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ON INTERSECTION PROBABILITIES OF FOUR LINES INSIDE A PLANAR CONVEX DOMAIN

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Abstract

Let $n \geq 2$ random lines intersect a planar convex domain D. Consider the probabilities p_{nk}, $k = 0, 1, \ldots, n(n-1)/2$ that the lines produce exactly k intersection points inside D. The objective is finding p_{nk} through geometric invariants of D. Using Ambartzumian's combinatorial algorithm, the known results are instantly reestablished for $n = 2, 3$. When $n = 4$, these probabilities are expressed by new invariants of D. When D is a disc of radius r, the simplest forms of all invariants are found. The exact values of p_{3k} and p_{4k} are established.

Keywords: Stochastic geometry; convex domain invariants; combinatorial algorithm; counting intersection points; random chord length moments.

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Secondary 52A22; 52A10

1. Introduction

The problem of finding relations between probabilistic and geometric characteristics of a convex domain that remain invariant under rigid motions goes far beyond theoretical interest. The field has been significantly developed during recent decades, when an increasing number of real-life applications required rigorous mathematical foundations (see [9]). For example, the reconstruction of a convex body by its random sections is the central problem of geometric tomography (introduced in [5]). Some recent results on finding the chord length distribution or the distance distribution between two random points in a convex domain can be viewed in [1], [2], [6], and [7].

The main results of this work concern the intersection probabilities of four random lines meeting a planar convex domain. This is a classical object in stochastic geometry, with a dominant geometric flavour. For a bounded open convex domain $D \subset \mathbb{R}^2$ we consider N_n, the number of intersection points of n random lines in D, given that all n lines meet D. We will assume that D contains the origin of the Cartesian plane, and for a line $g \subset \mathbb{R}^2$, we let (p, φ) denote the polar coordinates of the foot of the perpendicular from the origin onto g.

Let $p_{nk} = \mathbb{P}(N_n = k)$. It is easy to check that $p_{21} = 2\pi F/L^2$, where F and L are the area and the perimeter of D, respectively. Computation of p_{3k} requires more invariants of D besides the

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area and the perimeter. These are suggested in [8, Chapter 4] to be

$$I_2 = \int_{g\cap D\neq\varnothing} |\chi(g)|^2 \, \mathrm{d}g \quad \text{and} \quad U = \int_{g_1\cap g_2\in D} u(g_1, g_2)\, \mathrm{d}g_1\, \mathrm{d}g_2,$$

where $\chi(g) = g \cap D$ is the chord in D produced by the line g, $|\chi(g)|$ is the length of $\chi(g)$, and $u(g_1, g_2)$ denotes the perimeter of the convex quadrilateral whose vertices are at the points of intersections of the lines g_1 and g_2 with the boundary ∂D. The measure element $\mathrm{d}g$ is interpreted as $\mathrm{d}g = \mathrm{d}p\, \mathrm{d}\varphi$, where $\mathrm{d}p$ is the one-dimensional Lebesgue measure and $\mathrm{d}\varphi$ is the uniform measure on the unit circle.

The formulas for intersection probabilities p_{3k}, suggested in [8, Chapter 4], contain an error. The correct formulas are

$$p_{33} = \frac{8I_2 - U}{L^3}, \quad p_{32} = \frac{3U - 12I_2}{L^3}, \quad p_{31} = \frac{6\pi FL - 3U}{L^3}, \tag{1.1}$$

established earlier by R. Sulanke in [10]. These formulas imply

$$I_2 = \frac{L^3}{12}(p_{32} + 3p_{33}), \quad U = \frac{L^3}{3}(2p_{32} + 3p_{33}). \tag{1.2}$$

In this paper we obtain explicit formulas for probabilities p_{4k}, $k = 1, 2, \ldots, 6$ in terms of new invariants of D and find an analogue of (1.2) for those invariants. After the main results, in the final section we provide exact computations of all the new invariants for a disc of radius r. The simplest expressions in terms of r are reached. The exact values of intersection probabilities p_{3k}, $0 \le k \le 3$ and p_{4k}, $0 \le k \le 6$ are found.

Our computations are based on Ambartzumian's combinatorial algorithm (see [3, Chapter 5]). Before passing on to the main results, the algorithm is adapted to the new situation in Section 2.

2. The combinatorial algorithm

Let $\mathbb{G}$ be the space of all lines g in $\mathbb{R}^2$. We equip $\mathbb{G}$ with a measure μ invariant under Euclidean motions in $\mathbb{R}^2$. Then, up to a constant factor,

$$\mu(X) = \int_X \mathrm{d}g,$$

for the measurable subsets $X \subset \mathbb{G}$ (see [3]).

Let $\mathcal{P} = \{P_i\}_{i=1}^n$ be a finite set of points in the plane. For any line $g \in \mathbb{G}$ we consider $\Pi_1(g)$ and $\Pi_2(g)$, the two open half-planes generated by g. We call two lines g_1, g_2 *equivalent* if $\{\mathcal{P} \cap \Pi_1(g_1), \mathcal{P} \cap \Pi_2(g_1)\} = \{\mathcal{P} \cap \Pi_1(g_2), \mathcal{P} \cap \Pi_2(g_2)\}$. $\mathbb{G}$ is decomposed into subsets of equivalent lines, which we call *atoms*. We let $r(\mathcal{P})$ denote the minimal ring containing all bounded atoms.

If $g \cap \mathcal{P} = \varnothing$ and neither of the sets $\mathcal{P} \cap \Pi_1(g)$ and $\mathcal{P} \cap \Pi_2(g)$ are empty, then consider the atom B such that $g \in B$. We will say that the atom B *separates the points* $\mathcal{P} \cap \Pi_1(g)$ *from* $\mathcal{P} \cap \Pi_2(g)$.

Let ρ_{ij} be the Euclidean distance between points P_i and P_j. The combinatorial algorithm below aims to express the μ-measure of any set $B \in r(\mathcal{P})$ by linear combinations of ρ_{ij} with integer coefficients belonging to $\{0, \pm 1, \pm 2\}$. The algorithm is introduced [3] for the case where any three points from $\mathcal{P}$ are not collinear. If there are collinear triads, then (see [4]) the

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linear combinations should be taken over those indices (i, j), $i < j$, for which the segment P_iP_j contains no other points from $\mathcal{P}$. We call such points P_i and P_j *neighbour points*.

Let g_{ij} be the line passing through the neighbour points P_i and P_j. For sufficiently small positive numbers δ and θ, we define two types of displacements for g_{ij}.

δ-**translation.** This is a set of two lines which are parallel to g_{ij} and distant from g_{ij} by δ. The set is denoted by $T_\delta(g_{ij})$.

θ-**rotation.** This is a set of two lines each passing through the midpoint of P_iP_j and making angle θ with g_{ij}. The set is denoted by $R_\theta(g_{ij})$.

If $B \in r(\mathcal{P})$, then let us define the numbers

$$R_{ij}(B) = \lim_{\theta\to 0+} \#[R_\theta(g_{ij}) \cap B], \qquad T_{ij}(B) = \lim_{\delta\to 0+} \#[T_\delta(g_{ij}) \cap B],$$

where # stands for the cardinality of a set.

Obviously $R_{ij}(B), T_{ij}(B) \in \{0, 1, 2\}$. Ambartzumian's combinatorial algorithm/formula can now be reformulated as follows.

Theorem 2.1. *Let $\mathcal{P} = \{P_i\}_{i=1}^n$ be a finite set of points in the plane and $B \in r(\mathcal{P})$. Then*

$$\mu(B) = \sum_{(i,j)\in I} [R_{ij}(B) - T_{ij}(B)]\rho_{ij}, \tag{2.1}$$

where I is the set of pairs (i, j) for all neighbour points P_i and P_j, $i < j$.

As an application, one can easily re-obtain the formulas for p_{nk}, where $n = 2, 3$. For example, let us prove the second formula in (1.1).

Here and in the next sections, for any set $X \subset \mathbb{R}^2$ we let $[X]$ denote the set of all lines $g \in \mathbb{G}$ such that $g \cap X \neq \varnothing$. Two intersection points can occur when two of the lines g_1, g_2, g_3 have no intersection inside D and the third one intersects each of them inside D. The three events where either of g_i is the third line are equally probable, and therefore

$$p_{32} = \frac{3}{L^3} \int_{g_1\cap g_2\in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)]\cap[\chi(g_2)]^c} \mathrm{d}g_3 = \frac{3}{L^3} \int_{g_1\cap g_2\in D} \mu(B)\, \mathrm{d}g_1\, \mathrm{d}g_2,$$

where $B = [\chi(g_1)] \cap [\chi(g_2)]^c$ (the complement is taken over the sample space $[D]$). One can check that among $T_{ij}(B)$ and $R_{ij}(B)$ the only non-zero coefficients are $T_{24}(B) = 2$ and $R_{12}(B) = R_{23}(B) = R_{34}(B) = R_{14}(B) = 1$. Then (2.1) yields

$$p_{32} = \frac{3}{L^3} \int_{g_1\cap g_2\in D} (-2|\chi(g_2)| + u(g_1, g_2))\, \mathrm{d}g_1\, \mathrm{d}g_2.$$

It remains to notice that

$$\int_{g_1\cap g_2\in D} |\chi(g_2)|\, \mathrm{d}g_1\, \mathrm{d}g_2 = \int_{[D]} |\chi(g_2)|\, \mathrm{d}g_2 \int_{[\chi(g_2)]} \mathrm{d}g_1 = 2\int_{[D]} |\chi(g)|^2\, \mathrm{d}g = 2I_2.$$

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3. Introduction of new invariants: computation of p_{4k} for $k = 6, 5$

Definition 3.1. For any $g_1 \cap g_2 \in D$, we define

$$d(g_1, g_2) = |\chi(g_1)| + |\chi(g_2)|, \quad c(g_1, g_2) = \mu([\chi(g_1)] \cap [\chi(g_2)]),$$

$$u(g_1, g_2) = \left|\partial\left(\operatorname{conv}\left(\bigcup_{i=1}^{2} g_i \cap D\right)\right)\right|,$$

and for any three lines g_1, g_2, g_3 such that $g_i \cap g_j \in D$, $1 \le i < j \le 3$, we define

$$v(g_1, g_2, g_3) = \left|\partial\left(\operatorname{conv}\left(\bigcup_{i=1}^{3} g_i \cap D\right)\right)\right|,$$

where $\operatorname{conv}(X)$ denotes the convex hull of $X \subset \mathbb{R}^2$, and $|\partial Y|$ denotes the perimeter of a convex domain Y.

