Combinatorics, Probability and Computing (2017) 26, 267–284.
doi:10.1017/S0963548316000274

Judicious Partitioning of Hypergraphs with Edges of Size at Most 2†

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Received 12 December 2014; revised 29 April 2016; first published online 16 August 2016

Judicious partitioning problems on graphs and hypergraphs ask for partitions that optimize several quantities simultaneously. Let $k \geqslant 2$ be an integer and let G be a hypergraph with m_i edges of size i for $i = 1, 2$. Bollobás and Scott conjectured that G has a partition into k classes, each of which contains at most $m_1/k + m_2/k^2 + O(\sqrt{m_1 + m_2})$ edges. In this paper, we confirm the conjecture affirmatively by showing that G has a partition into k classes, each of which contains at most

$$m_1/k + m_2/k^2 + \frac{k-1}{2k^2}\sqrt{2(km_1 + m_2)} + O(1)$$

edges. This bound is tight up to $O(1)$.

2010 *Mathematics subject classification*: Primary 05C35
Secondary 05C75

1. Introduction

Classical graph or hypergraph partitioning problems often consider partitioning the vertex set of a graph or hypergraph into pairwise disjoint subsets that optimize a single quantity. For example, the well-known Max-Cut problem asks for a maximum bipartite subgraph of a graph, that is, a bipartition V_1, V_2 of a given graph maximizing the number of edges between V_1 and V_2. It is NP-hard even when restricted to triangle-free cubic graphs [22] and has been a very active research subject in both combinatorics and computer science.

It is easy to see that every graph with m edges contains a bipartite subgraph with at least $m/2$ edges. Edwards [9, 10] proved the essentially best possible result: a bipartite subgraph with at least $m/2 + (\sqrt{2m + 1/4} - 1/2)/4$ edges. An extension of Edwards' bound for partitions into more than two parts was proved in [6].

In practice, one often needs to find a partition of a given graph or hypergraph to optimize several quantities simultaneously. Such problems are called *judicious partitioning problems* by Bollobás and Scott [7]. In the Max-Cut setting, the canonical example is the

† This work is supported by research grant NSFC.
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beautiful result of Bollobás and Scott [4]: there is a cut (V_1, V_2) which not only achieves Edwards' bound, but also has few edges in each V_i for $i = 1, 2$.

In [4], Bollobás and Scott also considered the judicious k-partitions of graphs and proved that every graph G with m edges has a partition into k classes, each of which contains at most

$$\frac{1}{k^2}m + \frac{k-1}{2k^2}\left(\sqrt{2m + \frac{1}{4}} - \frac{1}{2}\right)$$

edges. The bound is tight for complete graphs K_{kn+1}.

While there are reasonable bounds for many judicious partitioning problems for graphs [1, 2, 8, 12, 16, 20, 21, 23], the analogous problems for hypergraphs seem to be much more difficult [3, 5, 14, 15, 18, 19]. In this paper, we consider the judicious partitioning of hypergraphs with edges of size at most 2.

Note that a *hypergraph* $G = (V, E)$ consists of a finite set $V := V(G)$ of vertices and a set $E := E(G)$ of edges, where each edge is a subset of V. For each edge $e \in E$, if e contains at most two elements of V, then G is a *hypergraph with edges of size at most 2*. For $i = 1, 2$, let E_i denote the set of edges of size i. For disjoint subsets X, Y of V, we use $f(X)$ (or $e(X)$) to denote the number of edges in E_1 (or E_2) that are contained in X, and $e(X, Y)$ to denote the number of edges in E_2 between X and Y. In particular, if $X = \{v\}$, we simply write $e(v, Y)$ for $e(\{v\}, Y)$. In a slight abuse of notation, we denote $v \in X$ and $\{v\} \in E_1$ (respectively, $\{v\} \notin E_1$) by $v \in X \cap E_1$ (respectively, $v \in X \setminus E_1$). Similarly, by $X \subseteq E_1$, we mean $\{v\} \in E_1$ for each $v \in X$. In addition, for each $v \in V$, we define the indicator function

$$\mathbf{1}_v = \begin{cases} 1 & \text{if } \{v\} \in E_1, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mu(X) = e(X) + f(X).$$

Obviously, $\mu(X)$ is the number of edges of G contained in X.

Let G be a hypergraph with m_i edges of size i, $i = 1, 2$. Although, in a random partition of G into k classes $V_1, \ldots, V_k$, we expect each V_i to have $m_1/k + m_2/k^2$ edges, bounding all k quantities simultaneously is much harder. Bollobás and Scott [7] posed the following conjecture.

Conjecture 1.1. *For fixed $k \geqslant 2$, every hypergraph with $m = m_1 + m_2$ edges, of which m_1 have size* 1 *and m_2 have size* 2*, has a partition into k classes, each of which contains at most*

$$\frac{1}{k}m_1 + \frac{1}{k^2}m_2 + O(\sqrt{m_1 + m_2})$$

edges, as $m \to \infty$.

Ma, Yen and Yu [17] first confirmed the conjecture asymptotically by showing that if G is a hypergraph with m_i edges of size i, $i = 1, 2$, then G admits a partition $V_1, \ldots, V_k$ such that each V_i contains at most $m_1/k + m_2/k^2 + O(m_2^{4/5})$ edges. In this paper, we confirm the conjecture completely with the following result.

Theorem 1.2. *For fixed $k \geqslant 2$, every hypergraph $G = (V, E)$ with m_i edges of size i, $i = 1, 2$, has a partition into k classes, each of which contains at most*

$$\frac{1}{k}m_1 + \frac{1}{k^2}m_2 + \frac{k-1}{2k^2}\sqrt{2(km_1 + m_2)} + O(1)$$

edges.

Note that complete graphs K_{kn+1} ($m_1 = 0$) show that the bound given in Theorem 1.2 is tight up to $O(1)$. We believe that the following conjecture is true.

Conjecture 1.3. *For fixed $k \geqslant 2$, every hypergraph with m_i edges of size i, $i = 1, 2$, has a partition into k classes, each of which contains at most*

$$\frac{1}{k}m_1 + \frac{1}{k^2}m_2 + \frac{k-1}{2k^2}\left(\sqrt{2(km_1 + m_2) + \left(k - \frac{1}{2}\right)^2} + k - \frac{1}{2}\right)$$

edges.

If Conjecture 1.3 holds, the hypergraph consisting of all edges and vertices of K_{kn+1} shows that the bound would be sharp. In this paper, we confirm the case when $k = 2$, as follows.

