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Dependability of Position Solutions in Celestial Sight-Run-Sight Cases – Part 1

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The sight-run-sight case involves a series of two or more sights in which each sight is 'run on' to the time of the last sight. There are several known techniques to account for the run, which is for convenience assumed to be at a constant course. It is argued that only the GHA-Dec updating technique (GD-UT) will give a correct position solution, whereas the pre-electronic Intercept Method (IM)-based technique and its modern least-squares variant (LSQ) and the Altitude updating technique (A-UT), are proxy techniques that may yield significant deviations in position solution. GD-UT can be used in combination with a double-sight solution method for two sights and with LSQ for two or more sights. IM, LSQ and a geometric or algebraic double sight solution method applied to simultaneous double sights give identical results but the simultaneous sights case is generally an abstraction and strictly not applicable on a moving vessel.

KEY WORD

1. Astronomical running fixes.

1. POSITION SOLUTIONS IN GENERAL. A fix (observed position) with two sights is determined by the intersection of their position circles. The most common methods to determine the relevant intersection point are the Intercept Method (IM), its modern variant the least-squares method (LSQ) and the double-sight or two-body method. The latter method is either geometric or algebraic. In this article we will use the algebraic or K-Z method[1]. When applied to two or more 'simultaneous' sights all of the above methods obtain identical results[2] in terms of the intersection point for each combination or pair of two sights. In terms of the resulting 'cocked hat' n-polygon ($n \geqslant 3$), with simultaneous sights all methods produce identical vertex coordinates. Of the above methods, only LSQ computes a fix for the n-polygon and much of the subsequent discussion in this paper deals with the interpretation of the triangulation implied by the cocked hat n-polygon.

Critical to the dependability of position solutions in the sight-run-sight case, which involves a series of two or more sights, is the manner in which the position circle of an earlier sight is transferred for the run to the time of the last sight in the series. The sight-run-sight problem is often simply sidestepped by assuming that all sights are simultaneous[3]. This is generally an inadmissible abstraction when a moving vessel is involved, even if the run is short.

Table 1. GD-UT illustrated with actual data[4].

d = 31′·5602; α = 79°·5949; Dec = 19°·8730; GHA = 15°·2600. The GP is denoted X and the transferred GP as X*; all transferred variables are indicated with *. From ΔXPX* (∠PXX* = α; ∠XPX* = β):

Cosα	= [SinDec*-SinDecCos(d/60)]/CosDecSin(d/60)
SinDec*	= CosαCosDecSin(d/60) + SinDecCos(d/60) = 0·3415
Dec*	= 19°·9672 (Cosine Formula)
Dec*	= Dec + (d/60)Cosα = 19·9680 (rhumbline equation)
Cosβ	= [Cos(d/60)-SinDecSinDec*]/CosDecCosDec* = 0·99995385
β	= 0·5504, or with Sine Rule:
Sinβ	= (Sin(d/60)Sinα/CosDec* = 0·0096
β	= 0·5504
GHA*	= GHA-β = 14·7096
GHA*	= GHA – (d/60)Sinα/Cos[$\frac{1}{2}$(Dec + Dec*)] = 14·7097 (with rhumbline equation)

The differences in Dec* and GHA* obtained with either the rhumbline formulas or the fundamental formula (Cosine Formula) are negligible and either method may be used.

There is actually only one correct transfer technique, which is the GHA ~ Dec updating technique (GD-UT) but several other techniques have been used. As will be argued in Sections 3 and 4, the latter transfer techniques are at best only 'proxy' methods, which may cause significant deviations in position solution, depending on the interaction of such factors as azimuth, zenith distance (Zd), magnitude and direction of the run displacement. The dependability of position solutions in the sight-run-sight case has in fact two main aspects:

- Application of the correct method for transferring the position circle of the earlier sight for the run data
- The error margin of the fix and the error sources

