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APPLICATIONS OF THE ALEXANDER IDEALS TO THE ISOMORPHISM PROBLEM FOR FAMILIES OF GROUPS

DO VIET HUNG[1] AND VU THE KHOI[2]*

[1] *Ha Giang College of Education, Ha Giang, Vietnam* (viethunghg81@gmail.com)
[2] *Institute of Mathematics (VAST), 18 Hoang Quoc Viet, 10307, Hanoi, Vietnam* (vtkhoi@math.ac.vn)

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Abstract In this paper we use the Alexander ideals of groups to solve the isomorphism problem for the Baumslag–Solitar groups and a family of parafree groups introduced by Baumslag and Cleary.

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1. Introduction and main results

1.1. The group isomorphism problem

The isomorphism problem is a fundamental problem in group theory in which we have to decide whether two finitely presented groups are isomorphic. In its most general form, the isomorphism problem was proved to be unsolvable. It therefore makes sense to restrict the problem to a special class of groups. The isomorphism problem for certain families of groups has been considered by many authors. For those works particularly related to ours, see [**8**–**11**, **15**, **17**]. The purpose of this paper is to use the Alexander ideal, an algebraic invariant of groups that originated from topology, to study the isomorphism problem for families of groups. In many cases, by computing the Alexander ideals of the groups in the family, we can deduce that two groups in that family are not isomorphic. Our main results are the solutions of the isomorphism problem for the Baumslag–Solitar groups and a family of parafree groups.

The paper consists of two sections. In the rest of this section we give some brief background and state the main results on the solutions of the isomorphism problem for the Baumslag–Solitar groups and a family of parafree groups. Section 2 is the technical heart of the paper. In the beginning of § 2 we review background on Alexander ideals and Fox's free differential calculus. In the rest of § 2 we compute the Alexander ideals and present the proofs of the main results.

* Corresponding author.

1.2. The isomorphism problem for the Baumslag–Solitar groups

The Baumslag–Solitar group $B(m, n)$, where m, n are non-zero integers, can be defined by

$$B(m, n) = \langle a, b \mid a^{-1}b^m a = b^n\rangle = \langle a, b \mid a^{-1}b^m ab^{-n} = 1\rangle.$$

The Baumslag–Solitar groups were first given by Baumslag and Solitar [**7**] as an example of one-relator non-Hopfian groups. The Baumslag–Solitar groups have attracted the attention of many authors since they serve as a rich source of examples and counterexamples for many questions in group theory.

The isomorphism problem for Baumslag–Solitar groups was first solved by Moldavanskii [**17**] in 1991. Later on, Clement [**10**] gave an alternative proof. The isomorphism problem for generalized Baumslag–Solitar groups has been studied by Clay and Forester [**9**].

As a first illustration of our method, we present another solution to the isomorphism problem for the Baumslag–Solitar groups by using the Alexander ideals.

Theorem 1.1 (Moldavanskii [17]). *The groups $B(m, n)$ and $B(p, q)$ are isomorphic if and only if for a suitable $\varepsilon = \pm 1$ either $m = p\varepsilon$ and $n = q\varepsilon$, or $m = q\varepsilon$ and $n = p\varepsilon$.*

1.3. The isomorphism problem for a family of parafree groups

A group G is called *parafree* if it is residually nilpotent and has the same nilpotent quotients as a given free group. Parafree groups were studied for the first time by Baumslag [**2**, **3**]. Since then, several explicit families of parafree groups have been introduced.

Baumslag (see [**3**, **4**]) introduced a family of parafree groups that is denoted by $G_{i,j}$ and presented as

$$G_{i,j} := \langle a, b, c \mid a = [c^i, a][c^j, b]\rangle;$$

here $[x, y] := x^{-1}y^{-1}xy$.

Later on, in [**5**], Baumslag introduced another family of parafree groups. This family is denoted by $H_{i,j}$ and is presented as

$$H_{i,j} := \langle a, s, t \mid a = [a^i, t^j][s, t]\rangle.$$

In 2006, Baumslag and Cleary [**6**] introduced several new families of parafree groups, including the following family, which is denoted by $K_{i,j}$:

$$K_{i,j} := \langle a, s, t \mid a^i[s, a] = t^j\rangle, \quad \text{where } i > 0,\ j > 1 \text{ are relatively prime.}$$

As parafree groups enjoy many common properties with free groups, the isomorphism problem for parafree groups is known to be difficult. There have been some partial results for the isomorphism problem for groups in the families $G_{i,j}$, $H_{i,j}$, $K_{i,j}$.

