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EQUIVALENT REPRESENTATIONS OF MAX-STABLE PROCESSES VIA ℓ^p-NORMS

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Abstract

While max-stable processes are typically written as pointwise maxima over an infinite number of stochastic processes, in this paper, we consider a family of representations based on ℓ^p-norms. This family includes both the construction of the Reich–Shaby model and the classical spectral representation by de Haan (1984) as special cases. As the representation of a max-stable process is not unique, we present formulae to switch between different equivalent representations. We further provide a necessary and sufficient condition for the existence of an ℓ^p-norm-based representation in terms of the stable tail dependence function of a max-stable process. Finally, we discuss several properties of the represented processes such as ergodicity or mixing.

Keywords: Extreme value theory; Reich–Shaby model; spectral representation; stable tail dependence function

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1. Introduction

Arising as limits of rescaled maxima of stochastic processes, max-stable processes play an important role in spatial and spatio-temporal extremes. A stochastic process $X = \{X(s),\ s \in S\}$ on a countable index set S is called max-stable if there exist sequences $\{a_n(\cdot)\}_{n\in\mathbb{N}}$ and $\{b_n(\cdot)\}_{n\in\mathbb{N}}$ of functions $a_n : S \to (0, \infty]$ and $b_n : S \to \mathbb{R}$ such that, for all $n \in \mathbb{N}$,

$$\mathcal{L}(X) = \mathcal{L}\left(\max_{i=1}^{n} \frac{X_i - b_n}{a_n}\right),$$

where X_i, $i \in \mathbb{N}$, are independent copies of X and the maximum is taken pointwise. From univariate extreme value theory, it is well known that the marginal distributions of X, if nondegenerate, are necessarily generalized extreme value distributions, i.e.

$$\mathbb{P}(X(s) \le x) = \exp\left(-\left(1 + \xi(s)\frac{x - \mu(s)}{\sigma(s)}\right)^{-1/\xi(s)}\right), \qquad 1 + \xi(s)\frac{x - \mu(s)}{\sigma(s)} > 0,$$

with $\xi(s) \in \mathbb{R}$, $\mu(s) \in \mathbb{R}$, and $\sigma(s) > 0$ for $s \in S$. As max-stability is preserved by marginal transformations, it is common practice in extreme value theory to consider only one type of marginal distributions, e.g. the case that the shape parameter ξ is positive. In this case, the marginal distributions are of α-Fréchet type, i.e. up to affine transformations, the marginal distribution functions are of the form

$$\Phi_\alpha(x) = \exp(-x^{-\alpha}), \qquad x > 0,$$

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for some $\alpha > 0$. Here, we will focus on the case of max-stable processes with unit Fréchet margins, i.e. $X(s) \sim \Phi_1$ for all $s \in S$. In this case, X is called a simple max-stable process.

From de Haan [3], the class of simple max-stable processes on S can be fully characterized: a stochastic process $\{X(s),\ s \in S\}$ is simple max-stable if and only if it possesses the spectral representation

$$X(s) = \max_{i \in \mathbb{N}} A_i V_i(s), \qquad s \in S, \tag{1.1}$$

where $\sum_{i \in \mathbb{N}} \delta_{A_i}$ is a Poisson point process on $(0, \infty)$ with intensity measure $a^{-2}\,\mathrm{d}a$ and $V_i = \{V_i(s),\ s \in S\}$ are independent copies of a stochastic process V such that $\mathbb{E}(V(s)) = 1$ for all $s \in S$ (see also [10] and [20]). It is important to note that this representation is not unique. As different representations of the same max-stable process might be convenient for different purposes such as estimation (see, e.g. [6] and [7]) or simulation (see, e.g. [4], [18], and [19]), finding novel representations is of interest.

Recently, Reich and Shaby [21] presented a class of max-stable processes written as a product:

$$X(s) = U^{(p)}(s) \cdot \left[\sum_{l=1}^{L} B_l w_l(s)^p\right]^{1/p}, \qquad s \in S, \tag{1.2}$$

where $\{U^{(p)}(s)\}_{s \in S}$ is a noise process with $U^{(p)}(s) \overset{\text{i.i.d.}}{\sim} \Phi_p$, the functions $w_l \colon S \to [0, \infty)$, $l = 1, \dots, L$, are deterministic weight functions such that $\sum_{l=1}^{L} w_l(s) = 1$ for all $s \in S$, and, independently from $\{U^{(p)}(s)\}_{s \in S}$, the independent random variables B_l, $l = 1, \dots, L$, follow a stable law given by the Laplace transform

$$\mathbb{E}\{\exp(-t B_l)\} = \exp(-t^{-1/p}), \qquad t > 0.$$

The parameter $p \in (1, \infty)$ determines the strength of the effect of the noise process which – analogously to the terminology in geostatistics – is also called a nugget effect. In [21], the weight functions w_l were chosen as shifted and appropriately rescaled Gaussian density functions yielding an approximation of the well-known Gaussian extreme value process [29] joined with a nugget effect. Similarly, Reich and Shaby [21] proposed analogues to popular max-stable processes such as extremal Gaussian processes [25] and Brown–Resnick processes [15] by choosing appropriately rescaled realizations of Gaussian and log-Gaussian processes, respectively, as weight functions. Due to the flexibility in modelling the strength of the nugget by the additional parameter p and the tractability of the likelihood which allows embedding the model in a hierarchical Bayesian model, the Reich–Shaby model (1.2) has found its way into several applications (see, e.g. [22], [27], [28], and [30]).

While a simple max-stable process in the spectral representation (1.1) is written as the pointwise supremum of an infinite number of processes, i.e. the pointwise ℓ_∞-norm of the random sequence $\{A_i V_i(s)\}_{i \in \mathbb{N}}$, the Reich–Shaby model (1.2) is represented as the pointwise p-norm of the finite random vector $(B_l^{1/p} w_l(s))_{l=1,\dots,L}$. In this paper we will present a more general class of representations of max-stable processes by writing them as pointwise ℓ^p-norms of sequences of stochastic processes, including, for instance, both de Haan's representation and the Reich–Shaby model as special cases. The finite-dimensional distributions of the resulting processes will turn out to be generalized logistic mixtures introduced in [8] and [9].

This paper is structured as follows. In Section 2 we will introduce the spectral representation based on ℓ^p-norms. As a single max-stable process might allow for equivalent ℓ^p-norm-based representations for different $p \in (1, \infty]$, we give formulae to switch between them in Section 3. Section 4 provides a full characterization of the resulting class of processes whose properties are finally discussed in Section 5.

2. Generalization of the spectral representation

Denoting by

$$\|\boldsymbol{A} \circ \boldsymbol{V}(s)\|_p = \begin{cases} \Big[\sum_{i\in\mathbb{N}} (A_i V_i(s))^p\Big]^{1/p}, & p \in (1,\infty), \\ \max_{i\in\mathbb{N}} A_i V_i(s), & p = \infty, \end{cases}$$

the ℓ^p-norm of the Hadamard product of the sequences $\boldsymbol{A} = \{A_i\}_{i\in\mathbb{N}}$ and $\boldsymbol{V}(s) = \{V_i(s)\}_{i\in\mathbb{N}}$, $s \in S$, the spectral representation (1.1) can be expressed as

$$X(s) = \|\boldsymbol{A} \circ \boldsymbol{V}(s)\|_\infty, \qquad s \in S.$$

We present a more general representation replacing the ℓ^∞-norm by a general ℓ^p-norm, $p \in (1,\infty]$, and multiplication by an independent noise process with Φ_p marginal distributions. Here, we use the convention that Φ_∞ denotes the weak limit of Φ_p as $p \to \infty$, i.e. $\Phi_\infty(x) = \mathbf{1}_{[1,\infty)}(x)$ is a degenerate distribution function.

