)\|_{E^1(\Omega)} \tag{4.12}$$

for some positive α. Moreover, due to estimates (2.1), (3.27), lemma 2.3 and the fact that $\xi_u(\tau) \in B_{\bar{R}}$, we have

$$\|h_u(t)\|_{H^2(\Omega)} \leqslant C, \tag{4.13}$$

where C is independent of ε, $\xi_u(\tau)$, t and ϕ_ε. Thus, applying proposition A.1 to equation (4.11), we derive that

$$\|\xi_w(t)\|_{E^{1+\delta}(\Omega)} \leqslant \bar{R}_1, \tag{4.14}$$

where $0 < \delta < \frac{1}{2}$ and the constant $\bar{R}_1$ is independent of ε, t, τ and ϕ_ε. Estimates (4.12) and (4.14) show that the set

$$B_{\bar{R}_1,\delta}(G) := \xi_G + \{\xi \in E^{1+\delta}(\Omega),\ \|\xi\|_{E^{1+\delta}(\Omega)} \leqslant \bar{R}_1\} \tag{4.15}$$

is a uniform (with respect to t, τ and ϕ_ε) attracting set for the family of processes (4.1). Thus, taking into account the fact that the hull $\mathcal{H}(f_\varepsilon)$ is compact in $C_b(\mathbb{R}, C^3_{\mathrm{loc}}(\mathbb{R}))$ (due to the almost-periodicity of f_ε; see definition 2.2), we finally derive that the set $\mathcal{B}_\varepsilon := B_{\bar{R}_1,\delta}(G) \times \mathcal{H}(f_\varepsilon)$ is a compact attracting set for semigroup (4.3) in Φ_ε.

Therefore, all the assumptions of the attractor existence theorem are verified for the semigroup (4.3) and, consequently, this semigroup indeed possesses the global attractor $\mathbb{A}_\varepsilon \subset \mathcal{B}_\varepsilon$. It remains to note that estimate (4.6) is an immediate corollary of this embedding and our construction of the set $\mathcal{B}_\varepsilon$, and description (4.7) is a standard corollary of the abstract attractors' existence theorem mentioned above (see [11]). Theorem 4.2 is proven. □

Remark 4.3. There exists an intrinsic definition of the uniform attractor $\mathcal{A}_\varepsilon$ that does not use the extended semigroup $\mathbb{S}^\varepsilon_t$; namely, the set $\mathcal{A}_\varepsilon$ is a uniform attractor for equation (1.1) if the following conditions are satisfied:

(i) $\mathcal{A}_\varepsilon$ is compact in $E^1(\Omega)$;

(ii) $\mathcal{A}_\varepsilon$ is a uniform (with respect to $\tau \in \mathbb{R}$) attracting set of (4.1), i.e. for every (bounded) $B \subset B_{R_0(\varepsilon)}$, we have

$$\lim_{t\to\infty} \sup_{\tau\in\mathbb{R}} \mathrm{dist}_{E^1(\Omega)}(U_{f_\varepsilon}(\tau + t, \tau)B, \mathcal{A}_\varepsilon) = 0,$$

where $\mathrm{dist}_V(X, Y)$ denotes the non-symmetric Hausdorff distance between sets X and Y in a metric space V;

(iii) the set $\mathcal{A}_\varepsilon$ is a minimal compact set that satisfies properties (i) and (ii).

The equivalence of this definition and definition 4.1 is proved in [11].

We now recall, that in the limit case $\varepsilon = 0$, we have the autonomous equation (3.1) which generates semigroup (3.29) on the whole space $E^1(\Omega)$ and, consequently, has a global attractor $\mathcal{A}_0$. The next result shows that, in a sense, this attractor can be interpreted as the average of attractors $\mathcal{A}_\varepsilon$.

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Corollary 4.4. *Let the assumptions of theorem 4.2 hold. Then the family* $\mathcal{A}_\varepsilon$*,* $\varepsilon \in [0, \varepsilon_0]$*, is upper semi-continuous at* $\varepsilon = 0$*, i.e. for every neighbourhood* $\mathcal{O}(\mathcal{A}_0)$ *in* $E^1(\Omega)$*, there exists* $\varepsilon' = \varepsilon'(\mathcal{O})$ *such that*

$$\mathcal{A}_\varepsilon \subset \mathcal{O}(\mathcal{A}_0) \quad \text{if } \varepsilon \leqslant \varepsilon'. \tag{4.16}$$

Indeed, due to proposition 3.8, family (4.1) of the dynamical processes associated with equation (3.3) tends uniformly (with respect to $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$) as $\varepsilon \to 0$ to the limit semigroup (3.29) associated with the limit autonomous equation (3.1). Thus, semi-continuity (4.16) is an immediate corollary of estimate (4.6) and the abstract theorem on the upper semi-continuity of the global (uniform) attractors (see, for example, [3, 11] and also the proof of corollary 4.5, below).

We now discuss the rate of convergence of the non-averaged attractors $\mathcal{A}_\varepsilon$ to the averaged one $\mathcal{A}_0$ (in the sense of upper semi-continuity) as $\varepsilon \to 0$. We recall that this rate of convergence essentially depends on the rate of attraction to the limit attractor $\mathcal{A}_0$ and, since this rate of attraction can be arbitrarily slow in general, we cannot obtain the estimates of the rate of convergence of $\mathcal{A}_\varepsilon \to \mathcal{A}_0$ *without* the additional assumptions on the limit attractor $\mathcal{A}_0$. One of the most natural additional assumptions is that the limit global attractor $\mathcal{A}_0$ is *exponential* (see [3, 13, 14]). The latter means that there exists a positive constant $\alpha > 0$ and a monotonic function Q such that, for every bounded subset $B \subset E^1(\Omega)$, we have

$$\operatorname{dist}_{E^1(\Omega)}(S_t B, \mathcal{A}_0) \leqslant Q(\|B\|_{E^1(\Omega)})\mathrm{e}^{-\alpha t}. \tag{4.17}$$

Here and below, the symbol $\operatorname{dist}_V(X, Y)$ denotes the non-symmetric Hausdorff semi-distance between sets X and Y in a metric space V.

Corollary 4.5. *Let the assumptions of theorem 4.2 hold and, in addition, let estimate (4.17) be satisfied. The following estimate then holds:*

$$\operatorname{dist}_{E^1(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leqslant C\alpha_{C_{\bar{R}}}(\varepsilon)^\kappa, \tag{4.18}$$

where the positive constants C *and* κ *are independent of* ε*,* $\bar{R}$ *is the same as in theorem 4.2, and the function* $\alpha_{C_R}(\varepsilon)$ *is the same as in (3.33).*

Proof. Indeed, let $\varepsilon > 0$ be sufficiently small and $\xi \in \mathcal{A}_\varepsilon$ be an arbitrary point. Then, due to description (4.7), there exist $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ and a solution $u_\varepsilon(t)$, $t \in \mathbb{R}$, of equation (3.3) such that $\xi_{u_\varepsilon} \in \mathcal{K}_{\phi_\varepsilon}$ and $\xi_{u_\varepsilon}(0) = \xi$. Now fix an arbitrary $T > 0$ and consider a solution $\bar{u}(t)$, $t \geqslant -T$, such that $\xi_{u_\varepsilon}(-T) = \xi_{\bar{u}}(-T)$. Then, on the one hand, due to proposition 3.8 and estimate (4.6), we have

$$\|\xi - \xi_{\bar{u}}(0)\|_{E^1(\Omega)} \leqslant C_{\bar{R}}\mathrm{e}^{KT}\alpha_{C_{\bar{R}}}(\varepsilon)^\beta \tag{4.19}$$

and, on the other hand, due to (4.17), we have

$$\operatorname{dist}_{E^1(\Omega)}(\xi_{\bar{u}}, \mathcal{A}_0) \leqslant C_{\bar{R}}\mathrm{e}^{-\alpha T}. \tag{4.20}$$

Combining (4.19) and (4.20) and taking into account the fact that $\xi \in \mathcal{A}_\varepsilon$ is arbitrary, we derive

$$\operatorname{dist}_{E^1(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leqslant C_{\bar{R}}(\mathrm{e}^{-\alpha T} + \mathrm{e}^{KT}\alpha_{C_{\bar{R}}}(\varepsilon)^\beta). \tag{4.21}$$

Optimizing the right-hand side of (4.21) with respect to T (i.e. fixing $T = T(\varepsilon)$ as a solution of $\mathrm{e}^{-\alpha T} = \mathrm{e}^{KT}\alpha_{C_{\bar{R}}}(\varepsilon)^{\beta}$), we derive estimate (4.18) and finish the proof of corollary 4.5. □

We now recall that assumption (4.17) is satisfied for generic external forces $g \in L^2(\Omega)$ (for which all of the equilibria of equation (3.1) are hyperbolic [3]). In this case, the limit attractor $\mathcal{A}_0$ has a specific structure (the so-called *regular* attractor in the terminology of [3]; see also § 6, below) which allows us to prove, for instance, that the family of attractors $\mathcal{A}_\varepsilon$ is also *lower* semi-continuous as $\varepsilon = 0$ and to obtain the analogue of (4.18) for the *symmetric* Hausdorff distance. However, in order to study the non-autonomous perturbations of regular attractors, it is more convenient to use the alternative concept of the *pull-back* attractor.

Definition 4.6. Let $\{U(t,\tau), \tau \in \mathbb{R}, t \geqslant \tau\}$ be a dynamical process in a metric space Ψ that satisfies the co-cycle property

$$U(t,\tau) \circ U(\tau,s) = U(t,s), \quad t \geqslant s \geqslant \tau. \tag{4.22}$$

The *set-valued* function $t \to \mathcal{A}(t)$ is then a pull-back attractor of this process if the following conditions hold:

(i) the sets $\mathcal{A}(t) \subset \Psi$ are compact for every $t \in \mathbb{R}$;

(ii) the sets $\mathcal{A}(t)$ are strictly invariant, i.e. $U(t,\tau)\mathcal{A}(\tau) = \mathcal{A}(t)$;

(iii) the pull-back attraction property is satisfied, i.e. for every bounded subset $B \subset \Psi$ and every $t \in \mathbb{R}$, we have

$$\lim_{T\to+\infty} \operatorname{dist}_{E^1(\Omega)}(U(t,t-T)B, \mathcal{A}(t)) = 0. \tag{4.23}$$

Corollary 4.7. *Let the assumptions of theorem 4.2 hold. Then, for every $\varepsilon \leqslant \varepsilon_0$ and every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, dynamical process (4.1) possesses the pull-back attractor $\mathcal{A}_{\phi_\varepsilon}(t)$ which has the following structure:*

$$\mathcal{A}_{\phi_\varepsilon}(\tau) = \Pi_{t=\tau}\mathcal{K}_{\phi_\varepsilon}$$

where the sets $\mathcal{K}_{\phi_\varepsilon}$ are defined in theorem 4.2.

Indeed, according to the general theory (see [11]), the existence of the *uniform* attractor $\mathcal{A}_\varepsilon$ implies the existence of the pull-back attractors $\mathcal{A}_{\phi_\varepsilon}(t)$ for every dynamical process $U_{\phi_\varepsilon}(t,\tau)$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, and relation (4.7).

Remark 4.8. In general, the convergence in (4.23) is not uniform with respect to $t \in \mathbb{R}$ and (consequently) the sets $U_{\phi_\varepsilon}(t+T,t)B$ are not necessarily convergent to $\mathcal{A}_{\phi_\varepsilon}(t+T)$ as $T \to +\infty$. Nevertheless, in contrast to what generally happens, we prove in § 6, below, that we have this convergence (which will be even exponential) in the case where the limit attractor $\mathcal{A}_0$ is regular and $\varepsilon > 0$ is sufficiently small.

We also mention the following obvious, but useful, relation between the uniform and pull-back attractors:

$$\mathcal{A}_\varepsilon = \bigcup_{\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)} \mathcal{A}_{\phi_\varepsilon}(0), \tag{4.24}$$

which is an immediate corollary of (4.7) and (4.7).

5. Averaging near the hyperbolic equilibrium

In this section, we investigate the behaviour of solutions of the non-averaged system (3.3) in a small neighbourhood of the hyperbolic equilibrium of the averaged system (3.1) if $\varepsilon > 0$ is sufficiently small. In particular, we construct here the non-autonomous unstable manifold associated with this equilibrium, which is necessary for studying (in the next section) the non-autonomous perturbations of the averaged regular attractor $\mathcal{A}_0$. Since all the results of this section are more or less standard corollaries of propositions 3.7 and 3.8 and the implicit function theorem, we give below only the necessary definitions and statements and indicate the main ideas of their proofs, leaving the details to the reader (see also [14, 18]).