It was shown in [**11**] that $G_{i,1} \ncong G_{1,1}$ for $i > 1$ and $G_{i,1} \ncong G_{j,1}$ for distinct primes i, j. Computational approaches aimed at distinguishing parafree groups by enumerating homomorphisms to a fixed finite group have also been used: in [**15**] the groups $G_{i,j}$ were considered, in [**8**] the groups $G_{i,j}$, $H_{i,j}$, $K_{i,j}$ were considered. This method can only distinguish finitely many groups in each class. In this paper we use Alexander ideals to distinguish parafree groups in the family $K_{i,j}$.

Theorem 1.2. *For $(i, j) = 1$, $(i', j') = 1$, the groups $K_{i,j}$ and $K_{i',j'}$ are isomorphic if and only if $i = i'$ and $j = j'$.*

We remark that our methods do no work for the families of groups $G_{i,j}$ and $H_{i,j}$ as we have found that their Alexander ideals are trivial.

2. Proofs of the main results

In this section we compute the Alexander ideals and derive the proofs of the main results. We begin by giving a brief review on Alexander ideals and Fox's free differential calculus.

2.1. Alexander ideals

In this section we present some background on the Alexander ideals of finitely presented groups. For more details, the reader is referred to [**1**, **12**–**14**, **16**, **18**].

Let $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ be a finitely presented group and let

$$\mathrm{ab}(G) := H_1(G; \mathbb{Z})/(\text{torsion})$$

be its maximal free abelian quotient. Suppose that (X, p) is a pointed CW-complex such that $\pi_1(X, p) = G$. Let $\pi\colon \tilde{X} \to X$ be the covering corresponding to $\phi\colon G \to \mathrm{ab}(G)$ and let $\tilde{p} := \pi^{-1}(p)$. We denote by $H_1(\tilde{X}, \tilde{p}; \mathbb{Z})$ the relative homology. Then the deck transformation action of $\mathrm{ab}(G)$ on $\tilde{X}$ makes $H_1(\tilde{X}, \tilde{p}; \mathbb{Z})$ an $\mathbb{Z}[\mathrm{ab}(G)]$-module, which is called the *Alexander module* of G.

Suppose that we fix an isomorphism $\chi\colon \mathrm{ab}(G) \to \mathbb{Z}^n$, then the group ring $\Lambda := \mathbb{Z}[\mathrm{ab}(G)]$ can be identified with $\mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}, \ldots, t_n^{\pm 1}]$. It is well known that the Alexander module $M = H_1(\tilde{X}, \tilde{p}; \mathbb{Z})$ is a finitely generated Λ-module, so we choose a presentation

$$\Lambda^l \xrightarrow{A} \Lambda^k \to M. \qquad (*)$$

Let A be the presentation matrix of the Alexander module as above. The *ith Alexander ideal* of G is the ideal generated by all the $(k-i) \times (k-i)$ minors of the presentation matrix A.

The Alexander ideals do not depend on the choices of the CW-complex X or the presentation $(*)$. However, we have freedom in choosing the isomorphism χ above, so the Alexander ideals are invariants of the group G up to a *monomial automorphism* of Λ. That is, an automorphism such that $\varphi(t_i) = t_1^{a_{i1}} t_2^{a_{i2}} \cdots t_n^{a_{in}}$, $i = 1, \ldots, n$, where (a_{ij}) is a matrix belonging to $\mathrm{GL}(n, \mathbb{Z})$.

There is an effective algorithm to compute the Alexander modules and ideals by using Fox's free differential calculus [**12**, **13**], which we will describe briefly below.