Theorem 2.1. *Let $p \in (1,\infty]$ and $\{U^{(p)}(s)\}_{s\in S}$ be a collection of independent Φ_p random variables. Further, let $\sum_{i\in\mathbb{N}} \delta_{A_i}$ be a Poisson process on $(0,\infty)$ with intensity $a^{-2}\,\mathrm{d}a$ and $W_i^{(p)}$, $i \in \mathbb{N}$, be independent copies of a stochastic process $\{W^{(p)}(s),\ s \in S\}$ with $\mathbb{E}\{W^{(p)}(s)\} = 1$ for all $s \in S$. Then the process X defined by*

$$X(s) = \frac{U^{(p)}(s)}{\Gamma(1-p^{-1})} \|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s)\|_p, \qquad s \in S, \tag{2.1}$$

is simple max-stable.

Proof. For $p = \infty$, we have $U^{(p)}(s) = 1$ almost surely (a.s.) and, thus, (2.1) is of the same form as (1.1). Consequently, max-stability follows from [3].

For $p \in (1,\infty)$, we first show that $\|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s)\|_p < \infty$ a.s. According to Campbell's theorem (see [16, p. 28]), this holds if and only if

$$\mathbb{E}\bigg(\int_0^\infty \min\{|aW^{(p)}(s)|^p, 1\} a^{-2}\,\mathrm{d}a\bigg) < \infty. \tag{2.2}$$

Substituting $v = aW(s)$, we can easily see that the left-hand side of (2.2) is equal to

$$\mathbb{E}(W^{(p)}(s)) \int_0^\infty \min\{|v|^p, 1\} v^{-2}\,\mathrm{d}v = 1 + \frac{1}{p-1}.$$

Thus, $\|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s)\|_p < \infty$ a.s. Then, for $s_1,\dots,s_n \in S$, $x_1,\dots,x_n > 0$, $n \in \mathbb{N}$, we obtain

$$\begin{aligned} &\mathbb{P}(X(s_i) \le x_i,\ i = 1,\dots,n) \\ &= \mathbb{E}\bigg(\mathbb{P}\bigg(U(s_i) \le \frac{\Gamma(1-p^{-1})x_i}{\|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s_i)\|_p},\ i = 1,\dots,n \ \bigg|\ \boldsymbol{A}, \boldsymbol{W}^{(p)}\bigg)\bigg) \\ &= \mathbb{E}\bigg(\exp\bigg(-\sum_{i=1}^n \bigg(\frac{\Gamma(1-p^{-1})x_i}{\|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s_i)\|_p}\bigg)^{-p}\bigg)\bigg). \end{aligned}$$

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Using well-known results on the Laplace functional of Poisson point processes, this yields

$$
\begin{aligned}
&\mathbb{P}(X(s_i) \le x_i,\ i = 1, \dots, n) \\
&\quad = \exp\biggl(\mathbb{E}\biggl(\int_0^\infty \biggl\{\exp\biggl(-\sum_{i=1}^n \biggl(\frac{aW^{(p)}(s_i)}{\Gamma(1-p^{-1})x_i}\biggr)^p\biggr) - 1\biggr\} a^{-2}\,\mathrm{d}a\biggr)\biggr) \\
&\quad = \exp\biggl(\mathbb{E}\biggl(\biggl\| \biggl(\frac{W^{(p)}(s_i)}{x_i}\biggr)_{i=1}^n \biggr\|_p\biggr) \frac{1}{p\Gamma(1-p^{-1})} \int_0^\infty (\mathrm{e}^{-a} - 1) a^{-1-p^{-1}}\,\mathrm{d}a\biggr) \\
&\quad = \exp\biggl(-\mathbb{E}\biggl(\biggl\| \biggl(\frac{W^{(p)}(s_i)}{x_i}\biggr)_{i=1}^n \biggr\|_p\biggr)\biggr), \qquad (2.3)
\end{aligned}
$$

where we used Equation (3.478.2) of [11]. Thus, for m independent copies $X_1, \dots, X_m$ of X, $m \in \mathbb{N}$, the homogeneity of the ℓ^p-norm yields

$$
\mathbb{P}\biggl(\frac{1}{m} \max_{j=1}^m X_j(s_i) \le x_i,\ i = 1, \dots, n\biggr) = \mathbb{P}(X(s_i) \le x_i,\ i = 1, \dots, n),
$$

i.e. X is simple max-stable. □

Remark 2.1. Theorem 2.1 could alternatively be verified by observing that the process $T(s) = \|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s)\|_p^p$, $s \in S$, is α-stable with $\alpha = 1/p$ (see also the proof of Theorem 4.1). Thus, all the finite-dimensional distributions of X are generalized logistic mixtures (see [8] and [9]) and, consequently, are max-stable distributions.

Noting that the finite-dimensional distributions of the Reich–Shaby model (1.2) are given by

$$
\mathbb{P}(X(s_i) \le x_i,\ i = 1, \dots, n) = \exp\biggl(-\sum_{j=1}^L \biggl\| \biggl(\frac{w_j(s_i)}{x_i}\biggr)_{i=1}^n \biggr\|_p\biggr),
$$

it can be easily seen that (1.2) is a special case of (2.1), where W follows the discrete distribution $\mathbb{P}(W = Lw_i) = 1/L$, $i = 1, \dots, L$. Further, the classical spectral representation (1.1) by de Haan [3] can be recovered from (2.1) with $p = \infty$.

Analogously to the law of the spectral processes $\{V_i(s),\ s \in S\}_{i\in\mathbb{N}}$ in (1.1), the law of the processes $\{W_i^{(p)}(s),\ s \in S\}_{i\in\mathbb{N}}$ in the ℓ^p-norm-based representation of a given process $\{X(s),\ i \in S\}$ is not unique. Let Y_i, $i \in \mathbb{N}$, be independent and identically distributed random variables with $\mathbb{E}(Y_i) = 1$ which are independent from $\sum_{i\in\mathbb{N}} \delta_{A_i}$ and $\{W^{(p)}(s),\ s \in S\}$. Then the processes $\{U^{(p)}(s)/\Gamma(1 - p^{-1})\|\boldsymbol{A} \circ \boldsymbol{W}^{(p)}(s)\|_p, s \in S\}$ and $\{U^{(p)}(s)/\Gamma(1 - p^{-1})\|\boldsymbol{A} \circ \boldsymbol{Y} \circ \boldsymbol{W}^{(p)}(s)\|_p,\ s \in S\}$ are equal in distribution.

Consequently, even for some fixed $p \in (1, \infty]$, representation (2.1) for a simple max-stable process X is not unique. Furthermore, there might be representations of type (2.1) with different p for the same process X. Such equivalent representations are discussed in the following section.

3. Equivalent representations

From [3], the class of simple max-stable processes is fully covered by the class of processes which allow for the spectral representation (1.1), i.e. (2.1) with $p = \infty$. Thus, any ℓ^p-norm-based representation (2.1) with $p < \infty$ of a simple max-stable process can be transformed to an equivalent representation of type (1.1). This transformation is presented in the following proposition. Even more generally, it is shown how an ℓ^q-norm-based representation can be derived from a ℓ^p-norm-based representation with $p < q < \infty$.