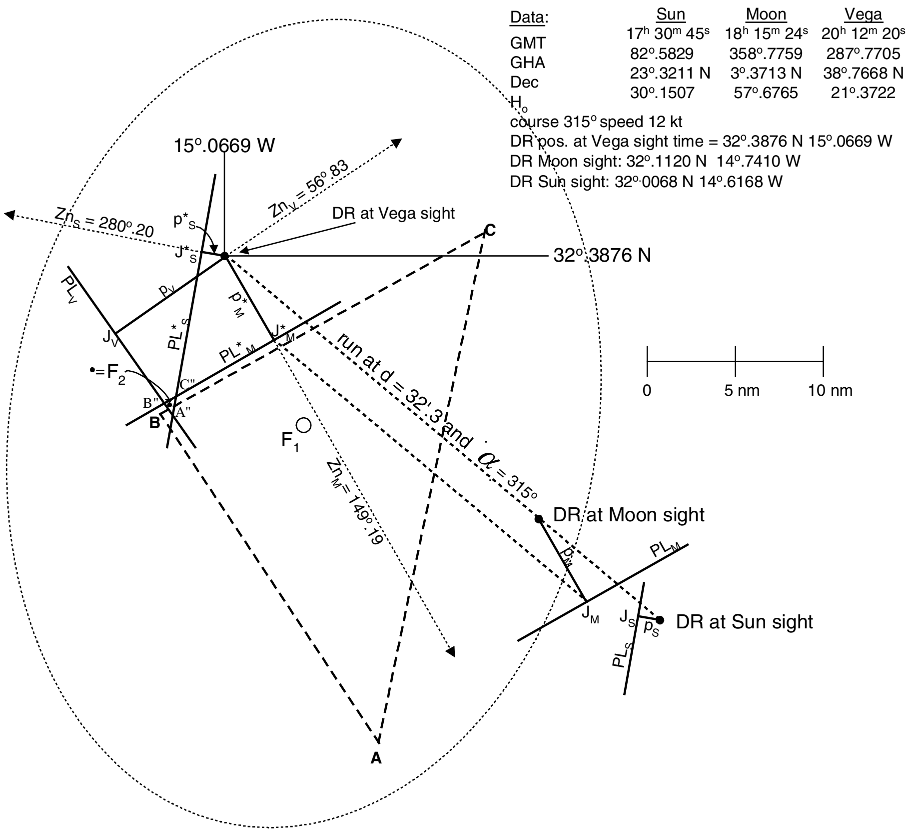
This part of the paper addresses the first aspect. The error issue is discussed in Part 2 which will appear in the May edition of *The Journal*.

2. THE GHA-DEC UPDATING TECHNIQUE (GD-UT).

With GD-UT, the geographic position (GP) of the earlier sight is displaced for the distance (d) and true course bearing (α) of the run, i.e. by 'updating' the GHA and Dec of the earlier sight for the run's displacement (see Table 1). In all examples it is for simplicity assumed that the run between sights is at a constant bearing (α). Of course, this assumption needs to and can be relaxed in practical applications.

The GD-UT method was of course known in the pre-electronic era but could effectively be applied only if the radii (zenith distances) of the intersecting position circles were short enough to be drawn on the chart, as most pre-electronic celestial navigation methods had to rely heavily on chart plotting[5].

When applied to two sights or to the cocked hat vertex combinations of three or more sights, GD-UT can be used with two methods, with K-Z (i.e. GD-UT + K-Z) or with LSQ (i.e. LSQ + GD-UT). Both methods will give identical vertex coordinates. As already explained, with either method the GHA and Dec of earlier sights, Sun and Moon in the example of Figure 1[6], are updated for the run with GD-UT to the time of the last sight. With three or more sights, only LSQ + GD-UT will obtain a fix.



	Sun	Moon
GHA* (run-on)	82°.9982	359°.0520
Dec* (run-on)	23°.7020 N	3°.6469 N

IM results:

	Sun	Moon	Vega
Zn	280°.20	149°.19	56°.83
Intercept (p)	p_S =1'.25	p_M = 5'.47	p_V = -7'.49

Vertex coordinates: (with vertex program)
Sun-Vega (A") = 32°.2342 N 15°.1246 W
Moon-Vega (B") = 32°.2474 N 15°.1348 W
Sun-Moon (C") = 32°.2547 N 15°.1202 W

LSQ results:

	Sun	Moon	Vega
Zn	280°.25	148°.97	56°.76
e	-0'.33	-0'.23	-0'.25

• Fix F_2 = 32°.2459 N 15°.1278 W

Confidence ellipse: θ = 10°; a = 1'.1; b = 0'.8

GD-UT + K-Z results:

Vertex coordinates:
Sun-Vega (A) = 31°.9252 N 14°.8854 W
Moon-Vega (B) = 32°.2387 N 15°.1280 W
Sun-Moon (C) = 32°.4151 N 14°.7772 W

LSQ + GD-UT results:

	Sun	Moon	Vega
Zn	280°.50	149°.12	56°.81
e	-7'.6	-5.'.3	-5.'.7

○ Fix F_1 = 32°.2143 N 14°.9742 W

Confidence ellipse: θ = 10°; a =25'.7; b = 18'.7

Figure 1. Position solutions from multiple sights with different methods. (Yallop-Hohenkerk Sun-run-Moon-run-Vega case)

3. SIGHT-RUN-SIGHT POSITION SOLUTIONS WITH IM/LSQ. With IM, the position line (PL) of an earlier sight in a series of two or more sights is transferred for the run as shown in Figure 1 for PL_{Moon} and PL_{Sun}. On the chart this was done with the parallel ruler[7], so that the transferred PLs and intercepts become PL^*_{Moon}, PL^*_{Sun}, with intercepts p^*_{Moon} and p^*_{Sun}. The PL of the last sight (PL_{Vega}) and the two transferred PL*s form the cocked hat triangle A″B″C″. The assumptions on which the IM rests[8], define the quadrangle

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DR_{Moon}-J_M -J^*_M -DR_{Vega} for the Moon sight as a plane parallelogram; similarly with DR_{Sun} -J_S -J^*_S -DR_{Vega} for the Sun sight.