We assume from now on that $\xi_{z_0} := (z_0, 0)$, $z_0 = z_0(x)$, is a hyperbolic equilibrium of equation (3.1), i.e.

$$-\Delta_x z_0 + \lambda_0 z_0 + \bar{f}(z_0) = g, \quad z_0|_{\partial\Omega} = 0, \tag{5.1}$$

and the spectrum of the linearization of (5.1) near ξ_{z_0} does not contain zero:

$$0 \notin \sigma(-\Delta_x + \lambda_0 + \bar{f}'(z_0), L^2(\Omega)). \tag{5.2}$$

Then, according to proposition 3.7, the Frechet derivative $D_{\xi^0} S_t(\xi_{z_0})$ of the semigroup S_t generated by the averaged equation (3.1) at ξ_{z_0} satisfies the linear equation

$$\partial_t^2 v_\theta + \gamma\partial_t v_\theta - \Delta_x v_\theta + \lambda_0 v_\theta + \bar{f}'(z_0)v_\theta = 0, \quad \xi_{v_\theta}|_{t=0} = \theta \tag{5.3}$$

where $\theta \in E^1(\Omega)$ and $v_\theta := [D_{\xi^0} S_t(\xi_{z_0})]\theta$. Moreover, hyperbolicity assumption (5.2) implies the existence of an exponential dichotomy for equation (5.3), i.e. there exist two subspaces $E_\pm$ of $E^1(\Omega)$ such that

$$E^1(\Omega) = E_+ \oplus E_-, \qquad D_{\xi^0} S_t(\xi_{z_0})E_\pm = E_\pm, \quad \dim E_+ = \operatorname{ind}^+(z_0) < \infty \tag{5.4}$$

and there exist positive constants C and α such that, for every $t \geqslant 0$,

$$\|D_{\xi^0} S_t(z_0)\theta\|_{E^1(\Omega)} \begin{cases} \leqslant C\mathrm{e}^{-\alpha t}\|\theta\|_{E^1(\Omega)}, & \forall\theta \in E_-, \\ \geqslant C^{-1}\mathrm{e}^{+\alpha t}\|\theta\|_{E^1(\Omega)}, & \forall\theta \in E_+ \end{cases} \tag{5.5}$$

(see, for example, [3]). We denote by $\Pi_\pm : E^1(\Omega) \to E_\pm$ the spectral projectors associated with decomposition (5.5).

The main task of this section is to obtain the nonlinear and non-autonomous analogue of decomposition (5.4) for the case of equation (3.3) in a small neighbourhood of z_0. To this end, we first construct the analogue of the equilibrium z_0 for equation (3.3).

Theorem 5.1. *Let the assumptions of theorem 4.2 hold and let ξ_{z_0} be a hyperbolic equilibrium of equation (3.1). There then exist $\varepsilon_0 > 0$ and a small neighbourhood V_{z_0} of the equilibrium ξ_{z_0} in $E^1(\Omega)$ such that, for every $\varepsilon < \varepsilon_0$ and $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, equation (3.3) possesses a unique solution $u_{\phi_\varepsilon, z_0}(t)$, $t \in \mathbb{R}$ such that*

$$\xi_{u_{\phi_\varepsilon, z_0}}(t) \in V_{z_0}, \quad \forall t \in \mathbb{R}. \tag{5.6}$$

Moreover, this solution is almost periodic with respect to t (as an $E^1(\Omega)$-valued function) with the same frequency basis as the nonlinearity f_ε and tends to ξ_{z_0}

as $\varepsilon \to 0$*:*

$$\|\xi_{u_{\phi_\varepsilon,z_0}}(t) - \xi_{z_0}\|_{E^1(\Omega)} \leqslant C\alpha_{C_{\bar R}}(\varepsilon)^\beta, \quad \forall t \in \mathbb{R}, \tag{5.7}$$

where $\bar R$ *is the same as in theorem 4.2 the function* $\alpha_{C_{\bar R}}(\varepsilon)^\beta$ *is the same as in proposition 3.8 and the constant* C *is independent of* ε*,* t *and* ϕ_ε*.*

Sketch of the proof. Following [14], instead of solving (3.3), we solve the equivalent difference equation

$$\xi(n) = U_{\phi_\varepsilon}(n, n-1)\xi(n-1), \quad n \in \mathbb{Z}, \tag{5.8}$$

in the space $\mathbb{L}(E^1) := L^\infty(\mathbb{Z}, E^1(\Omega))$ of $E^1(\Omega)$-valued sequences. To this end, we introduce an operator

$$\Phi : \mathbb{L}(E^1) \times \mathbb{R}_+ \to \mathbb{L}(E^1), \qquad \Phi(\xi, \varepsilon)(n) := \xi(n) - U_{\phi_\varepsilon}(n, n-1)\xi(n-1). \tag{5.9}$$

Then, due to propositions 3.7 and 3.8, the operators $U_{\phi_\varepsilon}(n, n-1)$, tend together with the Frechet derivative, to the solution operator S_1 of the averaged equation (3.1) (uniformly with respect to n and ϕ_ε). Thus, $\Phi(\xi_{z_0}, 0) = 0$. Consequently, in order to apply the implicit function theorem to (5.9), it remains to verify that the derivative

$$(D_\xi \Phi(\xi_{z_0}, 0)\psi)(n) = \psi(n) - D_{\xi^0} S_1(\xi_{z_0})\psi(n-1) \tag{5.10}$$

is invertible in $\mathbb{L}(E^1)$, but this fact is a standard corollary of exponential dichotomy (5.5). Thus, due to the implicit function theorem, there exists a neighbourhood V_{z_0} of ξ_{z_0} and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ and $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, there exists a unique sequence $\xi(n) = \xi_{\phi_\varepsilon,z_0}(n)$ that belongs to V_{z_0}, solves (5.8), satisfies (5.7) for $t \in \mathbb{Z}$ and depends continuously on $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ (the detailed derivation of this fact is given in [14]). Moreover, since $\xi_{\phi_\varepsilon,z_0}$ is unique, the translation identity (3.28) implies that

$$U_{\phi_\varepsilon}(n+t, n)\xi_{\phi_\varepsilon,z_0}(n) = \xi_{T_t\phi_\varepsilon,z_0}(n), \quad \forall t \in \mathbb{R}_+. \tag{5.11}$$

The required continuous solution $u_{\phi_\varepsilon,z_0}(t)$ can then be defined via

$$u_{\phi_\varepsilon,z_0}(t) := u_{T_t\phi_\varepsilon,z_0}(0), \quad t \in \mathbb{R}. \tag{5.12}$$

Indeed, the fact that (5.12) solves (3.3) follows from (5.11), and the almost-periodicity of (5.12) is an immediate corollary of the fact that the flow T_t is almost periodic on the hull $\mathcal{H}(f_\varepsilon)$ (see [24]). Theorem 5.1 is proven. □

We are now ready to define the nonlinear analogue of the space E_+ for equation (3.3).

Definition 5.2. Let the assumptions of theorem 5.1 hold and let $\tilde V_{z_0}$ be a sufficiently small neighbourhood of ξ_{z_0} in $E^1(\Omega)$. Then, for every $\varepsilon \leqslant \varepsilon_0$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ and $\tau \in \mathbb{R}$, we define the unstable set $\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(t)$ as follows:

$$\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau) := \{\xi^\tau \in E^1(\Omega),\ \exists \xi_{u_\varepsilon} \in \mathcal{K}_{\phi_\varepsilon} \text{ such that } \xi_{u_\varepsilon}(\tau) = \xi^\tau \text{ and } \xi_{u_\varepsilon}(t) \in \tilde V_{z_0},\ \forall t \leqslant \tau\}. \tag{5.13}$$

The following theorem shows that sets (5.13) are finite-dimensional manifolds if ε is sufficiently small.

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THEOREM 5.3. *Let the assumptions of theorem 5.1 hold. There then exists a neighbourhood $\tilde V_{z_0}$ of the equilibrium ξ_{z_0} and $\varepsilon_0'>0$ such that the sets (5.13) are finite-dimensional submanifolds of $E^1(\Omega)$, for every $\varepsilon\leqslant\varepsilon_0'$, $\phi_\varepsilon\in\mathcal{H}(f_\varepsilon)$ and $\tau\in\mathbb{R}$. To be more precise, there exist a neighbourhood $W^+\subset E_+$ of zero in E_+ (which is independent of ε, τ and ϕ_ε), a family of neighbourhoods $W^+_{\phi_\varepsilon}(\tau)$ of zero in E_+ such that $W^+\subset W^+_{\phi_\varepsilon}(\tau)$ and a family of C^1-maps*

$$\mathbb{M}^+_{\phi_\varepsilon,\tau}:W^+_{\phi_\varepsilon,\tau}\to E_- \tag{5.14}$$

such that

$$\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau)=u_{\phi_\varepsilon,z_0}(\tau)+\{\xi^++\mathbb{M}^+_{\phi_\varepsilon,\tau}(\xi^+),\ \xi^+\in W^+_{\phi_\varepsilon}(\tau)\} \tag{5.15}$$

and

$$\|\mathbb{M}^+_{\phi_\varepsilon,\tau}\|_{C^1}\leqslant C,\qquad \mathbb{M}^+_{\phi_\varepsilon,\tau}(0)=0, \tag{5.16}$$

where the constant C is independent of ε, τ and ϕ_ε. Moreover, for every $\xi^\tau\in\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau)$, the corresponding solution $\xi_{u_\varepsilon}\in\mathcal{K}_{\phi_\varepsilon}$ (which exists due to definition (5.13)) tends exponentially, as $t\to-\infty$, to the almost-periodic solution $\xi_{u_{\phi_\varepsilon,z_0}}$:

$$\|\xi_{u_\varepsilon}(t)-\xi_{u_{\phi_\varepsilon,z_0}}(t)\|_{E^1(\Omega)}\leqslant C\mathrm{e}^{\alpha(t-\tau)}\|\xi-\xi_{u_{\phi_\varepsilon,z_0}}\|_{E^1(\Omega)},\quad t\leqslant\tau, \tag{5.17}$$

where the positive constants C and α are independent of ε, ξ, ϕ_ε and τ.

Sketch of the proof. By definition (5.13), in order to construct the unstable manifold $\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau)$, it is sufficient to find all the backward solutions of equation (3.3) defined for $t\leqslant\tau$ and belonging to the small neighbourhood $\tilde V_{z_0}$ of ξ_{z_0}, and to prove that the set of all these solutions generates a manifold. Moreover, without loss of generality, we may assume that $\tau=0$. The general case reduces to this particular one using the obvious translation formula

$$\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau)=\mathcal{M}^{+,\mathrm{loc}}_{T_\tau\phi_\varepsilon,z_0}(0). \tag{5.18}$$

As in the proof of theorem 5.1, instead of finding the backward solutions of problem (3.3), it is more convenient to solve the equivalent difference equation (5.8) on the space of one-sided sequences $\mathbb{L}^-(E^1):=L^\infty(\mathbb{Z}_-,E^1(\Omega))$, but, in contrast to the proof of theorem 5.1, we now need to endow equation (5.1) with the appropriate initial condition at $n=0$. To be more precise, we make the change of variables $\tilde\xi(n):=\xi(n)-\xi_{u_{\phi_\varepsilon,z_0}}(n)$ and we consider, for every $\xi^+\in E_+$, the following problem in the space $\mathbb{L}^-(E^1)$:

$$\begin{aligned}\tilde\xi(n)=U_{\phi_\varepsilon}(n,n-1)(\xi_{u_{\phi_\varepsilon,z_0}}(n-1)+\tilde\xi(n-1))\\ -U_{\phi_\varepsilon}(n,n-1)(\xi_{u_{\phi_\varepsilon,z_0}}(n-1)),\quad \Pi_+\tilde\xi(0)=\xi^+.\end{aligned} \tag{5.19}$$

As in the proof of theorem 5.1, the uniform convergence of operators $U_{\phi_\varepsilon}(n,n-1)$ to the limit semigroup S_1 established in propositions 3.7 and 3.8 and the exponential dichotomy (5.5) allow us to prove, using the implicit function theorem, that, for sufficiently small $\varepsilon\leqslant\varepsilon_0'$, equation (5.19) possesses a unique solution $\xi_{v_{\phi_\varepsilon,\xi^+}}(n)$, $n\leqslant 0$, which belongs to $\tilde V_{z_0}$ and this solution depends smoothly (C^1) on the initial data ξ^+ belonging to some small neighbourhood $W^+_{\phi_\varepsilon}$ of zero in E^+ (the detailed

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proof of this is given in [14]). Thus, the desired maps $\mathbb{M}^+_{\phi_\varepsilon,0}(\xi^+)$ can now be defined via

$$\mathbb{M}^+_{\phi_\varepsilon,0}(\xi^+) := \Pi_-\xi_{v_{\phi_\varepsilon,\xi^+}}(0), \quad \xi^+ \in W^+_{\phi_\varepsilon}. \tag{5.20}$$

Indeed, the representation (5.15) is an immediate corollary of the construction of the solution $\xi_{v_{\phi_\varepsilon,\xi^+}}$ and estimate (5.16) follows from (5.19) and the implicit function theorem. Thus, it only remains to verify (5.17) or (analogously) that the constructed solution $v_{\phi_\varepsilon,\xi^+}(n)$ of equation (5.19) decays exponentially as $n \to -\infty$. To this end, following [14], it is sufficient to consider equation (5.19) in the weighted space $\mathbb{L}^-_\beta(E^1)$ of sequences decaying exponentially as $n \to -\infty$ (the norm of this space is given by $\|\xi\|_{\mathbb{L}^-_\beta(E^1)} := \sup_{n\leqslant 0} \mathrm{e}^{-\beta n}\|\xi(n)\|_{E^1(\Omega)}$, $\beta > 0$). As shown in [14], the exponential dichotomy (5.5) allows us to apply the implicit function theorem to equation (5.19) not only in the space $\mathbb{L}^-(E^1)$, but also in $\mathbb{L}^-_\beta$ for $\beta > 0$ sufficiently small, and obtain a solution $\xi_{\tilde v_{\phi_\varepsilon,\xi^+}} \in \mathbb{L}^-_\beta(E^1)$. Finally, the uniqueness part of the implicit function theorem implies that $\xi_{\tilde v_{\phi_\varepsilon,\xi^+}} = \xi_{v_{\phi_\varepsilon,\xi^+}}$. Therefore, (5.17) is verified and theorem 5.3 is proven. □

Remark 5.4. According to theorem 5.3,

$$\dim \mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau) = \dim E_+ = \mathrm{ind}^+(z_0).$$

Moreover, it follows from the proof of theorem 5.3 that these manifolds are C^1-diffeomorphic to $\mathbb{R}^{\mathrm{ind}^+(z_0)}$. We also note that, analogously to theorem 5.3, we may also construct the local *stable* manifolds $\mathcal{M}^{-,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau)$, which are diffeomorphic to E_- and consist of all solutions of (3.3) stabilizing to $u_{\phi_\varepsilon,z_0}(t)$ as $t \to +\infty$, but these manifolds are not necessary for the construction of regular attractors and, therefore, we do not consider them here.