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Proposition 3.1. *Let X be a simple max-stable process with representation (2.1) for some $p \in (1, \infty)$. Then the following hold.*

(i) *The process X allows for the spectral representation (1.1) with*

$$V(\cdot) \stackrel{\mathrm{D}}{=} \frac{U^{(p)}(\cdot)}{\Gamma(1-p^{-1})} W^{(p)}(\cdot). \tag{3.1}$$

(ii) *For $q \in (p, \infty)$, the process X satisfies*

$$X(\cdot) \stackrel{\mathrm{D}}{=} \frac{U^{(q)}(\cdot)}{\Gamma(1-q^{-1})} \|\boldsymbol{A} \circ \boldsymbol{W}^{(q)}(\cdot)\|_q, \tag{3.2}$$

where $\{U^{(q)}(s)\}_{s \in S}$ is a collection of independent Φ_q random variables and $W_i^{(q)}$, $i \in \mathbb{N}$, are independent copies of a stochastic process $\{W^{(q)}(s),\ s \in S\}$ given by

$$W^{(q)}(s) = \frac{\Gamma(1-q^{-1})}{\Gamma(1-p^{-1})} (T_{(p/q)}(s))^{p/q} W^{(p)}(s), \qquad s \in S.$$

Here, independently from the process $W^{(p)}$, the collection $\{T_{(p/q)}(s)\}_{s \in S}$ consists of independent stable random variables whose law is given by the Laplace transform

$$\mathbb{E}(\mathrm{e}^{-tT_{(p/q)}(s)}) = \mathrm{e}^{-t^{p/q}}, \qquad t \geq 0.$$

Proof. (i) By comparing the finite-dimensional distributions of the processes defined via (1.1) and (2.1), it suffices to show that

$$\frac{1}{\Gamma(1-p^{-1})} \mathbb{E}\left(\left\| \left(\frac{U^{(p)}(s_i) W^{(p)}(s_i)}{x_i} \right)_{i=1}^n \right\|_\infty \right) = \mathbb{E}\left(\left\| \left(\frac{W^{(p)}(s_i)}{x_i} \right)_{i=1}^n \right\|_p \right) \tag{3.3}$$

for all $s_1, \ldots, s_n \in S$, $x_1, \ldots, x_n > 0$, $n \in \mathbb{N}$. To this end, we first note that, for $y > 0$,

$$\mathbb{P}\left(\left\| \left(\frac{U^{(p)}(s_i) W^{(p)}(s_i)}{x_i} \right)_{i=1}^n \right\|_\infty \leq y \;\middle|\; \boldsymbol{W}^{(p)} \right) = \exp\left(-\frac{1}{y^p} \sum_{i=1}^n \left(\frac{W^{(p)}(s_i)}{x_i} \right)^p \right),$$

i.e. conditionally on $\boldsymbol{W}^{(p)}$, the norm $\|(U^{(p)}(s_i) W^{(p)}(s_i)/x_i)_{i=1}^n\|_\infty$ follows a p-Fréchet distribution with scale parameter $\|(W^{(p)}(s_i)/x_i)_{i=1}^n\|_p$. Thus,

$$\begin{aligned}
\mathbb{E}\left(\left\| \left(\frac{U^{(p)}(s_i) W^{(p)}(s_i)}{x_i} \right)_{i=1}^n \right\|_\infty \right) &= \mathbb{E}_W \left\{ \mathbb{E}\left(\left\| \left(\frac{U^{(p)}(s_i) W^{(p)}(s_i)}{x_i} \right)_{i=1}^n \right\|_\infty \;\middle|\; \boldsymbol{W}^{(p)} \right) \right\} \\
&= \mathbb{E}_W \left\{ \Gamma(1-p^{-1}) \left\| \left(\frac{W^{(p)}(s_i)}{x_i} \right)_{i=1}^n \right\|_p \right\},
\end{aligned}$$

i.e. (3.3).

(ii) From the first part of the proposition, it follows that the right-hand side of (3.2) allows for a spectral representation (1.1) where the spectral functions are independent copies of the process $\widetilde{V}$ given by

$$\widetilde{V}(\cdot) = \frac{U^{(q)}(\cdot)(T_{p/q}(\cdot))^{1/q}}{\Gamma(1-p^{-1})} W^{(p)}(\cdot),$$

while the spectral functions of the process X on the left-hand side of (3.2) are independent copies of the process V given in (3.1). Conditioning on the value of the stable random variable $T_{(p/q)}(s)$, it can be shown that the product $U^{(q)}(s) \cdot T_{(p/q)}(s)$ has the distribution function Φ_p for all $s \in S$ (see [9]) and, thus, $\widetilde{V}(\cdot) \stackrel{\mathrm{D}}{=} V(\cdot)$. □

Remark 3.1. Even though the transformation in the second part of the proposition requires $p < q < \infty$, the two cases $p = q$ and $q = \infty$ can be regarded as limiting cases.

As $q \searrow p$, it follows that $U^{(q)}(\cdot) \xrightarrow{\mathrm{D}} U^{(p)}(\cdot)$ and $\{T_{(p/q)}(s)\}_{s\in S}$ converges in distribution to a collection of random variables which equal 1 a.s. Thus, in the limit $p = q$, there is no transformation.

As $q \to \infty$, it follows that $\Gamma(1 - q^{-1}) \to 1$ and each $U^{(q)}(s)$, $s \in S$, converges to 1 a.s. Further, by Theorem 1.4.5 of [24], for each $s \in S$, the random variable $T_{(p/q)}(s)$ can be represented as $(1/\Gamma(1 - p/q))\sum_{i\in\mathbb{N}}(\tilde{A}_i Y_i)^{q/p}$, where $\{\tilde{A}_i\}_{i\in\mathbb{N}}$ are the points of a Poisson point process on $(0, \infty)$ with intensity $\tilde{a}^{-2}\,\mathrm{d}\tilde{a}$ and Y_i, $i \in \mathbb{N}$, are independent and identically distributed nonnegative random variables with expectation 1. Thus, as $q \to \infty$,

$$(T_{(p/q)}(s))^{1/q} \stackrel{\mathrm{D}}{=} \bigg(\frac{1}{\Gamma(1 - p/q)} \sum\nolimits_{i\in\mathbb{N}}(\tilde{A}_i Y_i)^{q/p}\bigg)^{1/q} \xrightarrow{\mathrm{D}} \max_{i\in\mathbb{N}}(\tilde{A}_i Y_i)^{1/p},$$

which has the distribution function Φ_p. Consequently, $(T_{(p/q)}(\cdot))^{1/q} \xrightarrow{\mathrm{D}} U^{(p)}(\cdot)$.

Denoting by MS the class of all simple max-stable processes and by MS_p the class of simple max-stable processes allowing for a ℓ^p-norm-based spectral representation (2.1), Proposition 3.1 yields

$$\mathrm{MS}_p \subset \mathrm{MS}_q \subset \mathrm{MS}_\infty = \mathrm{MS}, \qquad 1 < p < q < \infty.$$

A full characterization of the class MS_p is given in the following section.