LSQ replicates the 'parallel-ruler' method[9]. The first iteration with LSQ, using an initial assumed (DR) position defines the cocked hat triangle that an electronic version of IM would obtain, e.g. A″B″C″ in Figure 1. Thus, the cocked hat obtainable with traditional IM is generally identical to the cocked hat triangle obtained with LSQ. Additional iterations define the point (fix F_2) inside the cocked hat where the PLs would intersect if there were no observation errors so that intercepts like p^*_{Sun}, p^*_{Moon}, p_{Vega}, respectively 1′·25, 5′·47 and −7′·49, would only reflect corrections for the (incorrect) assumed (DR) position. The cocked hat therefore reflects observation errors only and the intercepts from F_2 may be called 'error intercepts' (**e**), which are respectively −0′·31, −0′·23 and −0′·25. Similar error intercepts are of course computed with LSQ + GD-UT. For the cocked hat triangle ABC in Figure 1 they are respectively −7′·6, −5′·3 and −5′·7.

The fundamental assumption underlying the IM/LSQ transfer technique and also the Altitude-updating technique (A-UT; see Section 4), is that the displacement of the earlier position circle's GP according to d and α for all practical purposes translates in a displacement of the same magnitude and direction at the observer position on the position circle's circumference, like at J_M, J_S. The assumption that DR_{Moon}-J_M -J^*_M -DR_{Vega} forms a plane parallelogram is in fact based on the more fundamental assumption that spherical quadrangles like XX*J*J and XX*Z*Z (see Figure 2) can be treated as plane parallelograms too. The great circles through respectively X and Z and X* and Z* are what may be called 'parallel' great circles in spherical trigonometry: they are not parallel[10]. Treating XX*Z*Z as a parallelogram has therefore no general validity and it is in fact the chief cause of the deviations mentioned earlier.

This can be shown in various ways. One way is to compare actual position solutions obtained with the various techniques, as we will do later. It can also be shown that LSQ does not obtain intermediate results that are mathematically equivalent to those obtained with GD-UT + K-Z. For instance, the Zn (true calculated azimuth) of the sight transferred with LSQ defines GHA and Dec values that differ from the GHA and Dec values updated with GD-UT + K-Z. A third way is using a "due North" model where α is 360° (due North) and the observer's position Z is predetermined for a given azimuth. If the GP is at X, the displaced GP at X* and the displaced observer position at Z*, XX* may significantly differ from ZZ*, depending on the combination of azimuth, H_o and run distance. It also follows indirectly from this model that α at X is unequal to α′ at Z. If the parallel-ruler analogy would hold, this should not happen, i.e. XX* ≈ ZZ* and α ≈ α′.

A final point is, although the transfer concept underlying IM/LSQ and A-UT is identical, they may produce significantly different position solutions and intermediate results, which should not happen if they are based on a common theory. The difference between fix F_1 obtained with LSQ + GD-UT and fix F_2 with IM/LSQ indicates the deviation introduced with the latter methods. In the case of Figure 1 the deviation in absolute terms is:

difference in d'Lat = 1′·9
difference in Dep = 7′·8

Similar deviations in position solution can be demonstrated by using two sights only. This is shown for a number of cases analyzed in Table 2. In cases (1) and (3) the

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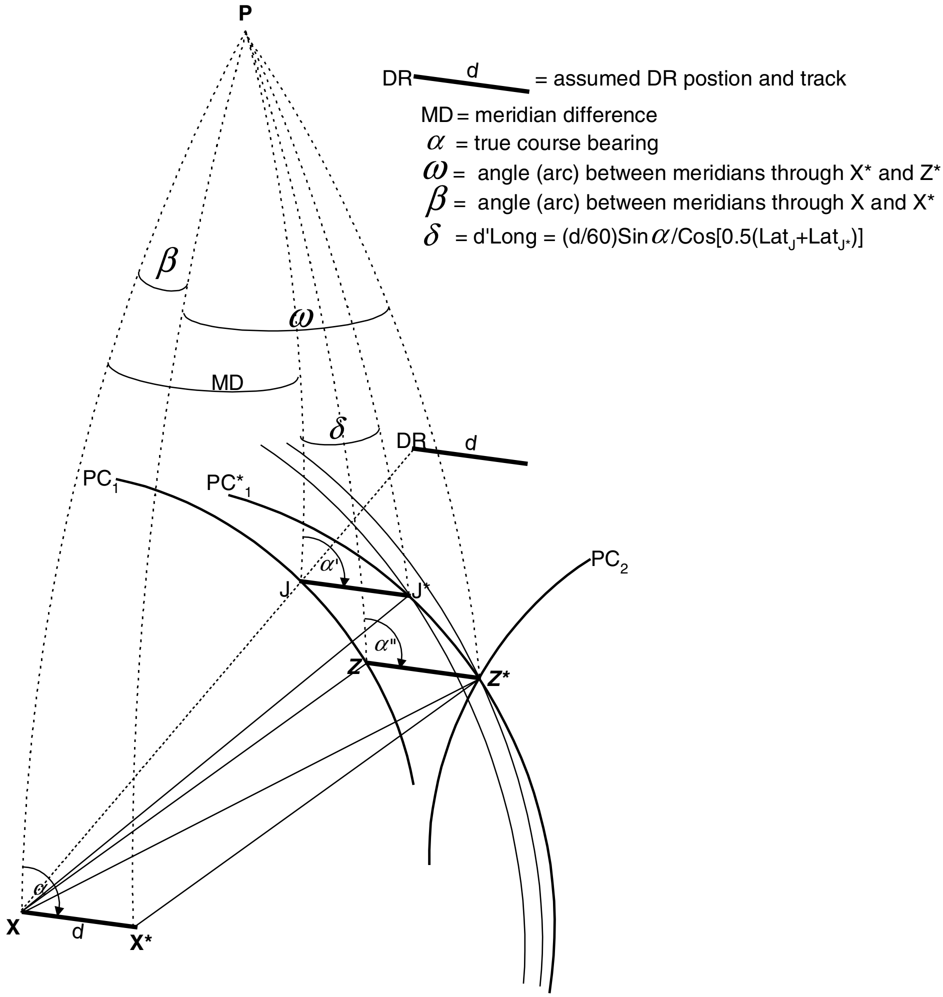

