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Two-dimensional planar plumes: non-Boussinesq effects

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In an accompanying paper (van den Bremer & Hunt, *J. Fluid Mech.*, vol. 750, 2014, pp. 210–244) closed-form solutions, describing the behaviour of two-dimensional planar turbulent rising plumes from horizontal planar area and line sources in unconfined quiescent environments of uniform density, that are universally applicable to Boussinesq and non-Boussinesq plumes, are proposed. This universality relies on an entrainment velocity unmodified by non-Boussinesq effects, an assumption that is derived in the literature based on similarity arguments and is, in fact, in contradiction with the axisymmetric case, in which entrainment is modified by non-Boussinesq effects. Exploring these solutions, we show that a non-Boussinesq plume model predicts exactly the same behaviour with height for a pure plume as would a Boussinesq model, whereas the effects on forced and lazy plumes are opposing. Non-intuitively, the non-Boussinesq model predicts larger fluxes of volume and mass for lazy plumes, but smaller fluxes for forced plumes at any given height compared to the Boussinesq model. This raises significant questions regarding the validity of the unmodified entrainment model for planar non-Boussinesq plumes based on similarity arguments and calls for detailed experiments to resolve this debate.

Key words: convection, double diffusive convection

1. Introduction

The Boussinesq approximation, which ignores density variation except where it is responsible for the existence of the buoyancy force itself, has been ubiquitously and successfully applied throughout the buoyancy-driven-flow literature, and within the literature on plumes and fountains in particular. Relaxing this assumption and thus considering so-called non-Boussinesq plumes changes the way variations in density affect variations in the fluxes of mass and momentum and alters the form of their respective conservation equations. The significant density variations between plumes and ambient that drive the non-Boussinesq modelling approach may also affect the entrainment process and necessitate a different form of parametric turbulence closure. For axisymmetric plumes, several authors including Rooney & Linden (1996), Fanneløp & Webber (2003) and Carlotti & Hunt (2005) have examined how

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the Boussinesq assumption can be relaxed. In addition to including a variation of the density in the conservation equations, where it was ignored under the Boussinesq approximation, the radial entrainment velocity u_e is reduced by a factor proportional to the square root of the local density contrast with the ambient ($u_e = \alpha w\sqrt{\rho/\rho_a}$), where α is the (constant) entrainment coefficient, ρ and ρ_a are the densities of the plume (locally) and ambient fluids, and w the local vertical (time-averaged) velocity in the plume. These authors have relied largely on limited experimental evidence by Ricou & Spalding (1961) to support this modified entrainment hypothesis. The main rationale behind the modification is consistency of the modified entrainment assumption with similarity solutions, as shown by Rooney & Linden (1996). A physical motivation is that turbulent entrainment is driven by the Reynolds stresses, which are proportional to ρu^2 and, hence, naturally provide a velocity scale for entrainment proportional to $\sqrt{\rho u^2}$; van den Bremer & Hunt (2010) have then shown that, when scaled appropriately, the solutions for non-Boussinesq and Boussinesq (axisymmetric) plumes take the same mathematical form.

Although non-Boussinesq planar plumes have received considerably less attention, Rooney (1997) and Delichatsios (1988) showed that an entrainment hypothesis in which the entrainment rate is not modified by non-Boussinesq effects ($u_e = \alpha w$) is consistent with similarity solutions. As recognized by Delichatsios (1988), there is no experimental evidence to either support or contradict this claim and little has changed since then. In an analogous fashion to van den Bremer & Hunt (2010), we show in an accompanying paper, van den Bremer & Hunt (2014), hereinafter referred to as vdB&H (2014), that universal solutions also exist for planar plumes encompassing both the Boussinesq and the non-Boussinesq cases. This universality quintessentially relies on the assumption that the entrainment velocity is unmodified by non-Boussinesq effects.

By comparing the predictions of equivalent physical quantities by a non-Boussinesq model and a Boussinesq model, this paper explores the implications of this assumption for non-Boussinesq effects. It is worthy of note that 'non-Boussinesq effects' can be interpreted in two ways: the change in prediction for a given model as a lower value of the source density contrast $\eta_0 = \rho_0/\rho_a$ is chosen or the difference in prediction between a non-Boussinesq model and a Boussinesq model for a given value of η_0. Herein, our intention is a focus solely on the latter.

This paper is laid out as follows. Starting from the governing equations and their solutions written in universal notation from vdB&H (2014), we return to physical quantities with invariant definitions across Boussinesq and non-Boussinesq plumes in § 2. The predictions of Boussinesq and non-Boussinesq models are compared in §§ 3 and 4. Pure plumes are examined first (§ 3), followed by the contrasting behaviour of forced plumes and lazy plumes (§ 4). Finally, conclusions are drawn in § 5 regarding the *a priori* validity of the model for planar non-Boussinesq plumes.

2. Plume equations and universal solutions

2.1. *Universal solutions*

As shown in vdB&H (2014), for top-hat profiles of density and vertical velocity variation across horizontal sections, conservation of the fluxes of mass, volume and vertical momentum can be rewritten in terms of conservation of three quantities that are universally valid for Boussinesq and non-Boussinesq plumes:

$$\frac{\mathrm{d}\mathscr{G}}{\mathrm{d}z} = 2u_e, \quad \frac{\mathrm{d}\mathscr{M}}{\mathrm{d}z} = \frac{B\mathscr{G}}{\mathscr{M}}, \quad \frac{\mathrm{d}B}{\mathrm{d}z} = 0, \tag{2.1a–c}$$

where the vertical coordinate z is measured from the source upwards and $u_e = \alpha w$ denotes the horizontal entrainment velocity measured at the edge of the plume, which is unmodified by non-Boussinesq effects. We define the plume width as $2b$. In the Boussinesq case, $\mathscr{G} = Q = 2bw$ and $\mathscr{M} = M_B = 2bw^2$ denote the volume flux and the Boussinesq approximation to the momentum flux, respectively. In the non-Boussinesq case, $\mathscr{G} = G = 2\eta bw$ and $\mathscr{M} = M = 2\eta bw^2$ denote the mass flux and the momentum flux, respectively. The density deficit flux $B = 2g(1-\eta)bw$ denotes the same quantity in both Boussinesq and non-Boussinesq cases and B is equal to the buoyancy flux in the Boussinesq case only. All fluxes are per unit length. As for all the relevant quantities in this paper, the horizontal entrainment velocity and the fluxes are averaged in time and in the direction along the axis of symmetry of the plume and have been normalized by the (uniform) ambient density ρ_a, where appropriate.