4. Existence of ℓ^p-norm-based representations

In the following, we will present a necessary and sufficient criterion for the existence of an ℓ^p-norm-based representation of a simple max-stable process X in terms of the stable tail dependence functions of its finite-dimensional distributions. For a simple max-stable distribution $(X(s_1), \dots, X(s_n))^\top$, its stable tail dependence function $l_{s_1,\dots,s_n}$ is defined via

$$\begin{gathered} l_{s_1,\dots,s_n}\colon [0, \infty)^n \to [0, \infty), \\ (x_1, \dots, x_n) \mapsto -\log\bigg\{\mathbb{P}\bigg(X(s_1) \le \frac{1}{x_1}, \dots, X(s_n) \le \frac{1}{x_n}\bigg)\bigg\}. \end{gathered}$$

From the spectral representation (1.1), we obtain the form

$$l_{s_1,\dots,s_n}(x) = \mathbb{E}\Big(\max_{i=1,\dots,n} x_i W(s_i)\Big), \qquad x \in [0, \infty)^n. \tag{4.1}$$

The stable tail dependence function is homogeneous and convex (see, e.g. [1]). Further, from (4.1) together with dominated convergence, we can deduce that the stable tail dependence function is continuous.

Theorem 4.1. *Let $\{X(s),\ s \in S\}$ be a simple max-stable process and $p \in (1, \infty)$. Then the following statements are equivalent.*

(i) *Process X possesses an ℓ^p-norm-based representation (2.1).*

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(ii) *For all pairwise distinct* $s_1, \ldots, s_n \in S$ *and* $n \in \mathbb{N}$, *the function* $f^{(p)}_{s_1,\ldots,s_n}$, *defined by*

$$f^{(p)}_{s_1,\ldots,s_n}(x) = l_{s_1,\ldots,s_n}(x_1^{1/p}, \ldots, x_n^{1/p}), \qquad x = (x_1, \ldots, x_n) \in [0, \infty)^n,$$

is conditionally negative definite on the additive semigroup $[0, \infty)^n$, *i.e. for all* $x^{(1)}, \ldots, x^{(m)} \in [0, \infty)^n$ *and* $a_1, \ldots, a_m \in \mathbb{R}$ *such that* $\sum_{i=1}^m a_i = 0$, *we have*

$$\sum_{i=1}^m \sum_{j=1}^m a_i a_j f^{(p)}_{s_1,\ldots,s_n}(x^{(i)} + x^{(j)}) \leq 0. \tag{4.2}$$

Proof. First, we show that (i) implies (ii). To this end, let X be a simple max-stable process with representation (2.1). Then, from (2.3), we obtain

$$\begin{aligned} f^{(p)}_{s_1,\ldots,s_n}(x) &= -\log\left\{\mathbb{P}\left(X(s_1) \leq \frac{1}{x_1^{1/p}}, \ldots, X(s_n) \leq \frac{1}{x_n^{1/p}}\right)\right\} \\ &= \mathbb{E}\left\{\left(\sum_{i=1}^n x_i W^{(p)}(s_i)^p\right)^{1/p}\right\}, \qquad x = (x_1, \ldots, x_n) \in [0, \infty)^n. \end{aligned}$$

Now let $w(s_1), \ldots, w(s_n) \geq 0$ be fixed. Then, by a straightforward computation, it can be seen that the function $x \mapsto \sum_{k=1}^n x_k w(s_k)^p$ is conditionally negative definite on $[0, \infty)^n$. As the function $y \mapsto y^{1/p}$ is a Bernstein function, and the composition of a conditionally negative function and a Bernstein function yields a conditionally negative definite function [2, Theorem 3.2.9], the function $x \mapsto (\sum_{k=1}^n x_k w(s_k)^p)^{1/p}$ is conditionally negative definite as well. Being a mixture, the same holds for $f^{(p)}_{s_1,\ldots,s_n}$.

Second, we show that (ii) implies (i). From the conditionally negative definiteness of $f^{(p)}_{s_1,\ldots,s_n}$, it follows that $\exp(-f^{(p)}_{s_1,\ldots,s_n})$ is positive definite on $[0, \infty)^n$ [2, Theorem 3.2.2]. As $l_{s_1,\ldots,s_n}$ is nonnegative and continuous, $\exp(-f^{(p)}_{s_1,\ldots,s_n})$ is further bounded by 1 and continuous. Thus, by Theorem 4.4.7 of [2], there exists a unique finite measure $\mu_{s_1,\ldots,s_n}$ on $[0, \infty)^n$ with Laplace transform

$$\mathcal{L}\mu_{s_1,\ldots,s_n}(x) = \int_{[0,\infty)^n} \exp(-\langle x, a\rangle)\mu(\mathrm{d}a) = \exp(-f_{s_1,\ldots,s_n}(x)), \qquad x \in [0, \infty)^n. \tag{4.3}$$

Due to $\mu_{s_1,\ldots,s_n}([0, \infty)^n) = \exp(-l_{s_1,\ldots,s_n}(0, \ldots, 0)) = 1$, $\mu_{s_1,\ldots,s_n}$ is a probability measure. Further,

$$\begin{aligned} &l_{s_1,\ldots,s_n}(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \\ &\qquad = l_{s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \end{aligned} \tag{4.4}$$

for all $x = (x_1, \ldots, x_n) \in [0, \infty)^n$ and $i \in \{1, \ldots, n\}$ implies that

$$\begin{aligned} &\mu_{s_1,\ldots,s_n}(A_1 \times \cdots \times A_{i-1} \times [0, \infty) \times A_{i+1} \times \cdots \times A_n) \\ &\qquad = \mu_{s_1,\ldots,s_{i-1},s_{i+1},\ldots,s_n}(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n) \end{aligned}$$

for all Borel sets $A_1, \ldots, A_n \subset [0, \infty)$ and $i \in \{1, \ldots, n\}$, i.e. the family $\{\mu_{s_1,\ldots,s_n} : s_1, \ldots, s_n \in S,\ n \in \mathbb{N}\}$ of probability measures satisfies the consistency conditions from Kolmogorov's existence theorem. Thus, there exists a stochastic process $\{T(s),\ s \in S\}$ with finite-dimensional distributions $\mu_{\cdot}$.

Now let $\{U^{(p)}(s)\}_{s\in S}$ be a collection of independent Φ_p random variables and

$$\tilde{X}(s) = U^{(p)}(s)T(s)^{1/p}, \qquad s \in S.$$

Then, for all pairwise distinct $s_1, \dots, s_n \in S$ and $x_1, \dots, x_n > 0$, we have

$$\begin{aligned}
&\mathbb{P}(\tilde{X}(s_1) \le x_1, \dots, \tilde{X}(s_n) \le x_n) \\
&\quad = \mathbb{E}\bigg\{\mathbb{P}\bigg(U^{(p)}(s_1) \le \frac{x_1}{T^{1/p}(s_1)}, \dots, U^{(p)}(s_n) \le \frac{x_1}{T^{1/p}(s_n)} \;\bigg|\; T(s_1), \dots, T(s_n)\bigg)\bigg\} \\
&\quad = \mathbb{E}\bigg\{\exp\bigg(-\sum_{i=1}^{n} \frac{T(s_i)}{x_i^p}\bigg)\bigg\}.
\end{aligned}$$

From (4.3), we obtain

$$\begin{aligned}
\mathbb{P}(\tilde{X}(s_1) \le x_1, \dots, \tilde{X}(s_n) \le x_n) &= \exp(-f^{(p)}_{s_1,\dots,s_n}(x_1^{-p}, \dots, x_n^{-p})) \\
&= \mathbb{P}(X(s_1) \le x_1, \dots, X(s_n) \le x_n).
\end{aligned}$$