The new definition of $u(g_1, g_2)$ coincides with the one we have used so far. Also, by (2.1), we have $c(g_1, g_2) = 2d(g_1, g_2) - u(g_1, g_2)$.

Along with the well-known invariants $I_k = \int_{[D]} |\chi(g)|^k \, dg$, $k = 0, 1, 2, \ldots,$ let us consider the following moments of the functions introduced in Definition 3.1:

$$D_k = \int_{g_1 \cap g_2 \in D} d^k(g_1, g_2) \, dg_1 \, dg_2, \quad C_k = \int_{g_1 \cap g_2 \in D} c^k(g_1, g_2) \, dg_1 \, dg_2,$$

$$U_k = \int_{g_1 \cap g_2 \in D} u^k(g_1, g_2) \, dg_1 \, dg_2, \quad V_k = \int_{g_i \cap g_j \in D,\ 1 \le i < j \le 3} v^k(g_1, g_2, g_3) \, dg_1 \, dg_2 \, dg_3.$$

It is easy to verify that

$$I_0 = L, \quad D_0 = C_0 = U_0 = 2I_1 = 2\pi F, \quad V_0 = C_1 = 2D_1 - U_1 = 8I_2 - U_1$$

and

$$p_{21} = \frac{U_0}{L^2}, \quad p_{33} = \frac{C_1}{L^3}, \quad p_{32} = \frac{3(U_1 - D_1)}{L^3}, \quad p_{31} = \frac{3(U_0 L - U_1)}{L^3}. \tag{3.1}$$

In this section we aim to express the probabilities p_{46} and p_{45} in terms of the new invariants. In this way we first obtain expressions for two useful integrals.

Proposition 3.1. *We have*

$$\int_{g_1 \cap g_2 \in D} |\chi(g_1)||\chi(g_2)| \, dg_1 \, dg_2 = \frac{D_2 - 4I_3}{2},$$

$$\int_{g_1 \cap g_2 \in D} |\chi(g_1)| u(g_1, g_2) \, dg_1 \, dg_2 = \frac{4D_2 + U_2 - C_2}{8}.$$

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Proof. Direct computation of D_2 leads to

$$D_2 = \int_{g_1 \cap g_2 \in D} (|\chi(g_1)|^2 + |\chi(g_2)|^2)\, \mathrm{d}g_1\, \mathrm{d}g_2 + 2\int_{g_1 \cap g_2 \in D} |\chi(g_1)||\chi(g_2)|\, \mathrm{d}g_1\, \mathrm{d}g_2$$

$$= 2\int_{[D]} |\chi(g_1)|^2 \int_{[\chi(g_1)]} \mathrm{d}g_2\, \mathrm{d}g_1 + 2\int_{g_1 \cap g_2 \in D} |\chi(g_1)||\chi(g_2)|\, \mathrm{d}g_1\, \mathrm{d}g_2$$

$$= 4\int_{[D]} |\chi(g_1)|^3\, \mathrm{d}g_1 + 2\int_{g_1 \cap g_2 \in D} |\chi(g_1)||\chi(g_2)|\, \mathrm{d}g_1\, \mathrm{d}g_2,$$

which is equivalent to the first identity.

To prove the second identity we expand the integrand of C_2 and obtain

$$C_2 = 4D_2 + U_2 - 4\int_{g_1 \cap g_2 \in D} |\chi(g_1)|u(g_1, g_2)\, \mathrm{d}g_1\, \mathrm{d}g_2 - 4\int_{g_1 \cap g_2 \in D} |\chi(g_2)|u(g_1, g_2)\, \mathrm{d}g_1\, \mathrm{d}g_2.$$

It remains to notice that the last two integrals coincide due to symmetry. □

Definition 3.2. For $g_1, g_2, \ldots, g_n \in [D]$, let $\langle g_1, g_2, \ldots, g_n\rangle$ be the set of all chords χ_{12} that join an endpoint of $\chi(g_i)$ to an endpoint of $\chi(g_j)$, $1 \le i < j \le n$. Then, for any integer k, $0 \le k \le n-2$, we define the function $I_k\colon \langle g_1, g_2, \ldots, g_n\rangle \to \{0, 1\}$ by

$$I_k(\chi_{12}) = \begin{cases} 1 & \text{if } \#\big(\chi_{12} \cap \big(\bigcup_{i=1}^n \overline{\chi(g_i)}\big)\big) = k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{\chi(g_i)}$ is the closure of $\chi(g_i)$ in $\mathbb{R}^2$.

The following two integrals are essential for our further work.

Lemma 3.1. *We have*

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} |\chi(g_1)|\, \mathrm{d}g_3 = \frac{4D_2 - U_2 + C_2}{8}, \tag{3.2}$$

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \sum_{\chi_{12} \in \langle g_1, g_2, g_3\rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 = \frac{12D_2 - 3C_2 - 3U_2 - 2V_1}{2}. \tag{3.3}$$

Proof. The left-hand side of (3.2) is equal to

$$\int_{g_1 \cap g_2 \in D} |\chi(g_1)|(2d(g_1, g_2) - u(g_1, g_2))\, \mathrm{d}g_1\, \mathrm{d}g_2$$

$$= 2\int_{g_1 \cap g_2 \in D} |\chi(g_1)|^2\, \mathrm{d}g_1\, \mathrm{d}g_2 + 2\int_{g_1 \cap g_2 \in D} |\chi(g_1)||\chi(g_2)|\, \mathrm{d}g_1\, \mathrm{d}g_2$$

$$- \int_{g_1 \cap g_2 \in D} |\chi(g_1)|u(g_1, g_2)\, \mathrm{d}g_1\, \mathrm{d}g_2$$

$$= 4I_3 + (D_2 - 4I_3) - \frac{4D_2 + U_2 - C_2}{8},$$

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which coincides with the right-hand side of (3.2). To prove (3.3) we first notice that

$$\sum_{\chi_{12}\in\langle g_1,g_2,g_3\rangle} |\chi_{12}| I_1(\chi_{12}) = \sum_{1\le i<j\le 3} u(g_i, g_j) - v(g_1, g_2, g_3).$$

Consequently, due to symmetry, the left-hand side of (3.3) becomes equal to

$$\begin{aligned} &3\int_{g_1\cap g_2\in D} \mathrm{d}g_1\,\mathrm{d}g_2 \int_{[\chi(g_1)]\cap[\chi(g_2)]} u(g_1, g_2)\,\mathrm{d}g_3 - V_1 \\ &\quad = 3\int_{g_1\cap g_2\in D} u(g_1, g_2)(2d(g_1, g_2) - u(g_1, g_2))\,\mathrm{d}g_1\,\mathrm{d}g_2 - V_1 \\ &\quad = 12\int_{g_1\cap g_2\in D} |\chi(g_1)| u(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2 - 3U_2 - V_1 \\ &\quad = \frac{12(4D_2 + U_2 - C_2)}{8} - 3U_2 - V_1, \end{aligned}$$

which coincides with the right-hand side of (3.3). □

We are now ready to compute p_{46} and p_{45}.

Theorem 3.1. *We have*

$$p_{46} = \frac{3U_2 + 9C_2 - 12D_2 + 4V_1}{4L^4}, \tag{3.4}$$

$$p_{45} = \frac{36D_2 - 9U_2 - 15C_2 - 12V_1}{2L^4}. \tag{3.5}$$

Proof. Four lines $g_i \in [D]$, $i = 1, 2, 3, 4$ generate six intersection points inside D if and only if g_1, g_2, g_3 generate three intersections and $g_4 \in \bigcap_{i=1}^{3}[\chi(g_i)]$. Therefore

$$p_{46} = \frac{1}{L^4}\int_{g_1\cap g_2\in D} \mathrm{d}g_1\,\mathrm{d}g_2 \int_{[\chi(g_1)]\cap[\chi(g_2)]} \mu(B_6)\,\mathrm{d}g_3, \tag{3.6}$$

where $B_6 = \bigcap_{i=1}^{3}[\chi(g_i)]$.

Let us fix g_1, g_2, g_3 and consider the set of points $\mathcal{P} = (g_1 \cup g_2 \cup g_3) \cap \partial D$. Without loss of generality one can assume that $g_1 \cap \partial D = \{P_1, P_4\}$, $g_2 \cap \partial D = \{P_2, P_5\}$, and $g_3 \cap \partial D = \{P_3, P_6\}$, where the points P_i are consecutively distributed over the boundary ∂D.