Theorem 1.4. *Every hypergraph $G = (V, E)$ with m_i edges of size i, $i = 1, 2$, admits a bipartition V_1, V_2 such that*

$$\mu(V_i) \leqslant \frac{m_1}{2} + \frac{m_2}{4} + \frac{1}{8}\left(\sqrt{4m_1 + 2m_2 + \frac{9}{4}} + \frac{3}{2}\right)$$

for $i = 1, 2$.

Remark. Let V_1, V_2 be a bipartition of a hypergraph G with m_i edges of size i, $i = 1, 2$. Let $d(V_i)$ denote the number of edges of G meeting V_i (*i.e.*, containing at least one vertex of V_i). Bollobás and Scott [7] conjectured that G has a bipartition V_1, V_2 such that

$$d(V_i) \geqslant \frac{m_1 - 1}{2} + \frac{2m_2}{3}$$

for $i = 1, 2$. Note that the bound is sharp for the hypergraph consisting of all edges and vertices of K_3. Recently, the conjecture has been proved by Haslegrave [13].

It is easy to see that $d(V_i) = m_1 + m_2 - \mu(V_{3-i})$ for $i = 1, 2$. By Theorem 1.4, we know that G admits a bipartition V_1, V_2 such that

$$d(V_i) \geqslant \frac{m_1}{2} + \frac{3}{4}m_2 - \frac{1}{8}\left(\sqrt{4m_1 + 2m_2 + \frac{9}{4}} + \frac{3}{2}\right)$$

for $i = 1, 2$, which gives a better bound of the above conjecture for G with $m_2 \geqslant 6$.

2. Bipartition of hypergraphs

In this section we prove Theorem 1.4. For convenience, let

$$\alpha := \sqrt{\frac{m_1}{2} + \frac{m_2}{4} + 3c},$$

where $c = 3/32$. It suffices to show that G admits a bipartition V_1, V_2 such that, for $i = 1, 2$,

$$\mu(V_i) \leqslant \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c. \tag{2.1}$$

Let V_1, V_2 be a partition of G maximizing $e(V_1, V_2)$, and subject to this, we assume that $|f(V_1) - f(V_2)|$ is minimal. Without loss of generality, suppose $\mu(V_1) \geqslant \mu(V_2)$. Subject to these, we may assume that $\mu(V_1)$ is minimal.

If

$$\mu(V_1) \leqslant \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c,$$

then we are done. Otherwise,

$$\mu(V_1) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c.$$

As mentioned in the Introduction, we have $e(V_1, V_2) \geqslant m_2/2$. Thus,

$$\begin{aligned}\mu(V_2) &= m_1 + m_2 - e(V_1, V_2) - \mu(V_1) \\ &< m_1 + m_2 - \frac{m_2}{2} - \left(\alpha^2 + \frac{\sqrt{2}}{4}\alpha - c\right) \\ &< \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c.\end{aligned}$$

In the following, we show that we may move some vertices from V_1 to V_2 to get a partition satisfying (2.1). Let W_2 be the maximal subset of V that satisfies the following conditions:

(i) $W_2 \supseteq V_2$, and

(ii) $\mu(W_2) \leqslant \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c$.

Let $W_1 = V \backslash W_2$. If $|W_1| \leqslant \sqrt{2}\alpha - 1/4$, then

$$e(W_1) \leqslant \binom{|W_1|}{2} \leqslant \alpha^2 - \frac{3\sqrt{2}}{4}\alpha + \frac{5}{32},$$

which together with $f(W_1) \leqslant |W_1|$ and $c = 3/32$ yields

$$\mu(W_1) = e(W_1) + f(W_1) \leqslant \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c.$$

This together with (ii) implies the required result.

Suppose that

$$|W_1| > \sqrt{2}\alpha - \frac{1}{4}. \tag{2.2}$$

In the following, we show that

$$\mu(W_1) \leqslant \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c.$$

By contradiction, assume that

$$\mu(W_1) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c. \tag{2.3}$$

By the choice of W_2, for each $w \in W_1$, we have

$$\mu(W_2 \cup \{w\}) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c. \tag{2.4}$$

Thus, by the fact that $\mu(W_2 \cup \{w\}) = \mu(W_2) + e(w, W_2) + \mathbf{1}_w$, we conclude that

$$\mu(W_2) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c - e(w, W_2) - \mathbf{1}_w. \tag{2.5}$$

Claim 2.1. *For each* $w \in W_1$,

$$e(w, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_w.$$

For convenience, let

$$\Theta := e(W_1, V_1 \backslash W_1) = \sum_{w \in W_1} e(w, V_1 \backslash W_1)$$

and

$$\Lambda := \sum_{w \in W_1} (e(w, V_2) - e(w, V_1)).$$

Note that

$$\begin{aligned} e(w, W_2) &= e(w, V_2) + e(w, V_1 \backslash W_1) \\ &= e(w, W_1) + 2e(w, V_1 \backslash W_1) + (e(w, V_2) - e(w, V_1)). \end{aligned}$$

Summing over all $w \in W_1$ yields that

$$e(W_1, W_2) = 2e(W_1) + 2\Theta + \Lambda. \tag{2.6}$$

Note that $m_1 = f(W_1) + f(W_2)$ and $m_2 = e(W_1) + e(W_1, W_2) + e(W_2)$. Adding $e(W_1) + 3f(W_1)$ to both sides of (2.6), we have

$$\begin{aligned} \mu(W_1) &= \frac{1}{3}(m_2 + 3f(W_1) - e(W_2) - 2\Theta - \Lambda) \\ &= \frac{1}{3}(4\alpha^2 - 12c + f(W_1) - f(W_2) - \mu(W_2) - 2\Theta - \Lambda), \end{aligned}$$

which, together with the fact that

$$\mu(W_1) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c,$$

https://doi.org/10.1017/S0963548316000274 Published online by Cambridge University Press

establishes

$$\mu(W_2) < \alpha^2 - \frac{3\sqrt{2}}{4}\alpha + f(W_1) - f(W_2) - 9c - 2\Theta - \Lambda. \tag{2.7}$$

Combining (2.5) and (2.7), we obtain

$$e(w, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_w + f(W_2) - f(W_1) + 2\Theta + \Lambda. \tag{2.8}$$

Case 1. $f(V_1) - f(V_2) \leqslant 0$. Since Θ and Λ are non-negative integers, it follows from the fact $f(W_2) - f(W_1) \geqslant f(V_2) - f(V_1) \geqslant 0$ that $e(w, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_w$, as desired.