Figure 2. A-UT and GD-UT in the sight-run-sight case.

deviations are very significant, in cases (2), (4) and (5) less significant. The deviations in all cases are caused by the fact that LSQ depends for its solution model on the 'parallel-ruler' analogy to represent the run data and therefore gives unreliable results.

4. GD-UT VERSUS A-UT. The A-UT method was already mentioned and characterized as a proxy method. The rationale of A-UT is illustrated with the sketch in Figure 2. The position estimated from the DR would be in J, transferred as a rhumbline to J*. The actual position is in Z*, the intersection of PC^*_1 (transferred earlier sight's PC) and PC_2. In plane geometry the points J* and Z* would lie on the transferred circle. In spherical trigonometry this 'parallel-ruler' analogy does not hold, i.e. the points J* and Z* would lie on the transferred position circle but the distances JJ* and ZZ* and the bearings α′ at J and α″ at Z would differ from respectively d and α.

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Table 2. Deviations in position solution with LSQ+GD-UT and LSQ.

	Sun-run-Moon (Y-H) (1)		Moon-run-Vega (Y-H) (2)		Sun-run-Vega (Y-H) (3)		Moon-run-Sun (ANM) (4)		Sun-run-Sun (G.Keys) (5)	
	Sun	Moon	Moon	Vega	Sun	Vega	Moon	Sun	Sun	Sun
GHA	82·583	358·776	358·776	287·771	82·583	287·771	7·995	309·178	15·260	89·500
Dec	23·321	3·371	3·371	38·767	23·321	38·767	−22·040	5·205	19·873	19·832
H_o	30·151	57·677	57·677	21·372	30·151	21·372	17·412	19·945	62·455	22·972
GHA*	82·698	—	359·052	—	82·998	—	7·453	—	14·710	—
Dec*	23·426	—	3·647	—	23·702	—	−21·857	—	19·968	—
d	8′·9		23′·5		32′·3		32′·1		31′·6	
α	315°		315°		315°		70°		79°·6	
Initial	32°·0068 N		32°·1120 N		32°·0068 N		50°·1667 N		47°·3300 N	
DR	14°·6168 W		14°·7410 W		14°·6168 W		14°·8333 W		13°·1170 W	
LSQ+GD-UT:										
Fix	32·0269/−14·6995		32·2387/−15·1280		31·9252/−14·8854		50·5117/−13·8323		47·4402/−12·2474	
Zn	280·310	149·189	148·880	56·761	280·709	56·743	173·796	106·776	185·009	274·771
LSQ:										
Fix	31·9789/−14·7940		32·2469/−15·1344		32·2334/−15·1239		50·4930/−13·8412		47·4494/−12·2462	
Zn	280·249	148·991	148·960	56·761	280·259	56·7603	173·556	106·762	184·578	274·768
Dev	d'Lat 2′·9		d'Lat 0′·5		d'Lat 18′·5		d'Lat 1′·1		d'Lat 0′·6	
	Dep 4′·8		Dep 0′·3		Dep 12′·1		Dep 0′·3		Dep 0′·1	

Notes to Table 2:

– The data for the cases marked "Y-H" are taken from Yallop Hohenkerk, op.cit. The ANM case data are from the ANM, op. cit., p 191–195; the G. Keys' case is from op.cit., p 140.

– The initial DR Position is given for the first sight in all cases.