The conservation equations for $\mathrm{d}\mathscr{G}/\mathrm{d}z$ and $\mathrm{d}B/\mathrm{d}z$ in (2.1) imply that the fluxes of volume and mass are conserved in both Boussinesq and non-Boussinesq plumes. In the Boussinesq case conservation of volume is directly implied by conservation of mass. In the non-Boussinesq case, on the other hand, conservation of volume relies on the assumption that the plume fluid behaves as an ideal gas, there is no external heat input into the plume, the pressure variation in the ambient is hydrostatic and by restricting attention to length scales much smaller than the length scale associated with the hydrostatic pressure distribution of the ambient ($L_H = p_a/g\rho_a$, where p_a is the ambient pressure). The derivation originally made by Rooney & Linden (1996) is reproduced in the review by Hunt & van den Bremer (2011) in its simplest form for the axisymmetric case.

In universal notation ($\beta = b$ and $\Delta = 1-\eta$ in the Boussinesq case and $\beta = b\eta$ and $\Delta = (1-\eta)/\eta$ in the non-Boussinesq case), the flux balance parameter or Richardson number is given by (see vdB&H 2014):

$$\Gamma = \frac{B\mathscr{G}^3}{2\alpha\mathscr{M}^3} = \frac{g\beta\Delta}{\alpha w^2}. \tag{2.2}$$

From the definition of Γ (2.2), we note that the value of Γ is equivalent under the Boussinesq and the non-Boussinesq definition of $\mathscr{G}$ and $\mathscr{M}$ (see also § 4.3); the additional density contrast factors in the non-Boussinesq case cancel each other out so that $\mathscr{G}/\mathscr{M} = 1/w$ in both the Boussinesq and the non-Boussinesq cases. In other words, the Richardson number Γ for a plume with known fluxes at its source (or at a given height above its source) takes an identical value as an input parameter in either Boussinesq or non-Boussinesq models. This allows a direct assessment of non-Boussinesq effects in planar plumes with the same source Richardson number Γ_0 (in contrast to the axisymmetric case, see van den Bremer & Hunt 2010).

Using a hat to denote variables scaled on their source values, the system of conservation equation (2.1) can be expressed in terms of Γ, the non-dimensional effective half-width $\hat{\beta} = \beta/\beta_0$, the non-dimensional velocity $\hat{w} = w/w_0$ and a non-dimensional height $\zeta = \alpha z/\beta_0$:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta} = \frac{3\Gamma(1-\Gamma)}{\hat{\beta}}, \quad \frac{\mathrm{d}\hat{\beta}}{\mathrm{d}\zeta} = 2-\Gamma, \quad \frac{\mathrm{d}\hat{w}}{\mathrm{d}\zeta} = \frac{\hat{w}}{\hat{\beta}}(\Gamma - 1). \tag{2.3a–c}$$

2.2. *Non-Boussinesq versus Boussinesq model for planar plumes*

Assuming the entrainment model is unmodified by non-Boussinesq effects, the only difference between a Boussinesq and a non-Boussinesq model is that the density variation in the equation for momentum flux conservation, which is neglected under the Boussinesq approximation, is properly taken into account, i.e. $M_B = 2w^2b$ is replaced by $M = 2\eta w^2 b = M_B\eta$. Decomposition of the variation of the momentum flux M with height z, thus reveals the only effect of making the Boussinesq approximation:

$$\frac{\mathrm{d}M/\mathrm{d}z}{M} = \frac{\mathrm{d}M_B/\mathrm{d}z}{M_B} + \frac{\mathrm{d}\eta/\mathrm{d}z}{\eta}. \tag{2.4}$$

For plumes, η always increases monotonically from its source value $\eta_0(=\rho_0/\rho_a < 1)$ and reaches $\eta = 1$ asymptotically with height (as $\rho \to \rho_a$). Therefore, $\mathrm{d}\eta/\mathrm{d}z > 0$ and the rate of growth of M, $(\mathrm{d}M/\mathrm{d}z)/M$, always exceeds the rate of growth of M_B, $(\mathrm{d}M_B/\mathrm{d}z)/M_B$. Differences between the predictions of a non-Boussinesq model and a Boussinesq model can therefore be explained by noting that the growth rate of the 'effective' momentum flux is larger in a non-Boussinesq model.

To aid the discussions that follow, we introduce a scaled height ξ, the definition of which does not depend on whether a Boussinesq or a non-Boussinesq model is considered:

$$\xi = \frac{\alpha z}{b_0} \quad \text{for Boussinesq and non-Boussinesq plumes.} \tag{2.5}$$

3. Non-Boussinesq effects on pure plumes ($\varGamma_0 = 1$)

By combining the solutions for β and $\varDelta$ in § 3.2 of vdB&H (2014) it can be shown that the half-width, the vertical velocity and the density contrast of pure planar plumes ($\varGamma_0$) are unaffected by non-Boussinesq effects. In both the Boussinesq and the non-Boussinesq cases, these are given by:

$$\frac{b}{b_0} = 1 + \xi, \quad \frac{w}{w_0} = 1, \quad \frac{1-\eta}{1-\eta_0} = \frac{1}{1+\xi} \quad \text{(Boussinesq and non-Boussinesq).} \tag{3.1a–c}$$

From (3.1), the fluxes of volume, mass and momentum can be evaluated:

$$\frac{Q}{Q_0} = 1 + \xi, \quad \frac{G}{G_0} = 1 + \frac{\xi}{\eta_0}, \quad \frac{M}{M_0} = 1 + \frac{\xi}{\eta_0} \quad \text{(Boussinesq and non-Boussinesq).} \tag{3.2a–c}$$

The rationale for the equivalence for pure plumes is straightforward. Since the velocity does not change with height ($\mathrm{d}\hat{w}/\mathrm{d}\xi = 0$, $\hat{w} = 1$ giving $u_e/w_0 = \alpha$), an unmodified entrainment model leads to equivalent conservation equations for the fluxes of volume and mass: $(\mathrm{d}\hat{Q}/\mathrm{d}\xi)_{BM} = (\mathrm{d}\hat{Q}/\mathrm{d}\xi)_{NBM}(= u_e/\alpha w_0 = 1)$ and $(\mathrm{d}\hat{G}/\mathrm{d}\xi)_{BM} = (\mathrm{d}\hat{G}/\mathrm{d}\xi)_{NBM}(= u_e/\alpha w_0 \eta_0 = 1/\eta_0)$. The subscripts BM and NBM denote the predictions of a Boussinesq model and a non-Boussinesq model, respectively. Noting that $\hat{M} = \hat{w}\hat{G}$, where $\hat{w}(z)$ is constant, this also implies equivalent conservation equations for the momentum flux: $(\mathrm{d}\hat{M}/\mathrm{d}\xi)_{BM} = (\mathrm{d}\hat{M}/\mathrm{d}\xi)_{NBM}(= 1/\eta_0)$. The equivalence is noteworthy and would imply, if non-Boussinesq entrainment is unmodified, that no differences between a pure Boussinesq and a pure non-Boussinesq plume could be observed experimentally.