Case 2. $f(V_1) - f(V_2) \geqslant 2$. For each $v \in V_1$, we have $e(v, V_1) \leqslant e(v, V_2)$ by the maximality of $e(V_1, V_2)$. We show that if $\{v\} \in E_1$, then

$$e(v, V_1) + 1 \leqslant e(v, V_2). \tag{2.9}$$

Otherwise, we have $e(v, V_1) = e(v, V_2)$. Let $V_1' = V_1 \backslash \{v\}$ and $V_2' = V_2 \cup \{v\}$. Note that

$$e(V_1', V_2') = e(V_1, V_2), \quad f(V_1') - f(V_2') = f(V_1) - f(V_2) - 2.$$

This together with the fact that $f(V_1) - f(V_2) \geqslant 2$ yields

$$|f(V_1') - f(V_2')| < |f(V_1) - f(V_2)|,$$

a contradiction to the minimality of $|f(V_1) - f(V_2)|$.

By the definition of Λ and inequality (2.9), we derive

$$\Lambda \geqslant \sum_{v \in W_1 \cap E_1} (e(v, V_2) - e(v, V_1)) \geqslant f(W_1).$$

This together with (2.8) yields that

$$e(w, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_w + f(W_2) + 2\Theta,$$

which implies the desired result.

Case 3. $f(V_1) - f(V_2) = 1$. Noting that $f(V_1) + f(V_2) = m_1$, we have $f(V_1) = (m_1 + 1)/2$. For convenience, let

$$\Omega := e(V_1, V_2) - 2e(V_1).$$

This implies

$$e(V_1, V_2) = 2\mu(V_1) - m_1 - 1 + \Omega.$$

Since $\mu(V_1) + \mu(V_2) + e(V_1, V_2) = m_1 + m_2$, we know that

$$3\mu(V_1) + \mu(V_2) = 2m_1 + m_2 + 1 - \Omega. \tag{2.10}$$

Write

$$\mu(V_1) = \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c + \eta, \tag{2.11}$$

where $\eta > 0$.

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Note that $f(W_2) - f(W_1) \geqslant f(V_2) - f(V_1) = -1$. By (2.8), we have

$$e(w, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_w - 1. \tag{2.12}$$

Furthermore, we may assume $f(W_2) - f(W_1) = -1$ and $\Theta = \Lambda = 0$, since otherwise we are done by (2.8). Let

$$\mathfrak{D} := \{u \in W_1 : \sqrt{2}\alpha + 8c - \mathbf{1}_u - 1 < e(u, W_2) \leqslant \sqrt{2}\alpha + 8c - \mathbf{1}_u\}.$$

It suffices to show that $\mathfrak{D} = \emptyset$. Otherwise, for each $u \in \mathfrak{D}$, let $V_1' = V_2 \cup \{u\}$ and $V_2' = V_1 \backslash \{u\}$. If we want to specify u explicitly, we will write $V_{i,u}'$ instead of V_i' for $i = 1, 2$. However, we drop the indices when they are not necessary.

It follows from the fact $\Theta = \Lambda = 0$ that $e(w, W_1) = e(w, W_2)$ for each $w \in W_1$. Thus, for each $u \in \mathfrak{D} \subset W_1$, $e(V_1', V_2') = e(V_1, V_2)$. Additionally, since $f(V_1) - f(V_2) = 1$, it follows that $|f(V_1') - f(V_2')| = |f(V_1) - f(V_2)|$. Note that $\mu(V_1') \geqslant \mu(V_2')$; otherwise,

$$\mu(V_1') < \mu(V_2') = \mu(V_1 \backslash \{u\}) < \mu(V_1),$$

which contradicts the minimality of $\mu(V_1)$. Thus, for some $\lambda \geqslant 0$, we may assume that

$$\mu(V_1') = \mu(V_1) + \lambda. \tag{2.13}$$

Proposition 2.2. $\Omega = \lambda = 0$ *and* $0 < \eta \leqslant 1/4$.

Otherwise, by the integrality of Ω and λ, we have $\Omega + \lambda + 4\eta > 1$. It follows from (2.13) that

$$\mu(V_1') = \mu(V_2) + e(u, V_2) + \mathbf{1}_u = \mu(V_1) + \lambda,$$

which implies

$$e(u, W_2) \geqslant e(u, V_2) = \mu(V_1) - \mu(V_2) + \lambda - \mathbf{1}_u.$$

This together with (2.10) and (2.11) yields

$$\begin{aligned} e(u, W_2) &\geqslant 4\mu(V_1) - 2m_1 - m_2 - 1 + \Omega + \lambda - \mathbf{1}_u \\ &= 4\left(\alpha^2 + \frac{\sqrt{2}}{4}\alpha - c + \eta\right) - 2m_1 - m_2 - 1 + \Omega + \lambda - \mathbf{1}_u \\ &= \sqrt{2}\alpha + 8c - \mathbf{1}_u + \Omega + \lambda + 4\eta - 1 \\ &> \sqrt{2}\alpha + 8c - \mathbf{1}_u. \end{aligned}$$

This contradicts the choice of u, completing the proof of Proposition 2.2.

The fact $\lambda = 0$ by Proposition 2.2 implies $\mu(V_1') = \mu(V_1)$ for each $u \in \mathfrak{D}$. Thus, we can move some vertices from V_1' to V_2' to get a partition W_1', W_2' of G such that W_2' is the maximal subset of V satisfying

(i) $W_2' \supseteq V_2'$, and

(ii) $\mu(W_2') \leqslant \alpha^2 + \dfrac{\sqrt{2}}{4}\alpha - c$.

Similarly, we let

$$\Theta' := e(W'_1, V'_1\backslash W'_1) = \sum_{w'\in W'_1} e(w', V'_1\backslash W'_1)$$

and

$$\Lambda' := \sum_{w'\in W'_1} (e(w', V'_2) - e(w', V'_1)).$$

Substituting V'_1, V'_2, W'_1, W'_2 for V_1, V_2, W_1, W_2, respectively, with a similar calculation as (2.8), for each $w' \in W'_1$, we deduce

$$e(w', W'_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_{w'} + f(W'_2) - f(W'_1) + 2\Theta' + \Lambda'. \tag{2.14}$$

Let

$$\theta' := e(u, V'_1\backslash W'_1).$$

Note that $u \in W'_1$ by the choice of W'_2. Thus, we have $\theta' = e(u, W'_2) - e(u, V_1)$. This together with (2.14) implies

$$e(u, V_1) > \sqrt{2}\alpha + 8c - \mathbf{1}_u + f(W'_2) - f(W'_1) + (2\Theta' - \theta') + \Lambda'. \tag{2.15}$$

Proposition 2.3. $V_1 = W_1 = \mathfrak{D} \subseteq E_1$.