– The d'Lat and Dep deviations are the absolute deviation in the fixes

GD-UT as explained finds the locus of the transferred position circle by moving X (=GP) to X* for the magnitude and direction of the displacement (d and α) and by projecting the circle from X* with its given radius ($Zd=90°-H_o=XZ=X^*Z^*$). With A-UT this logic is for some reason sidestepped. The point J* is seen as the intersection of the transferred position circle (PC^*_1) and a concentric circle with radius XJ* drawn from X. The reasoning appears to be that if J* and Z* are close enough, the distances XJ* and XZ* would be about the same, so that J* and Z* would lie approximately on a concentric circle with centre X and radius XJ*. As mentioned, A-UT also subscribes to the parallel-ruler analogy, so that it is assumed that $JJ^*=ZZ^*=d$ and $\alpha'=\alpha''=\alpha$.

There are to my knowledge two versions of A-UT in the literature. One version is A-UT as applied by G. Keys[11]. In this version, computed first is MD, with Dec_1, $H_{o,1}$ and Lat_J (see Fig 2). The radius XJ* ($=90°-H^*_{o,1}$) is computed next, from $MD\pm\delta$, Dec_1 and Lat_{J^*}[12]. In applying a double-sight solution method, $H^*_{o,1}$ is then substituted for $H_{o,1}$.

G. G Bennett uses another version of A-UT[13]. Assuming that the approximate true azimuth at J is known from observation or from a star finder, Bennett's 'short-cut' formula for finding the surrogate altitude ($H^*_{o,1}=90°$ - XJ*) is $H^*_{o,1}=H_{o,1}+(d/60)\mathrm{Cos}(Zn-\alpha)$. For short run-distances this formula yields practically the same result as the cosine formula applied to the spherical triangle XJJ* (see Figure 2)[14]. The technique may have been used in Surveyors' circles. It is for example not found in the Admiralty Navigation Manual or in Bowditch[15].

Taking as observed azimuth (Zn) 55° as implied by Bennett's data and analyzing the data with the A-UT versions and GD-UT gives the results in Table 3.

The difference in position solution obtained with the two A-UT versions compared to GD-UT is negligible in the case of Bennett's version and 5′·0 in d'Lat and 1′·1 in Dep in respect of Keys' version. The A-UT methods also do not obtain the same position solutions and in this example differ in d'Lat by 5′·0 and in Dep by 1′·2. The A-UT versions are evidently not based on an identical interpretation of the properties of the triangle PXJ* in Figure 2.

Keys' version requires an assumed initial (DR) position. Bennett's A-UT version is seemingly free of assumed position but the catch is that MD, Lat_J and $Long_J$ (or Lat_{DR} and $Long_{DR}$, i.e. one's initial DR position) are automatically determined once Zn is established. Only GD-UT is really free from the assumed-position odium.

Bennett subsequently revised the A-UT version mentioned above, apparently with the aim to account for long run distances and to allay criticism that the run being a rhumbline distance is being equated with a great-circle distance[16]. Again, the refinements with the revised version do not change the A-UT rationale, nor do they significantly affect the position solution from one A-UT version to another. Nevertheless, the analysis of Bennett's numerical example of the long run distance shows that the position solution with A-UT is significantly different from the position solution obtained with GD-UT. The data of this case are:

Lat_{DR}–$Long_{DR}$ at 1st sight: 54° N/46° W; Course: 205°; Speed: 20 kt
Time difference between sights: 5.1100 hrs

	Body 1		Body 2
GHA_1	39·2933	GHA_2	80·6400
Dec_1	26·7367	Dec_2	51·4900
$H_{o,1}$	62·4083	$H_{o,2}$	69·4117

Table 3. Comparison of A-UT and GD-UT results with Bennett's data.

	A-UT + K-Z method G. G Bennett		A-UT + K-Z method G.Keys		GD-UT + K-Z	
			Sun	Moon		
	GHA		140·3100	263·8083		
	Dec		22·7333	−2·5967		
	H$_o$		10·2850	26·3333		
Lat$_{Fix}$	29·7187 S		29·6349 S		29·7180 S	
Long$_{Fix}$	157·1522 E		157·1743 E		157·1523 E	
Difference (1)	negligible		d'Lat 5′·0 Dep 1′·1			
Difference (2)	d'Lat 5′·0 Dep 1′·2					
	Sun	Moon	Sun	Moon	Sun	Moon
Z	56·2672	77·0368	56·2669	77·0903	56·2585	77·0372
Zn	56·2672	282·9632	56·2669	282·9097	56·2585	282·9628
LHA	297·4622	60·9605	297·4843	60·9827	297·5037	60·9607
	time diff (hr) = 0·3672; d = 2′·5706; α = 243°					
	Zn	55·00	MD	60·9125	d'Lat	−1′·17
	Zn-α	−188·00	CosZ	0·5736	Dep	−2′·29
	dCos(Zn-α)	−2′·5455	Z	55·0000	Dec$_{Sun}$	22·7333
	H*$_{o,1}$	**10·2426**	SinH*$_{o,1}$	0·1789	Dec*$_{Sun}$	22·7139
			H*$_{o,1}$	**10·3051**	GHA$_{Sun}$	140·3100
			Lat$_{DR1}$	−31·6874	GHA*$_{Sun}$	140·3513
			Long$_{DR1}$	158·7775		
			Lat$_{DR2}$	−31·7068		
			Long$_{DR2}$	158·7326		
			dCosα	−1′·17		
			d'Long$_{DR}$	−2′·69		