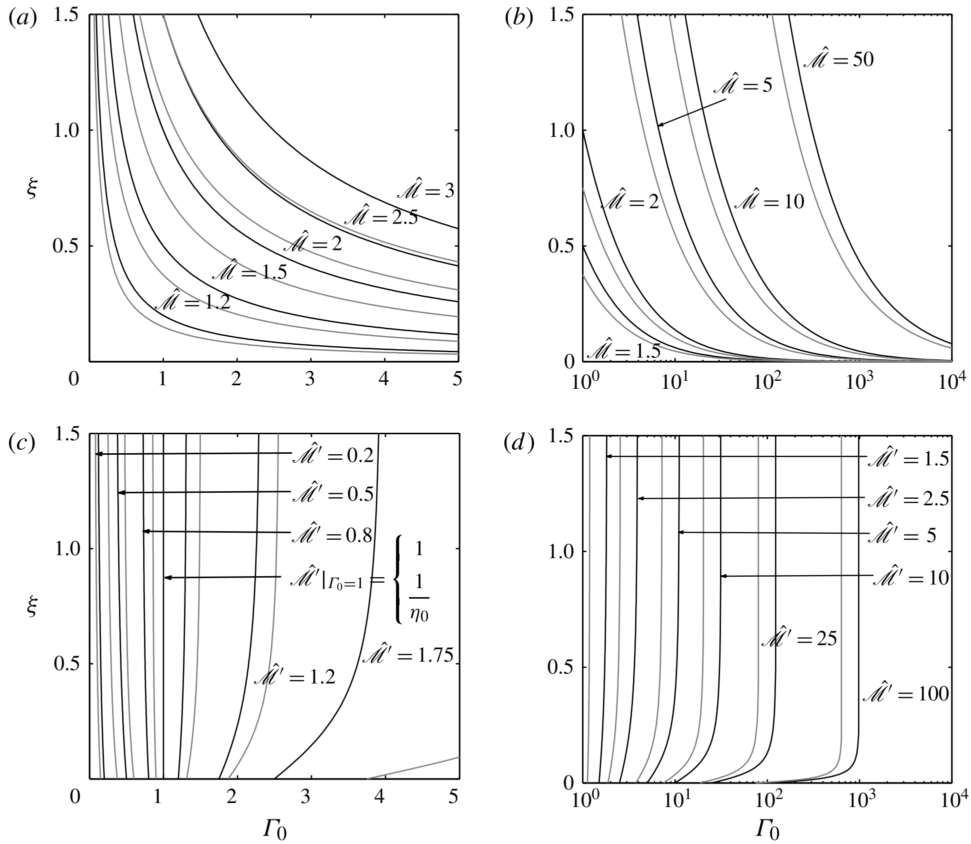

FIGURE 1. Contours of (a,b) constant $\hat{\mathscr{M}}$ and (c,d) constant $\hat{\mathscr{M}}'=\mathrm{d}\hat{\mathscr{M}}/\mathrm{d}\xi$ for plumes as predicted by a Boussinesq model (black lines) and a non-Boussinesq model (grey lines) with $\eta_0=0.75$. Similar trends are predicted for other values of η_0. At the source ($\xi=0$), we have $\mathrm{d}\hat{M}_B/\mathrm{d}\xi=\Gamma_0$ in the Boussinesq case and $\mathrm{d}\hat{M}/\mathrm{d}\xi=\Gamma_0/\eta_0$ in the non-Boussinesq case.

4. Non-Boussinesq effects on forced ($0<\Gamma_0<1$) and lazy ($\Gamma_0>1$) plumes

Contrary to pure plumes, forced and lazy plumes are affected by non-Boussinesq effects. To disentangle the effects on the different variables that are coupled in a system of simultaneous differential equations, we distinguish between direct and indirect effects. Direct effects are the direct result of modification to the conservation equations, whereas indirect effects are only the result of changes in the values of the variables induced by the former. As explained in § 2.2, the only direct effect is the modification of the conservation equation for momentum flux. We now examine direct and indirect effects on the relevant quantities in turn (§§ 4.1–4.5). The source values of the rates of change with height of the different quantities and their values in the far-field limit ($\xi\rightarrow\infty$) are summarized in table 1 to further aid the discussion and complement figures 1–6 that follow, which depict how the quantities of primary interest vary with height (and with Γ_0). Whilst not all aspects of the trends shown in these detailed figures are discussed in the accompanying text, they are retained as they provide a useful reference.

	Γ	$\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\hat{w}}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\hat{b}}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\eta}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\hat{Q}}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\hat{G}}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\hat{M}_B}{\mathrm{d}\xi}$	$\frac{\mathrm{d}\hat{M}}{\mathrm{d}\xi}$
$BM:\xi=0$	Γ_0	$3\Gamma_0(1-\Gamma_0)$	Γ_0-1	$2-\Gamma_0$	$1-\eta_0$	1	$\frac{1}{\eta_0}$	Γ_0	$\frac{\Gamma_0+(1-\eta_0)(1-\Gamma_0)}{\eta_0}$
$NBM:\xi=0$	Γ_0	$\frac{3\Gamma_0(1-\Gamma_0)}{\eta_0}$	$\frac{\Gamma_0-1}{\eta_0}$	$2-\Gamma_0+\frac{1-\eta_0}{\eta_0}(1-\Gamma_0)$	$1-\eta_0$	1	$\frac{1}{\eta_0}$	—	$\frac{\Gamma_0}{\eta_0}$
$BM:\xi\rightarrow\infty$	1	0	0	1	0	$\Gamma_0^{1/3}$	$\frac{\Gamma_0^{1/3}}{\eta_0}$	$\Gamma_0^{2/3}$	$\frac{\Gamma_0^{2/3}}{\eta_0}$
$NBM:\xi\rightarrow\infty$	1	0	0	1	0	$\Gamma_0^{1/3}$	$\frac{\Gamma_0^{1/3}}{\eta_0}$	—	$\frac{\Gamma_0^{2/3}}{\eta_0}$

TABLE 1. Values of the derivatives of plume quantities and fluxes at the sources ($\xi=0$) and in the limit $\xi\rightarrow\infty$ (or equivalently $\Gamma\rightarrow 1$) for Boussinesq (*BM*) and non-Boussinesq (*NBM*) models.