The set B_6 belongs to the ring $r(\mathcal{P})$ and can be written as a union of three atoms B_{61}, B_{62}, and B_{63}, where B_{6i} separates the points P_i, P_{i+1}, P_{i+2} from the other points of $\mathcal{P}$. By Theorem 2.1,

$$\mu(B_{61}) = \rho_{14} + \rho_{36} - \rho_{13} - \rho_{46}, \quad \mu(B_{62}) = \rho_{25} + \rho_{14} - \rho_{24} - \rho_{15},$$

$$\mu(B_{63}) = \rho_{36} + \rho_{25} - \rho_{35} - \rho_{26},$$

where we notice that $\rho_{14} = |\chi(g_1)|$, $\rho_{25} = |\chi(g_2)|$, $\rho_{36} = |\chi(g_3)|$, and the six subtracted terms represent the lengths of all chords $\chi_{12} \in \langle g_1, g_2, g_2\rangle$ that meet exactly one of the closed chords $\overline{\chi(g_i)}$, $i = 1, 2, 3$. Thus

$$\mu(B_6) = \sum_{i=1}^{3}\mu(B_{6i}) = 2\sum_{i=1}^{3}|\chi(g_i)| - \sum_{\chi_{12}\in\langle g_1,g_2,g_3\rangle} |\chi_{12}| I_1(\chi_{12}). \tag{3.7}$$

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Now (3.6), (3.7), and Lemma 3.1 imply

$$p_{46} = 6 \cdot \frac{4D_2 - U_2 + C_2}{8L^4} - \frac{12D_2 - 3C_2 - 3U_2 - 2V_1}{2L^4} = \frac{3U_2 + 9C_2 - 12D_2 + 4V_1}{4L^4}.$$

Five intersection points may occur when three lines, e.g. g_1, g_1, g_3, produce three intersections, and the fourth line, g_4, intersects only g_1 and g_2 inside D. In this scenario, the roles of g_3 and g_4 are interchangeable. Therefore the probability p_{45} can be written as

$$p_{45} = \frac{1}{2} \cdot \binom{4}{3} \cdot \frac{1}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \mu(B_5)\, \mathrm{d}g_3, \tag{3.8}$$

where $B_5 = ([\chi(g_1)] \cap [\chi(g_2)] \cap [\chi(g_3)]^c) \cup ([\chi(g_1)] \cap [\chi(g_2)]^c \cap [\chi(g_3)]) \cup ([\chi(g_1)]^c \cap [\chi(g_2)] \cap [\chi(g_3)])$.

Using the same set $\mathcal{P}$ as in the case of six intersection points, one can represent B_5 as a union of six atoms $B_{5i} \in r(\mathcal{P})$, $i = 1, \ldots, 6$, where B_{5i} separates $\{P_i, P_{i+1}\}$ from the other points of $\mathcal{P}$ (when $i = 6$ we replace $i + 1$ with 1).

By Theorem 2.1,

$$\mu(B_{51}) = \rho_{13} + \rho_{26} - \rho_{12} - \rho_{36}, \quad \mu(B_{52}) = \rho_{24} + \rho_{31} - \rho_{23} - \rho_{41},$$

$$\mu(B_{53}) = \rho_{35} + \rho_{42} - \rho_{34} - \rho_{52}, \quad \mu(B_{54}) = \rho_{46} + \rho_{53} - \rho_{45} - \rho_{63},$$

$$\mu(B_{55}) = \rho_{51} + \rho_{64} - \rho_{56} - \rho_{14}, \quad \mu(B_{56}) = \rho_{62} + \rho_{15} - \rho_{61} - \rho_{25}.$$

Taking into account that $\rho_{ji} = \rho_{ij}$ and recognizing the type of each ρ_{ij} participating in the above formulas, we come up with

$$\mu(B_5) = \sum_{i=1}^{6} \mu(B_{5i}) = 2 \cdot \sum_{\chi_{12} \in \langle g_1, g_2, g_3 \rangle} |\chi_{12}| I_1(\chi_{12}) - 2 \sum_{i=1}^{3} |\chi(g_i)| - \nu(g_1, g_2, g_3). \tag{3.9}$$

Due to (3.8) and (3.9),

$$p_{45} = \frac{4}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \sum_{\chi_{12} \in \langle g_1, g_2, g_3 \rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3$$
$$- \frac{12}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} |\chi(g_1)|\, \mathrm{d}g_3 - \frac{2V_1}{L^4}.$$

It remains to apply Lemma 3.1 and establish (3.5) by combining the like terms. □

4. Computation of p_{4k} for $k = 4, 3, 2, 1, 0$

We will use further notation in this section to make relevant calculations in all the scenarios that may occur when four lines meet inside D at less than five points.

Given $g_1 \cap g_2 \in D$, we let $\rho_1, \rho_2, \rho_3, \rho_4$ denote the lengths of four consecutive sides of the quadrilateral $\mathrm{conv}((g_1 \cup g_2) \cap \partial D)$. To avoid ambiguity, we will always assume that the first two sides lie in different half-planes with respect to g_1. If two lines, e.g. g_2 and g_3, are from $[D]$ but do not meet inside D, then d_1, d_2 will stand for the lengths of the diagonals of $\mathrm{conv}((g_2 \cup g_3) \cap \partial D)$, and s_1, s_2 will represent the lengths of the sides of the quadrilateral which are different from $\chi(g_2), \chi(g_3)$.

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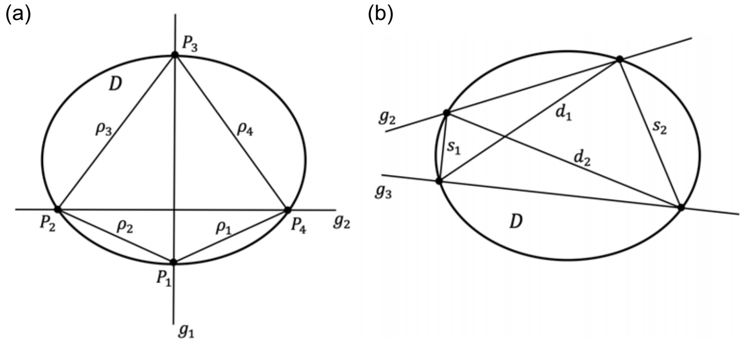

FIGURE 1. Scenarios of two lines intersecting D. (a) The case $g_1 \cap g_2 \in D$. (b) The case $g_2 \cap g_3 \notin D$.

The new notation is illustrated in Figure 1. These are used to define the following new invariants of D:

$$R = \int_{g_1 \cap g_2 \in D} ((\rho_1 + \rho_2)(\rho_3 + \rho_4) + (\rho_2 + \rho_3)(\rho_4 + \rho_1))\, \mathrm{d}g_1\, \mathrm{d}g_2,$$

$$Q_s = \int_{g_2 \cap g_3 \notin D} (s_1 + s_2)(d_1 + d_2 - s_1 - s_2)\, \mathrm{d}g_2\, \mathrm{d}g_3,$$

$$Q_d = \int_{g_2 \cap g_3 \notin D} (d_1 + d_2)(d_1 + d_2 - s_1 - s_2)\, \mathrm{d}g_2\, \mathrm{d}g_3.$$

To make upcoming formulas shorter, for the given pair of lines $g_1 \cap g_2 \in D$ we will use $\mathcal{S}$ to denote the symmetric difference of $[\chi(g_1)]$ and $[\chi(g_2)]$. $\mathcal{S}_1$ and $\mathcal{S}_2$ will stand for $[\chi(g_1)] \cap [\chi(g_2)]^c$ and $[\chi(g_1)]^c \cap [\chi(g_2)]$, respectively. Use of parentheses in integrands will be avoided if it does not lead to misreading.

Lemma 4.1. *Let $I_{\mathcal{S}_1}$ and $I_{\mathcal{S}_2}$ be the indicator functions of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Then the following six identities hold:*

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{\mathcal{S}} I_{\mathcal{S}_1} |\chi(g_1)| + I_{\mathcal{S}_2} |\chi(g_2)|\, \mathrm{d}g_3 = \frac{U_2 - C_2 - 4D_2}{4} + 8I_3, \tag{4.1}$$

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{\mathcal{S}} I_{\mathcal{S}_1} \sum_{\langle g_2, g_3 \rangle} |\chi_{12}| I_0(\chi_{12}) + I_{\mathcal{S}_2} \sum_{\langle g_1, g_3 \rangle} |\chi_{12}| I_0(\chi_{12})\, \mathrm{d}g_3 = 2Q_s, \tag{4.2}$$

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{\mathcal{S}} \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 = \frac{7U_2 - 4D_2 + C_2}{4} - 2R, \tag{4.3}$$

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{\mathcal{S}} \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_0(\chi_{12})\, \mathrm{d}g_3 = \frac{C_2 - 4D_2 - U_2}{4} + 2R, \tag{4.4}$$

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$$\int_{g_1\cap g_2\in D} \mathrm{d}g_1\,\mathrm{d}g_2 \int_{\mathcal{S}} I_{\mathcal{S}_1} \sum_{\langle g_2,g_3\rangle} |\chi_{12}| I_1(\chi_{12}) + I_{\mathcal{S}_2} \sum_{\langle g_1,g_3\rangle} |\chi_{12}| I_1(\chi_{12})\,\mathrm{d}g_3 = 2Q_d, \tag{4.5}$$

$$\int_{g_1\cap g_2\in D} \mathrm{d}g_1\,\mathrm{d}g_2 \int_{\mathcal{S}} I_{\mathcal{S}_1}|\chi(g_2)| + I_{\mathcal{S}_2}|\chi(g_1)|\,\mathrm{d}g_3 = \frac{U_2 - C_2 + 4D_2}{4} - 8I_3. \tag{4.6}$$

Proof. Since $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \varnothing$,

$$\int_{\mathcal{S}} I_{\mathcal{S}_1}|\chi(g_1)| + I_{\mathcal{S}_2}|\chi(g_2)|\,\mathrm{d}g_3 = |\chi(g_1)|\mu(\mathcal{S}_1) + |\chi(g_2)|\mu(\mathcal{S}_2).$$