Note to Table 3: (1) Abs. difference with GD-UT results; (2) Abs. difference between A-UT (Bennett) and A-UT (Keys)

The position solution obtained with A-UT (Bennett's Method A) is: Lat$_{fix}$ 52·3733 N; Long$_{fix}$ 46·9750 W. The application of GD-UT + K-Z to this case proceeds as follows:

	time diff	distance	course	d'Lat	d'Long	Initial Dec	New Dec	Initial GHA	New GHA
Body 1	5·1100	102′·2000	205	−1·5437	−1·8007	26·7367	25·1929	39·2933	40·0940

	Body 1		Body 2
GHA*$_1$	40·0940	GHA$_2$	80·6400
Dec*$_1$	25·1929	Dec$_2$	51·4900
H$_{o,1}$	62·4083	H$_{o,2}$	69·4117
Lat1(selected)	52·2829		
Lat2	32·2494		
Long1(selected)	47·0046 W		
Long2	70·6325 W		
CosZ	−0·9720		0·1951
Z	166·4044		78·7467
Zn	166·4044		281·2533
LHA	353·0894		33·6354
	LHA > 180		LHA < 180
	Zn = Z		Zn = 360-Z

When comparing the respective position solutions it is seen that the (absolute) difference in Lat_{fix} between A-UT and GD-UT is 5′·4, in $Long_{fix}$ 1′·8. This is another example of the fact that A-UT (Bennett's methods in this case) may give comparable results in some instances, but not in others. In this particular case, the run distance is 102′·2, while in the case analyzed in Table 2, the run distance is only 2′·6. In this latter instance, the position solution with A-UT and GD-UT is virtually the same. It therefore appears that run distance can be a contributory factor in causing a difference in position solution. Further noted is, Bennett's revised A-UT versions simply require an initial assumed (DR) position, just like Keys' A-UT method. Also, azimuth (Zn) used in the formulas is calculated and no longer obtained from an approximate bearing or a star finder as with Bennett's 1979 "standard method".

Table 4 compares the results from four double sight cases analyzed with A-UT and GD-UT. As may be seen, a large difference in position solution between A-UT and GD-UT as in case (1) tends to correspond to a large difference between XZ* and XJ* (Figure 2). In case (1) the difference in position is d'Lat 18′·5/Dep 12′·1 and the difference between XZ* and XJ* is 56′·7. In the other three cases, relatively small differences in |XZ*-XJ*| correspond to small differences in position solution. The differences in position solution are in general too significant to be ignored and no case can be made for the general validity of A-UT.

5. CONCLUSIONS.

The various methods that may be used for finding the relevant point of intersection of two position circles, the Intercept Method (IM), LSQ and the double sight solution methods, all give identical results in the simultaneous sights' case. This also applies to the cocked hat n-polygons ($n \geqslant 3$) that are obtained with these methods when more than two sights are involved: the coordinates obtained for the vertices of such polygons will be identical.

In the sight-run-sight case, however, the Lat and Long coordinates of a fix obtained with two sights or of the polygon's vertices in the case of three or more sights can be significantly influenced by the methods employed to account for the run between sights. It is argued that the correct method (GHA-Dec Updating Technique; GD-UT) for transferring the position circle of an earlier sight is to transfer the coordinates of its GP (GHA and Dec) for the run data, i.e. distance (d) and course (α). The effect is that an observer's position on this position circle will not be transferred according to the distance d and course α as is implied by other methods used to account for the run between sights, such as the transfer method with IM/LSQ and the Altitude Updating Technique (A-UT). In all sight-run-sight situations the correct vertex coordinates are found by updating the GHA and Dec of each sight for the run and to the time when the last sight in the series was taken. The recommended approach in the multiple sight-run-sight situation is therefore to first adjust the GHA and Dec of earlier sights for the run and then apply LSQ to the adjusted data (GHA*, Dec* and H_o) by bypassing LSQ's run-subroutine.

Part 2 of this paper which will follow in the next issue will examine the errors and error margins as another aspect of position solution dependability.

Table 4. Comparison of results obtained with A-UT and GD-UT for four double sight cases.