We are now ready to define the *global* unstable sets $\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau)$ via

$$\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau) := \Big\{\xi^\tau \in E^1(\Omega),\ \exists \xi_{u_\varepsilon} \in \mathcal{K}_{\phi_\varepsilon} \text{ such that } \xi_{u_\varepsilon}(\tau) = \xi^\tau \text{ and } \lim_{t\to-\infty} \|\xi_{u_\varepsilon}(t) - \xi_{u_{\phi_\varepsilon,z_0}}(t)\|_{E^1(\Omega)} = 0\Big\}, \tag{5.21}$$

which consist of values at $t = \tau$ of all solutions $\xi_u \in \mathcal{K}_{\phi_\varepsilon}$ that *stabilize* to the 'equilibrium' $\xi_{u_{\phi_\varepsilon,z_0}}(t)$ as $t \to -\infty$. Then, obviously, the sets $\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau)$, $\tau \in \mathbb{R}$, are strictly invariant with respect to $U_{\phi_\varepsilon}(t,\tau)$, i.e.

$$U_{\phi_\varepsilon}(t,\tau)\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau) = \mathcal{M}^+_{\phi_\varepsilon,z_0}(t), \quad t \geqslant \tau. \tag{5.22}$$

Moreover, due to definition 5.2 and theorem 5.3, the global unstable sets can be expressed in terms of the local sets via

$$\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau) = \bigcup_{n=1}^{\infty} U_{\phi_\varepsilon}(\tau,\tau-n)\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau-n) \tag{5.23}$$

if ε is sufficiently small. It is also worth mentioning that, in the limit case $\varepsilon = 0$, we have the *autonomous* equation (3.1) and, consequently, the limit unstable sets $\mathcal{M}^+_{\bar f,z_0}(\tau)$ that correspond to equation (3.1) are independent of τ, i.e.

$$\mathcal{M}^+_{\bar f,z_0}(\tau) \equiv \mathcal{M}^+_{\bar f,z_0}, \quad \forall \tau \in \mathbb{R} \quad \text{and} \quad S_t\mathcal{M}^+_{\bar f,z_0} = \mathcal{M}^+_{\bar f,z_0}. \tag{5.24}$$

REMARK 5.5. We recall that (5.23) and the fact that

$$\mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(\tau) \sim \mathbb{R}^{\mathrm{ind}^+(z_0)}$$

allows us to endow the set $\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau)$ with the structure of a C^1-manifold diffeomorphic to $\mathbb{R}^{\mathrm{ind}^+(z_0)}$. But, in contrast to the local sets, generally these sets may not be *submanifolds* of $E^1(\Omega)$, since the recurrent motions (e.g. homoclinic orbits to $u_{\phi_\varepsilon,z_0}(t)$) may exist near $u_{\phi_\varepsilon,z_0}(t)$. Nevertheless, in our case, the limit equation (3.1) possesses a global Lyapunov function, which does not allow the motions mentioned above to exist if $\varepsilon \geqslant 0$ is sufficiently small (see lemma 6.4, below). Thus, as proved for example, in [14] (see also [3, 18]), the sets (5.21) are indeed C^1-submanifolds of $E^1(\Omega)$ diffeomorphic to $\mathbb{R}^{\mathrm{ind}^+(z_0)}$ if ε is sufficiently small.

To conclude this section, we formulate the standard fact that every trajectory of equation (3.3) is exponentially attracted to $\mathcal{M}^+_{\phi_\varepsilon}(t)$ while staying in the neighbourhood of ξ_{z_0}. This is the main technical tool in the proof of the exponential rate of the attraction to the regular attractor (see [3, 14]).

THEOREM 5.6. *Let the assumptions of theorem 5.1 hold. Then, there exist $\varepsilon_0'' > 0$ and a neighbourhood V_{z_0} of the equilibrium ξ_{z_0} in $E^1(\Omega)$ such that if $\varepsilon \leqslant \varepsilon_0''$, $\tau \in \mathbb{R}$ and $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ is arbitrary and $\xi_u(t)$, $t \geqslant \tau$ is an arbitrary solution of (3.3) which satisfies*

$$\xi_u(t) \in V_{z_0}, \quad \forall t \in [\tau, \tau + N], \tag{5.25}$$

for some $N \in \bar{\mathbb{N}}$ ($N = +\infty$ is allowed), then there exists a solution $\xi_{u^+}(t)$ of equation (3.3) such that $\xi_{u^+}(t) \in \mathcal{M}^{+,\mathrm{loc}}_{\phi_\varepsilon,z_0}(t)$, $t \leqslant \tau + N$, and

$$\|\xi_u(t) - \xi_{u^+}(t)\|_{E^1(\Omega)} \leqslant C\mathrm{e}^{-\beta(t-\tau)}\|\xi_u(\tau) - \xi_{u^+}(\tau)\|_{E^1(\Omega)}, \quad t \in [\tau, \tau + N], \tag{5.26}$$

where positive constants C and β are independent of ε, N, τ, ξ_u and ϕ_ε.

The detailed proof of this theorem (which is based on propositions 3.7 and 3.8 and the dichotomy (5.5)) is given in [14] (in fact, in an abstract setting). This is the reason why we only mention here that the desired solution $\xi_{u^+}(t)$ of (3.3) or (analogously) its discrete analogue $\xi(n) = \xi_{u^+}(\tau + n)$, $n \in \{0, \dots, N\}$, can be obtained by applying the implicit function theorem to the following problem:

$$\left.\begin{aligned} \xi(n) = U_{T_\tau\phi_\varepsilon}(n, n-1)\xi(n-1), \quad n = 1, \dots, N, \\ \Pi_-(\xi(0) - \xi_{u_{\phi_\varepsilon,z_0}}(\tau)) = \mathbb{M}^+_{\phi_\varepsilon,\tau}(\Pi_+(\xi(0) - \xi_{u_{\phi_\varepsilon,z_0}}(\tau))), \\ \Pi_-\xi(N) = \Pi_-\xi_u(\tau + N), \end{aligned}\right\} \tag{5.27}$$

and the remaining details are left to the reader.

REMARK 5.7. We note that, in the case $N = +\infty$ in theorem 5.6, we necessarily have $\xi_{u^+}(t) \equiv \xi_{u_{\phi_\varepsilon,z_0}}(t)$ (since, due to (5.17), this is the only solution belonging to the unstable manifold $\mathcal{M}^+_{\phi_\varepsilon,z_0}(t)$, which remains in a small neighbourhood of ξ_{z_0} for all $t \geqslant \tau$). Thus, thanks to theorem 5.6, every solution $\xi_u(t)$ of equation (3.3) which belongs to V_{z_0} for every $t \geqslant \tau$ stabilizes exponentially to $\xi_{u_{\phi_\varepsilon,z_0}}(t)$ as $t \to \infty$.

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6. The regular pull-back attractor and its averaging

This section is devoted to the detailed study of the pull-back attractors $\mathcal{A}_{\phi_\varepsilon}(t)$ in the case where $\varepsilon > 0$ is sufficiently small and the limit attractor $\mathcal{A}_0$ is *regular* using the theory of the non-autonomous perturbations of regular attractors developed in [14] (see also [18]).

We start with the limit case $\varepsilon = 0$. In this case, as known, equation (3.1) possesses a global Lyapunov function of the form

$$L(\xi_u) := \int_\Omega [|\partial_t u(x)|^2 + |\nabla_x u(x)|^2 + \lambda_0 |u(x)|^2 + 2F(u(x)) - 2g(x)u(x)]\,\mathrm{d}x, \quad (6.1)$$

where $F(u) := \int_0^u \bar{f}(v)\,\mathrm{d}v$ (see, for example, [3]).

The main additional assumption of this section is that all of the equilibria of equation (3.1) are hyperbolic, i.e. that all the solutions of equation (5.1) satisfy condition (5.2). In this case, obviously, the set $\mathcal{R}_0$ of all the equilibria of (3.1) is finite:

$$\mathcal{R}_0 = \{\xi_{z_i}\}_{i=1}^N \quad \text{and } z_i \text{ satisfies (5.1) and (5.2).} \quad (6.2)$$

Under the above assumptions, the limit attractor $\mathcal{A}_0$ possesses the following description.

THEOREM 6.1. *Let the assumptions of theorem 3.1 hold and, in addition, let (6.2) be satisfied. The global attractor $\mathcal{A}_0$ is then a finite collection of the finite-dimensional unstable manifolds $\mathcal{M}^+_{\bar{f},z_0}$ associated with the equilibria (6.2):*

$$\mathcal{A}_0 = \bigcup_{z_0 \in \mathcal{R}_0} \mathcal{M}^+_{\bar{f},z_0}, \quad \mathcal{M}^+_{\bar{f},z_0} \sim \mathbb{R}^{\mathrm{ind}^+(z_0)}. \quad (6.3)$$

Furthermore, every solution $\xi_u \in \mathcal{K}_{\bar{f}}$ is a heteroclinic orbit between two different equilibria $\xi_{z_0^+}$ and $\xi_{z_0^-}$ belonging to $\mathcal{R}_0$ and every solution $\xi_u(t)$ of equation (3.1) defined on a semi-interval $[\tau, +\infty)$ tends, as $t \to \infty$, to one of the equilibria $\xi_{z_0} \in \mathcal{R}_0$. Moreover, the attractor $\mathcal{A}_0$ attracts exponentially all bounded subsets of $E^1(\Omega)$, i.e. estimate (4.17) is satisfied.

The proof of this theorem can be found in [3]; see also the explanations in the proof of theorem 6.3 below.

REMARK 6.2. We recall that the hyperbolicity assumption (6.2) is generic in the sense that it is satisfied for all external forces $g(x)$ belonging to an open and dense subset of $L^2(\Omega)$ (see [3]).

The main result of this section is the following theorem which gives the analogous description of the *pull-back* attractors $\mathcal{A}_{\phi_\varepsilon}(\tau)$ of equations (3.3) for sufficiently small, but positive ε and establish the upper and lower semi-continuity of them as $\varepsilon \to 0$.

THEOREM 6.3. *Let the assumptions of theorems 4.2 and 6.1 hold. Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon \leqslant \varepsilon_0$, the following assertions are satisfied.*

(i) *For every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, equation (3.3) possesses exactly $N = \#\mathcal{R}_0$ different almost-periodic solutions $\xi_{u_{\phi_\varepsilon,z_i}}(t)$, $i = 1, \dots, N$, in the ball $B_{R_0(\varepsilon)}$, which are constructed in theorem 5.1.*

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(ii) *Every complete bounded solution* $\xi_u \in \mathcal{K}_{\phi_\varepsilon}$ *of equation (3.3) is a heteroclinic orbit between the two different almost-periodic solutions mentioned above, i.e.*

$$\lim_{t\to\pm\infty} \|\xi_u(t) - \xi_{u_{\phi_\varepsilon,z_\pm}}(t)\|_{E^1(\Omega)} = 0, \quad \xi_{z_\pm} \in \mathcal{R}_0, \quad z_+ \neq z_-, \tag{6.4}$$

and every solution $\xi_u(t)$ *of (3.3) defined on a semi-interval* $t \in [\tau, +\infty)$ (*which satisfies* $\xi_u(\tau) \in B_{R_0}(\varepsilon)$) *converges as* $t \to \infty$ *to one of these almost-periodic solutions.*

(iii) *The pull-back attractor* $\mathcal{A}_{\phi_\varepsilon}(\tau)$ *possesses the description*

$$\mathcal{A}_{\phi_\varepsilon}(\tau) = \bigcup_{z_0\in\mathcal{R}_0} \mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau), \quad \tau \in \mathbb{R}, \tag{6.5}$$

analogous to (6.3), where the C^1*-submanifolds* $\mathcal{M}^+_{\phi_\varepsilon,z_0}(\tau)$ *are the* (*global*) *unstable manifolds of the almost-periodic solution* $u_{\phi_\varepsilon,z_0}(t)$ *associated with the equilibrium* $\xi_{z_0} \in \mathcal{R}_0$ (*which are constructed in the previous section*).

(iv) *The pull-back attractors* $\mathcal{A}_{\phi_\varepsilon}$ *are uniformly* (*with respect to* $\tau \in \mathbb{R}$ *and* $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$) *exponential, i.e. there exist a positive constant* α *and a monotonic function* Q (*which is independent of* ε, τ *and* ϕ_ε) *such that, for every* (*bounded*) *subset* $B \subset B_{R_0(\varepsilon)}$, *we have*

$$\operatorname{dist}_{E^1(\Omega)}(U_{\phi_\varepsilon}(\tau+t,\tau)B, \mathcal{A}_{\phi_\varepsilon}(\tau+t)) \leqslant Q(\|B\|_{E^1(\Omega)})\mathrm{e}^{-\alpha t}. \tag{6.6}$$

(v) *The attractors* $\mathcal{A}_{\phi_\varepsilon}(\tau)$ *tend as* $\varepsilon \to 0$ *to the limit attractor* $\mathcal{A}_0$ *in the following sense:*

$$\operatorname{dist}^{\mathrm{symm}}_{E^1(\Omega)}(\mathcal{A}_{\phi_\varepsilon}(\tau), \mathcal{A}_0) \leqslant C_{\bar R}[\alpha_{C'_{\bar R}}(\varepsilon)]^\kappa, \tag{6.7}$$

where the function $\alpha_R(\varepsilon)$ *is the same as in proposition 3.8,* $\bar R$ *is the same as in theorem 4.2, the positive constants* κ, $C_{\bar R}$ *and* $C'_{\bar R}$ *are independent of* τ, ε, B *and* ϕ_ε, *and* $\operatorname{dist}^{\mathrm{symm}}_V(X,Y)$ *denotes the symmetric Hausdorff distance between subsets* X *and* Y *of* V.