First, we show $V_1 = W_1$, for otherwise, let $v_0 \in V_1\backslash W_1$. It follows from the fact $\Omega = 0$ by Proposition 2.2 that $e(v, V_1) = e(v, V_2)$ for each $v \in V_1$. Clearly, $V_1\backslash\{v_0\}, V_2 \cup \{v_0\}$ is a partition of G with

$$e(V_1\backslash\{v_0\}, V_2 \cup \{v_0\}) = e(V_1, V_2) \quad \text{and} \quad |f(V_1\backslash\{v_0\}) - f(V_2 \cup \{v_0\})| = |f(V_1) - f(V_2)|.$$

By the definition of W_1 and W_2, we know that $\mu(V_1\backslash\{v_0\}) > \mu(V_2 \cup \{v_0\})$. Clearly,

$$\mu(V_1\backslash\{v_0\}) < \mu(V_1),$$

which contradicts the minimality of $\mu(V_1)$.

Then, we prove $\mathfrak{D} \subseteq E_1$. Otherwise, there exists $u \in \mathfrak{D}\backslash E_1$. Thus,

$$f(V'_2) - f(V'_1) = f(V_1) - f(V_2) = 1.$$

It follows that

$$f(W'_2) - f(W'_1) \geqslant f(V'_2) - f(V'_1) = 1.$$

Note that $\Theta' \geqslant \theta'$ and $e(u, W_2) = e(u, W_1) = e(u, V_1)$. By (2.15), we deduce

$$e(u, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_u,$$

a contradiction to the choice of u.

Finally, we show $W_1 = \mathfrak{D}$. Suppose that there exists $w_0 \in W_1$ such that

$$e(w_0, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_{w_0}.$$

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It follows from $e(w_0, W_1) = e(w_0, W_2)$ that

$$|W_1| \geqslant e(w_0, W_1) + 1 > \sqrt{2}\alpha + 8c - \mathbf{1}_{w_0} + 1. \tag{2.16}$$

Since $e(x, W_2) > \sqrt{2}\alpha + 8c - 2$ for each $x \in W_1$ by (2.12) and $V_1 = W_1$, we have

$$\begin{aligned}\mu(V_1) &= \frac{1}{2}\sum_{x\in W_1} e(x, W_1) + f(W_1)\\ &> \frac{1}{2}\big((\sqrt{2}\alpha + 8c - 2)(|W_1| - 1) + \sqrt{2}\alpha + 8c - \mathbf{1}_{w_0}\big) + f(W_1)\\ &> \frac{1}{2}(\sqrt{2}\alpha + 8c - 1)(\sqrt{2}\alpha + 8c - \mathbf{1}_{w_0}) + f(W_1)\\ &= \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c - \frac{1}{2}(\sqrt{2}\alpha + 8c - 1)\cdot \mathbf{1}_{w_0} + f(W_1). \end{aligned} \tag{2.17}$$

The last equality holds since $c = 3/32$. If $\{w_0\} \notin E_1$, then we have

$$\mu(V_1) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c + f(W_1),$$

which contradicts $\eta \leqslant 1/4$ by Proposition 2.2. This means that $W_1\backslash\mathfrak{D} \subseteq E_1$, which together with $\mathfrak{D} \subseteq E_1$ implies $f(W_1) = |W_1|$. Combining (2.16) and (2.17), we derive

$$\mu(V_1) > \alpha^2 + \frac{\sqrt{2}}{4}\alpha - c + \frac{\sqrt{2}\alpha + 8c + 1}{2},$$

also a contradiction. Thus, we complete the proof of Proposition 2.3.

The fact $\mathfrak{D} \subseteq E_1$ implies

$$f(W_2') - f(W_1') \geqslant f(V_2') - f(V_1') = -1.$$

Note that $2\Theta' - \theta' \geqslant \Theta'$ and $e(u, W_2) = e(u, W_1) = e(u, V_1)$. By (2.15), we may assume

$$f(W_2') - f(W_1') = -1 \quad \text{and} \quad \Theta' = \theta' = \Lambda' = 0.$$

Otherwise, for each $u \in W_1 = \mathfrak{D}$, we have

$$e(u, W_2) > \sqrt{2}\alpha + 8c - \mathbf{1}_u,$$

a contradiction. Thus, by (2.14), we have $e(w', W_2') > \sqrt{2}\alpha + 8c - \mathbf{1}_{w'} - 1$ for each $w' \in W_1'$. Let

$$\mathfrak{D}' := \{u' \in W_1' : \sqrt{2}\alpha + 8c - \mathbf{1}_{u'} - 1 < e(u', W_2') \leqslant \sqrt{2}\alpha + 8c - \mathbf{1}_{u'}\}.$$

An argument similar to that used in Proposition 2.3 gives the following proposition, whose proof details are omitted.

Proposition 2.4. $V_1' = W_1' = \mathfrak{D}' \subseteq E_1$.

Now, we establish the next proposition by characterizing the hypergraph G according to Propositions 2.3 and 2.4.

Proposition 2.5. *G is the hypergraph consisting of all edges and vertices of K_{m_1}.*

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First, we show that $e(v, V_2) = (m_1 - 1)/2$ for each $v \in V_1$. It follows from Propositions 2.3 and 2.4 that $V_i \subset E_1$ for $i = 1, 2$. Suppose that there exists $v_2 \in V_2$ such that $v_2 \notin N(v)$, where $N(v)$ is the set of the neighbours of v in G. Clearly, there exists $v_1 \in V_1$ such that $v_1 \in N(v_2)$, since the cut (V_1, V_2) is maximal and G is connected. Note that, for each $v' \in V_1'$, we have $e(v', V_1') = e(v', V_2')$. Recall that $V_1' = V_{1,u}' = V_2 \cup \{u\}$ and $V_2' = V_{2,u}' = V_1 \backslash \{u\}$ for each $u \in \mathfrak{D} = V_1$. Substituting v for u, and noting that $v_2 \in V_{1,v}'$, we have $e(v_2, V_{1,v}') = e(v_2, V_{2,v}')$, that is, $e(v_2, V_2) = e(v_2, V_1)$. Similarly, substituting v_1 for u, we obtain $e(v_2, V_2) + 1 = e(v_2, V_1) - 1$, a contradiction.