4.1. *Momentum flux*

Figure 1(*a*,*b*) confirms that the flux $\hat{\mathscr{M}}$, which denotes the approximation to the momentum flux in the Boussinesq case and the momentum flux in the non-Boussinesq case, increases more rapidly when predicted by a non-Boussinesq model. This is a direct implication of the Boussinesq approximation (cf. (2.4)). It is also instructive to consider the form for the rates of change of $\hat{\mathscr{M}}$ (from (3.12c) in vdB&H 2014):

$$\frac{\mathrm{d}\hat{\mathscr{M}}}{\mathrm{d}\xi} = \begin{cases} \Gamma\left(\dfrac{\Gamma_0}{\Gamma}\right)^{2/3} & \text{(Boussinesq)}, \\ \dfrac{1}{\eta_0}\Gamma\left(\dfrac{\Gamma_0}{\Gamma}\right)^{2/3} & \text{(non-Boussinesq)}. \end{cases} \tag{4.1}$$

At a given height the non-Boussinesq model predicts a larger momentum flux (figure 1*a*,*b*) and a larger gradient (figure 1*c*,*d*). At the source ($\Gamma = \Gamma_0$), the contours of constant $\mathrm{d}\hat{\mathscr{M}}/\mathrm{d}\xi$ are scaled by a constant factor $1/\eta_0 (= \rho_a/\rho_0 > 1)$; for given values of $\mathrm{d}\hat{\mathscr{M}}/\mathrm{d}\xi$ the contours in figure 1(*c*,*d*) for the non-Boussinesq model are shifted to the left. The effect of this shift is felt particularly strongly by highly forced plumes ($\Gamma_0 \ll 1$) as the discrepancy between $\hat{M}$ and $\hat{M}_B$ grows as Γ_0 decreases. We return in § 4.3 to this 'enhanced jet effect'.

4.2. *Volume and mass fluxes*

The inclusion of non-Boussinesq effects, on the other hand, does not have a direct effect on the rate of change of the fluxes of mass $\hat{G}$ and volume $\hat{Q}$ in the absence of modifications to the entrainment model (from (3.12b) in vdB&H 2014):

$$\frac{\mathrm{d}\hat{G}}{\mathrm{d}\xi} = \begin{cases} \dfrac{1}{\eta_0}\left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3} & \text{(Boussinesq)}, \\ \dfrac{1}{\eta_0}\left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3} & \text{(non-Boussinesq)}, \end{cases} \quad \frac{\mathrm{d}\hat{Q}}{\mathrm{d}\xi} = \begin{cases} \left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3} & \text{(Boussinesq)}, \\ \left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3} & \text{(non-Boussinesq)}. \end{cases} \tag{4.2a,b}$$

After all, the fluxes of mass and volume are conserved in both the Boussinesq and the non-Boussinesq cases and entrainment is unmodified. Direct effects can therefore not explain the trends in figure 2(*a*,*b*) (showing contours of constant volume flux) and 3(*a*,*b*) (showing contours of constant mass flux), namely, at any given height a non-Boussinesq model predicts the fluxes of volume and mass to be larger for lazy plumes ($\Gamma_0 > 1$) and smaller for forced plumes ($\Gamma_0 < 1$). This result may not have been entirely expected.

To begin to explain this non-intuitive result, consider the rate of increase with height of the volume (mass) flux, which takes a value $\mathrm{d}\hat{Q}/\mathrm{d}\xi|_{\xi=0} = 1$ ($\mathrm{d}\hat{G}/\mathrm{d}\xi|_{\xi=0} = 1/\eta_0$) at the source and approaches a value of $\mathrm{d}\hat{Q}/\mathrm{d}\xi = \Gamma_0^{1/3}$ ($\mathrm{d}\hat{G}/\mathrm{d}\xi = \Gamma_0^{1/3}/\eta_0$) asymptotically with height (cf. table 1). For forced plumes this amounts to a monotonic decrease: rapid mixing and consequent volume increases take place near the source and decay as the plume rises and becomes more pure. For lazy plumes, on the other hand, the rate of volume increase becomes larger with height: mixing only becomes rapid as the plume rises and becomes more pure.

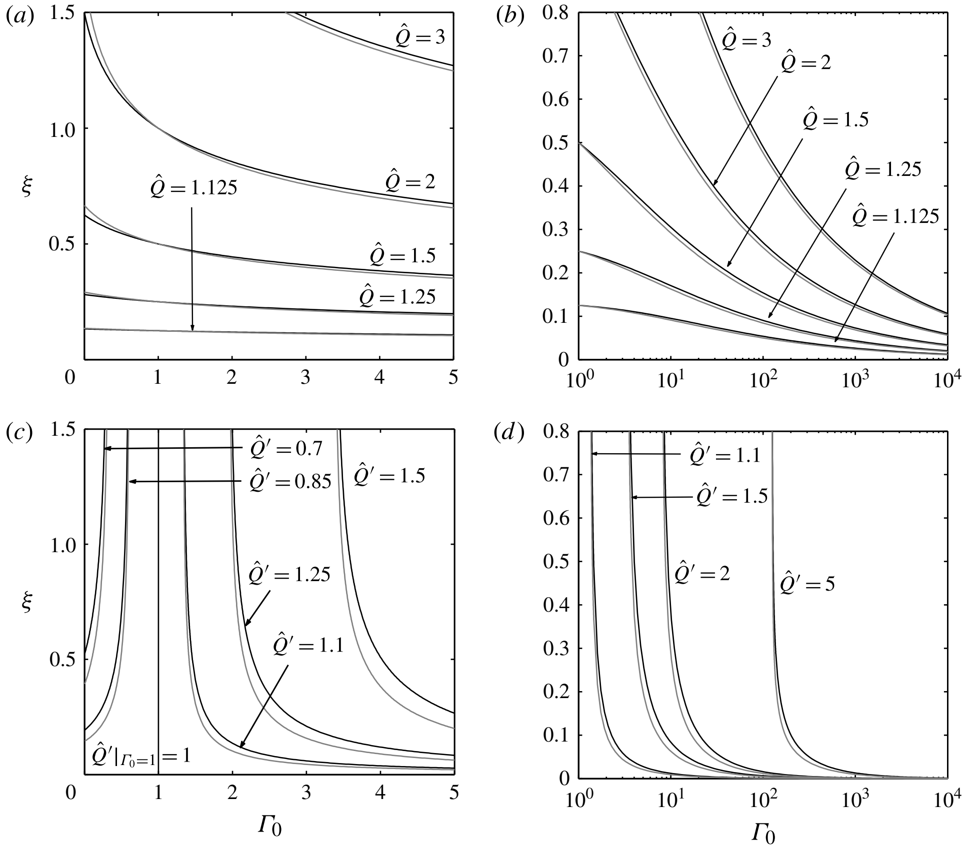

FIGURE 2. Contours of (*a*,*b*) constant $\hat{Q}$ and (*c*,*d*) constant $\hat{Q}' = \mathrm{d}\hat{Q}/\mathrm{d}\xi$ for plumes as predicted by a Boussinesq model (black lines) and a non-Boussinesq model (grey lines) with $\eta_0 = 0.75$. For pure plumes ($\Gamma_0 = 1$) model predictions are identical (§ 3). At the source ($\xi = 0$), we have $\mathrm{d}\hat{Q}/\mathrm{d}\xi = 1$ in the Boussinesq and the non-Boussinesq cases.