Thus, X allows for the spectral representation

$$X(s) = U(s)T^{1/p}(s), \qquad s \in S. \tag{4.5}$$

Now let $T^{(1)}, \dots, T^{(m)}$ be m independent copies of T for $m \in \mathbb{N}$. Then, for all $s_1, \dots, s_n \in S$ and $x = (x_1, \dots, x_n) \in [0, \infty)^n$, we have

$$\begin{aligned}
\mathbb{E}\bigg\{\exp\bigg(-\bigg\langle x, \bigg(\sum_{k=1}^{m} T^{(k)}(s_i)\bigg)_{i=1}^{n}\bigg\rangle\bigg)\bigg\} &= [\mathbb{E}\{\exp(-\langle x, (T(s_i))_{i=1}^n\rangle)\}]^m \\
&= \exp(-ml_{s_1,\dots,s_m}(x_1^{1/p}, \dots, x_n^{1/p})) \\
&= \exp(-l_{s_1,\dots,s_m}((m^p x_1)^{1/p}, \dots, (m^p x_n)^{1/p})) \\
&= \mathbb{E}\{\exp(\langle x, m^p(T(s_i))_{i=1}^n\rangle)\},
\end{aligned}$$

where we used the homogeneity of the stable tail dependence function. Hence, for all $s_1, \dots, s_n \in S$, the vectors $(\sum_{k=1}^m T^{(k)}(s_i))_{i=1}^n$ and $m^p(T(s_i))_{i=1}^n$ have the same distribution, i.e. $\{T(s),\ s \in S\}$ is an α-stable process with $\alpha = 1/p$. Thus, from Theorem 13.1.2 and Theorem 3.10.1 of [24], we can deduce that $\{T(s),\ s \in S\}$ allows for the representation

$$T(s) = \frac{1}{\Gamma(1-p^{-1})^p} \sum_{i\in\mathbb{N}} A_i^p \tilde{W}_i(s), \qquad s \in S, \tag{4.6}$$

where $\{A_i\}_{i\in\mathbb{N}}$ are the points of a Poisson point process on $[0, \infty)$ with intensity $a^{-2}\,\mathrm{d}a$ and $\{\tilde{W}_i(s),\ s \in S\}$ are independent and identically distributed stochastic processes which are independent from $\{A_i\}_{i\in\mathbb{N}}$ and satisfy $\mathbb{E}(\tilde{W}_i(s)^{1/p}) = l_s(1) = 1$ for all $s \in S$. Defining $W_i^{(p)}(s) = \tilde{W}_i(s)^{1/p}$, $s \in S$, $i \in \mathbb{N}$, and substituting (4.6) into (4.5), we obtain (2.1). □

Remark 4.1. Note that in Theorem 4.1 we assume that, for each $s_1, \dots, s_n \in S$, $\ell_{s_1,\dots,s_n}$ is the stable tail dependence function of the simple max-stable vector $(X(s_1), \dots, X(s_n))^\top$. The conditional negative definiteness of the function $f^{(p)}_{s_1,\dots,s_n}$ is an additional condition. In particular, it is always satisfied for $p = \infty$ – i.e. any simple max-stable process allows for de Haan's [3] spectral representation (1.1) – as $f^{(\infty)}_{s_1,\dots,s_n} \equiv l_{s_1,\dots,s_n}(1, \dots, 1)$ is always conditionally negative definite.

In order to check whether a function $l_{s_1,\dots,s_n}$ is the stable tail dependence function of some process X with an ℓ^p-norm-based representation, we first need to ensure that $l_{s_1,\dots,s_n}$ is a valid stable tail dependence function. This can be done by checking the necessary and sufficient conditions given in [17] and [23], for instance.

Using an integral representation of continuous conditionally negative definite functions on $[0,\infty)^n$ (see [2, Section 4.4.6]), Theorem 4.1(ii) can be reformulated yielding the following corollary.

Corollary 4.1. *For a simple max-stable process $\{X(s),\ s \in S\}$ and $p \in (1,\infty)$, the following statements are equivalent.*

(i) *Process X possesses an ℓ^p-norm-based representation (2.1).*

(ii) *For all pairwise distinct $s_1,\dots,s_n \in S$ and $n \in \mathbb{N}$, there exists a vector*

$$c(s_1,\dots,s_n) = (c_1(s_1,\dots,s_n),\dots,c_n(s_1,\dots,s_n))^\top \in [0,\infty)^n$$

and a Radon measure $\mu_{s_1,\dots,s_n}$ on $[0,\infty)^n$ such that the stable tail dependence function $l_{s_1,\dots,s_n}$ satisfies

$$l_{s_1,\dots,s_n}(x) = \sum_{i=1}^n c_i(s_1,\dots,s_n)x_i^p + \int_{[0,\infty)^n} \left\{1 - \exp\left(-\sum_{i=1}^n a_i x_i^p\right)\right\} \mu_{s_1,\dots,s_n}(\mathrm{d}a)$$

for all $x = (x_1,\dots,x_n)^\top \in [0,\infty)^n$.

From the characterization given in Theorem 4.1, we can deduce necessary conditions on the dependence structure of a max-stable process with an ℓ^p-norm-based representation (2.1) in terms of its extremal coefficients: For a general simple max-stable process $\{X(s),\ s \in S\}$ and a finite set $\tilde{S} = \{s_1,\dots,s_n\} \subset S$, let the extremal coefficient $\theta(\tilde{S})$ be defined via

$$\mathbb{P}\Big(\max_{s\in S} X(s) \le x\Big) = \exp\bigg(-\frac{\theta(\tilde{S})}{x}\bigg), \qquad x > 0.$$

Then we necessarily have $\theta(\tilde{S}) \in [1,n]$, where $\theta(\tilde{S}) = n$ if and only if $X(s_1),\dots,X(s_n)$ are independent and $\theta(\tilde{S}) = 1$ if and only if $X(s_1) = X(s_2) = \cdots = X(s_n)$ a.s. The extremal coefficient is closely connected to the stable tail dependence function via the relation

$$\theta(\{s_1,\dots,s_n\}) = l_{s_1,\dots,s_n}(1,\dots,1).$$

If X further allows for an ℓ^p-norm-based representation (2.1), we obtain the following condition.

Proposition 4.1. *Let $\{X(s),\ s \in S\}$ be a simple max-stable process with representation (2.1) and $S_1, S_2 \subset S$ be finite and disjoint. Then we have*

$$\theta(S_1 \cup S_2) \ge 2^{1/p}\frac{\theta(S_1)+\theta(S_2)}{2}.$$

Proof. Let $S_1 = \{s_1, s_2,\dots,s_{k_1}\}$ and $S_2 = \{s_{k_1+1},\dots,s_{k_1+k_2}\}$ and let further $\{e_1,\dots,e_{k_1+k_2}\}$ denote the standard basis in $\mathbb{R}^{k_1+k_2}$. As the function

$$(x_1,\dots,x_{k_1+k_2}) \mapsto l_{s_1,\dots,s_{k_1+k_2}}(x_1^{1/p},\dots,x_{k_1+k_2}^{1/p})$$

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is conditionally negative definite by Theorem 4.1, then (4.2) particularly holds for $n = 2$, $a_1 = 1$, $a_2 = -1$, $x^{(1)} = \sum_{i=1}^{k_1} e_i$, and $x^{(2)} = \sum_{i=k_1+1}^{k_1+k_2} e_i$, i.e.

$$l_{s_1,\dots,s_{k_1+k_2}}\left(2^{1/p}\sum_{i=1}^{k_1} e_i\right) + l_{s_1,\dots,s_{k_1+k_2}}\left(2^{1/p}\sum_{i=k_1+1}^{k_1+k_2} e_i\right) - 2l_{s_1,\dots,s_{k_1+k_2}}\left(\sum_{i=1}^{k_1+k_2} e_i\right) \le 0.$$

Using the homogeneity and property (4.4) of the stable tail dependence function, we obtain

$$2^{1/p} l_{s_1,\dots,s_{k_1}}(1,\dots,1) + 2^{1/p} l_{s_{k_1+1},\dots,s_{k_1+k_2}}(1,\dots,1) - 2l_{s_1,\dots,s_{k_1+k_2}}(1,\dots,1) \le 0.$$

As $\theta(\tilde{S}) = l_{\tilde{S}}(1,\dots,1)$ for any finite $\tilde{S} \subset S$, this yields the assertion. □

Of particular interest in extreme value analysis is the case of the pairwise extremal coefficient function (see [26] and [29]) where $\tilde{S} = \{s_1, s_2\}$. Then Proposition 4.1 provides the lower bound

$$\theta(\{s_1, s_2\}) \ge 2^{1/p} \quad \text{for all } s_1 \ne s_2 \in S. \tag{4.7}$$

For the particular case of model (1.2), this bound has already been found by Reich and Shaby [21] motivating their interpretation of model (1.2) as a max-stable process with nugget effect in analogy to the Gaussian case.