Sketch of the proof. We give below only an overview of the proof of theorem 6.3, (the details can be found in [14], see also [3,18]). As usual, this proof is based on the following lemma which allows us to reduce the analysis of the *global* behaviour of solutions of (3.3) to the *local* analysis of equation (3.3) near the equilibria $\xi_{z_0} \in \mathcal{R}_0$.

LEMMA 6.4. *Let the assumptions of theorem 6.1 hold. The following statements are then valid.*

(i) *For every neighbourhood* V *of zero in* $E^1(\Omega)$ *and every bounded subset* $B \subset E^1(\Omega)$, *there exist* $\varepsilon_0 = \varepsilon_0(B,V)$ *and* $T = T(B,V) > 0$ *such that every solution* $\xi_u(t)$ *of equation (3.3)* (*with* $\varepsilon \leqslant \varepsilon_0$, $\tau \in \mathbb{R}$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$), *such that* $\xi_u(\tau) \in B$, *visits the* V*-neighbourhood of the set* $\mathcal{R}_0$ *on every time-interval of length* T, *i.e. for every* $s \geqslant \tau$, *there exists* $T_s = T_s(u) \in [s, s+T]$ *and* $\xi_{z_0} \in \mathcal{R}_0$ *such that*

$$\xi_u(T_s) \in \xi_{z_0} + V.$$

(ii) *There exist small neighbourhoods W and W' of zero in $E^1(\Omega)$, $W' \subset W$, and positive ε_0 such that every solution $\xi_u(t)$, $t \geqslant \tau$, of equation (3.3) (with $\varepsilon \leqslant \varepsilon_0$, $\tau \in \mathbb{R}$ and $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$) satisfies the following condition: if $\xi_u(\tau) \in \xi_{z_0} + W'$ for some $\xi_{z_0} \in \mathcal{R}_0$ and $\xi_u(T) \notin \xi_{z_0} + W$ for some $T \geqslant \tau$, then this trajectory never returns to the W'-neighbourhood of ξ_{z_0}:*

$$\xi_u(t) \notin \xi_{z_0} + W', \quad \forall t \geqslant T. \tag{6.8}$$

Sketch of the proof. Assume that the first assertion of the lemma is false. There then exist a neighbourhood V_0 of zero, a sequence $T_n \to \infty$ and a sequence of solutions $\xi_{u_n}(t) := U_{\phi_{\varepsilon_n}}(t, \tau_n)\xi_n$ of equation (3.3) such that $\varepsilon_n \to 0$ as $n \to 0$ and

$$\xi_{u_n}(t) \notin V_0 + \mathcal{R}_0, \quad \forall t \in [\tau_n, \tau_n + T_n] \quad \text{and} \quad \xi_{u_n}(\tau_n) \in B. \tag{6.9}$$

Let us consider a new sequence of solutions $\xi_{\tilde{u}_n}(t) := \xi_{u_n}(t + \tau_n + \frac{1}{2}T_n)$. Then, (6.9) and (3.21) imply that these solutions are defined on $t \in [-\frac{1}{2}T_n, \frac{1}{2}T_n]$ and

$$\xi_{\tilde{u}}(t) \notin V_0 + \mathcal{R}_0, \qquad \|\xi_{\tilde{u}_n}(t)\|_{E^1(\Omega)} \leqslant C, \quad \forall t \in [-\tfrac{1}{2}T_n, \tfrac{1}{2}T_n]. \tag{6.10}$$

Using proposition 3.8 and the fact that the limit equation (3.1) possesses a global attractor, we can assume without loss of generality that, as $n \to \infty$, the sequence $\xi_{\tilde{u}_n}(t)$ tends to some complete solution $\xi_{\bar{u}} \in \mathcal{K}_{\bar{f}}$ of the limit equation (3.1) (e.g. in the space $L^\infty_{\mathrm{loc}}(\mathbb{R}, E^1(\Omega))$). Now, on passing to the limit $n \to \infty$ in (6.10), we deduce that $\xi_{\bar{u}}(t) \notin V_0 + \mathcal{R}$, for all $t \in \mathbb{R}$, which contradicts the fact that (3.1) possesses a global Lyapunov function (see theorem 6.1). Thus, the first assertion of the lemma is proven.

Analogously, assuming that the second assertion is wrong, we construct a homoclinic structure for the limit equation (3.1) that also contradicts the existence of a global Lyapunov function (see [14] for the details). Lemma 6.4 is proven. □

We are now ready to finish the proof of theorem 6.3. To this end, we fix a neighbourhood $V_0 \subset W'$ (where W' is the same as in lemma 6.4) such that the assertions of theorems 5.1, 5.3, 5.6 are satisfied for all neighbourhoods $\xi_{z_i} + V_0$, $i = 1, \dots, N$. We then fix $B = B_{\bar{R}}$ to be a uniform absorbing set for the processes $U_{\phi_\varepsilon}(t, \tau)$ in $E^1(\Omega)$ (which exists due to estimate (3.21)). Finally, we assume that $\varepsilon_0 > 0$ and $T > 0$ are such that assertions of theorems 5.1, 5.3, 5.6 and lemma 6.4 hold, for every $\varepsilon \leqslant \varepsilon_0$ and every $\xi_{z_0} \in \mathcal{R}_0$.

Then, the second statement of lemma 6.4 implies that every solution $\xi_u(t)$, $t \geqslant \tau$ (such that $\varepsilon \leqslant \varepsilon_0$ and $\xi_u(\tau) \in B$) can leave the neighbourhood $V_0 + \mathcal{R}_0$ only a finite number ($N_u \leqslant N$) of times and, consequently, due to the first assumption of lemma 6.4, there exists $\xi_{z_0^+} \in \mathcal{R}_0$ such that $\xi_u(t) \in \xi_{z_0^+} + V_0$ for all sufficiently large t. Now theorem 5.6 and remark 5.7 imply that $\xi_u(t)$ stabilizes exponentially to $\xi_{u_{\phi_\varepsilon, z_0^+}}(t)$ as $t \to +\infty$. Analogously, if $\xi_u \in \mathcal{K}_{\phi_\varepsilon}$ is a complete solution of (3.3), then, due to lemma 6.4 and the fact that the number of the equilibria is finite, we have $\xi_u(t) \in V_0 + \xi_{z_0^-}$ for all sufficiently small t. Now, theorem 5.3 implies that $\xi_u(t)$ stabilizes to $\xi_{u_{\phi_\varepsilon, z_0^-}}(t)$ as $t \to -\infty$. Thus, theorem 6.3(i) and (ii) are verified.

Description (6.5) is an immediate corollary of stabilization (6.4), definition (5.21) of the unstable manifolds and formula (4.7).

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The exponential attraction (6.6) is a standard corollary of the following facts: every trajectory of (3.1) spends only a finite time $\tilde{T} \leqslant T \cdot \#\mathcal{R}_0$ outside the neighbourhood $V_0 + \mathcal{R}_0$ (due to lemma 6.4); this trajectory is exponentially attracted to $\mathcal{A}_{\phi_\varepsilon}(t)$ while staying inside $V_0 + \mathcal{R}_0$ (due to theorem 5.6). Moreover, since the time $\tilde{T}$ and the rate of the attraction in theorem 5.6 are independent of ε, (6.6) will also be uniform with respect to ε (see [14] for the details).

Finally, estimate (6.7) is a formal corollary of the uniform exponential attraction (6.6) and proposition 3.8 (and can be obtained exactly as in corollary 4.5; see also [3, 14]). Theorem 6.3 is proven. □

REMARK 6.5. We recall that the constructed uniform ($\mathcal{A}_\varepsilon$) and pull-back ($\mathcal{A}_{\phi_\varepsilon}(\tau)$) attractors of equation (3.3) attract the solutions $u(t)$ whose initial data belong to the large ball $B_{R_0(\varepsilon)}$ in $E^1(\Omega)$ only (with $\lim_{\varepsilon\to 0} R_0(\varepsilon) = \infty$). We note, however, that, even in the case where we have the global solvability of problem (3.3) for every $\xi^\tau \in E^1(\Omega)$ and the associated family of processes has the attractor in the whole space $E^1(\Omega)$, it does not necessarily coincide with $\mathcal{A}_\varepsilon$ and may even diverge to infinity as $\varepsilon \to 0$. We will give the corresponding examples in § 8.

We now formulate two corollaries of theorem 6.3.

COROLLARY 6.6. *Let the assumptions of theorem 6.3 hold. Then the uniform attractors $\mathcal{A}_\varepsilon$ of problems (3.3) are upper and lower semi-continuous as $\varepsilon \to 0$. Moreover, the following estimate holds:*

$$\operatorname{dist}^{\mathrm{symm}}_{E^1(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leqslant C_{\bar{R}}[\alpha_{C'_{\bar{R}}}(\varepsilon)]^\kappa, \tag{6.11}$$

where the right-hand side of (6.11) is the same as in (6.7).

Indeed, estimate (6.11) is an immediate corollary of (6.7) and (4.24).

COROLLARY 6.7. *Let the assumptions of theorem 6.3 and lemma 2.6 hold (e.g. let f_ε be periodic with respect to t). Then, estimate (6.7) can be improved as follows:*

$$\operatorname{dist}^{\mathrm{symm}}_{E^1(\Omega)}(\mathcal{A}_{\phi_\varepsilon}(\tau), \mathcal{A}_0) \leqslant C''_{\bar{R}}\varepsilon^\kappa, \tag{6.12}$$

where $\bar{R}$ and $\kappa > 0$ are the same as in (6.7) and the positive constant $C''_{\bar{R}}$ is independent of independent of τ, ε and ϕ_ε.

Indeed, on inserting estimate (2.30) into the right-hand side of (6.7), we may derive (6.12).

REMARK 6.8. It is worth noting that, under the assumptions of theorem 6.3, we have a simpler relation between the uniform and pull-back attractors, namely,

$$\mathcal{A}_\varepsilon = \left[\bigcup_{t\in\mathbb{R}} \mathcal{A}_{\phi_\varepsilon}(t)\right]_{E^1(\Omega)}, \quad \text{for every fixed } \phi_\varepsilon \in \mathcal{H}(f_\varepsilon). \tag{6.13}$$

Indeed, description (6.13) follows from the uniform attraction (6.6) and the alternative definition of the uniform attractor (see remark 4.3).

It is also worth noting that, under the assumptions of theorem 6.3, the pull-back attractors $\mathcal{A}_{\phi_\varepsilon}(t)$ are *almost periodic* with respect to t as the set-valued functions $t \to \mathcal{A}_{\phi_\varepsilon}(t)$ for every $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ (see [14] for details).

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7. The subordinated oscillations

In this section, we study the case of the so-called subordinated oscillations where we have the global existence of solutions and uniform (with respect to $\varepsilon \to 0$) dissipativity of system (3.3) not only for $\xi^\tau \in B_{R_0(\varepsilon)}$ but for every $\xi^\tau \in E^1(\Omega)$ (and even for every $\xi^\tau \in E(\Omega)$).

We first recall that in the previous sections we imposed the dissipativity and growth assumptions (see (2.8)) to the averaged function $\bar{f}$ only, so, if we want to have the global solvability of problems (3.3) *for arbitrary* $\xi^\tau \in E^1(\Omega)$, we need to impose some assumptions to the functions $\phi_\varepsilon(u,t)$ for *positive* ε. It seems natural to require, analogously to the case of autonomous equation (3.1), the nonlinearity $f_\varepsilon(u,t)$ to satisfy conditions (2.8) *uniformly* with respect to t and ε. We note, however, that in the non-autonomous case the sole assumption (2.8) is not sufficient to obtain the dissipative estimate for the solutions of (3.3) (see, for example, example 8.4 below). The standard additional assumption (see, for example, [8,11]), which guarantees the dissipativity of the non-autonomous equation (3.3), is as follows:

$$\partial_t \phi_\varepsilon(u,t) \leqslant \delta \phi_\varepsilon(u,t) \cdot u + C_\delta, \quad \forall t, u \in \mathbb{R}, \tag{7.1}$$

where $\delta = \delta(\gamma)$ is a sufficiently small positive number. We note, however, that the function $\phi_\varepsilon(u,t)$ contains the rapidly oscillating term t/ε, so the derivative $\partial_t \phi_\varepsilon$ is of order $1/\varepsilon$ as $\varepsilon \to 0$ and, consequently, estimate (7.1) cannot be uniform with respect to ε. Thus, using (7.1), we cannot obtain uniform (with respect to ε) bounds for the corresponding attractors.

This is why, instead of (7.1), we use below the following (in a sense, more restrictive) assumption that

$$|f_\varepsilon(u,t) - \bar{f}(u)|^2 \leqslant \delta \bar{f}(u) \cdot u + C_\delta, \quad \forall t, u \in \mathbb{R}, \quad \varepsilon \geqslant 0, \tag{7.2}$$

where δ and C_δ are independent of ε and t, and $\delta = \delta(\gamma)$ is sufficiently small. In particular, (7.2) implies that the leading part of the nonlinearity $f_\varepsilon(u,t)$ is autonomous (which justifies the title 'subordinated oscillations' of this section). We start from the following theorem, which gives the uniform (with respect to ε) dissipative estimate in the space $E(\Omega)$ for the solutions of (3.3).