Using the technique developed in the previous two sections, one can check that $\mu(\mathcal{S}_1) = u(g_1, g_2) - 2|\chi(g_2)|$, $\mu(\mathcal{S}_2) = u(g_1, g_2) - 2|\chi(g_1)|$, and consequently the left-hand side of (4.1) becomes equal to

$$\begin{aligned}
&2\int_{g_1\cap g_2\in D} |\chi(g_1)| u(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2 - 4\int_{g_1\cap g_2\in D} |\chi(g_1)||\chi(g_2)|\,\mathrm{d}g_1\,\mathrm{d}g_2 \\
&\quad = \frac{4D_2 + U_2 - C_2}{4} - 2D_2 + 8I_3 = \frac{U_2 - C_2 - 4D_2}{4} + 8I_3.
\end{aligned}$$

To prove (4.2), we change the order of integration. The left-hand side of (4.2) becomes equal to

$$\begin{aligned}
&\int_{g_2\cap g_3\notin D} \mathrm{d}g_2\,\mathrm{d}g_3 \int_{[\chi(g_2)]\cap[\chi(g_3)]} \sum_{\langle g_2,g_3\rangle} |\chi_{12}| I_0(\chi_{12})\,\mathrm{d}g_1 \\
&\qquad + \int_{g_1\cap g_3\notin D} \mathrm{d}g_1\,\mathrm{d}g_3 \int_{[\chi(g_1)]\cap[\chi(g_3)]} \sum_{\langle g_1,g_3\rangle} |\chi_{12}| I_0(\chi_{12})\,\mathrm{d}g_2 \\
&\quad = 2\int_{g_2\cap g_3\notin D} \sum_{\langle g_2,g_3\rangle} |\chi_{12}| I_0(\chi_{12}) \mu([\chi(g_2)]\cap[\chi(g_3)])\,\mathrm{d}g_2\,\mathrm{d}g_3 \\
&\quad = 2Q_s,
\end{aligned}$$

where the last equality holds due to $\sum_{\langle g_2,g_3\rangle} |\chi_{12}| I_0(\chi_{12}) = s_1 + s_2$ and $\mu([\chi(g_2)]\cap[\chi(g_3)]) = d_1 + d_2 - s_1 - s_2$ (see Figure 1(b)).

Let us prove (4.3). The inner integral in (4.3) can be written in the form of the sum

$$\int_{\mathcal{S}_1} \sum_{\langle g_1,g_2\rangle} |\chi_{12}| I_1(\chi_{12})\,\mathrm{d}g_3 + \int_{\mathcal{S}_2} \sum_{\langle g_1,g_2\rangle} |\chi_{12}| I_1(\chi_{12})\,\mathrm{d}g_3. \tag{4.7}$$

The integrand of the first integral in (4.7) is piecewise constant over the set of lines $g_3 \in \mathcal{S}_1$. Indeed, look at Figure 1(a). The function is equal to $\rho_3 + \rho_4$ over the atom $B^+ \in r(\mathcal{P})$ that separates the point P_3 from the other points of $\mathcal{P} = \{P_1, P_2, P_3, P_4\}$. On the other hand, it is equal to $\rho_1 + \rho_2$ over B^-, the atom that separates P_1 from $\mathcal{P}\setminus P_1$. Taking into account that $\mathcal{S}_1 = B^+ \cup B^-$, we obtain

$$\int_{\mathcal{S}_1} \sum_{\langle g_1,g_2\rangle} |\chi_{12}| I_1(\chi_{12})\,\mathrm{d}g_3 = (\rho_3 + \rho_4)\mu(B^+) + (\rho_1 + \rho_2)\mu(B^-).$$

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By our main computational engine (2.1), it is easy to verify that $\mu(B^+) = \rho_3 + \rho_4 - |\chi(g_2)|$ and $\mu(B^-) = \rho_1 + \rho_2 - |\chi(g_2)|$. Substitution of these values in the right-hand side of the last formula yields

$$\int_{\mathcal{S}_1} \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 = (\rho_3 + \rho_4)^2 + (\rho_1 + \rho_2)^2 - |\chi(g_2)| u(g_1, g_2),$$

which is equivalent to

$$\int_{\mathcal{S}_1} \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 = u^2(g_1, g_2) - |\chi(g_2)| u(g_1, g_2) - 2(\rho_1 + \rho_2)(\rho_3 + \rho_4). \tag{4.8}$$

The second integral in (4.7) can be obtained from (4.8) by interchanging g_1 and g_2:

$$\int_{\mathcal{S}_2} \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 = u^2(g_1, g_2) - |\chi(g_1)| u(g_1, g_2) - 2(\rho_2 + \rho_3)(\rho_4 + \rho_1). \tag{4.9}$$

Based on (4.7), (4.8), (4.9), and Proposition 3.1, we conclude that

$$\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{\mathcal{S}} \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 = 2U_2 - \frac{4D_2 + U_2 - C_2}{4} - 2R,$$

which is equal to the right-hand side of (4.3). The proofs of (4.4), (4.5), and (4.6) are very similar to the ones provided for (4.3), (4.2), and (4.1) respectively, and are thus omitted. □

Theorem 4.1. *Let $p_{44}^{(1)}$ be the probability that $g_1, g_2, g_3, g_4 \in [D]$ produce four intersection points inside D and three of them intersect each other inside D. Then*

$$p_{44} = p_{44}^{(1)} + p_{44}^{(2)},$$

where

$$p_{44}^{(1)} = \frac{6}{L^4}(2V_1 - 4D_2 + C_2 + U_2), \tag{4.10}$$

$$p_{44}^{(2)} = \frac{3}{L^4}\left(\frac{3U_2 + C_2}{2} - 8I_3 - 2R - Q_s\right). \tag{4.11}$$

Proof. There are two scenarios where four lines $g_i \in [D]$, $i = 1, 2, 3, 4$ can generate four intersection points inside D. In the first scenario we require three of the lines to make three intersection points inside D (enclose a triangle inside D) and the fourth to cut exactly one of those three inside D. Otherwise, in the second scenario, we require any three of the four lines to make exactly two intersections inside D (the four lines enclose a convex quadrilateral inside D). We denote the two mutually exclusive events by E_1 and E_2, respectively, and need to prove that

$$p(E_1) = \frac{6}{L^4}(2V_1 - 4D_2 + C_2 + U_2), \quad p(E_2) = \frac{3}{L^4}\left(\frac{3U_2 + C_2}{2} - 8I_3 - 2R - Q_s\right).$$

In the first scenario there are four choices to select three out of four lines to enclose a triangle inside D. Let those three be g_1, g_2, g_3. Consider the set of points $\mathcal{P} = (g_1 \cup g_2 \cup g_3) \cap \partial D$.

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Again, without loss of generality one can assume that $g_1 \cap \partial D = \{P_1, P_4\}$, $g_2 \cap \partial D = \{P_2, P_5\}$, and $g_3 \cap \partial D = \{P_3, P_6\}$, where the points P_i are consecutively distributed over the boundary ∂D. Consider the set $B_4^{(1)} \in r(\mathcal{P})$ comprising six atoms $B_{4i}^{(1)}$, $i = 1, 2, \dots, 6$, where $B_{4i}^{(1)}$ separates the point P_i from the other five points of $\mathcal{P}$. Then

$$p(E_1) = \frac{4}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \mu\big(B_4^{(1)}\big) \, \mathrm{d}g_3.$$

The measure of each of the six atoms is computed below by Theorem 2.1:

$$\mu\big(B_{41}^{(1)}\big) = \rho_{12} + \rho_{16} - \rho_{26}, \quad \mu\big(B_{42}^{(1)}\big) = \rho_{23} + \rho_{21} - \rho_{31},$$
$$\mu\big(B_{43}^{(1)}\big) = \rho_{34} + \rho_{32} - \rho_{42}, \quad \mu\big(B_{44}^{(1)}\big) = \rho_{45} + \rho_{43} - \rho_{53},$$
$$\mu\big(B_{45}^{(1)}\big) = \rho_{56} + \rho_{54} - \rho_{64}, \quad \mu\big(B_{46}^{(1)}\big) = \rho_{61} + \rho_{65} - \rho_{15}.$$

As a result, we obtain

$$\begin{aligned} p(E_1) &= \frac{4}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \sum_{i=1}^{6} \mu\big(B_{4i}^{(1)}\big) \, \mathrm{d}g_3 \\ &= \frac{4}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} 2v(g_1, g_2, g_3) - \sum_{\langle g_1, g_2, g_3 \rangle} |\chi_{12}| I_1(\chi_{12}) \, \mathrm{d}g_3. \end{aligned}$$

Finally, using Lemma 3.1, we arrive at

$$p(E_1) = \frac{4}{L^4} \bigg(2V_1 - \frac{12D_2 - 3C_2 - 3U_2 - 2V_1}{2} \bigg) = \frac{6}{L^4} (2V_1 - 4D_2 + C_2 + U_2).$$

In the second scenario we have

$$p(E_2) = \frac{3}{2L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{\mathcal{S}} \mu\big(B_4^{(2)}\big) \, \mathrm{d}g_3, \tag{4.12}$$

where

$$B_4^{(2)} = \begin{cases} \mathcal{S}_2 \cap [\chi(g_3)] & \text{if } g_3 \in \mathcal{S}_1, \\ \mathcal{S}_1 \cap [\chi(g_3)] & \text{if } g_3 \in \mathcal{S}_2. \end{cases}$$

The measure $\mu\big(B_4^{(2)}\big)$ can be computed by Theorem 2.1 with reference to Figure 2.