The bound (4.7) and the characterization of simple max-stable processes with an ℓ^p-norm-based representation given in Theorem 4.1 can be used to show the existence of a *minimal* ℓ^p-norm-based representation of a simple max-stable process X, i.e. the existence of some $p_{\min}(X)$ such that $X \in \mathrm{MS}_p$ if and only if $p \ge p_{\min}(X)$.

Corollary 4.2. *Let $\{X(s),\ s \in S\}$ be a simple max-stable process such that not all random variables $\{X(s)\}_{s\in S}$ are independent. Then, there exists a number $p_{\min}(X) \in (1,\infty]$ such that $X \in \mathrm{MS}_p$ if and only if $p \ge p_{\min}(X)$.*

Proof. From [3], any simple max-stable process X satisfies $X \in \mathrm{MS}_\infty$. Thus, the assertion follows directly if

$$p_{\min}(X) = \inf\{p > 1 \colon X \in \mathrm{MS}_p\} = \infty.$$

Thus, we restrict ourselves to the case that $p_{\min}(X) < \infty$. As not all the random variables $\{X(s)\}_{s\in S}$ are independent, there exist $s_1, s_2 \in S$ and $\varepsilon > 0$ such that $\theta(\{s_1, s_2\}) < 2^{1/(1+\varepsilon)}$. Hence, from (4.7), we obtain $p_{\min}(X) \ge 1 + \varepsilon$. Using the fact that $\mathrm{MS}_p \subset \mathrm{MS}_q$ for $p < q$, it remains to show that $X \in \mathrm{MS}_{p_{\min}(X)}$. From Theorem 4.1, for all pairwise distinct $s_1, \dots, s_n \in S$, $n \in \mathbb{N}$, $a_1, \dots, a_m \in \mathbb{R}$ such that $\sum_{i=1}^m a_i = 0$, $x^{(1)}, \dots, x^{(m)} \in [0,\infty)^n$ and $m \in \mathbb{N}$, we have

$$\sum_{i=1}^m \sum_{j=1}^n a_i a_j l_{s_1,\dots,s_n}((x_1^{(i)} + x_1^{(j)})^{1/p}, \dots, (x_n^{(i)} + x_n^{(j)})^{1/p}) \le 0 \quad \text{for all } p > p_{\min}(X).$$

By the continuity of $l_{s_1,\dots,s_m}$, the same holds for $p = p_{\min}(X)$, and, thus, by Theorem 4.1, $X \in \mathrm{MS}_{p_{\min}(X)}$. □

For any $p \in (1,\infty]$, we now give an example for a simple max-stable process $X^{(p)}$ such that $p_{\min}(X^{(p)}) = p$. Thus, we will also see that

$$\mathrm{MS}_p \subsetneq \mathrm{MS}_q \subsetneq \mathrm{MS}_\infty = \mathrm{MS}, \qquad 1 < p < q < \infty.$$

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We consider the process $X_{\log}^{(p)} \in \mathrm{MS}_p$ which possesses an ℓ^p-norm-based representation (2.1) with $W(s) = 1$ a.s. for all $s \in S$. From (2.3), for pairwise distinct $s_1, \dots, s_n \in S$, we obtain the finite-dimensional distributions

$$\mathbb{P}(X_{\log}^{(p)}(s_i) \leq x_i,\ 1 \leq i \leq n) = \exp\biggl\{-\biggl(\sum_{i=1}^n x_i^{-p}\biggr)^{1/p}\biggr\}, \qquad x_1, \dots, x_n > 0,$$

i.e. all the multivariate distributions are multivariate logistic distributions [12]. Thus, the process $X_{\log}^{(p)}$ has pairwise extremal coefficients $\theta(s,t) = 2^{1/p}$ for all $s, t \in S$, $s \neq t$. From (4.7), it follows that $X_{\log}^{(p)} \notin \mathrm{MS}_{p'}$ for $p' < p$. Consequently, we have $p_{\min}(X_{\log}^{(p)}) = p$.

While we have $\theta(s,t) = 2^{1/p_{\min}(X)}$ for the process $X = X_{\log}^{(p)}$, the connection between $p_{\min}(X)$ and the pairwise extremal coefficients $\theta(s,t)$ is more involved in general. To see this, we consider the $S = \{s_1, s_2\}$ case. In this case, for a process $X \in \mathrm{MS}_p$, the condition $\theta(s_1, s_2) = 2^{1/p}$ implies that $W^{(p)}(s_1) = W^{(p)}(s_2)$ a.s., i.e. X necessarily follows a bivariate logistic distribution. For any other bivariate simple max-stable distribution, we have $\theta(s_1, s_2) > 2^{1/p_{\min}(X)}$.

5. Properties of processes with an ℓ^p-norm-based representation

In this section we will analyze several properties of simple max-stable processes with an ℓ^p-norm-based representation in more detail. We will particularly focus on properties related to the dependence structure of the process such as stationarity, ergodicity, and mixing. A characteristic feature of a process X with an ℓ^p-norm-based representation (2.1) is the additional noise introduced via the process $\{U^{(p)}(s),\ s \in S\}$. Thus, we will compare the process X to a 'denoised' reference process

$$X^*(s) = \max_{i \in \mathbb{N}} A_i W_i^{(p)}(s), \qquad s \in S,$$

i.e. the simple max-stable process constructed via the same spectral functions used in the original (ℓ^∞-norm-based) spectral representation (1.1). As the processes X and X^* just differ by the Fréchet noise process $U^{(p)}$, we will call X^* the denoised max-stable process associated to X.

The following proposition relates the extremal coefficients $\theta(\{s_1, s_2\})$, $s_1, s_2 \in S$, of X to the extremal coefficients $\theta^*(\{s_1, s_2\}) = \mathbb{E}(\max\{W^{(p)}(s_1), W^{(p)}(s_2)\})$ of the associated denoised process X^*. We find that the extremal dependence of the process X is always weaker than dependence of the associated denoised process – as expected.

Proposition 5.1. *Let $\{X(s),\ s \in S\}$ be a simple max-stable process with an ℓ^p-norm-based representation (2.1) with $p \in (1, \infty]$. Then, for the pairwise extremal coefficients $\theta(\{s_1, s_2\})$, we obtain the bounds*

$$\theta^*(\{s_1, s_2\}) \leq \theta(\{s_1, s_2\}) \leq 2^{1/p} \theta^*(\{s_1, s_2\})^{1-1/p},$$

where $\theta^*(\{s_1, s_2\})$ are the pairwise extremal coefficients of the associated denoised process X^*.