Due to the above arguments, we know that each vertex in V_1 has $m_1 - 1$ neighbours in G and $e(v_2, V_2) + 1 = e(v_2, V_1) - 1$ for each $v_2 \in V_2$. Since v_2 is adjacent to each vertex in V_1, we have $e(v_2, V_1) = (m_1 + 1)/2$. With the help of the preceding two equalities, we conclude $e(v_2, V_2) = (m_1 - 3)/2$. This implies that each vertex of G has $m_1 - 1$ neighbours, completing the proof of Proposition 2.5.

By Proposition 2.5, we have $m_2 = \binom{m_1}{2}$. This implies

$$|W_1| = |V_1| = \frac{m_1 + 1}{2} = \sqrt{2}\alpha - \frac{1}{4}.$$

Recall that $|W_1| > \sqrt{2}\alpha - 1/4$ by (2.2); this leads to a contradiction. Thus, we conclude that $\mathfrak{D} = \emptyset$, completing the proof of Claim 2.1.

By Claim 2.1, for $w_0 \in W_1$, summing over all $w \in W_1 \backslash \{w_0\}$ gives that

$$\begin{aligned} e(W_1 \backslash \{w_0\}, W_2) &= \sum_{w \in W_1 \backslash \{w_0\}} e(w, W_2) \\ &> (\sqrt{2}\alpha + 8c)(|W_1| - 1) - f(W_1) + \mathbf{1}_{w_0}. \end{aligned}$$

This together with (2.3) and (2.4) yields

$$\begin{aligned} m_2 &= e(W_1) + e(W_2 \cup \{w_0\}) + e(W_1 \backslash \{w_0\}, W_2) \\ &= \mu(W_1) + \mu(W_2 \cup \{w_0\}) - m_1 - \mathbf{1}_{w_0} + e(W_1 \backslash \{w_0\}, W_2) \\ &> 2\alpha^2 + \frac{\sqrt{2}}{2}\alpha - 2c + (\sqrt{2}\alpha + 8c)(|W_1| - 1) - m_1 - f(W_1). \end{aligned}$$

Recall that $|W_1| > \sqrt{2}\alpha - 1/4$, $f(W_1) \leqslant m_1$ and $c = 3/32$. We have

$$\begin{aligned} m_2 &> 2\alpha^2 + \frac{\sqrt{2}}{2}\alpha - 2c + (\sqrt{2}\alpha + 8c)\left(\sqrt{2}\alpha - \frac{5}{4}\right) - 2m_1 \\ &= 4\alpha^2 - 2m_1 - 12c \\ &= m_2, \end{aligned}$$

a contradiction. This completes the proof of Theorem 1.4. □

3. Partitioning hypergraphs into k sets

In this section we aim to prove Theorem 1.2. Before proving the result, we should make a few definitions and lemmas.

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Let G be a hypergraph with m_i edges of size i for $i = 1, 2$, and let $\mathcal{P} := \{V_1, \dots, V_k\}$ be a k-partition of G. For each $i \in [k]$ and $v \in V_i$, we define

$$S^i_{\mathcal{P}}(v) := \{j \in [k]\backslash\{i\} : e(v, V_i) = e(v, V_j), v \in V_i\},$$

and

$$S^i_{\mathcal{P}} := \bigcup_{v \in V_i \cap E_1} S^i_{\mathcal{P}}(v).$$

Let $s^i_{\mathcal{P}}(v) := |S^i_{\mathcal{P}}(v)|$ and $s^i_{\mathcal{P}} := |S^i_{\mathcal{P}}|$. Clearly, for each $v \in V_i \cap E_1$, we have $0 \leqslant s^i_{\mathcal{P}}(v) \leqslant s^i_{\mathcal{P}} \leqslant k - 1$.

Furthermore, if $\mathcal{P}$ is a partition maximizing $e(V_1, \dots, V_k)$, then for each $j \in [k]\backslash\{i\}$ and $v \in V_i$, we have $e(v, V_i) + \mathbf{1}_j \leqslant e(v, V_j)$, where $\mathbf{1}_j = 1$ if and only if $j \notin S^i_{\mathcal{P}}(v)$. Note that

$$\sum_{j \in [k]\backslash\{i\}} \mathbf{1}_j = k - 1 - s^i_{\mathcal{P}}(v).$$

Thus, for each $v \in V_i$, we have

$$(k-1)e(v, V_i) + k - 1 - s^i_{\mathcal{P}}(v) \leqslant e(v, \overline{V_i}). \tag{3.1}$$

The following lemmas play important roles in our proof of Theorem 1.2.

Lemma 3.1. *Let G be a hypergraph with m_i edges of size i for $i = 1, 2$, and $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of G maximizing $e(V_1, \dots, V_k)$. Suppose $\mathcal{Q} = \{W_1, \dots, W_k\}$ is another partition of G with $W_i \subseteq V_i$ and $W_j \supseteq V_j$ for $j \in [k]\backslash\{i\}$. Then, for each $w \in W_i$,*

$$(k-1)e(w, W_i) + k - 1 - s^i_{\mathcal{Q}}(w) \leqslant e(w, \overline{W_i}).$$

Proof. Note that, for each $w \in W_i \subseteq V_i$, inequality (3.1) holds by substituting w for v. Thus, we have

$$e(w, \overline{W_i}) \geqslant e(w, \overline{V_i}) \geqslant (k-1)e(w, V_i) + k - 1 - s^i_{\mathcal{P}}(w).$$

It suffices to show that

$$(k-1)(e(w, W_i) - e(w, V_i)) \leqslant s^i_{\mathcal{Q}}(w) - s^i_{\mathcal{P}}(w). \tag{3.2}$$

Let $N(w)$ be the set of the neighbours of w in G. If $N(w) \cap (V_i \backslash W_i) = \emptyset$, then we have $e(w, W_i) = e(w, V_i)$ and $s^i_{\mathcal{Q}}(w) = s^i_{\mathcal{P}}(w)$. Otherwise, $e(w, W_i) \leqslant e(w, V_i) - 1$ and $s^i_{\mathcal{Q}}(w) = 0$. Note that $0 \leqslant s^i_{\mathcal{P}}(w) \leqslant k - 1$. In either case, inequality (3.2) holds, as desired. □

For each partition $\mathcal{P} = \{V_1, \dots, V_k\}$ of G, let $\boldsymbol{f}_{\mathcal{P}} = (f(V_1), \dots, f(V_k))$ be a vector with k coordinates. Write the Euclidean norm

$$\|\boldsymbol{f}_{\mathcal{P}}\| = \sqrt{\sum_{i=1}^{k} f(V_i)^2}.$$

The following lemma shows that $f(V_i)$ can be bounded by m_1 and $s^i_{\mathcal{P}}$ for each $i \in [k]$ under certain assumptions.