The formula (2.1) implies

$$\mu\big(B_4^{(2)}\big) = \sum_{\langle g_1, g_2 \rangle \cup \langle g_1, g_3 \rangle} |\chi_{12}| I_1(\chi_{12}) - \sum_{\langle g_2, g_3 \rangle} |\chi_{12}| I_0(\chi_{12}) - 2|\chi(g_1)| \quad \text{if } g_3 \in \mathcal{S}_1$$

and

$$\mu\big(B_4^{(2)}\big) = \sum_{\langle g_1, g_2 \rangle \cup \langle g_2, g_3 \rangle} |\chi_{12}| I_1(\chi_{12}) - \sum_{\langle g_1, g_3 \rangle} |\chi_{12}| I_0(\chi_{12}) - 2|\chi(g_2)| \quad \text{if } g_3 \in \mathcal{S}_2.$$

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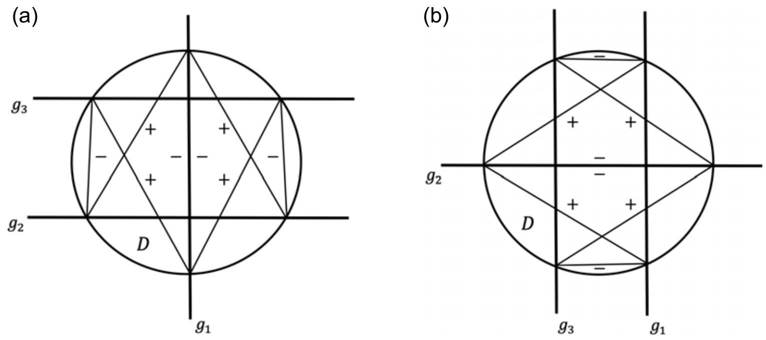

FIGURE 2. The distribution of signs over the chords participating in the combinatorial decomposition of $\mu\left(B_4^{(2)}\right)$. (a) The case $g_1 \cap g_2 \in D$ and g_3 meets $\chi(g_1)$ but not $\chi(g_2)$. (b) The case $g_1 \cap g_2 \in D$ and g_3 meets $\chi(g_2)$ but not $\chi(g_1)$.

By incorporating the indicator functions $I_{\mathcal{S}_i}$, we plug the obtained expressions into (4.12):

$$\begin{aligned} p(E_2) = {} & \frac{3}{2L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{\mathcal{S}} 2 \cdot \sum_{\langle g_1, g_2 \rangle} |\chi_{12}| I_1(\chi_{12}) \, \mathrm{d}g_3 \\ & - \frac{3}{2L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{\mathcal{S}} I_{\mathcal{S}_1} \sum_{\langle g_2, g_3 \rangle} |\chi_{12}| I_0(\chi_{12}) + I_{\mathcal{S}_2} \sum_{\langle g_1, g_3 \rangle} |\chi_{12}| I_0(\chi_{12}) \, \mathrm{d}g_3 \\ & - \frac{3}{2L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{\mathcal{S}} 2 \cdot I_{\mathcal{S}_1} |\chi(g_1)| + 2 \cdot I_{\mathcal{S}_2} |\chi(g_2)| \, \mathrm{d}g_3. \end{aligned}$$

Finally, due to Lemma 4.1 (the first three identities), we come up with

$$\begin{aligned} p(E_2) & = \frac{3}{2L^4} \left(2 \cdot \left(\frac{7U_2 - 4D_2 + C_2}{4} - 2R \right) - 2Q_s - 2 \cdot \left(\frac{U_2 - C_2 - 4D_2}{4} + 8I_3 \right) \right) \\ & = \frac{3}{L^4} \left(\frac{3U_2 + C_2}{2} - 8I_3 - 2R - Q_s \right). \end{aligned}$$

The proof is thus complete. □

Three intersection points made by four lines from [D] can occur in three ways.

Event 1. The lines produce three chords each possessing two intersection points, and one containing no intersection point.

Event 2. The lines produce two chords each possessing two intersection points, and the other two each possessing one intersection point.

Event 3. The lines produce three chords each possessing one intersection point, and one possessing three intersection points.

We let $p_{43}^{(1)}$, $p_{43}^{(2)}$, and $p_{43}^{(3)}$ denote the probabilities of Event 1, Event 2, and Event 3, respectively.

Theorem 4.2. *We have*

$$p_{43} = p_{43}^{(1)} + p_{43}^{(2)} + p_{43}^{(3)},$$

where

$$p_{43}^{(1)} = \frac{4}{L^4}(C_1 L - V_1), \tag{4.13}$$

$$p_{43}^{(2)} = \frac{12}{L^4}(Q_s + 2R - U_2), \tag{4.14}$$

$$p_{43}^{(3)} = \frac{3}{L^4}(C_2 - 4D_2 - U_2) + \frac{4}{L^4}(Q_d + 2R) + \frac{64}{L^4} I_3. \tag{4.15}$$

Proof. Events 1, 2, and 3 are mutually exclusive and cover all the cases of three intersections inside D.

Let us now define an undirected graph T with the vertex set $\{1, 2, 3, 4\}$, where the vertices i and j are adjacent if and only if $g_i \cap g_j \in D$. If g_1, g_2, g_3, g_4 are placed as described in Event 1, then there are four possibilities to make a graph T (three vertices of degree 2, and one vertex of degree 0). Thus

$$p_{43}^{(1)} = \frac{4}{L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \mu\big(B_3^{(1)}\big) \, \mathrm{d}g_3,$$

where $B_3^{(1)} = [\chi(g_1)]^c \cap [\chi(g_2)]^c \cap [\chi(g_3)]^c$.

Since $\mu\big(B_3^{(1)}\big) = L - \nu(g_1, g_2, g_3)$, we obtain

$$p_{43}^{(1)} = \frac{4}{L^3} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \mathrm{d}g_3 - \frac{4V_1}{L^4} = 4p_{33} - \frac{4V_1}{L^4}.$$

As by (3.1), $p_{33} = C_1/L^3$, we establish (4.13).

If g_1, g_2, g_3, g_4 are placed as described in Event 2, then the number of possible graphs T (with two vertices of degree 2 and two vertices of degree 1) is 12. This number will be reduced to 4 if we also require vertices 1 and 2 to be adjacent, and vertex 3 to be adjacent to either 1 or 2 but not both (see Figure 3). Hence we acquire

$$p_{43}^{(2)} = \frac{12}{4L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{\mathcal{S}} \mu\big(B_3^{(2)}\big) \, \mathrm{d}g_3, \tag{4.16}$$

where

$$B_3^{(2)} = \begin{cases} [\chi(g_1)]^c \cap \big([\chi(g_2)] \Delta [\chi(g_3)]\big) & \text{if } g_3 \in \mathcal{S}_1, \\ [\chi(g_2)]^c \cap \big([\chi(g_1)] \Delta [\chi(g_3)]\big) & \text{if } g_3 \in \mathcal{S}_2, \end{cases}$$

and Δ stands for the symmetric difference operation.

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FIGURE 3. Versions of T where 1 is adjacent to 2, and 3 is adjacent to either 2 or 1 but not both (Event 2).

FIGURE 4. Versions of T where 1 is adjacent to 2, and 3 is adjacent to either 2 or 1 but not both (Event 3).

In the last scenario, if g_1, g_2, g_3, g_4 are placed as described in Event 3, then there are only four graphs T (three vertices of degree 1 and one vertex of degree 3). An extra restriction requiring vertices 1 and 2 to be adjacent, and vertex 3 to be adjacent to either 1 or 2 but not both, allows us to construct only two graphs T. These are displayed in Figure 4. Consequently

$$p_{43}^{(3)} = \frac{4}{2L^4} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1 \, \mathrm{d}g_2 \int_{\mathcal{S}} \mu\big(B_3^{(3)}\big) \, \mathrm{d}g_3, \tag{4.17}$$

where

$$B_3^{(3)} = \begin{cases} [\chi(g_1)] \cap [\chi(g_2)]^c \cap [\chi(g_3)]^c & \text{if } g_3 \in \mathcal{S}_1, \\ [\chi(g_1)]^c \cap [\chi(g_2)] \cap [\chi(g_3)]^c & \text{if } g_3 \in \mathcal{S}_2. \end{cases}$$

It remains to calculate $\mu\big(B_3^{(2)}\big)$ and $\mu\big(B_3^{(3)}\big)$ and then the integrals (4.16) and (4.17) accordingly. For the measures, we apply the combinatorial formula (2.1), while the integrals need the Lemma 4.1 to be used. Computation is similar to that used for proving (4.11) and is therefore omitted. □

Two intersection points generated by four lines are possible in two scenarios.

Event 1. One chord possesses two intersection points, two of the chords possess one intersection point each, and one chord does not possess any intersection point.

Event 2. Each chord of the four lines possesses exactly one intersection point.

Let the probabilities of the above-mentioned events be $p_{42}^{(1)}$ and $p_{42}^{(2)}$ respectively.