Proof. In the $p = \infty$ case, we have

$$\theta(\{s_1, s_2\}) = \mathbb{E}(\max\{W^{(p)}(s_1), W^{(p)}(s_2)\}) = \theta^*(\{s_1, s_2\}),$$

which is equal to both the lower and the upper bound given in the assertion.

Now let $p \in (1, \infty)$. Then we have the lower bound

$$\theta(\{s_1, s_2\}) = \mathbb{E}\{(W^{(p)}(s_1)^p + W^{(p)}(s_2)^p)^{1/p}\} \geq \mathbb{E}(\max\{W^{(p)}(s_1), W^{(p)}(s_2)\}) = \theta^*(\{s_1, s_2\}).$$

Further, for any $p < r < \infty$ and $\boldsymbol{w} \in [0, \infty)^2$, we obtain

$$\|\boldsymbol{w}\|_p^p \le \|\boldsymbol{w}\|_1^{(r-p)/(r-1)} \cdot \|\boldsymbol{w}\|_r^{r(p-1)/(r-1)}$$

(see Theorem 18 of [13]) or, equivalently,

$$\|\boldsymbol{w}\|_p \le \|\boldsymbol{w}\|_1^{(r-p)/p(r-1)} \|\boldsymbol{w}\|_r^{(1-p^{-1})/(1-r^{-1})}.$$

As $r \to \infty$, this yields

$$\|\boldsymbol{w}\|_p \le \|\boldsymbol{w}\|_1^{1/p} \|\boldsymbol{w}\|_\infty^{1-p^{-1}}.$$

Taking the expectation of $\boldsymbol{w}$ with respect to the joint distribution of $W^{(p)}(s_1)$ and $W^{(p)}(s_2)$, and applying Hölder's inequality, we obtain the upper bound

$$\begin{aligned}\theta(\{s_1, s_2\}) &= \mathbb{E}\{(W^{(p)}(s_1)^p + W^{(p)}(s_2)^p)^{1/p}\} \\ &\le \mathbb{E}\{(W^{(p)}(s_1) + W^{(p)}(s_2))^{1/p} \max\{W^{(p)}(s_1), W^{(p)}(s_2)\}^{1-p^{-1}}\} \\ &\le [\mathbb{E}\{W^{(p)}(s_1) + W^{(p)}(s_2)\}]^{1/p} [\mathbb{E}(\max\{W^{(p)}(s_1), W^{(p)}(s_2)\})]^{1-p^{-1}}.\end{aligned}$$

The assertion follows from $\mathbb{E}\{W^{(p)}(s_1)\} = \mathbb{E}\{W^{(p)}(s_2)\} = 1$. □

In the following, we will consider the $S = \mathbb{Z}$ case. In this case, properties such as stationarity, ergodicity, or mixing are of interest. For a simple max-stable process $\{X(s),\ s \in \mathbb{Z}\}$ with representation (1.1), necessary and sufficient conditions for these properties can be expressed in terms of the distribution of the spectral function V. From [15], X is stationary if and only if

$$\mathbb{E}\{V(s_1)^{u_1} \cdots V(s_n)^{u_n}\} = \mathbb{E}\{V(s_1 + s)^{u_1} \cdots V(s_n + s)^{u_n}\} \tag{5.1}$$

for all $n \in \mathbb{N}$, $s, s_1, \dots, s_n \in \mathbb{Z}$, and $u_1, \dots, u_n \in [0, 1]$ such that $\sum_{i=1}^n u_i = 1$. For stationary simple max-stable processes, Kabluchko and Schlather [14] gave conditions for ergodicity and mixing in terms of the pairwise extremal coefficients $\theta(\{s_1, s_2\}) = \mathbb{E}(\max\{V(s_1), V(s_2)\})$, stating that X is mixing if and only

$$\lim_{r\to\infty} \theta(\{0, r\}) = 2, \tag{5.2}$$

and X is ergodic if and only if

$$\lim_{r\to\infty} \frac{1}{r} \sum\nolimits_{k=1}^r \theta(\{0, k\}) = 2, \tag{5.3}$$

respectively.

Now we transfer these results to a max-stable process X with an ℓ^p-norm-based representation (2.1) giving necessary and sufficient conditions in terms of $W^{(p)}$. For the associated denoised process X^*, (5.1)–(5.3) depend on the distribution $W^{(p)} = V$ only, while the structure of the process X is more difficult as we have $V(\cdot) = [\Gamma(1 - p^{-1})]^{-1} U^{(p)}(\cdot) W^{(p)}(\cdot)$ (see Proposition 3.1). In the following result, however, we show that those conditions simplify to the conditions for the associated denoised process X^*.

Proposition 5.2. *Let $\{X(s),\ s \in \mathbb{Z}\}$ be a simple max-stable process with an ℓ^p-norm-based representation (2.1) and let X^* be the denoised process associated to X. Then the following holds.*

(i) *Process X is stationary if and only if X^* is stationary.*

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If X is stationary, we further have

(ii) *X is mixing if and only if X^* is mixing;*

(iii) *X is ergodic if and only if X^* is ergodic.*

Proof. (i) From [15] and Proposition 3.1, the process X is stationary if and only if (5.1) holds for $V(\cdot) = [\Gamma(1-p^{-1})]^{-1} U^{(p)}(\cdot) W^{(p)}(\cdot)$. The left-hand side of (5.1) can be expressed as

$$\begin{aligned}
\mathbb{E}\{V(s_1)^{u_1} \cdots V(s_n)^{u_n}\} &= \frac{1}{\Gamma(1-p^{-1})} \mathbb{E}\bigg\{\prod_{i=1}^{n} U^{(p)}(s_i)^{u_i} W^{(p)}(s_i)^{u_i}\bigg\} \\
&= \frac{1}{\Gamma(1-p^{-1})} \mathbb{E}\bigg\{\prod_{i=1}^{n} U^{(p)}(s_i)^{u_i}\bigg\} \mathbb{E}\bigg\{\prod_{i=1}^{n} W^{(p)}(s_i)^{u_i}\bigg\} \\
&= \frac{\prod_{i=1}^{n} \Gamma(1-u_i p^{-1})}{\Gamma(1-p^{-1})} \mathbb{E}\bigg\{\prod_{i=1}^{n} W^{(p)}(s_i)^{u_i}\bigg\},
\end{aligned}$$

where we used the fact that $U^{(p)}(s_i)^{u_i}$, $i = 1, \ldots, n$, are independent Φ_{p/u_i} random variables. Thus, X is stationary if and only if (5.1) holds for $V = W^{(p)}$, i.e. if and only if X^* is stationary.

(ii) From [14], the process X is mixing if and only if (5.2) holds where θ denotes the pairwise extremal coefficient of X. Proposition 5.1 yields the bounds

$$\lim_{r\to\infty} \theta^*(\{0, r\})) \le \lim_{r\to\infty} \theta(\{0, r\}) \le 2^{1/p} \lim_{r\to\infty} \theta^*(\{0, r\})^{1-p^{-1}} \le 2.$$

Thus, $\lim_{r\to\infty} \theta(\{0, r\}) = 2$ if and only if $\lim_{r\to\infty} \theta^*(\{0, r\}) = 2$ which is equivalent to X^* being mixing.