THEOREM 7.1. *Let the functions $f(\varepsilon,u,z)$ satisfy the assumptions of lemma 2.1. Assume, in addition that estimate (7.2) holds, where the average $\bar{f}$ satisfies assumptions (2.8) and that the growth restriction*

$$|\partial_u^2 f_\varepsilon(u,t)| \leqslant C(1+|u|), \quad \forall t, u \in \mathbb{R}, \tag{7.3}$$

holds, where C is independent of ε. Then, for every $\varepsilon \geqslant 0$, $\tau \in \mathbb{R}$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ and $\xi^\tau \in E(\Omega)$, equation (3.3) possesses a unique solution $\xi_u(t) \in E(\Omega)$ for every $t \geqslant \tau$, and the following estimate holds:

$$\|\xi_u(t)\|_{E(\Omega)} \leqslant Q(\|\xi^\tau\|_{E(\Omega)}) \mathrm{e}^{-\alpha(t-\tau)} + Q(\|g\|_{L^2(\Omega)}), \tag{7.4}$$

where the positive constant α and the monotonic function Q are independent of ε, τ, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$ and $\xi^\tau \in E(\Omega)$.

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Proof. Although the assertion of the theorem is more or less standard, we give below the derivation of (7.4), in order to show that it is indeed uniform with respect to ε. For simplicity, we consider below only the case $\phi_\varepsilon = f_\varepsilon$ and $\tau = 0$ (the general case is analogous due to lemma 2.3). Moreover, we give only the formal derivation of estimate (7.4), which can be easily justified using, for example, the Galerkin approximations method (we recall that assumption (7.3) guarantees the uniqueness of a solution to (3.3) in the three-dimensional case; see [3]). To this end, following the standard procedure (see, for example, [3, 29]), we multiply equation (3.3) by $\partial_t u + \beta u$, for some positive β, and integrate over Ω. Then, after the integration by parts, we have

$$\begin{aligned}\partial_t[\|\xi_u(t)\|^2_{E(\Omega)} &+ 2\|F(u(t))\|_{L^1(\Omega)} + \lambda_0\|u(t)\|^2_{L^2(\Omega)} + 2\beta(u(t),\partial_t u(t))]\\ &+ 2(\gamma-\beta)\|\partial_t u(t)\|^2_{L^2(\Omega)} + 2\beta\|\nabla_x u(t)\|^2_{L^2(\Omega)}\\ &+ 2\lambda_0\|u(t)\|^2_{L^2(\Omega)} + 2\beta(\bar f(u(t)),u(t))\\ &\quad= 2(g,\partial_t u(t) - \beta u(t)) - 2(\tilde f_\varepsilon(u(t),t),\partial_t u(t)+\beta u(t)),\end{aligned} \tag{7.5}$$

where

$$F(v) := \int_0^v \bar f(u)\,\mathrm{d}u$$

and $\tilde f_\varepsilon(u,t) := f_\varepsilon(u,t) - \bar f(u)$. We recall that dissipativity assumption $(2.8)_3$ implies that

$$F(v) \leqslant f(v)\cdot v + C, \quad f(v)\cdot v \geqslant -C_\mu - \mu|v|^2 \quad \text{and} \quad F(v) \geqslant -C_\mu - \mu|v|^2, \tag{7.6}$$

where $\mu > 0$ can be arbitrarily small and the constants C and C_μ are independent of $v\in\mathbb{R}$. Now, using estimates (7.2), (7.6) and the Cauchy–Schwarz inequality, we find from (7.5) that there exist sufficiently small (but independent of ε) positive constants $\beta = \beta(\gamma)$, $\delta = \delta(\gamma)$ (which is the same as in assumption (7.2)) and $\alpha = \alpha(\gamma)$ such that

$$\begin{aligned}\partial_t[\|\xi_u(t)\|^2_{E(\Omega)} &+ 2\|F(u(t))\|_{L^1(\Omega)} + \lambda_0\|u(t)\|^2_{L^2(\Omega)} + 2\beta(u(t),\partial_t u(t))]\\ &+ \alpha[\|\xi_u(t)\|^2_{E(\Omega)} + 2\|F(u(t))\|_{L^1(\Omega)} + \lambda_0\|u(t)\|^2_{L^2(\Omega)} + 2\beta(u(t),\partial_t u(t))]\\ &\quad\leqslant C(1+\|g\|^2_{L^2(\Omega)}).\end{aligned} \tag{7.7}$$

By applying the Gronwall inequality to this relation, we derive estimate (7.4), and theorem 7.1 is proven. □

Thus, under the assumptions of theorem 7.1, equations (3.3) define a family of dynamical processes which are defined *globally* on $E(\Omega)$:

$$U_{\phi_\varepsilon}(t,\tau) : E(\Omega) \to E(\Omega), \quad \phi_\varepsilon \subset \mathcal{H}(f_\varepsilon), \quad \tau\in\mathbb{R},\ t\geqslant\tau. \tag{7.8}$$

The main result of this section is the following theorem, which establishes the existence of a uniform attractor for (7.8) and verifies that, for small ε, this attractor coincides with the one constructed in theorem 4.2 starting from the ball $B_{R_0(\varepsilon)}$ of the space $E^1(\Omega)$.

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THEOREM 7.2. *Let the assumptions of theorem 7.1 hold. Then, for every $\varepsilon > 0$, family (7.8) possesses a uniform attractor $\mathcal{A}_\varepsilon$ that is compact in $E(\Omega)$ and is uniformly bounded with respect to ε:*

$$\|\mathcal{A}_\varepsilon\|_{E(\Omega)} \leqslant C. \tag{7.9}$$

Moreover, there exists a small positive ε_0 such that, for every $\varepsilon \leqslant \varepsilon_0$, the attractors $\mathcal{A}_\varepsilon$ are compact in $E^1(\Omega)$ and

$$\|\mathcal{A}_\varepsilon\|_{E^1(\Omega)} \leqslant C_1, \tag{7.10}$$

where the constant C_1 is independent of ε.

Proof. We first consider the case of *small* ε, where we have regularity (7.10) of the attractor. We also recall that, in the subcritical case, where the growth rate of f_ε is strictly less than cubic (see assumptions (2.21)) this regularity can be obtained using, for example, bootstrap arguments, exactly as in the autonomous case (see [3]). Therefore, we mainly consider the critical case of a cubic rate of growth. In this case, the derivation of dissipative estimate (3.2) in the $E^1(\Omega)$-norm in the autonomous case (see [3]) essentially uses the so-called *dissipation integral* which equals infinity in the non-autonomous case. Therefore, the methods of [3] cannot be directly applied in order to obtain regularity (7.10). Nevertheless, there is a possibility to adapt these methods to equation (3.3) with *small* $\varepsilon > 0$. Since we are mainly interested in the limit $\varepsilon \to 0$, this is sufficient for our purposes (see [35] for the case of damped wave equations with general non-autonomous external forces).

We give below the proof only of the $E^\alpha(\Omega)$-regularity of the attractor $\mathcal{A}_\varepsilon$ for some *positive* α that is the most difficult part of the derivation of regularity (7.10) in the critical cubic rate of growth, leaving the proof of E^1-regularity to the reader (since the cubic rate of growth is *subcritical* with respect to the $E^\alpha(\Omega)$-norm, the bootstrap arguments work *starting with the $E^\alpha(\Omega)$-energy* and allow us to deduce estimate (7.10) exactly as in the autonomous case; see [3,29] for the details). Thus, due to the standard theorem on the existence of a global attractor (see the proof of theorem 4.2), we need only to prove the following proposition, which gives the uniform (with respect to ε) attracting set in $E^\alpha(\Omega)$.

PROPOSITION 7.3. *Let the assumptions of theorem 7.2 hold. There then exist $\varepsilon_0 > 0$, $\alpha > 0$ and a sufficiently large ball $B(E^\alpha)$ of the space $E^\alpha(\Omega)$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ and every bounded subset $B \subset E(\Omega)$, we have*

$$\operatorname{dist}_{E(\Omega)}(U_{\phi_\varepsilon}(t+\tau,\tau)B, B(E^\alpha)) \leqslant Q(\|B\|_{E(\Omega)})\mathrm{e}^{-\beta t}, \tag{7.11}$$

where the positive constant β and the monotonic function Q are independent of t, τ, ε and ϕ_ε.

Proof. As before, we consider below only the case $\phi_\varepsilon = f_\varepsilon$ and $\tau = 0$ (the general case is analogous due to lemma 2.3). Moreover, due to estimate (7.4), we may prove (7.11) only for the ball $B = B_R$ of a sufficiently large radius R in $E(\Omega)$.

In order to handle the rapid oscillations in time in equation (3.3), it is convenient to introduce the auxiliary function $w(t) = w_u(t)$ which solves the following equation

$$\partial_t^2 w + \gamma\partial_t w - \Delta_x w + \lambda_0 w = h_u(t) := -\tilde{f}_\varepsilon(u(t),t), \quad \xi_w|_{t=0} = 0. \tag{7.12}$$

Then, due to assumptions (7.2) and (7.3), the function $\tilde{f}_\varepsilon$ satisfies condition (2.21) (with $\delta = 1$). Consequently, thanks to lemma 2.5, estimate (7.4) and proposition A.2, we have

$$\|\xi_w(t)\|_{E^{-1}(\Omega)} \leqslant \alpha(\varepsilon), \quad \forall t \geqslant 0,\ \xi^0 \in B_R, \tag{7.13}$$

where the monotonic function α tends to zero as $\varepsilon \to 0$. Moreover, it follows from (7.2)–(7.4) and the Hölder inequality that

$$\|\nabla_x h_u(t)\|_{L^{3/2}(\Omega)} + \|h_u(t)\|_{L^3(\Omega)} \leqslant C. \tag{7.14}$$

Applying the appropriate interpolation inequality to (7.14), we derive that there exists a positive α, such that

$$\|h_u(t)\|_{H^{2\alpha}(\Omega)} \leqslant C_1. \tag{7.15}$$

Applying proposition A.1 to equation (7.12) and using (7.13) together with the interpolation inequality, we finally derive that

$$\|\xi_w(t)\|_{E^\alpha(\Omega)} \leqslant \tilde{\alpha}(\varepsilon), \quad t \in \mathbb{R}_+, \tag{7.16}$$

where the monotonic function $\tilde{\alpha}$ is independent of t and $\xi^0 \in B_R$ and tends to zero as $\varepsilon \to 0$.

We now set $v(t) := u(t) - w(t)$, where $u(t)$ solves (3.3). This function then satisfies the equation

$$\partial_t^2 v + \gamma\partial_t v - \Delta_x v + \lambda_0 v + \bar{f}(v + w(t)) = g, \quad \xi_v|_{t=0} = \xi_u|_{t=0}. \tag{7.17}$$

Thus, instead of equation (3.3), we will prove the existence of an exponentially attracting set in $E^\alpha(\Omega)$ for equation (7.17). To this end, we split (following [3]) the solution $v(t)$ as follows: $v(t) = v_0(t) + \theta(t)$, where $v_0(t)$ is a solution of the *autonomous* equation

$$\partial_t^2 v_0 + \gamma\partial_t v_0 - \Delta_x v_0 + \lambda v_0 + \bar{f}(v_0) + L v_0 = 0, \quad \xi_{v_0}|_{t=0} = \xi_u|_{t=0}, \tag{7.18}$$

where L is a sufficiently large positive number and the remainder $\theta(t)$ satisfies the equation

$$\partial_t^2 \theta + \gamma\partial_t\theta - \Delta_x\theta + \lambda_0\theta + [\bar{f}(\theta + v_0 + w) - \bar{f}(v_0)] = g + L v_0(t), \quad \xi_\theta|_{t=0} = 0, \tag{7.19}$$

where we may assume without loss of generality that $\bar{f}(0) = 0$. Then, arguing analogously to the proof of theorem 7.1, we derive that the solution $v_0(t)$ decays exponentially if $L = L(\bar{f})$ is sufficiently large:

$$\|\xi_{v_0}(t)\|_{E(\Omega)} \leqslant Q(\|\xi_u(0)\|_{E(\Omega)})\mathrm{e}^{-\beta t}, \quad t \geqslant 0, \tag{7.20}$$

where the positive constant β and the monotonic function Q is independent of ξ^0 and t (see [3] for details). Thus, in order to finish the proof of proposition 7.3, it only remains to verify that the solution $\theta(t)$ is uniformly bounded in $E^\alpha(\Omega)$. To this end, we need the following lemma, which plays the role of a 'dissipation integral' in the case of small positive ε.

Lemma 7.4. *Let the above assumptions hold. The following estimate is then valid for the solution $v(t)$ of equation (7.17):*

$$\int_0^T \|\partial_t v(t)\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant C(T+1)\tilde{\alpha}(\varepsilon), \quad T \in \mathbb{R}_+, \tag{7.21}$$

where the function $\tilde{\alpha}$ is the same as in (7.16) and the constant C is independent of ε, $\xi^0 \in B_R$ and T.

Indeed, estimate (7.21) can be obtained in a standard way by multiplying equation (7.17) by $\partial_t v(t)$, integrating over $[0,T]\times\Omega$ and using estimates (7.4) and (7.16).