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Lemma 3.2. *Let G be a hypergraph with m_i edges of size i for $i = 1, 2$. Let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of G maximizing $e(V_1, \dots, V_k)$, and subject to this, assume that $\|\boldsymbol{f}_{\mathcal{P}}\|$ is minimal. Then, for each $i \in [k]$, we have*

$$f(V_i) \leqslant \frac{m_1 + s^i_{\mathcal{P}}}{1 + s^i_{\mathcal{P}}}.$$

Proof. It is trivial if $S^i_{\mathcal{P}} = \emptyset$. Assume that $S^i_{\mathcal{P}} \neq \emptyset$. Suppose that there exists $j \in S^i_{\mathcal{P}}$ such that $f(V_j) < f(V_i) - 1$. Let $v \in V_i \cap E_1$ be a vertex satisfying $e(v, V_i) = e(v, V_j)$. Moving v from V_i to V_j gives another partition $\mathcal{P}' = \{V'_1, \dots, V'_k\}$ with

$$e(V'_1, \dots, V'_k) = e(V_1, \dots, V_k) - e(v, V_j) + e(v, V_i) = e(V_1, \dots, V_k).$$

Meanwhile,

$$\begin{aligned}
\|\boldsymbol{f}_{\mathcal{P}'}\|^2 - \|\boldsymbol{f}_{\mathcal{P}}\|^2 &= f(V'_i)^2 + f(V'_j)^2 - f(V_i)^2 - f(V_j)^2 \\
&= (f(V_i) - 1)^2 + (f(V_j) + 1)^2 - f(V_i)^2 - f(V_j)^2 \\
&= 2(f(V_j) - f(V_i) + 1) \\
&< 0,
\end{aligned}$$

which contradicts the minimality of $\|\boldsymbol{f}_{\mathcal{P}}\|$. Thus, $f(V_j) \geqslant f(V_i) - 1$ for each $j \in S^i_{\mathcal{P}}$.

Note that $f(\overline{V_i}) \geqslant \sum_{j \in S^i_{\mathcal{P}}} f(V_j)$. We have

$$m_1 = f(V_i) + f(\overline{V_i}) \geqslant f(V_i) + s^i_{\mathcal{P}}(f(V_i) - 1),$$

which implies the desired result. □

Now, we are ready to prove Theorem 1.2 by showing the following result.

Theorem 3.3. *Every hypergraph G with m_i edges of size i, $i = 1, 2$, admits a k-partition $V_1, \dots, V_k$ such that*

$$\mu(V_i) \leqslant \frac{m_1}{k} + \frac{m_2}{k^2} + \frac{k-1}{2k^2}\left(\sqrt{2(km_1 + m_2) + \left(k - \frac{1}{2}\right)^2 - k} + 2k - \frac{1}{2}\right)$$

for $i = 1, \dots, k$.

Proof. For convenience, let

$$\alpha_k := \sqrt{\frac{m_1}{k} + \frac{m_2}{k^2} + \beta_k},$$

where

$$\beta_k := \frac{(2k-1)^2}{8k^2} - \frac{1}{2k}.$$

It suffices to show that G has a partition $V_1, \dots, V_k$ such that

$$\mu(V_i) \leqslant \alpha_k^2 + \frac{k-1}{2k}\sqrt{2}\alpha_k + c_k$$

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for $i = 1, \dots, k$, where

$$c_k := \frac{1}{2} - \frac{2k-1}{8k^2}.$$

Simple calculations show that

$$\begin{aligned}\alpha_{k-1}^2 &= \frac{k^2}{(k-1)^2}\alpha_k^2 - \frac{m_1}{(k-1)^2} - \frac{2k-3}{2(k-1)^2} \\ &\leqslant \frac{k^2}{(k-1)^2}\alpha_k^2 - \frac{m_1}{(k-1)^2} - \frac{1}{k-1} + \frac{2k-1}{(k-1)^2}c_k. \end{aligned} \tag{3.3}$$

The proof proceeds by induction on k. The result holds when $k = 2$ by Theorem 1.4. Assume that $k \geqslant 3$. Let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of G maximizing $e(V_1, \dots, V_k)$. Subject to this, we assume that $\|\boldsymbol{f}_{\mathcal{P}}\|$ is minimal. Without loss of generality, we may suppose that $\mu(V_1) = \max_{1 \leqslant i \leqslant k} \mu(V_i)$.

If

$$\mu(V_1) \leqslant \alpha_k^2 + \frac{k-1}{2k}\sqrt{2}\alpha_k + c_k,$$

we are done. Otherwise,

$$\mu(V_1) > \alpha_k^2 + \frac{k-1}{2k}\sqrt{2}\alpha_k + c_k. \tag{3.4}$$

Since there is no danger of confusion, the reference to 1 in the superscript of $s_{\mathcal{P}}^1(v)$ and $s_{\mathcal{P}}^1$ will be dropped in the following proof.

Claim 3.4. *The hypergraph G' induced by $\overline{V_1}$ admits a partition into $k-1$ classes, each of which contains at most*

$$\alpha_k^2 + \frac{k-1}{2k}\sqrt{2}\alpha_k + c_k$$

edges.

By induction hypothesis, G' admits a partition $X_2, \dots, X_k$ such that, for $i = 2, \dots, k$,

$$\mu(X_i) \leqslant \Lambda_1 + \frac{k-2}{2(k-1)}\sqrt{2\Lambda_1} + c_{k-1},$$

where

$$\Lambda_1 := \frac{f(\overline{V_1})}{k-1} + \frac{e(\overline{V_1})}{(k-1)^2} + \beta_{k-1}.$$

Thus, it suffices to prove that $\Lambda_1 < \alpha_k^2$.