It is evident from figure 2(*a*,*b*) (and figure 3*a*,*b* for the mass flux) that in unconfined environments the change in volume flux experienced by forced plumes is less than for a pure plume, which, in turn, is less than for lazy plumes. These differences are accentuated in the non-Boussinesq model. Crucial to these differences between Boussinesq and non-Boussinesq approaches is the rate at which the plume becomes pure. To understand this behaviour, we turn to the flux balance parameter Γ.

4.3. *Richardson number* Γ

The implications of a non-Boussinesq model for the flux balance Γ can be noted from observing that the definition of Γ (2.2) is equivalent in both models:

$$\hat{\Gamma} = \frac{\Gamma}{\Gamma_0} = \left(\frac{\hat{\mathscr{G}}}{\hat{\mathscr{M}}}\right)^3 = \begin{cases} \left(\dfrac{\hat{Q}}{\hat{M}_B}\right)^3 = \left(\dfrac{\hat{G}}{\hat{M}}\right)^3 & \text{(Boussinesq)}, \\ \left(\dfrac{\hat{G}}{\hat{M}}\right)^3 = \left(\dfrac{\hat{Q}}{\hat{M}_B}\right)^3 & \text{(non-Boussinesq)}. \end{cases} \tag{4.3}$$

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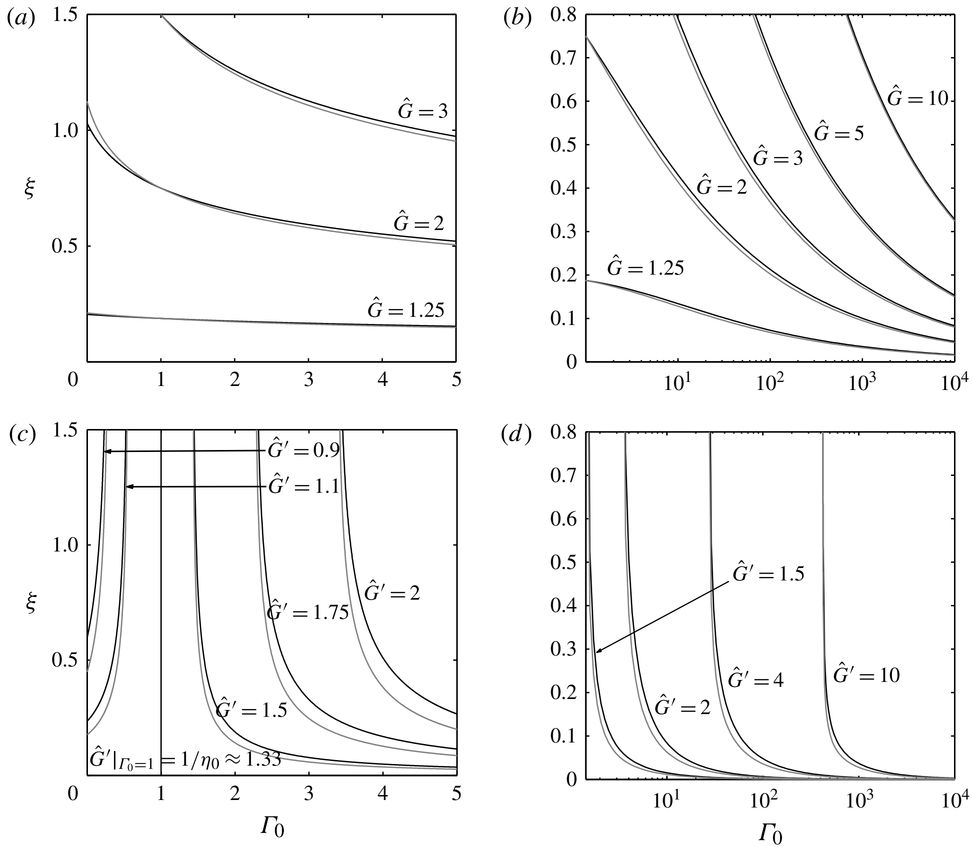

FIGURE 3. Contours of (*a*,*b*) constant $\hat{G}$ and (*c*,*d*) constant $\hat{G}'=\mathrm{d}\hat{G}/\mathrm{d}\xi$ for plumes as predicted by a Boussinesq model (black lines) and a non-Boussinesq model (grey lines) with $\eta_0=0.75$. At the source ($\xi=0$), we have $\mathrm{d}\hat{G}/\mathrm{d}\xi=1/\eta_0$ in the Boussinesq and the non-Boussinesq cases.

In turn, (4.3) can be differentiated to give (from (3.6) in vdB&H 2014):

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\xi}=\begin{cases} 3\Gamma(1-\Gamma)\left(\dfrac{\Gamma_0}{\Gamma}\right)^{2/3}\left(\dfrac{1-\Gamma}{1-\Gamma_0}\right)^{1/3} & \text{(Boussinesq)},\\ \dfrac{1}{\eta_0}3\Gamma(1-\Gamma)\left(\dfrac{\Gamma_0}{\Gamma}\right)^{2/3}\left(\dfrac{1-\Gamma}{1-\Gamma_0}\right)^{1/3} & \text{(non-Boussinesq)}.\end{cases} \tag{4.4}$$

The transition to pure plume behaviour is thus more rapid ($1/\eta_0>1$) in a non-Boussinesq model for both forced and lazy plumes, as illustrated by figure 4(*a*,*b*). This is explained by direct effects and the interpretation of Γ as a flux balance parameter. Noting that $\mathrm{d}\hat{\Gamma}/\mathrm{d}\xi=3\hat{\Gamma}((1/\hat{\mathscr{G}})\mathrm{d}\mathscr{G}/\mathrm{d}\xi-(1/\hat{\mathscr{M}})\mathrm{d}\mathscr{M}/\mathrm{d}\xi)$, larger $\mathrm{d}\hat{\mathscr{M}}/\mathrm{d}\xi$ (§ 4.1) or larger $\mathrm{d}\hat{\mathscr{G}}/\mathrm{d}\xi$ (§ 4.2) in the non-Boussinesq case leads to larger $|\mathrm{d}\Gamma/\mathrm{d}\xi|$ and, thus, more rapid restoration to unity of the balance of fluxes either from above (lazy plumes) or from below (forced plumes).