Let us now differentiate equation (7.19) with respect to t and denote $W(t) := \partial_t\theta(t)$. We then have

$$\begin{aligned}\partial_t^2 W + \gamma\partial_t W - \Delta_x W + \lambda_0 W \\ &= -\bar{f}'(v_0)(W + \partial_t w) + (\bar{f}'(v_0+\theta+w) - \bar{f}'(v_0))\partial_t u(t) + L\partial_t v_0(t) \\ &:= h_1(t) + h_2(t) + h_3(t), \quad \xi_W(0) = (0, g).\end{aligned} \tag{7.22}$$

On multiplying equation (7.22) by $(-\Delta_x+\lambda_0)_N^{\alpha-1}(\partial_t W + \beta W)$, where $\alpha > 0$ is the same as in estimate (7.16) (without loss of generality, we may assume that $\alpha < \frac{1}{2}$) and β is a sufficiently small positive number, we derive

$$\begin{aligned}\partial_t[\|\xi_W(t)\|^2_{E^{\alpha-1}} + 2\beta(\partial_t W(t), W(t))_{H^{\alpha-1}}] \\ + 2(\gamma-\beta)\|\partial_t W(t)\|^2_{H^{\alpha-1}} + 2\beta\|W(t)\|^2_{H^\alpha} \\ = 2(h_1(t), (-\Delta_x+\lambda_0)_N^{\alpha-1}(\partial_t W(t) + \beta W(t))) \\ + 2(h_2(t), (-\Delta_x+\lambda_0)_N^{\alpha-1}(\partial_t W(t) + \beta W(t))) \\ + 2(h_3(t), (-\Delta_x+\lambda_0)_N^{\alpha-1}(\partial_t W(t) + \beta W(t))).\end{aligned} \tag{7.23}$$

In order to estimate the right-hand side of (7.23), we need the following standard inequalities:

$$\|u_1\cdot(-\Delta_x+\lambda_0)_N^{\alpha-1}u_2\|_{L^3(\Omega)} \leqslant C\|u_1\|_{H^{1+\alpha}}\|u_2\|_{H^{\alpha-1}}, \tag{7.24 a}$$

$$\|u_3\cdot(-\Delta_x+\lambda_0)_N^{\alpha-1}u_2\|_{L^{3/2}(\Omega)} \leqslant C\|u_3\|_{H^{\alpha}}\|u_2\|_{H^{\alpha-1}}, \tag{7.24 b}$$

which hold for every $u_1 \in H^{\alpha+1}(\Omega)$, $u_2 \in H^{\alpha-1}(\Omega)$ and $u_3 \in H^\alpha(\Omega)$ and every $0 \leqslant \alpha < \frac{1}{2}$ (indeed, these estimate can be easily verified using (2.13), Sobolev's embedding theorem and the appropriate Hölder's inequality [35]).

Now by applying Hölder's inequality to the first term on the right-hand side of (7.23) and using estimate (7.24 *b*), we have

$$\begin{aligned}|(h_1, (-\Delta_x+\lambda_0)_N^{\alpha-1}(\partial_t W + \beta W))| \\ &\leqslant C\|\bar{f}(v_0(t))\|_{L^3(\Omega)}\|W + \beta\partial_t w\|_{H^\alpha}\|\partial_t W + \beta W\|_{H^{\alpha-1}(\Omega)} \\ &\leqslant C_1\|\bar{f}(v_0(t))\|_{L^3(\Omega)}(\|\xi_W(t)\|^2_{E^{\alpha-1}} + \|\xi_w(t)\|^2_{E^\alpha}).\end{aligned} \tag{7.25}$$

In order to estimate the second term, we first note that, expressing the term $\Delta_x\theta$ from equation (7.19) and using the elliptic regularity theorem for the Laplacian and the fact that the $E(\Omega)$-norm of ξ_θ and ξ_{v_0} are uniformly bounded, we derive

$$\|\theta(t)\|_{H^{1+\alpha}} \leqslant C_2(\|\xi_W(t)\|_{E^{\alpha-1}(\Omega)} + 1). \tag{7.26}$$

By applying the Hölder inequality to the second term on the right-hand side of (7.23) and using (7.24 a), (7.26) and the growth restriction (7.3), we have

$$\begin{aligned}|(h_2, (-\Delta_x + \lambda_0)_N^{\alpha-1}(\partial_t W + \beta W))| \\ &\leqslant C'(\|u\|_{L^6(\Omega)} + \|v_0\|_{L^6(\Omega)})\|\partial_t u\|_{L^2(\Omega)}\|\theta + w\|_{H^{1+\alpha}}\|\partial_t W + \beta W\|_{H^{\alpha-1}} \\ &\leqslant C''\|\partial_t u(t)\|_{L^2(\Omega)}(\|\xi_W(t)\|^2_{E^{\alpha-1}} + \|\xi_w(t)\|^2_{E^\alpha} + 1). \qquad (7.27)\end{aligned}$$

Inserting estimates (7.25) and (7.27) into the right-hand of (7.23) and using that the E^α-norm of ξ_w is uniformly bounded (due to (7.16)), we finally derive

$$\partial_t E_W(t) + c(t)E_W(t) \leqslant C''', \qquad (7.28)$$

where $E_W(t) := \|\xi_W(t)\|^2_{E^{\alpha-1}} + 2\beta\lambda_0\|W(t)\|^2_{H^{\alpha-1}} + 2\beta(\partial_t W(t), W(t))_{H^{\alpha-1}}$ and the function $c(t)$ has the form

$$c(t) = \gamma_0 - C'''(\|\bar{f}(v_0(t))\|_{L^3(\Omega)} + \|\partial_t u(t)\|_{L^2(\Omega)}) \qquad (7.29)$$

for some *positive* constant γ_0. It remains only to note that estimates (7.16), (7.20) and (7.21) imply that there exists a small positive ε_0 such that, for every $\varepsilon \leqslant \varepsilon_0$, we have

$$\int_0^T c(t)\,\mathrm{d}t \geqslant \tfrac{1}{2}\gamma_0 T - C_3, \quad \forall T \in \mathbb{R}_+, \qquad (7.30)$$

where the constant C_3 is independent of T and ε. Thus, applying Gronwall's inequality to (7.28) gives

$$\|\xi_W(t)\|_{E^{\alpha-1}(\Omega)} \leqslant C_4, \quad \forall t \in \mathbb{R}_+ \qquad (7.31)$$

and, returning to the variable $\theta(t)$ (using (7.26)), we prove that

$$\|\xi_\theta(t)\|_{E^\alpha(\Omega)} \leqslant C_5, \quad \forall t \in \mathbb{R}_+, \qquad (7.32)$$

where the constant C_5 is independent of $\varepsilon \leqslant \varepsilon_0$, t and $\xi^0 \in B_R$. Estimates (7.16), (7.20) and (7.32) give (7.11) and finish the proof of proposition 7.3. □

Therefore, we have proven that, for $\varepsilon \leqslant \varepsilon_0$, family (7.8) of the dynamical processes associated with equation (3.3) possesses a uniform attractor $\mathcal{A}_\varepsilon$, which is uniformly (with respect to ε) bounded in the space $E^\alpha(\Omega)$ for some *positive* exponent $\alpha < \frac{1}{2}$. Since the cubical rate of growth of the nonlinearity is subcritical with respect to the $E^\alpha(\Omega)$-norm, then, starting with this $E^\alpha(\Omega)$-estimate and using the bootstrap arguments (exactly as in the subcritical case; see, for example, [3]), we obtain the required estimate (7.10). Thus, the second part of theorem 7.2 is proven.

Let us now consider the case of an arbitrary (not necessarily small) $\varepsilon > 0$. In this case, 'dissipation integral' (7.21) is not necessarily small and we cannot obtain estimate (7.30). Therefore, instead of estimate (7.32), we have only that

$$\|\xi_\theta(t)\|_{E^\alpha(\Omega)} \leqslant C\mathrm{e}^{Kt}, \qquad (7.33)$$

for some *positive* constant K. Although estimate (7.33) is not strong enough in order to construct a bounded attracting set in $E^\alpha(\Omega)$ for positive α, it obviously (since $E(\Omega) \subset\subset E^\alpha(\Omega)$) implies that

$$\mathbb{K}_{E(\Omega)}(U_{\phi_\varepsilon}(t+\tau,\tau)B) \leqslant C\mathrm{e}^{-\beta t}, \qquad (7.34)$$

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where $\mathbb{K}_V(X)$ is a Kuratowski measure of non-compactness of the set X in the space V (i.e. the infimum over all $\mu > 0$ for which the set X possesses the *finite* covering by μ-balls of V) and the positive constants C and β are independent of ϕ_ε, t and τ. This estimate implies the analogous estimate for the extended semigroup $\mathbb{S}_t^\varepsilon$ on $E(\Omega) \times \mathcal{H}(f_\varepsilon)$ associated with family (7.8):

$$\mathbb{K}_{E(\Omega)\times\mathcal{H}(f_\varepsilon)}(\mathbb{S}_t^\varepsilon(B \times \mathcal{H}(f_\varepsilon))) \leqslant C\mathrm{e}^{-\beta t}, \tag{7.35}$$

which is sufficient in order to conclude that this semigroup possesses a global attractor in $E(\Omega) \times \mathcal{H}(f_\varepsilon)$ (see, for example, [19]). Thus, theorem 7.2 is proven. □

Remark 7.5. Arguing as in the proof of theorem 7.2, we may prove that, in the case $\varepsilon \leqslant \varepsilon_0$, equation (3.3) possesses a global solution $\xi_u(t) \in E^1(\Omega)$, for every $\xi^\tau \in E^1(\Omega)$ that satisfies the dissipative estimate

$$\|\xi_u(t)\|_{E^1(\Omega)} \leqslant Q(\|\xi^\tau\|_{E^1(\Omega)})\mathrm{e}^{-\alpha(t-\tau)} + Q(\|g\|_{L^2(\Omega)}), \tag{7.36}$$

where the positive constant α and the monotonic function Q are independent of ε.

Corollary 7.6. *Let the assumptions of theorem 7.1 hold. There then exists an* $\varepsilon_0' > 0$ *such that, for any* $\varepsilon \in (0, \varepsilon_0')$*, the uniform attractor of the processes (4.1) (defined on the phase space* $B_{R_0(\varepsilon)} \subset E^1(\Omega)$*), which is constructed in theorem 4.2, coincides with the uniform attractor of processes (7.8) (defined on* $E(\Omega)$*), which exists due to theorem 7.2. In particular, if, in addition, assumptions of theorem 6.3 are satisfied, then the pull-back attractors* $\mathcal{A}_{\phi_\varepsilon}(\tau)$ *of processes (7.8) are regular (i.e. they satisfy the properties formulated in theorem 6.3).*

Remark 7.7. We note that the assertion of corollary 7.6 may be false if the conditions of theorem 7.1 are violated, even in the case where equation (3.3) has globally defined and bounded solutions, for every $\xi^\tau \in E(\Omega)$, and the corresponding uniform attractor is bounded in $E^1(\Omega)$, for every fixed $\varepsilon > 0$. We give the corresponding example in the next section.

Remark 7.8. To conclude, we recall that, according to corollary 7.6 and theorem 6.1, the pull-back attractors $\mathcal{A}_{\phi_\varepsilon}(\tau)$ attract subsets exponentially only bounded in $E^1(\Omega)$ (see estimate (6.6)). Nevertheless, it is not difficult to deduce from (6.6) and (7.11), using the so-called transitivity of the exponential attraction (see [15]), that every bounded in $E(\Omega)$ subsets are also attracted exponentially to these attractors, i.e. for every bounded $B \subset E(\Omega)$, we have

$$\mathrm{dist}_{E(\Omega)}(U_{\phi_\varepsilon}(t+\tau,\tau)B, \mathcal{A}_{\phi_\varepsilon}(t+\tau)) \leqslant Q(\|B\|_{E(\Omega)})\mathrm{e}^{-\alpha t}, \tag{7.37}$$

where the constant $\alpha > 0$ and the monotonic function Q are independent of $\varepsilon \leqslant \varepsilon_0'$, $\phi_\varepsilon \in \mathcal{H}(f_\varepsilon)$, $\tau \in \mathbb{R}$ and $t \geqslant 0$.

8. Examples and concluding remarks

In this concluding section, we illustrate the results obtained above by several concrete examples of equations of the form (1.1). We start with the most natural example of the subordinated oscillations.

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Example 8.1. Let us consider the following semilinear hyperbolic problem in a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + u^3 - a\left(\frac{t}{\varepsilon}\right)u = g, \quad \xi_u|_{t=\tau} = \xi^\tau, \qquad \partial_n u|_{\partial\Omega} = 0, \tag{8.1}$$

where $a(z)$ is an almost-periodic (in the sense of Bohr) real-valued function. Then, the averaged equation for (8.1) obviously has the form

$$\partial_t^2 \bar{u} + \gamma\partial_t \bar{u} - \Delta_x \bar{u} + \bar{u}^3 - \bar{a}\bar{u} = g, \qquad \xi_{\bar{u}}|_{t=\tau} = \xi^\tau, \quad \partial_n \bar{u}|_{\partial\Omega} = 0, \tag{8.2}$$

where $\bar{a} := \mathbb{M}(a)$ is the average of the almost-periodic function $a(z)$.

It is not difficult to verify that equation (8.1) satisfies all of the assumptions of theorem 7.2 and, consequently, for every $\varepsilon \geqslant 0$, equation (8.1) possesses a uniform attractor $\mathcal{A}_\varepsilon$ in the whole phase space $E(\Omega)$ which is uniformly (with respect to ε) bounded in it. Furthermore, according to the second part of theorem 7.2, these attractors $\mathcal{A}_\varepsilon$ are uniformly bounded in $E^1(\Omega)$ if $\varepsilon \leqslant \varepsilon_0$ is sufficiently small (we recall that the nonlinearity in equation (8.1) has a critical cubic growth where the higher regularity of the attractors is a rather delicate problem, especially in the non-autonomous case in absence of the dissipation integral and, to the best of the author's knowledge, the $E^1(\Omega)$-regularity of the attractors of equation (8.1) (even for small positive ε) has not previously been known).