Suppose that $F_k = \langle x_1, \ldots, x_k | \cdot \rangle$ is the free group on k generators. Let $\varepsilon\colon \mathbb{Z}F_k \to \mathbb{Z}$ be the augmentation homomorphism defined by $\varepsilon(\sum n_i g_i) = \sum n_i$. The jth partial Fox derivative is a linear operator $\partial/\partial x_j\colon \mathbb{Z}F_k \to \mathbb{Z}F_k$ that is uniquely determined by the

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following rules:

$$\frac{\partial}{\partial x_j}(1) = 0;$$

$$\frac{\partial}{\partial x_j}(x_i) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j; \end{cases}$$

$$\frac{\partial}{\partial x_j}(uv) = \frac{\partial}{\partial x_j}(u)\varepsilon(v) + u\frac{\partial}{\partial x_j}(v).$$

As consequences of the above rules we obtain

(i) $\dfrac{\partial}{\partial x_i}(x_i^n) = 1 + x_i + x_i^2 + \cdots + x_i^{n-1}$ for all $n \geqslant 1$,

(ii) $\dfrac{\partial}{\partial x_i}(x_i^{-n}) = -x_i^{-1} - x_i^{-2} - \cdots - x_i^{-n}$ for all $n \geqslant 1$.

Let $G = \langle x_1, \ldots, x_k \mid r_1, \ldots, r_l \rangle$ be a group as above. From the quotient map

$$F_k \to G \xrightarrow{\phi} \mathrm{ab}(G)$$

we get a map $\Phi \colon \mathbb{Z}F_k \to \Lambda$.

The results of Fox's free differential calculus (see [**13**]) say that the Jacobian matrix

$$J := \left(\frac{\partial}{\partial x_j} r_i\right)^{\Phi} : \Lambda^l \to \Lambda^k$$

is a presentation matrix for the Alexander module of G. Thus, we have an effective method to find the Alexander ideals.

2.2. Proofs of the main results

By using Fox's free differential calculus, we can find the Alexander ideals of the Baumslag–Solitar groups.

Proposition 2.1. *The first Alexander ideal of the group* $B(m, n)$ *is*

(i) $I = ((1 + t_2 + \cdots + t_2^{|n|-1})(1 - t_1), 1 - t_2^{|n|}) \subseteq \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$ *in the* $m = n$ *case,*

(ii) $I = (m - nt_1) \subseteq \mathbb{Z}[t_1]$ *in the* $m \neq n$ *case.*

Proof. To shorten the computations, we note without proof the following fact:

$$B(m, n) \cong B(n, m) \cong B(-m, -n) \cong B(-n, -m).$$

(i) By using the isomorphism $B(m, n) \cong B(-m, -n)$, it is enough to prove the result under the assumption that $m = n > 0$. It is easy to see that

$$\mathrm{ab}(B(n, n)) \cong \langle t_1 \rangle \oplus \langle t_2 \rangle,$$
$$a \mapsto t_1,$$
$$b \mapsto t_2.$$

Here $\langle t\rangle = \langle \dots, t^{-1}, 1, t, t^2, \dots\rangle$ is the infinite cyclic group generated by t. So the group ring is given by

$$\Lambda = \mathbb{Z}[\mathrm{ab}(B(n,n))] \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}].$$

Applying Fox's free differential calculus to the relation $r = a^{-1}b^n a b^{-n}$, we have

$$\frac{\partial r}{\partial a} = -a^{-1} + a^{-1}b^n,$$
$$\frac{\partial r}{\partial b} = a^{-1}(1 + b + b^2 + \cdots + b^{n-1}) - a^{-1}b^n a(b^{-1} + \cdots + b^{-n}).$$

Therefore,

$$\Phi\biggl(\frac{\partial r}{\partial a}\biggr) = t_1^{-1}(t_2^n - 1), \qquad \Phi\biggl(\frac{\partial r}{\partial b}\biggr) = (1 + t_2 + \cdots + t_2^{n-1})(t_1^{-1} - 1).$$

We see that the first Alexander ideal of $B(n,n)$ is $I = ((1+t_2+\cdots+t_2^{n-1})(1-t_1), 1-t_2^n)$.

(ii) We only give the computation of the Alexander ideal in the $m > 0$ case. For other cases, the reader can easily verify the results by using the isomorphisms noted at the beginning of the proof. So we consider two cases.