	(1)		(2)		(3)		(4)	
	Sun	Vega	Sun	Sun	Moon	Sun	Moon	Vega
GHA	82·5829	287·7705	15·2600	89·5000	7·9950	309·1783	358·7759	287·7705
Dec	23·3211	38·7668	19·8730	19·8320	−22·0400	5·2050	3·3713	38·7668
H_o	30·1507	21·3722	62·4550	22·9720	17·4117	19·9450	57·6765	21·3722
d	32′·3		31′·6		32′·1		18′·7	
α	315°		79°·6		70°		315°	
Initial DR	32·0068		47·3200		50·1667		32·1120	
(assumed)	−14·6168		−13·1170		−14·8333		14·7410	
	A-UT (Keys)	**A-UT** (Bennett)	**A-UT** (Keys)		**A-UT** (Bennett)		**A-UT** (Bennett)	
H^*_o	30·5918	30·5946	62·3130		17·2914		57·2983	
XJ*	59·4082	59·4054	27·6870		72·7086		32·7017	
Lat_{Fix}	32·2329	32·2363	47·4513		50·4842		32·2467	
$Long_{Fix}$	−15·1235	−15·1262	−12·2460		−13·8454		−15·1342	
Zn (approx.)		280°·5			173°		149°	
Zn-α		−34°·5			103°		−166°	
dCos(Zn-α)		26′·6335			−7′·2172		−22′·6920	
	GD-UT		**GD-UT**		**GD-UT**		**GD-UT**	
GHA*	82·9982		14·7096		7·4533		359·0520	
Dec*	23·7020		19·9672		−21·8571		3·6469	
ω	68·1128		2·4623		6·3791		16·0760	
β	0·4153		0·5503		0·5417		0·2761	
XZ*	60·3539		27·6758		72·7351		32·6922	
Lat_{Fix}	31·9252		47·4402		50·5117		32·2387	
$Long_{Fix}$	−14·8854		−12·2474		−13·8323		−15·1280	
|XZ* − XJ*|	56′·7		0′·7		1′·6		0′·6	
Diff d'Lat	18′·5	18′·7	0′·7		1′·7		0′·5	
Diff Dep	12′·1	12′·2	0′·1		0′·5		0′·3	

Data source: (1) – Yallop-Hohenkerk; (2) G.Keys; (3) ANM; (4) Yallop-Hohenkerk

REFERENCES

[1] The geometric solution is elegant if it is based on what may be called the "cosine method", using $Cos(GHA_1 - GHA_2)$ as argument, which is equal to $Cos(GHA_2 - GHA_1)$. An example is the derivation by Gerry Keys – "Practical Navigation by Calculator", 1982, p 135–138. If the solution method uses $Sin(GHA_1 - GHA_2)$ as argument it requires special sub-rules, switching from a rectangular to a polar function for certain but not all equations, not executing certain divisions and so on. An example of the latter geometric solution is found in G. G. Bennett, "General Conventions and Solutions – Their Use in Celestial Navigation", Journal of the Inst. of Nav. (USA), Vol 26–4, p 275–280. For the algebraic or K-Z method see K. H. Zevering "The K-Z Position Solution for the Double Sight" – European Journal of Navigation, Vol 1–3 p 43–46. Corrections for some typographic errors appeared in Vol 1–4. The co-ordinates of the intersections of two position circles, if real solutions exist, are found algebraically by solving Lat and Long from: $Cos(GHA_1 \pm Long) = SinAlt_1/CosLatCosDec_1 - TanDec_1TanLat$ (1) and $Cos(GHA_2 \pm Long) = SinAlt_2/CosLatCosDec_2 - TanDec_2TanLat$ (2)

[2] By "identical" is strictly meant that in using either of these methods in the simultaneous sights situation rounding will generally not affect the position coordinate values (in degrees) at the 3rd decimal place and only marginally so at the 4th decimal place. Only rather extreme assumed (DR) positions in the case of the IM may produce significant differences in position solution compared to the double-sight method, which is free from any assumptions regarding initial (DR) position.

[3] An example is M. Blewitt's widely read "Celestial Navigation for Yachtsmen". It simply sidesteps the fact that all of the author's examples show differences between the timing of successive sights, which are simply ignored. To the contrary, in the (British) Admiralty Navigation Manual (ANM), the title "Simultaneous Star and Planet Sights" (op. cit., ed. 1937, Vol II, p 201) appears as chapter heading but even the example in this chapter is converted into a sight-run-sight case. There are no examples in the ANM of truly simultaneous sights.

[4] Data derived from the Sun-run-Sun case in Gerry Keys. op. cit., p 140

[5] See for example the ANM, Vol III, p 43: "When two observations … are taken, two position circles may be drawn, and the observer's position is at one of their two points of intersection ….. If the observer is in a ship and there is a run between sights, the first position circle must be transferred for the run. This can be done by transferring the geographical position and then drawing the circle" (ANM Vol III, p 43). This statement is made in connection with a large altitude case because only in this case could the method be implemented on the chart.

[6] The data are from B. D Yallop & C. Y. Hohenkerk – "Compact Data for Navigation and Astronomy (1986–1990)", p xxiii

[7] The coordinates of all points J, DR, and the vertices of triangle A"B"C" (see Figure 1) can be found using rhumbline equations, in the past calculated with the Traverse Table. It is easy to devise an electronic "vertex program" that will calculate the coordinates of all these points. Needless to say the entire IM is programmable.