For forced plumes, the region of rapid mixing and volume (mass) flux increase near the source is therefore confined to an even smaller height, resulting in a smaller value of the volume (mass) flux being reached at any given height (figures 2*a* and 3*a*).

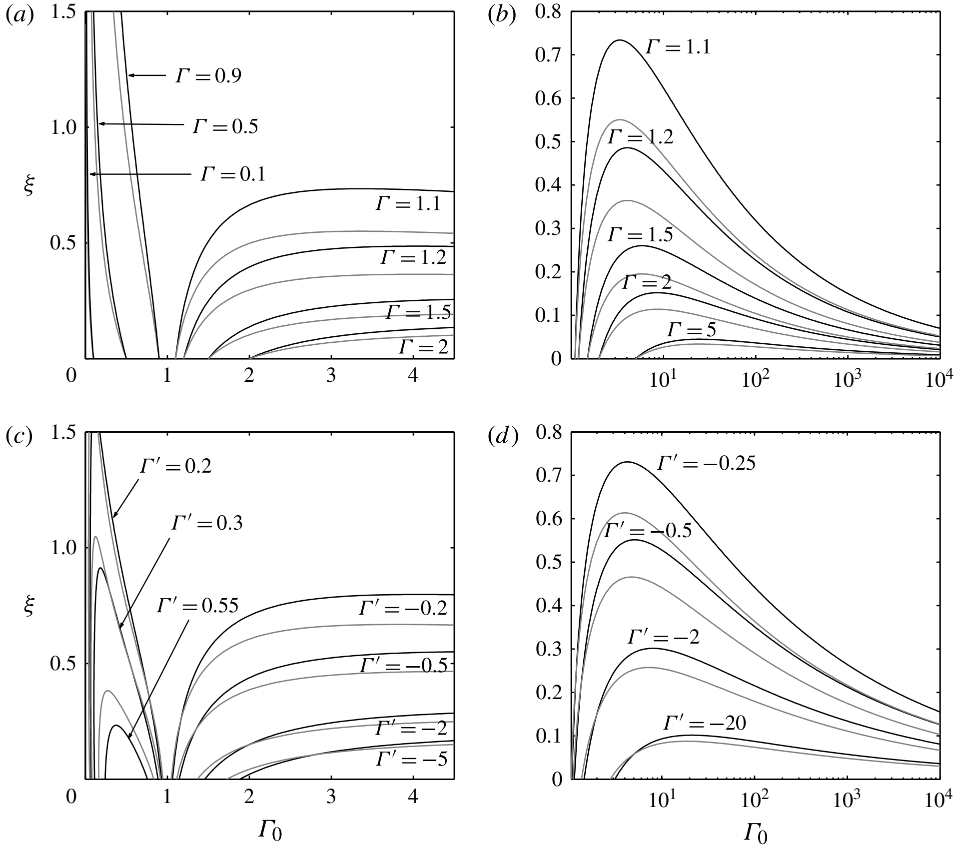

FIGURE 4. Contours of (a,b) constant Γ and (c,d) constant $\Gamma' = \mathrm{d}\Gamma/\mathrm{d}\xi$ for plumes as predicted by a Boussinesq model (black lines) and a non-Boussinesq model (grey lines) with $\eta_0 = 0.75$. At the source ($\xi = 0$), we have $\mathrm{d}\Gamma/\mathrm{d}\xi = 3\Gamma_0(1 - \Gamma_0)$ in the Boussinesq case and $\mathrm{d}\Gamma/\mathrm{d}\xi = 3\Gamma_0(1 - \Gamma_0)/\eta_0$ in the non-Boussinesq case.

For lazy plumes, on the other hand, the region of slow mixing and slow increase of volume (mass) flux near the source is passed through more rapidly. Thus, the observation that a non-Boussinesq model without modified entrainment predicts the fluxes of volume and mass to be larger for lazy plumes and smaller for forced plumes is explained.

It is worth summarizing here that should one wish to model a particular plume which has a source parameter value Γ_0, a value that is identical as an input in both the Boussinesq and the non-Boussinesq models, an immediate implication of the locally greater momentum flux in the non-Boussinesq model is as follows: if Γ_0 exceeds unity, the momentum flux deficit of the Boussinesq model (relative to the pure plume) exceeds that of the non-Boussinesq model; if Γ_0 is less than unity, the momentum flux excess of the non-Boussinesq model (an excess relative to the pure plume) exceeds that of the Boussinesq model. In other words, in terms of their momentum flux, lazy plumes may be regarded as 'less lazy' and forced plumes as 'more forced' (or, as we refer to previously, as having an 'enhanced jet effect'). Given that lazy plumes adjust to being pure more rapidly with height than forced plumes, the implications of a non-Boussinesq model are more strongly felt by forced plumes, the increased excess