Case 1 ($m \neq n$ and $m, n > 0$). We see that

$$\mathrm{ab}(B(m,n)) \cong \langle t_1\rangle,$$
$$a \mapsto t_1,$$
$$b \mapsto 1.$$

We get $\Lambda = \mathbb{Z}[\mathrm{ab}(B(m,n))] \cong \mathbb{Z}[t_1^{\pm 1}]$.

Applying Fox's free differential calculus to $r = a^{-1}b^m a b^{-n}$, we have

$$\frac{\partial r}{\partial a} = -a^{-1} + a^{-1}b^m,$$
$$\frac{\partial r}{\partial b} = a^{-1}(1 + b + b^2 + \cdots + b^{m-1}) - a^{-1}b^m a(b^{-1} + \cdots + b^{-n}).$$

From this computation we get

$$\Phi\biggl(\frac{\partial r}{\partial a}\biggr) = 0, \qquad \Phi\biggl(\frac{\partial r}{\partial b}\biggr) = mt_1^{-1} - n.$$

So the first Alexander ideal is $I = (m - nt_1)$.

Case 2 ($m > 0$ and $n < 0$). This case is almost identical to Case 1 except for a small change:

$$\frac{\partial r}{\partial b} = a^{-1}(1 + b + b^2 + \cdots + b^{m-1}) + a^{-1}b^m a(1 + b + \cdots + b^{-n-1}).$$

However, straightforward computations show that the first Alexander ideal is still $I = (m - nt_1)$. □

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We can now quickly deduce the proof of Theorem 1.1

Proof of Theorem 1.1. We only need to show the necessity since the sufficiency is obvious, as already noted above. Now suppose that $B(m,n) \cong B(p,q)$. As $\mathrm{ab}(B(m,n)) \cong \mathrm{ab}(B(p,q))$, we see that $m=n$ if and only if $p=q$. So we only need to prove the theorem in the following cases.

Case 1 ($m=n$ and $p=q$). We denote by

$$V(I) = \{(t_1,t_2) \in (\mathbb{C}^*)^2 \mid f(t_1,t_2)=0\ \forall f \in I\}$$

the zero locus of the first Alexander ideal I. By the results of Proposition 2.1 (i), for the group $B(n,n)$, $V(I)$ consists of $(|n|-1)$ lines $\{t_2 = \mathrm{e}^{2\pi\sqrt{-1}k/|n|}\}$, $k=1,\dots,|n|-1$, and an isolated point $\{t_1=1, t_2=1\}$. In particular, $V(I)$ has $|n|$ connected components. Therefore, $B(n,n) \cong B(q,q)$ implies that $|n|=|q|$. So the theorem is proved in this case.

Case 2 ($m \neq n$ and $p \neq q$). Using Proposition 2.1 (ii), we get the equality of ideals $(m-nt_1) = (p-qt_1)$ in $\mathbb{Z}[t_1]$ up to a monomial automorphism. In this case, the monomial automorphism is just changing $t \leftrightarrow t^{-1}$. So this implies that either $m-nt_1 = \pm(p-qt_1)$ or $m-nt_1 = \pm(pt_1-q)$. We thus obtain the required conclusion. □

We will compute the Alexander ideal of the group $K_{i,j}$ in the following proposition.

Proposition 2.2. *The second Alexander ideal of the group $K_{i,j}$ above is given by*

$$I = (1-t_2^j, (1+t_2^j+\cdots+t_2^{(i-1)j}) - t_2^{(i-1)j}t_1^{-1} + t_2^{(i-1)j}, 1+t_2^i+\cdots+t_2^{(j-1)i}).$$

Proof. By using the fact that $(i,j)=1$, we have the isomorphism

$$\begin{aligned}\mathrm{ab}(K_{i,j}) &\cong \langle t_1\rangle \oplus \langle t_2\rangle,\\ s &\mapsto t_1,\\ a &\mapsto t_2^j,\\ t &\mapsto t_2^i.\end{aligned}$$

We deduce that the group ring is given by $\Lambda = \mathbb{Z}[\mathrm{ab}(K_{i,j})] \cong \mathbb{Z}[t_1^{\pm 1}, t_2^{\pm 1}]$.