Note that

$$(k-1)e(v, V_1) + (k-1-s_{\mathcal{P}}(v)) \cdot \mathbf{1}_v \leqslant e(v, \overline{V_1})$$

for each $v \in V_1$ by (3.1). Summing over all $v \in V_1$ yields

$$(k-1)(2e(V_1) + f(V_1)) - \sum_{v \in V_1 \cap E_1} s_{\mathcal{P}}(v) \leqslant e(V_1, \overline{V_1}).$$

Noting that

$$\sum_{v\in V_1\cap E_1} s_{\mathcal{P}}(v) \leqslant s_{\mathcal{P}} f(V_1),$$

we deduce

$$2(k-1)e(V_1) + (k-1-s_{\mathcal{P}})f(V_1) \leqslant e(V_1,\overline{V_1}).$$

This implies

$$e(\overline{V_1}) = m_2 - e(V_1,\overline{V_1}) - e(V_1) \leqslant m_2 - (2k-1)e(V_1) - (k-1-s_{\mathcal{P}})f(V_1).$$

Therefore,

$$\begin{aligned}
\Lambda_1 &= \frac{f(\overline{V_1})}{k-1} + \frac{e(\overline{V_1})}{(k-1)^2} + \beta_{k-1}\\
&\leqslant \frac{m_1 - f(V_1)}{k-1} + \frac{m_2 - (2k-1)e(V_1) - (k-1-s_{\mathcal{P}})f(V_1)}{(k-1)^2} + \beta_{k-1}\\
&= \frac{m_1}{k-1} + \frac{m_2}{(k-1)^2} + \beta_{k-1} - \frac{2k-1}{(k-1)^2}\mu(V_1) + \frac{1+s_{\mathcal{P}}}{(k-1)^2} f(V_1)\\
&< \alpha_{k-1}^2 - \frac{2k-1}{(k-1)^2}(\alpha_k^2 + c_k) + \frac{1+s_{\mathcal{P}}}{(k-1)^2} f(V_1) \quad \text{(by (3.4))}\\
&\leqslant \alpha_k^2 - \frac{1}{(k-1)^2}\big(m_1 + s_{\mathcal{P}} - (1+s_{\mathcal{P}})f(V_1)\big) - \frac{k-1-s_{\mathcal{P}}}{(k-1)^2} \quad \text{(by (3.3))}\\
&\leqslant \alpha_k^2.
\end{aligned}$$

The last inequality holds because $m_1 + s_{\mathcal{P}} - (1+s_{\mathcal{P}})f(V_1) \geqslant 0$ by Lemma 3.2 and $0 \leqslant s_{\mathcal{P}} \leqslant k-1$. This completes the proof of Claim 3.4.

In the following, we simply write α for α_k for convenience. By Claim 3.4, we can take $\overline{W_1} \supseteq \overline{V_1}$ maximal such that there exists a $(k-1)$-partition $W_2,\dots,W_k$ of $\overline{W_1}$ satisfying

$$\mu(W_i) \leqslant \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + c_k$$

for $i = 2,\dots,k$. Let $W_1 = V\backslash\overline{W_1}$. If

$$|W_1| \leqslant \sqrt{2}\alpha - \frac{1}{2k},$$

then

$$e(W_1) \leqslant \binom{|W_1|}{2} \leqslant \alpha^2 - \frac{k+1}{2k}\sqrt{2}\alpha + \frac{2k+1}{8k^2},$$

which together with the fact $f(W_1) \leqslant |W_1|$ implies

$$\mu(W_1) = e(W_1) + f(W_1) \leqslant \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha - \frac{2k-1}{8k^2}.$$

Thus we are done unless (3.5).

Suppose that

$$|W_1| > \sqrt{2}\alpha - \frac{1}{2k}. \tag{3.5}$$

By the choice of $\overline{W_1}$, it suffices to prove that

$$\mu(W_1) \leqslant \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + c_k.$$

By contradiction, assume that

$$\mu(W_1) > \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + c_k.$$

Claim 3.5. *For each $w \in W_1$,*

$$e(\overline{W_1} \cup \{w\}) > (k-1)^2(\alpha^2 - \beta_k) + \frac{k-1}{2k}\sqrt{2}\alpha + \gamma_k - (k-1)(f(\overline{W_1}) + \mathbf{1}_w),$$

where $\gamma_k = \beta_k + c_k$.

Suppose that there exists $w \in W_1$ such that

$$e(\overline{W_1} \cup \{w\}) \leqslant (k-1)^2(\alpha^2 - \beta_k) + \frac{k-1}{2k}\sqrt{2}\alpha + \gamma_k - (k-1)(f(\overline{W_1}) + \mathbf{1}_w). \tag{3.6}$$

Consider the hypergraph G'' induced by $\overline{W_1} \cup \{w\}$. Assume that G'' has m_i' edges of size i for $i = 1, 2$. We have $m_1' = f(\overline{W_1}) + \mathbf{1}_w$ and $m_2' = e(\overline{W_1} \cup \{w\})$.

By induction hypothesis, there is a $(k-1)$-partition $U_2, \ldots, U_k$ of G'' such that

$$\mu(U_i) \leqslant \Lambda_2 + \frac{k-2}{2(k-1)}\sqrt{2\Lambda_2} + c_{k-1}$$

for $i = 2, \ldots, k$, where

$$\Lambda_2 := \frac{m_1'}{k-1} + \frac{m_2'}{(k-1)^2} + \beta_{k-1}.$$

It follows from (3.6) that

$$\begin{aligned}\Lambda_2 &\leqslant \alpha^2 + \frac{1}{2k(k-1)}\sqrt{2}\alpha - (\beta_k - \beta_{k-1}) + \frac{\gamma_k}{(k-1)^2} \\ &= \alpha^2 + \frac{1}{2k(k-1)}\sqrt{2}\alpha + \frac{1}{8k^2(k-1)^2} \\ &= \left(\alpha + \frac{\sqrt{2}}{4k(k-1)}\right)^2.\end{aligned}$$

Therefore,

$$\begin{aligned}\mu(U_i) &\leqslant \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + \frac{1}{8k^2(k-1)^2} + \frac{k-2}{4k(k-1)^2} + c_{k-1} \\ &= \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + c_k,\end{aligned}$$

a contradiction to the choice of $\overline{W_1}$. This completes the proof of Claim 3.5.