(iii) The proof runs analogously to the proof of the second assertion. The process X is ergodic if and only if (5.3) holds. From Proposition 5.1 and Jensen's inequality, we obtain

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \theta^*(\{0, k\}) &\le \lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \theta(\{0, k\}) \\
&\le 2^{1/p} \lim_{r\to\infty} \frac{1}{r} \sum_{k=1}^{r} \theta^*(\{0, k\})^{1-p^{-1}} \\
&\le 2^{1/p} \lim_{r\to\infty} \bigg[\frac{1}{r} \sum_{k=1}^{r} \theta^*(\{0, k\})\bigg]^{1-p^{-1}} \\
&\le 2.
\end{aligned}$$

Consequently, it holds that $\lim_{r\to\infty} r^{-1} \sum_{k=1}^{r} \theta(\{0, k\}) = 2$ if and only if

$$\lim_{r\to\infty} r^{-1} \sum_{k=1}^{r} \theta^*(\{0, k\}) = 2. \qquad \square$$

Remark 5.1. The mixing properties of a stochastic process $\{X(s),\ s \in S\}$ are described more precisely by its mixing coefficients. For two subsets $S_1, S_2 \subset S$, the β-mixing coefficient $\beta(S_1, S_2)$ is defined by

$$\beta(S_1, S_2) = \sup\{|\mathcal{P}_{S_1 \cup S_2}(C) - \mathcal{P}_{S_1} \otimes \mathcal{P}_{S_2}(C)|,\ C \in \mathcal{C}_{S_1 \cup S_2}\},$$

where, for each $\tilde{S} \subset S$, the probability measure $\mathcal{P}_{\tilde{S}}$ denotes the distribution of the restricted process $\{X(s),\ s \in \tilde{S}\}$ on the space of nonnegative functions on $\tilde{S}$ endowed with the Borel σ-algebra $\mathcal{C}_{\tilde{S}}$.

For the case of a max-stable process, Dombry and Eyi-Minko [5] provide the upper bound

$$\beta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta(s_1, s_2)].$$

Applying Proposition 5.1 , we obtain

$$\beta(S_1, S_2) \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta(s_1, s_2)] \leq 4 \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} [2 - \theta^*(s_1, s_2)],$$

i.e. the upper bound for a process with an ℓ^p-norm-based representation (2.1) is lower than the bound for the associated denoised process.

As Proposition 5.2 states, a max-stable process with ℓ^p-norm-based representation (2.1) shares properties such as stationarity, ergodicity, and mixing with the associated denoised process. In particular, the 'noisy' analogues of well-studied max-stable processes might be used without changing any of these properties.

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References

[1] Beirlant, J., Goegebeur, Y., Teugels, J. and Segers, J. (2004). *Statistics of Extremes: Theory and Applications*. John Wiley, Chichester.
[2] Berg, C., Christensen, J. P. R. and Ressel, P. (1984). *Harmonic Analysis on Semigroups*. Springer, New York.
[3] De Haan, L. (1984). A spectral representation for max-stable processes. *Ann. Prob.* **12,** 1194–1204.
[4] Dieker, A. B. and Mikosch, T. (2015). Exact simulation of Brown-Resnick random fields at a finite number of locations. *Extremes* **18,** 301–314.
[5] Dombry, C. and Eyi-Minko, F. (2012). Strong mixing properties of max-infinitely divisible random fields. *Stoch. Process. Appl.* **122,** 3790–3811.
[6] Engelke, S., Malinowski, A., Kabluchko, Z. and Schlather, M. (2015). Estimation of Hüsler–Reiss distributions and Brown–Resnick processes. *J. R. Statist. Soc. B* **77,** 239–265.
[7] Engelke, S., Malinowski, A., Oesting, M. and Schlather, M. (2014). Statistical inference for max-stable processes by conditioning on extreme events. *Adv. Appl. Prob.* **46,** 478–495.
[8] Fougères, A.-L., Mercadier, C. and Nolan, J. P. (2013). Dense classes of multivariate extreme value distributions. *J. Multivariate Anal.* **116,** 109–129.
[9] Fougères, A.-L., Nolan, J. P. and Rootzén, H. (2009). Models for dependent extremes using stable mixtures. *Scand. J. Statist.* **36,** 42–59.
[10] Giné, E., Hahn, M. and Vatan, P. (1990). Max-infinitely divisible and max-stable sample continuous processes. *Prob. Theory Relat. Fields* **87,** 139–165.
[11] Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Tables of Integrals, Series, and Products*. Academic Press, New York.
[12] Gumbel, É. J. (1960). Distributions des valeurs extrêmes en plusieurs dimensions. *Publ. Inst. Statist. Univ. Paris* **9,** 171–173.
[13] Hardy, G. H., Littlewood, J. E. and Pólya, G. (1952). *Inequalities*, 2nd edn. Cambridge University Press.
[14] Kabluchko, Z. and Schlather, M. (2010). Ergodic properties of max-infinitely divisible processes. *Stoch. Process. Appl.* **120,** 281–295.
[15] Kabluchko, Z., Schlather, M. and de Haan, L. (2009). Stationary max-stable fields associated to negative definite functions. *Ann. Prob.* **37,** 2042–2065.
[16] Kingman, J. F. C. (1993). *Poisson Processes*. Oxford University Press.

[17] MOLCHANOV, I. (2008). Convex geometry of max-stable distributions. *Extremes* **11,** 235–259.
[18] OESTING, M., KABLUCHKO, Z. AND SCHLATHER, M. (2012). Simulation of Brown-Resnick processes. *Extremes* **15,** 89–107.
[19] OESTING, M., SCHLATHER, M. AND ZHOU, C. (2018). Exact and fast simulation of max-stable processes on a compact set using the normalized spectral representation. *Bernoulli* **24,** 1497–1530.
[20] PENROSE, M. D. (1992). Semi-min-stable processes. *Ann. Prob.* **20,** 1450–1463.
[21] REICH, B. J. AND SHABY, B. A. (2012). A hierarchical max-stable spatial model for extreme precipitation. *Ann. Appl. Statist.* **6,** 1430–1451.
[22] REICH, B. J., SHABY, B. A. AND COOLEY, D. (2014). A hierarchical model for serially-dependent extremes: a study of heat waves in the western US. *J. Agric. Biol. Environ. Statist.* **19,** 119–135.
[23] RESSEL, P. (2013). Homogeneous distributions—and a spectral representation of classical mean values and stable tail dependence functions. *J. Multivariate Anal.* **117,** 246–256.
[24] SAMORODNITSKY, G. AND TAQQU, M. S. (1994). *Stable Non-Gaussian Random Processes: Stochastic Models with Infinite Variance*. Chapman & Hall, New York.
[25] SCHLATHER, M. (2002). Models for stationary max-stable random fields. *Extremes* **5,** 33–44.
[26] SCHLATHER, M. AND TAWN, J. A. (2003). A dependence measure for multivariate and spatial extreme values: properties and inference. *Biometrika* **90,** 139–156.
[27] SEBILLE, Q., FOUGÈRES, A.-L. AND MERCADIER, C. (2017). Modeling extreme rainfall: a comparative study of spatial extreme value models. *Spat. Statist.* **21,** 187–208.
[28] SHABY, B. A. AND REICH, B. J. (2012). Bayesian spatial extreme value analysis to assess the changing risk of concurrent high temperatures across large portions of European cropland. *Environmetrics* **23,** 638–648.
[29] SMITH, R. L. (1990). *Max–Stable Processes and Spatial Extremes.* Unpublished manuscript.
[30] STEPHENSON, A. G., SHABY, B. A., REICH, B. J. AND SULLIVAN, A. L. (2015). Estimating spatially varying severity thresholds of a forest fire danger rating system using max-stable extreme-event modeling. *J. Appl. Meteor. Climatol.* **54,** 395–407.