Applying Fox's free differential calculus to the relation $r = a^i s^{-1} a^{-1} s a t^{-j}$, we have

$$\begin{aligned}\frac{\partial r}{\partial s} &= -a^i s^{-1} + a^i s^{-1} a^{-1},\\ \frac{\partial r}{\partial a} &= (1+a+\cdots+a^{i-1}) - a^i s^{-1} a^{-1} + a^i s^{-1} a^{-1} s,\\ \frac{\partial r}{\partial t} &= -a^i s^{-1} a^{-1} s a (t^{-1}+\cdots+t^{-j}).\end{aligned}$$

So, we find that

$$\Phi\left(\frac{\partial r}{\partial s}\right) = -t_2^{ij}t_1^{-1} + t_2^{(i-1)j}t_1^{-1},$$
$$\Phi\left(\frac{\partial r}{\partial a}\right) = (1 + t_2^j + \cdots + t_2^{(i-1)j}) - t_2^{(i-1)j}t_1^{-1} + t_2^{(i-1)j},$$
$$\Phi\left(\frac{\partial r}{\partial t}\right) = -(1 + t_2^i + \cdots + t_2^{(j-1)i}).$$

From this, the proposition follows. □

Now we can solve the isomorphism problem for the family $K_{i,j}$.

Proof of Theorem 1.2. We first find the zero set $V(I)$ of the second Alexander ideal of $K_{i,j}$. It follows from Proposition 2.2 that $V(I)$ is the solution set of the system

$$1 - t_2^j = 0, \tag{2.1 a}$$
$$(1 + t_2^j + \cdots + t_2^{(i-1)j}) - t_2^{(i-1)j}t_1^{-1} + t_2^{(i-1)j} = 0, \tag{2.1 b}$$
$$1 + t_2^i + \cdots + t_2^{(j-1)i} = 0. \tag{2.1 c}$$

We first consider the solutions of (2.1 a) and (2.1 c). As $(i, j) = 1$, we deduce that $t_2^i \neq 1$ since otherwise we get

$$t_2 = t_2^{(i,j)} = t_2^{pi+qj} = t_2^{pi}t_2^{qj} = 1,$$

which contradicts (2.1 c). Therefore, by multiplying (2.1 c) with $(1 - t_2^i)$, we get

$$(2.1\,a) + (2.1\,c) \iff \left\{\begin{matrix} t_2^j = 1, \\ t_2^{ij} = 1, \\ t_2^i \neq 1, \end{matrix}\right\} \iff t_2 = \mathrm{e}^{2\pi\sqrt{-1}k/j}, \quad k = 1, \ldots, j-1.$$

Combining with (2.1 b), we find that $V(I)$ consists of $(j - 1)$ points

$$\left\{(t_1, t_2) = \left(\frac{1}{i+1}, \mathrm{e}^{2\pi\sqrt{-1}k/j}\right)\right\}_{k=1,\ldots,j-1}.$$

Since $K_{i,j}$ and $K_{i',j'}$ are isomorphic, by counting the cardinality of $V(I)$ we deduce that $j = j'$. Now, suppose that

$$\varphi\colon \mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^* \times \mathbb{C}^*,$$
$$t_1 \mapsto t_1^a t_2^b,$$
$$t_2 \mapsto t_1^c t_2^d$$

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is the monomial automorphism that maps the zero set of the second Alexander ideal of $K_{i,j}$ to that of $K_{i',j'}$. That is,

$$(\varphi(t_1), \varphi(t_2)) \in \left\{ \left(\frac{1}{i'+1}, \mathrm{e}^{2\pi\sqrt{-1}k/j'} \right) \right\}_{k=1,\dots,j'-1} \quad \forall (t_1, t_2) \in \left\{ \left(\frac{1}{i+1}, \mathrm{e}^{2\pi\sqrt{-1}k/j} \right) \right\}_{k=1,\dots,j-1}.$$

From this, we obtain $|\varphi(t_2)| = (1/(i+1))^c = 1$, and therefore $c = 0$. Because the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ belongs to $\mathrm{GL}(2, \mathbb{Z})$, from $c = 0$ we find that $a = \pm 1$. So we must have $|\varphi(t_1)| = (1/(i+1))^{\pm 1} = 1/(i'+1)$. It follows that $a = 1$ and $i = i'$. □

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