Moreover, due to theorem 4.2 and corollary 7.6, as $\varepsilon \to 0$, the uniform attractors $\mathcal{A}_\varepsilon$ tend to the global attractor $\mathcal{A}_0$ in the space $E^1(\Omega)$ (in the sense of the upper semi-continuity). Finally, under the additional generic assumption that all of the equilibria of the averaged problem (8.2) are hyperbolic, we (due to theorem 6.3) also have the lower semi-continuity of these uniform attractors as $\varepsilon \to 0$ and the associated pull-back attractor $\mathcal{A}_{f_\varepsilon}(\tau)$ is *regular* (and satisfies assertions (i)–(v) of theorem 6.3) if $\varepsilon \leqslant \varepsilon_0$ is sufficiently small.

In the next example we apply the results of previous sections to the *autonomous* equation with *supercritical* nonlinearity.

Example 8.2. Let us consider the following semilinear hyperbolic problem in a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + \varepsilon u|u|^p + u^3 - \bar{a}u = g, \quad \xi_u|_{t=0} = \xi^0, \qquad \partial_n u|_{\partial\Omega} = 0, \tag{8.3}$$

where $\bar{a} \in \mathbb{R}$ is an arbitrary and the exponent $p > 2$. In this case the nonlinearity has the supercritical rate of growth and, *a priori*, we only have the global existence (without uniqueness) of weak energy solutions of equation (8.3). Nevertheless, since the nonlinearity of equation (8.3) satisfies the assumptions of § 3, due to corollary 3.6, this equation possesses a (unique) global strong solution $\xi_u(t) \in E^1(\Omega)$ if $\varepsilon \leqslant \varepsilon_0$ and the initial $E^1(\Omega)$-energy is not very large, i.e. $\|\xi^0\|_{E^1(\Omega)} \leqslant R_0(\varepsilon)$, where the monotonic function $R_0(\varepsilon)$ tends to $+\infty$ as $\varepsilon \to 0$. Moreover, the semigroup S_t^ε generated by this equation on the ball $B_{R_0(\varepsilon)}$ of the space $E^1(\Omega)$ possesses (due to theorem 4.2) a global attractor $\mathcal{A}_\varepsilon$ which is uniformly bounded in $E^1(\Omega)$ and tends as $\varepsilon \to 0$ to the global attractor $\mathcal{A}_0$ of the 'averaged' equation (8.2). Finally, under the generic assumption that all the equilibria of equation (8.2) are

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hyperbolic, the global attractor $\mathcal{A}_\varepsilon$ is regular if $\varepsilon \leqslant \varepsilon_0$ (due to theorem 6.3) and tends to the limit attractor $\mathcal{A}_0$ as $\varepsilon \to 0$ in the following sense:

$$\operatorname{dist}^{\text{symm}}_{E^1(\Omega)}(\mathcal{A}_\varepsilon, \mathcal{A}_0) \leqslant C\varepsilon^\kappa, \tag{8.4}$$

for some positive C and κ (we recall that the considered nonlinearity obviously satisfies assumptions (2.28) and (2.29) and, thus, (8.4) follows from corollary 6.7).

Remark 8.3. The uniqueness problems with equation (8.3) can be partly overcome using the so-called trajectory approach (see [8,11,28,32]). We recall that, under this approach, instead of the classical way of constructing a dynamical system in a phase space $E(\Omega)$ associated with equation (8.3) (which can be defined as a semigroup of multi-valued maps only), one considers the so-called *trajectory dynamical system*

$$T_h^\varepsilon : \mathcal{K}_\varepsilon^+ \to \mathcal{K}_\varepsilon^+, \quad (T_h^\varepsilon u)(t) := u(t+h), \quad h \geqslant 0, \tag{8.5}$$

where the trajectory phase space $\mathcal{K}_\varepsilon^+ \subset L^\infty(\mathbb{R}_+, E(\Omega))$ is the set of all (properly defined) weak energy solutions of problem (8.3) endowed by the appropriate topology (see [11, 34]). Then, the global attractor $\mathcal{A}_\varepsilon^{\text{tr}}$ of shift semigroup (8.5) is called the *trajectory attractor* associated with problem (8.3). It is also worth emphasizing that, in the case where we have the uniqueness, the trajectory dynamical system is usually equivalent to the classical system (see [11, 34] for the details).

The existence of trajectory attractors $\mathcal{A}_\varepsilon^{\text{tr}}$ for equations (8.3) and their weak convergence to the attractor $\mathcal{A}_0^{\text{tr}}$ of the limit equation (8.2) (which is equivalent to the global attractor $\mathcal{A}_0$, since we have uniqueness for (8.2)) was established in [8, 9]. Moreover, as proved in [34], every complete bounded weak solution $\xi_u(t)$, $t \in \mathbb{R}$, of equation (8.3) belonging to the attractor $\mathcal{A}_\varepsilon^{\text{tr}}$ becomes regular as $t \to -\infty$ ($\xi_u(t) \in E^1(\Omega)$ if t is sufficiently small) and tends to the set $\mathcal{R}_\varepsilon$ of the equilibria of equation (8.3) (in fact, this result holds for every $\varepsilon > 0$). Since the set $\mathcal{R}_\varepsilon$ is uniformly bounded in $E^1(\Omega)$ as $\varepsilon \to 0$, we have the following result. For every complete weak solution $\xi_u(t)$ of equation (8.3), there exists a time $T = T_u$ such that

$$\|\xi_u(t)\|_{E^1(\Omega)} \leqslant C, \quad \forall t \leqslant T, \tag{8.6}$$

where the constant C is independent of ξ_u and $\varepsilon \leqslant \varepsilon_0$. Combining this result with corollary 3.6 and arguing as in [34], we then establish that $\xi_u(t) \in E^1(\Omega)$ for every $t \in \mathbb{R}$ if $\varepsilon > 0$ is sufficiently small (i.e. such that $R_0(\varepsilon) > C$). Therefore, we proved that, in this case,

$$\mathcal{A}_\varepsilon^{\text{tr}} \subset L^\infty(\mathbb{R}_+, E^1(\Omega)) \tag{8.7}$$

and is uniformly bounded (with respect to $\varepsilon \to 0$) in this space. Thus, the trajectory attractor $\mathcal{A}_\varepsilon^{\text{tr}}$ that describes the long-time behaviour of weak solutions of equation (8.3) with $\xi^0 \in E(\Omega)$ coincides with the attractor $\mathcal{A}_\varepsilon$ constructed in theorem 4.2; more precisely, we have the relation

$$\mathcal{A}_\varepsilon = \mathcal{A}_\varepsilon^{\text{tr}}|_{t=0}, \quad \varepsilon \leqslant \varepsilon_0. \tag{8.8}$$

Therefore, every weak energy solution $\xi_u(t)$, $t \geqslant 0$ of equation (8.3) with arbitrary initial data $\xi^0 \in E(\Omega)$ (and not necessarily $\xi^0 \in B_{R_0(\varepsilon)}$) is attracted as $t \to +\infty$ by the global attractor $\mathcal{A}_\varepsilon$ constructed in theorem 4.2.

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The next example shows that (3.3) may have additional complete bounded solutions outside of the ball $B_{R_0(\varepsilon)}$ for every $\varepsilon > 0$.

EXAMPLE 8.4. Let us consider the following semilinear hyperbolic equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$:

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + u^3\left(1 + \nu\cos\left(\frac{2t}{\varepsilon}\right)\right) - \bar{a}u = 0, \qquad \xi_u|_{t=\tau} = \xi^\tau, \quad \partial_n u|_{\partial\Omega} = 0, \tag{8.9}$$

where $\bar{a} \in \mathbb{R}$ and $\nu > 0$ is a small parameter. The averaged equation for (8.9) is, obviously, has the form of (8.2) with $g = 0$.

Then, thanks to corollary 3.6 and theorem 4.2, for every $\xi^\tau \in B_{R_0(\varepsilon)}$, equation (8.9) possesses a unique global bounded solution $\xi_u(t)$, $t \geqslant \tau$, and the dynamical processes generated by this equation on the ball $B_{R_0(\varepsilon)}$ possess the uniform attractors $\mathcal{A}_\varepsilon$ which are uniformly (with respect to $\varepsilon \to 0$) bounded in $E^1(\Omega)$ and tend to the global attractor $\mathcal{A}_0$ of the averaged equation (8.2) in the sense of the upper semi-continuity in $E^1(\Omega)$.

Moreover, under the additional generic assumption that all the equilibria of (8.2) are hyperbolic, the corresponding pull-back attractors $\mathcal{A}_{\phi_\varepsilon}(\tau)$ are regular (due to theorem 6.3) and we have estimate (8.4) for the symmetric distance between $\mathcal{A}_\varepsilon$ and $\mathcal{A}_0$ (due to corollary 6.7 and lemma 2.6).

Nevertheless, as shown in the next lemma, equation (8.9) possesses additional complete bounded solutions outside the ball $B_{R_0(\varepsilon)}$ which tend to infinity as $\varepsilon \to 0$.

LEMMA 8.5. *For all sufficiently small $\nu > 0$, $\varepsilon > 0$ and $\varepsilon \ll \nu$, equation (8.9) possesses at least one spatially homogeneous time-periodic solution $u_{\nu,\varepsilon}(t)$ (of period $2\pi\varepsilon$) which satisfies the following estimates:*

$$C_1\varepsilon^{-2} \leqslant \|\xi_{u_{\nu,\varepsilon}}(t)\|_{E(\Omega)} = \|\xi_{u_{\nu,\varepsilon}}(t)\|_{E^1(\Omega)} \leqslant C_2\varepsilon^{-2}, \tag{8.10}$$

where the positive constants C_1 and C_2 are independent of ε and ν.

Proof. Making the change of variables $z = t/\varepsilon$ and $v(z) = \varepsilon u(t/\varepsilon)$ in equation (8.9), we have

$$\partial_z^2 v + \varepsilon\gamma\partial_z v - \varepsilon^2\Delta_x v + v^3(1 + \nu\cos(2z)) - \varepsilon^2\bar{a}v = 0. \tag{8.11}$$

Recall that we seek for the spatially homogeneous solutions of (8.11) only, so we may forget about the Laplacian and obtain the second-order ordinary differential equation

$$V''(z) + V^3(z)(1 + \nu\cos(2z)) + \varepsilon\gamma V'(z) - \varepsilon^2\bar{a}V(z) = 0. \tag{8.12}$$

Moreover, since $u(t) = \varepsilon^{-1}V(t/\varepsilon)$, to prove the lemma, it is sufficient to construct the 2π-periodic solution of (8.12) of order $O(1)$ as $\varepsilon \to 0$. In order to do so, we first construct a 2π-periodic solution $V_\nu(z)$ of the equation

$$V_\nu''(z) + V_\nu^3(z)(1 + \nu\cos(2z)) = 0, \tag{8.13}$$

which corresponds to the case $\varepsilon = 0$ in (8.12). Then, if the constructed solution is non-degenerate (e.g. hyperbolic), it is preserved under small perturbations of equation (8.13) and, in particular, (8.12) possesses a limit cycle close to $V_\nu(z)$

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if $\varepsilon \ll \nu$ is sufficiently small. Thus, it only remains to find a hyperbolic 2π-periodic solution of equation (8.13) for small positive ν.

It is worth recalling now that equation (8.13) is a Hamiltonian system with $\frac{3}{2}$ degrees of freedom (see, for example, [1, 2, 7, 25] and the references therein). In particular, it is known that the desired periodic solution $V_\nu(z)$ can be computed in the form of a power series with respect to the parameter ν:

$$V_\nu(z) = V_0(z) + \nu V_1(z) + \nu^2 V_2(z) + \cdots, \tag{8.14}$$

where the $V_i(z)$ are the appropriate 2π-periodic functions that can be found by standard recursive procedure. In particular, $V_0(z)$ should satisfy the conservative Hamiltonian equation (8.13) with $\nu = 0$:

$$V_0''(z) + V_0^3(z) = 0. \tag{8.15}$$

Thus, we first need to find the 2π-periodic solution $V_0(z)$ of equation (8.15). To this end, we endow equation (8.15) with the initial conditions $V_{0,L}(0) = L$ and $V_{0,L}'(0) = 0$, where $L > 0$ is a parameter, and denote the obtained unique solution of equation (8.15) by $V_{0,L}(z)$. It is then not difficult, using the explicit integral formula for the solutions of (8.15), to verify that $V_{0,L}(z)$ is T-periodic, where

$$T = T(L) = CL^{-1} \tag{8.16}$$

and the constant $C > 0$ is independent of L. In particular, fixing $L = L_0 := C/(2\pi)$, we obtain the desired 2π-periodic solution $V_0(z) = V_{0,L_0}(z)$ of equation (8.15). We recall that, in a fact, we have the one-parametric family $\{V_0(z+h)\}_{h\in\mathbb{R}}$ of 2π-periodic solutions of equation (8.15), consequently, following the general procedure, in order to determine h, we should consider the so-called Poincaré integral, which has the form

$$P(h) = -\int_0^{2\pi} V_0(t+h)^3 \cos(2z) V_0'(z+h)\,\mathrm{d}z = 2\int_0^{2\pi} V_0(z+h)^4 \sin(2z)\,\mathrm{d}z \tag{8.17}$$

(see [25]). We claim that $P(0) = 0$. Indeed, the function $z \to V_0(z)^4$ is even with respect to z and $z \to \sin(2z)$ is odd. Consequently, the function $V_0^4(z)\sin(2z)$ is odd and its mean is equal to zero. The analogous arguments show that

$$P'(0) = 8\int_0^{2\pi} V_0(z)^3 V_0'(z)\sin(2z)\,\mathrm{d}z = 4\int_0^{2\pi} V_0(z)^4 \cos(2z)\,\mathrm{d}z > 0. \tag{8.18}$$

Indeed, due to the symmetries $V_0(z) \to -V_0(z)$ and $V_0(z) \to V_0(-z)$ of equation (8.15) and our assumption for the initial data, we see that the function $[V_0(z)]^4$ is π-periodic, has a unique zero on the interval $[0, \pi]$ at $z = \frac{1}{2}\pi$ and is symmetric with respect to $z \to \pi - z$. This allows us to rewrite (8.18) as

$$P'(0) = 16\int_0^{\pi/2} [V_0(z)]^4 \cos(2z)\,\mathrm{d}z = 16\int_0^{\pi/4} ([V_0(z)]^4 - [V_0(\tfrac{1}{2}\pi - z)]^4)\cos(2z)\,\mathrm{d}z.$$

Since $V_0(z)$ is monotonically decreasing on $[0, \frac{1}{2}\pi]$, $[V_0(z)]^4 - [V_0(\frac{1}{2}\pi - z)]^4 > 0$ for $z \in [0, \frac{1}{4}\pi)$, which confirms that $P'(0) > 0$.