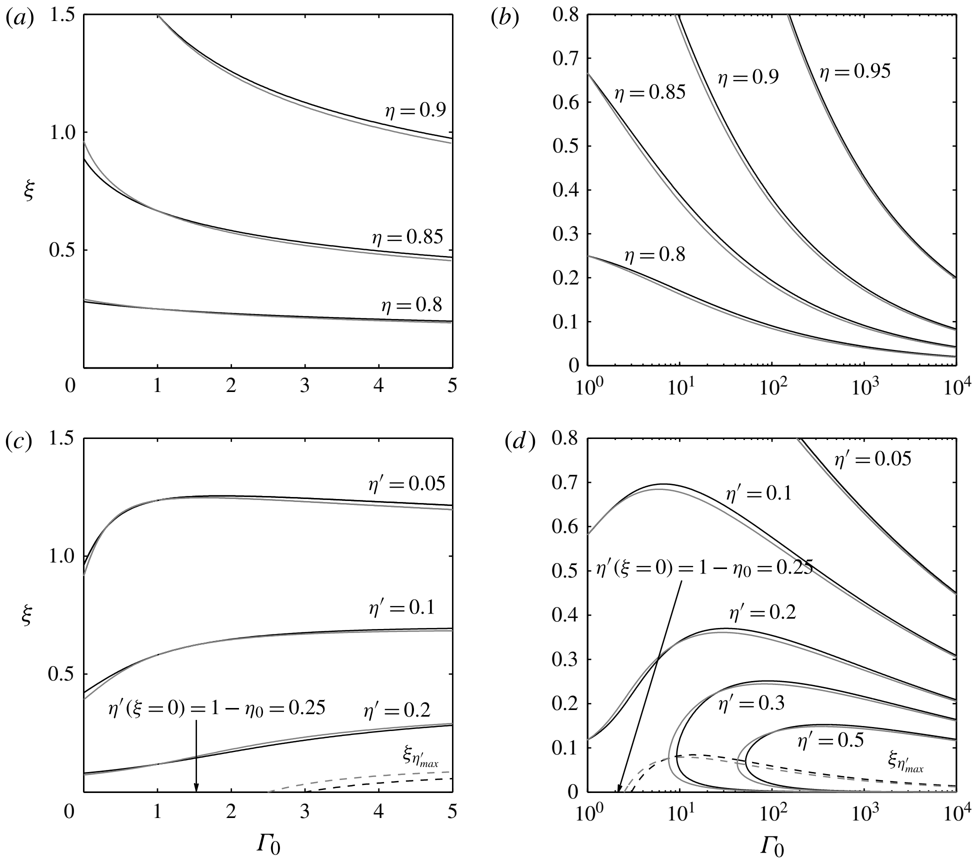

FIGURE 5. Contours of (*a*,*b*) constant η and (*c*,*d*) constant $\eta' = \mathrm{d}\eta/\mathrm{d}\xi$ for plumes as predicted by a Boussinesq model (black lines) and a non-Boussinesq model (grey lines) with $\eta_0 = 0.75$. At the source ($\xi = 0$), we have $\mathrm{d}\eta/\mathrm{d}\xi = 1 - \eta_0$ in both the Boussinesq and the non-Boussinesq cases.

of momentum flux requiring a greater vertical extent to 'correct' by the action of the buoyancy forces.

4.4. *Density contrast*

We turn to the contours of the density contrast η. The behaviour of η, which simply increases from its source value η_0 to reach $\eta = 1$ asymptotically with height as mixing takes place, can be easily explained once the behaviour of volume flux with height is understood. Dilution predicted by a non-Boussinesq model is simply less rapid for forced plumes and more rapid for lazy plumes (cf. § 4.3), corresponding to slower convergence to $\eta = 1$ in the forced case and more rapid convergence in the lazy case (figure 5*a*,*b*). From (3.11a) in vdB&H (2014):

$$\eta = \begin{cases} 1 - (1-\eta_0)\left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3}\left(\dfrac{\Gamma_0 - 1}{\Gamma - 1}\right)^{1/3} & \text{(Boussinesq),} \\ \dfrac{1}{1 + \dfrac{1-\eta_0}{\eta_0}\left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3}\left(\dfrac{\Gamma_0 - 1}{\Gamma - 1}\right)^{1/3}} & \text{(non-Boussinesq).} \end{cases} \tag{4.5}$$

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From (3.12a) in vdB&H (2014):

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \begin{cases} (1-\eta_0)\left(\dfrac{\Gamma_0}{\Gamma}\right)\left(\dfrac{\Gamma-1}{\Gamma_0-1}\right)^{2/3} & \text{(Boussinesq),} \\ \dfrac{(1-\eta_0)\left(\dfrac{\Gamma_0}{\Gamma}\right)\left(\dfrac{\Gamma-1}{\Gamma_0-1}\right)^{2/3}}{\left(\eta_0+(1-\eta_0)\left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3}\left(\dfrac{\Gamma-1}{\Gamma_0-1}\right)^{1/3}\right)^2} & \text{(non-Boussinesq).} \end{cases} \tag{4.6}$$

Despite their contrasting forms (4.6), the heights at which particular gradients in density contrast are attained (figure 5*c*,*d*) are similar and both gradients take identical values of $\mathrm{d}\eta/\mathrm{d}\xi = (1-\eta_0)$ at the source (cf. table 1). At a sufficiently large distance above the source (cf. figure 5*c*,*d*), contours of constant $\mathrm{d}\eta/\mathrm{d}\xi$ corresponding to a non-Boussinesq model lie below those corresponding to a Boussinesq model for forced and lazy plumes, but not for pure plumes, for which they lie at the same height. At a given height, a non-Boussinesq model thus predicts a lower value of the dilution rate $\mathrm{d}\eta/\mathrm{d}\xi$, explained by such plumes being effectively more pure-like due to the more rapid convergence to pure plume behaviour predicted by a non-Boussinesq model. Closer to the source and for larger values of Γ_0, the difference in behaviour is more subtle due to the maximum rate of dilution that is reached for sufficiently lazy plumes (cf. the dashed lines denoted by $\xi_{\eta'_{max}}$ in figure 5*c*,*d*).

4.5. *Vertical velocity*

Noting that $\hat{w} = \mathscr{M}/\mathscr{G}$, the different behaviour of $\hat{w}$ is also entirely explained by the differences in the behaviour of Γ (cf. § 4.3). From (3.8) in vdB&H (2014):

$$\hat{w} = \begin{cases} \left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3} & \text{(Boussinesq),} \\ \left(\dfrac{\Gamma_0}{\Gamma}\right)^{1/3} & \text{(non-Boussinesq).} \end{cases} \tag{4.7}$$

However, we note the dependence of $\mathrm{d}\Gamma/\mathrm{d}\xi$ on η_0 (4.4) in the non-Boussinesq case and thus the velocities (4.7) differ with height:

$$\frac{\mathrm{d}\hat{w}}{\mathrm{d}\xi} = \begin{cases} -(1-\Gamma)\left(\dfrac{\Gamma_0}{\Gamma}\right)\left(\dfrac{1-\Gamma}{1-\Gamma_0}\right)^{1/3} & \text{(Boussinesq),} \\ -\dfrac{1}{\eta_0}(1-\Gamma)\left(\dfrac{\Gamma_0}{\Gamma}\right)\left(\dfrac{1-\Gamma}{1-\Gamma_0}\right)^{1/3} & \text{(non-Boussinesq).} \end{cases} \tag{4.8}$$

Indeed, figure 6(*c*,*d*) confirms that a non-Boussinesq model predicts a more rapid increase in $\hat{w}$ for lazy plumes and a more rapid decrease in $\hat{w}$ for forced plumes. The result of differences between the models on the vertical velocity is straightforward with the non-Boussinesq model giving higher velocities at any given height for all plumes (figure 6*a*,*b*).