Theorem 4.3. *We have*

$$p_{42} = p_{42}^{(1)} + p_{42}^{(2)},$$

where

$$p_{42}^{(1)} = 4p_{32} - \frac{3}{L^4}(C_2 - 4D_2 - U_2 - 4(Q_s + 2R)), \tag{4.18}$$

$$p_{42}^{(2)} = \frac{12\pi^2 F^2}{L^4} + \frac{48 I_3}{L^4} - \frac{3}{4L^4}(U_2 - C_2 + 12D_2) - \frac{1}{4}\big(p_{44}^{(1)} + 2p_{43}^{(2)}\big). \tag{4.19}$$

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Proof. Events 1 and 2 are mutually exclusive, so it remains to verify (4.18) and (4.19). We first notice that

$$p_{42}^{(1)}=\frac{12}{2L^4}\int_{g_1\cap g_2\in D}\mathrm{d}g_1\,\mathrm{d}g_2\int_{\mathcal{S}}\mu\big(B_2^{(1)}\big)\,\mathrm{d}g_3,$$

where $B_2^{(1)}=[\chi(g_1)]^c\cap[\chi(g_2)]^c\cap[\chi(g_3)]^c$. Since $\mu\big(B_2^{(1)}\big)=L-v(g_1,g_2,g_3)$, we obtain

$$p_{42}^{(1)}=\frac{6}{L^3}\int_{g_1\cap g_2\in D}\mu(\mathcal{S})\,\mathrm{d}g_1\,\mathrm{d}g_2-\frac{6}{L^4}\int_{g_1\cap g_2\in D}\mathrm{d}g_1\,\mathrm{d}g_2\int_{\mathcal{S}}\sum_{\langle g_1,g_2,g_3\rangle}|\chi_{12}|I_0(\chi_{12})\,\mathrm{d}g_3.$$

We recognize that the first term above is equal to $6\cdot\frac{2}{3}p_{32}=4p_{32}$. Evaluation of the second term is based on (4.4) and (4.2). Indeed

$$\begin{aligned}&\int_{g_1\cap g_2\in D}\mathrm{d}g_1\,\mathrm{d}g_2\int_{\mathcal{S}}\sum_{\langle g_1,g_2,g_3\rangle}|\chi_{12}|I_0(\chi_{12})\,\mathrm{d}g_3\\
&\quad=\int_{g_1\cap g_2\in D}\mathrm{d}g_1\,\mathrm{d}g_2\int_{\mathcal{S}}I_{\mathcal{S}_1}\sum_{\langle g_1,g_2\rangle\cup\langle g_1,g_3\rangle}|\chi_{12}|I_0(\chi_{12})+I_{\mathcal{S}_2}\sum_{\langle g_1,g_2\rangle\cup\langle g_2,g_3\rangle}|\chi_{12}|I_0(\chi_{12})\\
&\qquad+I_{\mathcal{S}_1}\sum_{\langle g_2,g_3\rangle}|\chi_{12}|I_0(\chi_{12})+I_{\mathcal{S}_2}\sum_{\langle g_1,g_3\rangle}|\chi_{12}|I_0(\chi_{12})\,\mathrm{d}g_3\\
&\quad=\int_{g_1\cap g_2\in D}\mathrm{d}g_1\,\mathrm{d}g_2\int_{\mathcal{S}}2\sum_{\langle g_1,g_2\rangle}|\chi_{12}|I_0(\chi_{12})+I_{\mathcal{S}_1}\sum_{\langle g_2,g_3\rangle}|\chi_{12}|I_0(\chi_{12})\\
&\qquad+I_{\mathcal{S}_2}\sum_{\langle g_1,g_3\rangle}|\chi_{12}|I_0(\chi_{12})\,\mathrm{d}g_3=\frac{C_2-4D_2-U_2}{2}+4R+2Q_s,\end{aligned}$$

and (4.18) is proved.

The other ‘partial’ probability can be expressed by the following integral:

$$p_{42}^{(2)}=\frac{1}{2}\binom{4}{2}\frac{1}{L^4}\int_{g_1\cap g_3\notin D}\mathrm{d}g_1\,\mathrm{d}g_3\int_{[\chi(g_1)]\cap[\chi(g_3)]^c}\mu\big(B_2^{(2)}\big)\,\mathrm{d}g_2,\tag{4.20}$$

where $B_2^{(2)}=[\chi(g_3)]\cap[\chi(g_2)]^c\cap[\chi(g_1))]^c$. Then

$$\mu\big(B_2^{(2)}\big)=2|\chi(g_3)|-\mu\bigg(\bigcap_{i=1}^{3}[\chi(g_i)]\bigg)-\mu(([\chi(g_1)]\Delta[\chi(g_2)])\cap[\chi(g_3)]).\tag{4.21}$$

Since

$$\frac{1}{L^4}\int_{g_1\cap g_3\notin D}\mathrm{d}g_1\,\mathrm{d}g_3\int_{[\chi(g_1)]\cap[\chi(g_3)]^c}\mu\bigg(\bigcap_{i=1}^{3}[\chi(g_i)]\bigg)\,\mathrm{d}g_2=\frac{1}{12}p_{44}^{(1)}$$

and

$$\frac{1}{L^4}\int_{g_1\cap g_3\notin D}\mathrm{d}g_1\,\mathrm{d}g_3\int_{[\chi(g_1)]\cap[\chi(g_3)]^c}\mu(([\chi(g_1)]\Delta[\chi(g_2)])\cap[\chi(g_3)])\,\mathrm{d}g_2=\frac{1}{6}p_{43}^{(2)},$$

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then due to (4.20) the proof will be finished if we show that

$$\int_{g_1 \cap g_3 \notin D} \mathrm{d}g_1\, \mathrm{d}g_3 \int_{[\chi(g_1)] \cap [\chi(g_3)]^c} |\chi(g_3)|\, \mathrm{d}g_2 = 2\pi^2 F^2 + 8I_3 - \frac{U_2 - C_2 + 12D_2}{8}. \tag{4.22}$$

By the combinatorial formula,

$$\mu([\chi(g_1)] \cap [\chi(g_3)]^c) = s_1 + s_2 - d_1 - d_2 + 2|\chi(g_1)|.$$

Then the integral in (4.22) is equal to

$$2\int_{g_1 \cap g_3 \notin D} |\chi(g_1)||\chi(g_3)|\, \mathrm{d}g_1\, \mathrm{d}g_3 - \int_{g_1 \cap g_3 \notin D} |\chi(g_3)|(d_1 + d_2 - s_1 - s_2)\, \mathrm{d}g_1\, \mathrm{d}g_3. \tag{4.23}$$

The first term in (4.23) is equal to

$$2\int_{[D]} \mathrm{d}g_1 \int_{[D]} |\chi(g_1)||\chi(g_3)|\, \mathrm{d}g_3 - 2\int_{g_1 \cap g_3 \in D} |\chi(g_1)||\chi(g_3)|\, \mathrm{d}g_1\, \mathrm{d}g_3 = 2\pi^2 F^2 - D_2 + 4I_3,$$

while the second term is equal to

$$\begin{aligned}
&\int_{g_1 \cap g_3 \notin D} |\chi(g_3)|\, \mathrm{d}g_1\, \mathrm{d}g_3 \int_{[\chi(g_1)] \cap [\chi(g_3)]} \mathrm{d}g_2 \\
&\quad = \frac{1}{2} \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{\mathcal{S}} I_{\mathcal{S}_1} |\chi(g_2)| + I_{\mathcal{S}_2} |\chi(g_1)|\, \mathrm{d}g_3 \\
&\quad = \frac{U_2 - C_2 + 4D_2}{8} - 4I_3,
\end{aligned}$$

where the last equality holds due to (4.6).

To complete the proof, it remains to subtract the last expression from $2\pi^2 F^2 - D_2 + 4I_3$ and check that it is equal to the right-hand side of (4.22). □

Theorem 4.4. *We have*

$$p_{41} = 2p_{31} - 2p_{42}^{(2)} - \frac{6U_1}{L^3} + \frac{6U_2}{L^4}.$$

Proof. The only possible way to generate one intersection point inside D is having two chords possessing one intersection point each, and two more chords possessing no intersection points. The model yields

$$p_{41} = \binom{4}{2} \frac{1}{L^4} \int_{g_1 \cap g_3 \notin D} \mathrm{d}g_1\, \mathrm{d}g_3 \int_{[\chi(g_1)] \cap [\chi(g_3)]^c} \mu(B_1)\, \mathrm{d}g_2, \tag{4.24}$$

where $B_1 = [\chi(g_3)]^c \cap [\chi(g_2)]^c \cap [\chi(g_1))]^c$. We deliver the computation of $\mu(B_1)$ by the combinatorial algorithm through Figure 5.

Let us fix $g_1, g_2, g_3 \in [D]$ such that the first two intersect each other inside D, and g_3 intersects none of the chords $\chi(g_1)$, $\chi(g_2)$. Consider the set of points $\mathcal{P} = (g_1 \cup g_2 \cup g_3) \cap \partial D$. Without loss of generality, we assume that $g_1 \cap \partial D = \{P_1, P_5\}$, $g_2 \cap \partial D = \{P_2, P_6\}$, and

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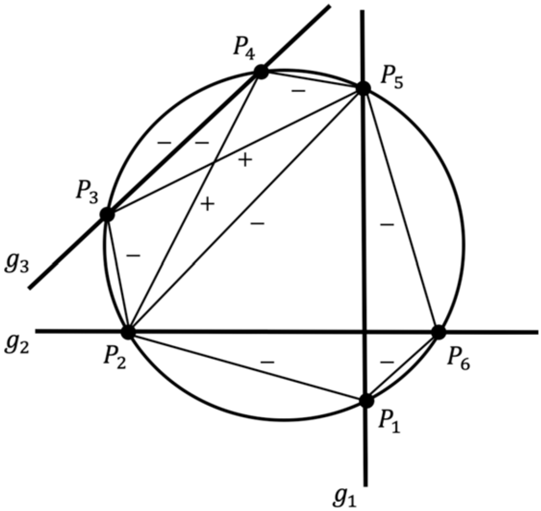

FIGURE 5. The distribution of signs over the chords participating in the combinatorial decomposition of $\mu(B_1)$.