Thus, according to the general theory (see, for example, [25, ch. 6, § 6]), the power series (8.14) indeed defines a unique 2π-periodic solution of equation (8.13)

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(for small ν) that tends to $V_0(z)$ as $\nu \to 0$. Thus, the desired solution $V_\nu(z)$ of equation (8.13) is constructed and it only remains to verify that this solution is non-degenerate. To this end, we use the standard expansions of the multipliers λ_1 and λ_2 of this solution:

$$\lambda_1 = \alpha_1 \nu^{1/2} + O(\nu), \qquad \lambda_2 = -\alpha_1 \nu^{1/2} + O(\nu), \tag{8.19}$$

where

$$\alpha_1^2 = \frac{1}{2\pi K} P'(0) \quad \text{and} \quad K = -\pi L_0^4 \left(\frac{\mathrm{d}T}{\mathrm{d}L}(L_0)\right)^{-1} \tag{8.20}$$

and the function $T = T(L)$ is defined by (8.16) (see [25, ch. 6, § 7]).

In our case, we obviously have $K > 0$ and, consequently, $\alpha_1^2 > 0$ and $\alpha_1 \in \mathbb{R}$. Thus, due to (8.19), the limit cycle $V_\nu(z)$ is indeed hyperbolic for small $\nu > 0$ and thus equation (8.12) possesses a limit cycle close to $V_\nu(z)$ if $\varepsilon \ll \nu$ is sufficiently small. Lemma 8.5 is proven. □

Hence, due to lemma 8.5, we have the time-periodic solution $u_{\nu,\varepsilon}(t)$ of equation (8.9) outside the ball $B_{R_0(\varepsilon)}$ if $\nu > 0$ and $\varepsilon \ll \nu$ are sufficiently small. Moreover, estimates (8.10) show that this solution tends to infinity as $\varepsilon \to 0$.

Remark 8.6. The existence of the solution $v_{\nu,\varepsilon}(z)$ in the last example is closely related with the so-called nonlinear parametric resonance phenomena which is typical for the hyperbolic equations (see [7,30] and the references therein) and which is usually not observed in reaction–diffusion equations. Indeed, according to lemma 8.5 the solution $u_{\nu,\varepsilon}(t)$ is close (for small ε and ν) to the solution $v_0(t) := (1/\varepsilon)V_0(t/\varepsilon)$ of the conservative equation

$$\partial_t^2 v_0 + v_0^3 = 0, \tag{8.21}$$

and the period $\pi\varepsilon$ of the parametrical exciting of system (8.9) differs by a factor two from the period $2\pi\varepsilon$ of the internal oscillations in conservative system (8.21) (solution $v_0(t)$), which is typical for the parametric resonance phenomena.

We also note that, in contrast to the linear equation, the period of oscillations in (8.21) *depends* on its energy E and decays as $E \to \infty$ ($T_{\mathrm{int}} = CE^{-1/2}$) and this is the main reason why the solution $u_{\nu,\varepsilon}(t) \to \infty$ as $\varepsilon \to 0$. Indeed, in order to compensate for the energy decay provided by the dissipation term $\gamma\partial_t u$ by the energy income provided by the parametrical resonance, we need to have $T_{\mathrm{int}} \sim T_{\mathrm{par}} = \pi\varepsilon$ and, consequently, $E \sim \varepsilon^{-2}$ in complete agreement with (8.10).

It is worth noting here that example 8.4 has an essential drawback, namely, we cannot construct the uniform attractor in the whole space $E^1(\Omega)$ for positive ε (in fact, we do not know whether or not every solution equation remains bounded as $t \to \infty$). In order to overcome this drawback, we conclude our exposition by the following modification of example 8.4.

Example 8.7. Let us consider the following semilinear hyperbolic problem in the bounded *two-dimensional* domain $\Omega \subset \mathbb{R}^2$:

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + \varepsilon^3 u^5 + u^3 \left(1 + \nu \cos\left(\frac{2t}{\varepsilon}\right)\right) - \bar{a}u = 0, \quad \xi_u|_{t=\tau} = \xi^\tau. \tag{8.22}$$

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Since the two-dimensional case can be considered as a particular case of the three-dimensional one, as in example 8.4, we have the global solvability of (8.22) for all $\xi^\tau \in B_{R_0(\varepsilon)}$, the existence of uniform attractors $\mathcal{A}_\varepsilon$ for the dynamical processes generated by this equation on the ball $B_{R_0(\varepsilon)}$ and their convergence as $\varepsilon \to 0$ to the global attractor of the limit equation (8.2) (with $g = 0$) in the space $E^1(\Omega)$. Moreover, the change of variables described in lemma 8.5 in equation (8.22) gives the following equation:

$$\partial_t^2 v + \varepsilon\gamma\partial_t v - \varepsilon^2\Delta_x v + v^3(1 + \cos(2z)) + \varepsilon v^5 - \varepsilon^2 \bar{a} v = 0, \tag{8.23}$$

which is also close (for small ε) to equation (8.13). Therefore, the assertion of lemma 8.5 also remains valid for equation (8.22).

On the other hand, the nonlinearity in equation (8.22) obviously satisfies estimate (7.1) (where the constant C_δ is *non-uniform* with respect to ε). Consequently, since every polynomial rate of growth is *subcritical* in the two-dimensional case [3, 29], by arguing in a standard way we can verify that equation (8.22) is dissipative for all $\xi^\tau \in E(\Omega)$ for every fixed $\varepsilon > 0$ and the corresponding dynamical process possesses a uniform attractor $\tilde{\mathcal{A}}_\varepsilon$, which is a compact set of $E^1(\Omega)$. Nevertheless, due to the existence of a solution $u_{\nu,\varepsilon}(z)$ constructed in lemma 8.5, this attractor *does not coincide* with $\mathcal{A}_\varepsilon$:

$$\tilde{\mathcal{A}}_\varepsilon \neq \mathcal{A}_\varepsilon, \quad \forall \varepsilon \ll 1. \tag{8.24}$$

Thus, in contrast to the previous results on averaging of global and uniform attractors (see [14, 21] and references therein), we now see that, roughly speaking, the uniform attractor $\tilde{\mathcal{A}}_\varepsilon$ consists of two basically different parts for small ε. The first part ($\mathcal{A}_\varepsilon$) is regular, has a large basin of attraction (which contains at least the ball $B_{R_0(\varepsilon)}$ with $R_0(\varepsilon) \to \infty$ as $\varepsilon \to 0$) and tends to the attractor $\mathcal{A}_0$ of the averaged equation as $\varepsilon \to 0$. In contrast to this, the irregular part which is provided by the parametric resonance phenomena (it is not empty, due to lemma 8.5) tends to infinity as $\varepsilon \to 0$. We believe that this picture is typical for the averaging of hyperbolic equations of the form (1.1) where the *leading part* of the nonlinearity contains the terms rapidly oscillating in time.

Remark 8.8. We note that it is not difficult to deduce from equation (8.11) and formula (8.19) for the multipliers that the instability index of the time periodic orbit $u_{\nu,\varepsilon}(t)$ tends to infinity as $\varepsilon \to 0$. Consequently, the fractal dimension of the attractor $\tilde{\mathcal{A}}_\varepsilon$ of equation (8.22) also tends to infinity as $\varepsilon \to 0$:

$$\lim_{\varepsilon \to 0} \dim_f \tilde{\mathcal{A}}_\varepsilon = \infty. \tag{8.25}$$

Moreover, it is also well known that, generically, equation (8.13) contains the so-called stochastic layers and chaotic hyperbolic sets inside them (see, for example, [2]). Since any hyperbolic set is preserved under the small perturbations, we may also construct the chaotic hyperbolic set for equation (8.22). Thus, in contrast to the regular part of the attractor $\hat{\mathcal{A}}_\varepsilon$ (the dynamic which is close to the gradient dynamics of the averaged system; see theorem 6.3), the dynamics on the irregular part of $\hat{\mathcal{A}}_\varepsilon$ are usually chaotic.

Appendix A. Auxiliary estimates for the linear wave equation

We formulate and prove some auxiliary results on the regularity of solutions of linear non-homogeneous wave equations that are essential for our study of averaging of the *nonlinear* equations. We start with the following standard result.

PROPOSITION A.1. *Let $u(t)$ be a solution of*

$$\partial_t^2 u + \gamma\partial_t u - \Delta_x u + \lambda_0 u = h(t), \quad \xi_u(0) = 0, \qquad \partial_n u|_{\partial\Omega} = 0, \tag{A 1}$$

in a smooth bounded domain Ω with positive constants γ and λ_0. Assume also that the right-hand side h belongs to the space $L^2([0,T], H^s(\Omega))$ for some $0 \leqslant s < \frac{3}{2}$ and $s \neq \frac{1}{2}$. Then, the solution $\xi_u \in L^\infty([0,T], E^s(\Omega))$ and the following estimate holds:

$$\|u(t)\|^2_{H^{s+1}(\Omega)} + \|\partial_t u(t)\|^2_{H^s(\Omega)} \leqslant C\int_0^t \mathrm{e}^{-\alpha(t-r)}\|h(r)\|^2_{H^s(\Omega)}\,\mathrm{d}r, \tag{A 2}$$

for some positive constants C and α.

Proof. Indeed, in the case $s = 0$, (A 2) is a well-known energy estimate (see, for example, [29]) and the general case $s \neq 0$ can easily be reduced to this particular one by applying the operator $(-\Delta_x)_N^{s/2}$ to both parts of equation (A 1) (which may be done since $D((-\Delta_x)_N^{s/2}) = H^s(\Omega)$ for $0 \leqslant s < \frac{3}{2}$ and $s \neq \frac{1}{2}$; see [31]) and proposition A.1 is proven. □

The next result, which gives the analogue of estimate (A 2) for the case where only the norm of h in $H^{-1}([0,T], H^s(\Omega))$ is known, is a basic technical tool in our averaging of semilinear hyperbolic equations.

PROPOSITION A.2. *Let $u(t)$ be a solution of equation (A 1) and let the external force h satisfy the condition*

$$\left\|\int_t^{t+\tau} h(z)\,\mathrm{d}z\right\|_{H^s(\Omega)} \leqslant M, \quad \forall t, t+\tau \in [0,T], \ \tau \in [0,1], \tag{A 3}$$

where the exponent s is the same as in proposition A.1. Then, the following estimate holds:

$$\|u(t)\|^2_{H^s(\Omega)} + \|\partial_t u(t)\|^2_{H_N^{s-1}(\Omega)} \leqslant CM^2, \tag{A 4}$$

where the constant C is independent of u, t and h.

Proof. Let us introduce a new unknown function $w(t)$ as follows:

$$w(r) := \int_0^r \mathrm{e}^{-\alpha(r-t)} u(t)\,\mathrm{d}t, \quad \text{i.e. } \partial_t w(t) + \alpha w(t) = u(t), \quad w(0) = 0, \tag{A 5}$$

where α is some fixed positive number. Then, by multiplying equation (A 1) by $\mathrm{e}^{-\alpha(r-t)}$ and then integrating over $[0,r]$ and by parts and using the fact that $\xi_u(0) = 0$, we derive that this new function satisfies

$$\partial_t^2 w + \gamma\partial_t w - \Delta_x w + \lambda_0 w = \tilde{H}(t), \quad \xi_w(0) = 0, \tag{A 6}$$

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where

$$\tilde{H}(t) := \int_0^t e^{-\alpha(t-r)} u(r) \, dr.$$

We now note that equation (A 6) has the form of (A 1). Consequently, due to proposition A.1, we have the following estimate:

$$\|\xi_w(t)\|_{E^s(\Omega)} \leqslant C' \|\tilde{H}\|_{L^\infty([0,T],H^s(\Omega))} \leqslant C'' M. \tag{A 7}$$

Moreover, expressing $\partial_t^2 w$ from equation (A 6) and by using (A 7), we derive that

$$\|\partial_t^2 w(t)\|_{H_N^{s-1}(\Omega)} \leqslant C_1 M. \tag{A 8}$$

Since $u(t) = \partial_t w(t) + \alpha w(t)$ and $\partial_t u(t) = \partial_t^2 w(t) + \alpha \partial_t w(t)$, estimates (A 7) and (A 8) imply (A 4) and complete the proof of proposition A.2. □

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