For forced plumes, results for the gradient of velocity (plume deceleration) are straightforward, the non-Boussinesq model yielding stronger deceleration. For lazy plumes, on the other hand, the implications for plume acceleration are quite subtle; near the source, accelerations are stronger in the non-Boussinesq model although they are smaller compared to the accelerations of the Boussinesq model at greater heights.

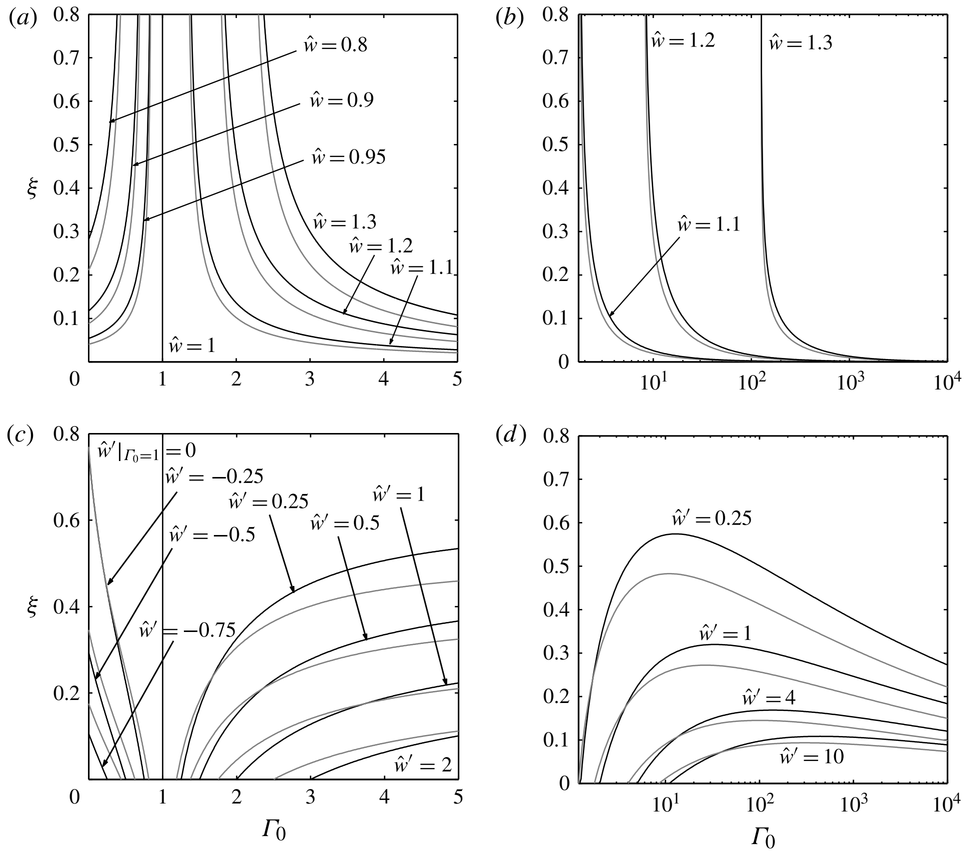

FIGURE 6. Contours of (*a*,*b*) constant $\hat{w}$ and (*c*,*d*) constant $\hat{w}' = \mathrm{d}\hat{w}/\mathrm{d}\xi$ for plumes as predicted by a Boussinesq model (black lines) and a non-Boussinesq model (grey lines) with $\eta_0 = 0.75$. At the source ($\xi = 0$), we have $\mathrm{d}\hat{w}/\mathrm{d}\xi = (\Gamma_0 - 1)$ in the Boussinesq case and $\mathrm{d}\hat{w}/\mathrm{d}\xi = (\Gamma_0 - 1)/\eta_0$ in the non-Boussinesq case.

The former is explained by the direct effect illustrated by (4.8), whereas the latter is due to the more rapid convergence to pure plume behaviour in a non-Boussinesq model ($\mathrm{d}\hat{w}/\mathrm{d}\xi \to 0$ as $\xi \to \infty$). The accelerations at the source increase with plume laziness as $\mathrm{d}\hat{w}/\mathrm{d}\xi|_{\xi=0} = (\Gamma_0 - 1)/\eta_0$ in the non-Boussinesq case (table 1) explaining the crossing of the constant-$\mathrm{d}\hat{w}/\mathrm{d}\xi$ contours in figure 6(*d*).

5. Conclusions

In an accompanying paper, van den Bremer & Hunt (2014) (vdB&H 2014) we have shown that if an (unmodified) entrainment model based on similarity is introduced for non-Boussinesq planar plumes, the solutions to the system of conservation equations for Boussinesq plumes and the solutions to the system of conservation equations for non-Boussinesq plumes take the same mathematical form. These 'universal solutions' relied on the introduction of an effective half-width, β, which is equal to the actual half-width, $\beta = b$, in the Boussinesq case and to the product of the half-width and the local density contrast, $\beta = b\eta$, in the non-Boussinesq case. We first and foremost emphasize the counter-intuitive nature, already noted by Rooney (1997), of the unmodified entrainment model suggested in the literature on the basis of similarity arguments, on which the validity of the solution for non-Boussinesq plumes and, thence, the universality demonstrated in vdB&H (2014) relies. Contrary to the axisymmetric case, an entrainment model consistent with similarity is not modified by non-Boussinesq effects: $u_e = \alpha w$ in the planar case versus $u_e = \alpha w \sqrt{\eta}$ in the axisymmetric case.

In this paper we have explicitly examined the implications for the differences in prediction between a Boussinesq and a non-Boussinesq model. We have done so by comparing the predicted behaviour of both models for plumes with equivalent source conditions. The solutions are evaluated for plumes that are moderately non-Boussinesq ($\eta_0 \not\approx 1$, $\eta_0 \not\ll 1$), but generalize to very non-Boussinesq plumes ($\eta_0 \ll 1$). We show that, compared to a Boussinesq model, a non-Boussinesq model leads to equivalent solutions for pure plumes and the prediction of larger fluxes of volume and mass for lazy plumes, but smaller fluxes for forced plumes at any given height. In particular, the equivalence of the solutions for pure plumes, which implies that varying η_0 *ceteris paribus* (notably $\varGamma_0 = 1$) would not result in any, let alone experimentally observable, differences in behaviour, leads to significant doubts regarding the validity of this unmodified entrainment model. Although similarity arguments are often a first port of call in such cases, they must be dismissed here, and only detailed measurements can resolve this debate. These measurements are unlikely to prove straightforward as, based on a recent study of axisymmetric plumes (Ezzamel 2011), sophisticated particle-image-velocimetry techniques alone may not be sufficient to establish the precise form of the entrainment.

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