$g_3 \cap \partial D = \{P_3, P_4\}$, where the points P_i are consecutively distributed over the boundary ∂D. As usual, we let ρ_{ij} denote the Euclidean distance between points P_i and P_j. Then B_1 consists of the lines that lie in the complement of the convex hull of $\mathcal{P}$ and the atom $A \in r(\mathcal{P})$ that separates the points P_3, P_4 from the other four. Hence $\mu(B_1) = L - \rho_{12} - \rho_{23} - \cdots - \rho_{61} + \mu(A)$. Since $\mu(A) = \rho_{24} + \rho_{35} - |\chi(g_3)| - \rho_{25}$, we obtain

$$\mu(B_1) = L - u(g_1, g_2) - 2|\chi(g_3)| + \rho_{24} + \rho_{35} - \rho_{23} - \rho_{45}.$$

On the other hand, $\rho_{24} + \rho_{35} - \rho_{23} - \rho_{45}$ can be interpreted as the measure of the set of lines that cut $\chi(g_3)$ and meet at least one of the chords $\chi(g_1)$, $\chi(g_2)$. From (4.21), that measure is equal to $2|\chi(g_3)| - \mu\big(B_2^{(2)}\big)$, and therefore we establish

$$\mu(B_1) = L - u(g_1, g_2) - \mu\big(B_2^{(2)}\big).$$

It is easy to verify that

$$\frac{1}{L^4} \int_{g_1 \cap g_3 \notin D} \mathrm{d}g_1 \, \mathrm{d}g_3 \int_{[\chi(g_1)] \cap [\chi(g_3)]^c} L \, \mathrm{d}g_2 = \frac{1}{3} p_{31}$$

and

$$\frac{1}{L^4} \int_{g_1 \cap g_3 \notin D} \mathrm{d}g_1 \, \mathrm{d}g_3 \int_{[\chi(g_1)] \cap [\chi(g_3)]^c} \mu\big(B_2^{(2)}\big) \, \mathrm{d}g_2 = \frac{1}{3} p_{42}^{(2)}.$$

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Integration of $u(g_1, g_2)$ requires a change of order:

$$\begin{aligned}
&\frac{1}{L^4}\int_{g_1\cap g_3\notin D} \mathrm{d}g_1\,\mathrm{d}g_3 \int_{[\chi(g_1)]\cap[\chi(g_3)]^c} u(g_1, g_2)\,\mathrm{d}g_2\\
&\quad= \frac{1}{L^4}\int_{g_1\cap g_2\in D} u(g_1, g_2)\,\mathrm{d}g_1\,\mathrm{d}g_2 \int_{[\chi(g_1)]^c\cap[\chi(g_2)]^c} \mathrm{d}g_3\\
&\quad= \frac{1}{L^4}\int_{g_1\cap g_2\in D} u(g_1, g_2)(L-u(g_1, g_2))\,\mathrm{d}g_1\,\mathrm{d}g_2\\
&\quad= \frac{U_1}{L^3} - \frac{U_2}{L^4}.
\end{aligned}$$

Now from (4.24) we conclude that

$$p_{41} = 6\cdot\left(\frac{1}{3}p_{31} - \frac{1}{3}p_{42}^{(2)} - \frac{U_1}{L^3} + \frac{U_2}{L^4}\right) = 2p_{31} - 2p_{42}^{(2)} - \frac{6U_1}{L^3} + \frac{6U_2}{L^4}. \qquad \square$$

Finally, the probability of having no intersection points inside D is $p_{40} = 1 - \sum_{k=1}^{6} p_{4k}$.

5. Representation of I_3 and V_1 by intersection probabilities

In this section we first aim to express the invariants V_1 and I_3 by intersection probabilities to establish an analogue of (1.2). Looking through the formulas of p_{4k} obtained in the previous two sections, we notice that the family of new invariants can be reduced to U_2, C_2, D_2, K_d, K_s, V_1, and I_3, where $K_s = Q_s + 2R$ and $K_d = Q_d + 2R$. I_3 is known (by Crofton; see e.g. [8]) to be equal to $3F^2$, but we will not be using this result below.

Theorem 5.1. *The following identities hold:*

$$V_1 = L^4\left(p_{33} - \frac{1}{4}p_{43}^{(1)}\right), \tag{5.1}$$

$$I_3 = \frac{L^4}{32}\left(4p_{33} + p_{43}^{(3)} - p_{43}^{(1)}\right). \tag{5.2}$$

Proof. Equation (5.1) immediately follows from (3.1) and (4.13).

Let us compute the mean number of the intersection points generated by four lines inside D. We directly use the formulas for intersection probabilities p_{4k}, $k = 6, 5, \ldots, 1$ obtained in the previous two sections. By combining the like terms accurately, we obtain

$$\sum_{k=1}^{6} kp_{4k} = \frac{12\pi F}{L^2} + \frac{3}{L^4}(-3U_2 + 3C_2 - 12D_2 + 4V_1 + 4K_d + 32I_3). \tag{5.3}$$

On the other hand, the mean number of intersection points generated by n lines inside D is known (see [8]) to be equal to $n(n-1)\pi F/L^2$. Thus, from (5.3), we obtain

$$-3U_2 + 3C_2 - 12D_2 + 4V_1 + 4K_d + 32I_3 = 0.$$

Based on (4.15), the last identity can be rewritten as $p_{43}^{(3)}\cdot L^4 + 4V_1 - 32I_3 = 0$, that is,

$$p_{43}^{(3)} = \frac{4}{L^4}(8I_3 - V_1). \tag{5.4}$$

Now (5.2) follows from (5.1) and (5.4). $\square$

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Below we check the results against the second moment of the number of intersection points generated by n lines inside D (for the formula below, see [8]):

$$E(v^2) = 2\pi \binom{n}{2} \frac{F}{L^2} + 24\pi^2 \binom{n}{4} \frac{F^2}{L^4} + 24 \binom{n}{3} \frac{I_2}{L^3}. \tag{5.5}$$

We start with the direct substitution of the obtained probabilities into the second moment formula.

$$\sum_{k=1}^{6} k^2 p_{4k} = 2p_{31} + 16p_{32} - \frac{1}{2}p_{44}^{(1)} - p_{43}^{(2)} + \frac{1}{L^4}(-36U_2 + 30C_2 - 120D_2$$
$$+ 42V_1 + 288I_3 + 12K_s + 36K_d + 36LC_1 + 24\pi^2F^2 - 6LU_1).$$

Further substitution of the known expressions for $p_{31}, p_{32}, p_{44}^{(1)}$, and $p_{43}^{(2)}$ results in

$$\sum_{k=1}^{6} k^2 p_{4k} = \frac{12\pi F}{L^2} + \frac{24\pi^2 F^2}{L^4} + \frac{36(U_1 + C_1) - 192I_2}{L^3}$$
$$+ \frac{1}{L^4}(27C_2 - 27U_2 - 108D_2 + 36K_d + 288I_3 + 36V_1). \tag{5.6}$$

Since $36(U_1 + C_1) = 72D_1 = 288I_2$ and, from (4.15),

$$27C_2 - 27U_2 - 108D_2 + 36K_d + 576I_3 = 9L^4 p_{43}^{(3)},$$

(5.6) implies

$$\sum_{k=1}^{6} k^2 p_{4k} = \frac{12\pi F}{L^2} + \frac{24\pi^2 F^2}{L^4} + \frac{96I_2}{L^3} + \frac{9}{L^4}\big(p_{43}^{(3)} \cdot L^4 + 4V_1 - 32I_3\big).$$

Due to (5.4), the last expression in the parentheses is equal to zero. This means that we have reached the right-hand side of (5.5) for $n = 4$.

6. Computation of intersection probabilities for a disc with radius r

The formulas obtained for intersection probabilities motivated us to compute invariants of D through simulations. For example, we used Python 3.8.8 software to approximate the values of I_2, U_1, I_3, and V_1 for the unit disc. The code for the simulations can be found here: http://rb.gy/1wei7h. Expressions of all the new invariants in terms of r for a disc of radius r are established in the current section.

Let D be the disc of radius r centred at the origin. For $g_1 \cap g_2 \in D$, we consider g_1 to be the horizontal line $y = -r \sin a$ $(0 < a < \pi/2)$ in the Cartesian plane, and g_2 the line that passes through the points $(r\cos w_1, r\sin w_1)$ and $(r\cos w_2, r\sin w_2)$, where $-a < w_1 < \pi + a < w_2 < 2\pi - a$ (see Figure 6).

Then

$$\mathrm{d}g_1 = 2\pi r \cos a \,\mathrm{d}a \quad \text{and} \quad \mathrm{d}g_2 = \frac{1}{2} r \sin \frac{w_2 - w_1}{2} \,\mathrm{d}w_1 \,\mathrm{d}w_2,$$

and therefore

$$\mathrm{d}g_1 \,\mathrm{d}g_2 = \pi r^2 \cos a \sin \frac{w_2 - w_1}{2} \,\mathrm{d}w_1 \,\mathrm{d}w_2. \tag{6.1}$$

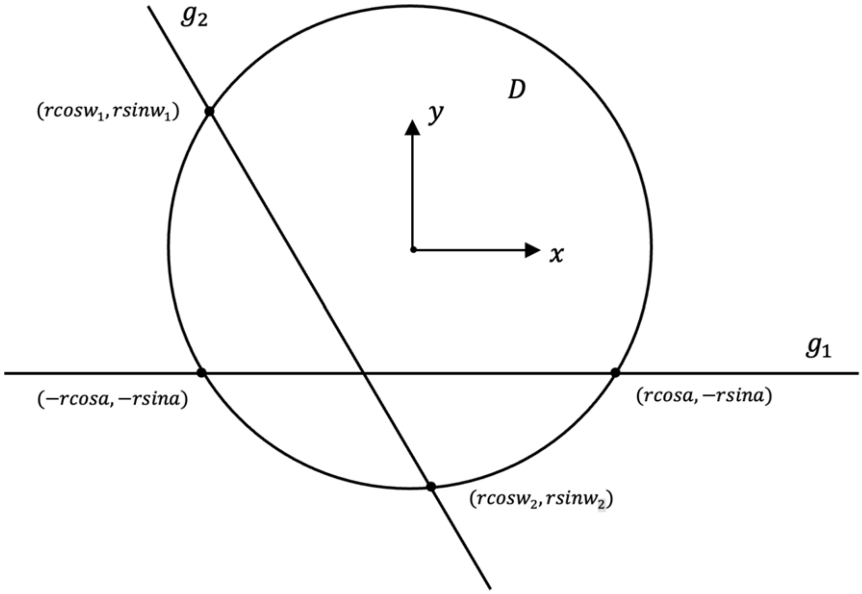

FIGURE 6. The model of two random lines g_1 and g_2 intersecting each other inside $D = \{(x, y) : x^2 + y^2 < r^2\}$.