[8] The assumptions may be found in the ANM, op. cit. Vol II, p 135, i.e constant azimuth in the neighbourhood of "J"; an intercept coincides with the zenith distance's line of bearing; the position circle may be represented as a straight line.

[9] An expose of the method is found in B. D Yallop & C. Y. Hohenkerk – "Compact Data for Navigation and Astronomy (1986–1990)". G. G. Bennett, op. cit., p 279 refers to his use of a "least-squares sub-routine", which includes a "sub-routine" for running on earlier sights. Yallop-Hohenkerk may not be the original authors of LSQ. LSQ has also been programmed as the foremost position solution algorithm in the package "Celestnav" (visit www.mobilegeographic.com).

[10] 'Parallel' great circles have two common points of intersection 180° apart. Meridians are a special 'family' of parallel great circles whose intersection points are P (North Pole) and P′ (South Pole). The equator intersects great circles belonging to a family other than the meridian family at varying angles (ε) that are all smaller than 90°. The translation of X (GP of an earlier sight) along direction **α** and distance **d** is to a point X* lying on a gr. circle through X*Z* that is a parallel great circle to the one through XZ. The spherical quadrangle XX*Z*Z begins to approach a parallelogram in terms of plane geometry only in large-altitude cases, so that the analysis of the large-altitude multiple sights case with IM/LSQ produces insignificant deviations in position solution compared to the analysis with LSQ + GD-UT.

[11] op. cit., p 139–140

[12] It is immaterial whether the calculations use $Lat_{DR} \sim Long_{DR}$ or $Lat_{EP} \sim Long_{EP}$, where $Lat_{EP} = Lat_{DR} + p_1CosZn_1$ and $Long_{EP} = Long_{DR} + p_1SinZn_1/Cos[1/2(Lat_{DR} + Lat_{EP})]$.

[13] op. cit., p 279–280

[14] Bennett calls his technique ''a standard method employed in sight reduction'' (ibid. p 279). The same result should in fact be obtainable from triangle XJJ* by applying the fundamental or cosine formula: $\text{Cos(Zn-}\alpha) = [\text{Cos XJ*} - \text{Cos}(d/60)\text{CosJX}]/\text{Sin}(d/60)\text{SinJX} = [\text{SinH*}_{o,1} - \text{Cos}(d/60)\text{SinH}_{o,1}]/\text{Sin}(d/60)\text{CosH}_{o,1}$, so that $\text{SinH*}_{o,1} = \text{Cos(Zn-}\alpha)\text{Sin}(d/60)\text{CosH}_{o,1} + \text{Cos}(d/60)\text{Sin H}_{o,1}$. Bennett's 'standard method' and the cosine formula-derived one tend to give significantly different results when the run distance becomes substantial. Regardless of these formulas, however, the displacement at J is assumed to be the same in magnitude and direction as the displacement at X, or if not the same then at least sufficiently close in magnitude and direction. This assumption has no general validity.

[15] N. Bowditch – The American Practical Navigator, 1977

[16] G. G. Bennett – The Two Body Fix Revisited, Navigators Newsletter (The Foundation for the Promotion of the Art of Navigation), Issue 44, 1994. The formulas for one revised method (''Method A'') corrected here for an error in Bennett's article are equivalent to:

$$\text{SinH*}_{o,1} = \text{SinH}_o\text{Cos}(d/60) + \text{CosH}_o\text{Sin}(d/60)\text{Cos(Zn-}\alpha\text{-CA)}..\text{(i)}$$
$$\text{CA} = -\tfrac{1}{2}(d/60)\text{SinTanMeanLat}..\ \text{(ii)}$$

where CA is a ''Conversion Angle'' to adjust triangle PJJ* (see Figure 2) to a spherical triangle. Zn is determined (with the quadrantal formula) from Dec_1, GHA_1 and initial $\text{Lat}_{DR} \sim \text{Long}_{DR}$. The resulting surrogate altitude $\text{H*}_{o,1}$ is consistent with $\text{H*}_{o,1}$ found when treating triangles PJJ* and XJJ* as spherical triangles with the cosine formula:

$$\text{Cos}(d'/60) = \text{Cos}\delta\text{CosLat}_J\text{CosLat}_{J*} + \text{SinLat}_J\text{SinLat}_{J*}..\text{(a)}$$
$$\text{Cos}\alpha^* = [\text{SinLat}_{J*} - \text{SinLat}_J\text{Cos}(d'/60)]/[\text{CosLat}_J\text{Sin}(d'/60)]..\text{(b)}$$
$$\text{SinH*}_{o,1} = \text{SinH}_o\text{Cos}(d'/60) + \text{CosH}_o\text{Sin}(d'/60)\text{Cos}(\text{Zn} - \alpha^*)..\text{(c)},$$

where for α^* in (c) is substituted either α^* from (b) or $360 - \alpha^*$ depending on compass bearing quadrant.