Let $\mathcal{P}'' = \{V_1'', \ldots, V_k''\}$ be a partition of G with $V_1'' = W_1 \subseteq V_1$, $V_i'' \supseteq V_i$ for $i = 2, \ldots, k$. For each $w \in V_1'' = W_1$, it is easy to see that $0 \leqslant s_{\mathcal{P}''}(w) \leqslant s_{\mathcal{P}}(w) \leqslant k-1$, which yields

$$0 \leqslant s_{\mathcal{P}''} \leqslant s_{\mathcal{P}} \leqslant k-1. \tag{3.7}$$

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Moreover, by Lemma 3.1, we deduce

$$(k-1)e(w, W_1) + (k-1-s_{\mathcal{P}''}(w)) \cdot \mathbf{1}_w \leqslant e(w, \overline{W_1}). \tag{3.8}$$

Noting that $e(\overline{W_1} \cup \{w\}) = e(\overline{W_1}) + e(w, \overline{W_1})$, we have

$$e(w, \overline{W_1}) = e(\overline{W_1} \cup \{w\}) + f(\overline{W_1}) - \mu(\overline{W_1}). \tag{3.9}$$

Claim 3.6. *For each* $w \in W_1$,

$$e(w, \overline{W_1}) > (k-1)(\sqrt{2}\alpha - 1 - \boldsymbol{1}_w) + 2k\gamma_k.$$

Summing over all $w \in W_1$ in (3.8) yields

$$(k-1)(2e(W_1) + f(W_1)) - \sum_{w \in W_1 \cap E_1} s_{\mathcal{P}''}(w) \leqslant e(W_1, \overline{W_1}).$$

In view of

$$\sum_{w \in W_1 \cap E_1} s_{\mathcal{P}''}(w) \leqslant s_{\mathcal{P}''} f(W_1),$$

we deduce

$$(k-1)(2e(W_1) + f(W_1)) - s_{\mathcal{P}''} f(W_1) \leqslant e(W_1, \overline{W_1}). \tag{3.10}$$

Note that $m_1 = f(W_1) + f(\overline{W_1})$ and $m_2 = e(W_1) + e(W_1, \overline{W_1}) + e(\overline{W_1})$. Adding $e(W_1) + kf(W_1)$ to both sides of (3.10) gives

$$\mu(W_1) \leqslant \frac{1}{2k-1}\big(k^2(\alpha^2 - \beta_k) - \mu(\overline{W_1}) - (k-1)f(\overline{W_1}) + s_{\mathcal{P}''} f(W_1)\big).$$

Since

$$\mu(W_1) > \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + c_k,$$

we have

$$\mu(\overline{W_1}) < (k-1)^2(\alpha^2 - \beta_k) - (2k-1)\left(\frac{k-1}{2k}\sqrt{2}\alpha + \gamma_k\right) - (k-1)f(\overline{W_1}) + s_{\mathcal{P}''} f(W_1).$$

This, together with Claim 3.5 and (3.9), implies that

$$e(w, \overline{W_1}) > (k-1)(\sqrt{2}\alpha - \mathbf{1}_w) + 2k\gamma_k + f(\overline{W_1}) - s_{\mathcal{P}''} f(W_1).$$

Note that

$$f(\overline{W_1}) - s_{\mathcal{P}''} f(W_1) = m_1 - (1 + s_{\mathcal{P}''}) f(W_1).$$

Since $s_{\mathcal{P}''} \leqslant s_{\mathcal{P}}$ by (3.7) and $f(W_1) \leqslant f(V_1)$, we obtain

$$f(\overline{W_1}) - s_{\mathcal{P}''} f(W_1) \geqslant m_1 - (1 + s_{\mathcal{P}}) f(V_1),$$

which together with Lemma 3.2 yields

$$f(\overline{W_1}) - s_{\mathcal{P}''} f(W_1) \geqslant -s_{\mathcal{P}} \geqslant -(k-1).$$

Thus, we have

$$e(w, \overline{W_1}) > (k-1)(\sqrt{2}\alpha - 1 - \mathbf{1}_w) + 2k\gamma_k,$$

as desired. This completes the proof of Claim 3.6.

By Claim 3.5, for $w_0 \in W_1$, we have

$$e(\overline{W_1} \cup \{w_0\}) > (k-1)^2(\alpha^2 - \beta_k) + \frac{k-1}{2k}\sqrt{2}\alpha + \gamma_k - (k-1)(f(\overline{W_1}) + \mathbf{1}_{w_0}).$$

By Claim 3.6, for $w_0 \in W_1$, summing over all $w \in W_1 \backslash \{w_0\}$ gives that

$$\begin{aligned} e(W_1\backslash\{w_0\}, \overline{W_1}) &= \sum_{w \in W_1\backslash\{w_0\}} e(w, \overline{W_1}) \\ &> ((k-1)(\sqrt{2}\alpha - 1) + 2k\gamma_k)(|W_1| - 1) - (k-1)(f(W_1) - \mathbf{1}_{w_0}). \end{aligned}$$

Recall that

$$e(W_1) = \mu(W_1) - f(W_1) > \alpha^2 + \frac{k-1}{2k}\sqrt{2}\alpha + c_k - m_1.$$

These, together with (3.5), establish that

$$\begin{aligned} m_2 &= e(W_1) + e(\overline{W_1} \cup \{w_0\}) + e(W_1\backslash\{w_0\}, \overline{W_1}) \\ &> k^2\alpha^2 - km_1 - k^2\beta_k + \left(2k\gamma_k - \frac{4k^2 - 5k + 1}{2k}\right)\sqrt{2}\alpha - \delta_k + \frac{2k^2 - k - 1}{2k}, \end{aligned}$$

where $\delta_k := \beta_k + (2k-1)c_k$. The fact that

$$\beta_k = \frac{(2k-1)^2}{8k^2} - \frac{1}{2k} \quad \text{and} \quad c_k = \frac{1}{2} - \frac{2k-1}{8k^2}$$

shows that

$$2k\gamma_k = \frac{4k^2 - 5k + 1}{2k} \quad \text{and} \quad \delta_k = \frac{2k^2 - k - 1}{2k}.$$

This implies that

$$m_2 > k^2\alpha^2 - km_1 - k^2\beta_k = m_2,$$

a contradiction. Thus, we complete the proof of Theorem 3.3. □

Acknowledgements

We would like to thank the anonymous referees for their helpful comments and suggestions. In particular, we are grateful to a referee for pointing out two serious gaps in the proofs of Theorems 1.4 and 3.3.

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