Lemma 6.1. *If D is a disc with radius r then*

$$U_1 = \left(2\pi^3 + \frac{32}{3}\pi\right) r^3.$$

Proof. It is easy to verify that

$$u(g_1, g_2) = 2r\left(\cos\frac{w_1 - a}{2} - \cos\frac{w_2 - a}{2} + \sin\frac{w_1 + a}{2} + \sin\frac{w_2 + a}{2}\right).$$

Then by (6.1) we obtain

$$U_1 = 2\pi r^3 \int_0^{\pi/2} \cos a \, \mathrm{d}a \int_{-a}^{\pi+a} \mathrm{d}w_1 \int_{\pi+a}^{2\pi-a} \left(\cos\frac{w_1 - a}{2} - \cos\frac{w_2 - a}{2} + \sin\frac{w_1 + a}{2} + \sin\frac{w_2 + a}{2}\right) \sin\frac{w_2 - w_1}{2} \, \mathrm{d}w_2,$$

and reduce it to

$$U_1 = 8\pi^2 r^3 \int_0^{\pi/2} \cos^2 a \, \mathrm{d}a + 16\pi r^3 \int_0^{\pi/2} \cos^3 a \, \mathrm{d}a = \left(2\pi^3 + \frac{32}{3}\pi\right) r^3, \tag{6.2}$$

thus completing the proof. □

Corollary 6.1. *If D is a disc with radius r then*

$$p_{33} = p_{30} = \frac{4}{\pi^2} - \frac{1}{4} \quad \textit{and} \quad p_{32} = p_{31} = \frac{3}{4} - \frac{4}{\pi^2}.$$

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Proof. The proof immediately follows from (1.1), the identity $I_2 = \frac{16}{3}\pi r^3$ (see [8]), and Lemma 6.1. □

The above technique can be applied to reveal the exact numerical values of further invariants of D.

Lemma 6.2. *If D is a disc with radius r then*

$$D_2 = \left(\frac{2}{3}\pi^4 + 17\pi^2\right) r^4, \tag{6.3}$$

$$U_2 = \left(\frac{2}{3}\pi^4 + 41\pi^2\right) r^4, \tag{6.4}$$

$$C_2 = \left(\frac{10}{3}\pi^4 - \frac{73}{3}\pi^2\right) r^4, \tag{6.5}$$

$$V_1 = (41\pi^2 - 2\pi^4) r^4, \tag{6.6}$$

$$R = \left(\frac{2}{3}\pi^4 + \frac{47}{3}\pi^2\right) r^4, \tag{6.7}$$

$$Q_s = \left(\frac{2}{3}\pi^4 - 2\pi^2\right) r^4, \tag{6.8}$$

$$Q_d = \left(\frac{2}{3}\pi^4 + \frac{11}{3}\pi^2\right) r^4. \tag{6.9}$$

Proof. For $g_1 \cap g_2 \in D$, the lengths of intersecting chords are

$$|\chi(g_1)| = 2r\cos a \quad \text{and} \quad |\chi(g_2)| = 2r\sin\frac{w_2 - w_1}{2}.$$

Since

$$D_2 = 4I_3 + 2\int_{g_1 \cap g_2 \in D} |\chi(g_1)||\chi(g_2)|\, dg_1\, dg_2,$$

then due to (6.1), we obtain (hereafter, long intermediate steps of integration are omitted)

$$\begin{aligned} D_2 &= 12\pi^2 r^4 + 8\pi r^4 \int_0^{\pi/2} \cos^2 a\, da \int_{-a}^{\pi+a} dw_1 \int_{\pi+a}^{2\pi-a} \sin^2\frac{w_2 - w_1}{2}\, dw_2 \\ &= \left(\frac{2}{3}\pi^4 + 17\pi^2\right) r^4. \end{aligned}$$

Evaluation of U_2 is provided by

$$\begin{aligned} U_2 = 4\pi r^4 &\int_0^{\pi/2} \cos a\, da \int_{-a}^{\pi+a} dw_1 \int_{\pi+a}^{2\pi-a} \left(\cos\frac{w_1 - a}{2} - \cos\frac{w_2 - a}{2}\right. \\ &\left. + \sin\frac{w_1 + a}{2} + \sin\frac{w_2 + a}{2}\right)^2 \sin\frac{w_2 - w_1}{2}\, dw_2 \\ &= \left(\frac{2}{3}\pi^4 + 41\pi^2\right) r^4. \end{aligned}$$

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Now (6.5) follows from (6.3), (6.4), the identity

$$C_2 = 4D_2 + U_2 - 8 \cdot \int_{g_1 \cap g_2 \in D} |\chi(g_1)| u(g_1, g_2)\, \mathrm{d}g_1\, \mathrm{d}g_2,$$

and (see (6.2))

$$\begin{aligned}\int_{g_1 \cap g_2 \in D} |\chi(g_1)| u(g_1, g_2)\, \mathrm{d}g_1\, \mathrm{d}g_2 &= 16r^4 \biggl(\pi^2 \int_0^{\pi/2} \cos^3 a\, \mathrm{d}a + 2\pi \int_0^{\pi/2} \cos^4 a\, \mathrm{d}a\biggr) \\ &= \frac{50}{3}\pi^2 r^4.\end{aligned}$$

V_1 is an integral over three lines that produce three intersection points inside D. To compute V_1 with the technique already developed, we need to represent it by an integral over $g_1 \cap g_2$. We complete this task in two steps. First we apply the combinatorial algorithm to suggest an alternative expression for the left-hand side of (3.3):

$$\begin{aligned}&\int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \sum_{\chi_{12} \in \langle g_1, g_2, g_3\rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 \\ &= 3 \int_{g_1 \cap g_2 \in D} \mathrm{d}g_1\, \mathrm{d}g_2 \int_{[\chi(g_1)] \cap [\chi(g_2)]} \sum_{\chi_{12} \in \langle g_1, g_2\rangle} |\chi_{12}| I_1(\chi_{12})\, \mathrm{d}g_3 \\ &= 3 \int_{g_1 \cap g_2 \in D} \{(\rho_1 + \rho_3)[|\chi(g_1)| + |\chi(g_2)| - \rho_2 - \rho_4] + (\rho_2 + \rho_4) \\ &\qquad \times [|\chi(g_1)| + |\chi(g_2)| - \rho_1 - \rho_3]\}\, \mathrm{d}g_1\, \mathrm{d}g_2 \\ &= 6 \int_{g_1 \cap g_2 \in D} \{|\chi(g_1)| u(g_1, g_2) - (\rho_1 + \rho_3)(\rho_2 + \rho_4)\}\, \mathrm{d}g_1\, \mathrm{d}g_2 \\ &= 100\pi^2 r^4 - 6 \int_{g_1 \cap g_2 \in D} (\rho_1 + \rho_3)(\rho_2 + \rho_4)\, \mathrm{d}g_1\, \mathrm{d}g_2.\end{aligned}$$

In the second step we make the obtained expression equal to the right-hand side of (3.3), substitute C_2, D_2, and U_2 with their known values, and calculate

$$V_1 = 6 \int_{g_1 \cap g_2 \in D} (\rho_1 + \rho_3)(\rho_2 + \rho_4)\, \mathrm{d}g_1\, \mathrm{d}g_2 - \bigl(23\pi^2 + 2\pi^4\bigr) r^4. \tag{6.10}$$

The pairs of the lengths of the opposite sides in the quadrilateral $\mathrm{conv}((g_1 \cup g_2) \cap \partial D)$ are

$$2r \cos \frac{w_1 - a}{2}, \quad 2r \sin \frac{w_2 + a}{2} \quad \text{and} \quad -2r \cos \frac{w_2 - a}{2}, \quad 2r \sin \frac{w_1 + a}{2}.$$

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This enables us to compute the integral in (6.10):

$$\begin{aligned}
&\int_{g_1\cap g_2\in D}(\rho_1+\rho_3)(\rho_2+\rho_4)\,\mathrm{d}g_1\,\mathrm{d}g_2\\
&=4\pi r^4\int_0^{\pi/2}\cos a\,\mathrm{d}a\int_{-a}^{\pi+a}\mathrm{d}w_1\\
&\quad\times\int_{\pi+a}^{2\pi-a}\bigg(\cos\frac{w_1-a}{2}+\sin\frac{w_2+a}{2}\bigg)\bigg(\sin\frac{w_1+a}{2}-\cos\frac{w_2-a}{2}\bigg)\sin\frac{w_2-w_1}{2}\,\mathrm{d}w_2\\
&=4\pi r^4\int_0^{\pi/2}\bigg(\frac{16}{3}\cos^2 a+2\pi\cos^3 a\bigg)\,\mathrm{d}a\\
&=\frac{32}{3}\pi^2 r^4.
\end{aligned}$$

Now (6.6) follows from (6.10). We omit the proof of (6.7) since we have already established the expressions of ρ_i in terms of parameters a, w_1, w_2.

The proofs for (6.8) and (6.9) are similar. Let us prove the first. To model the case $g_1\cap g_2\notin D$, we simply need to adjust the position of parameters w_1 and w_2. They must now satisfy either $-a<w_1<w_2<\pi+a$, or $\pi+a<w_1<w_2<2\pi-a$.

If $-a<w_1<w_2<\pi+a$, then

$$s_1+s_2=2r\bigg(\sin\frac{w_1+a}{2}+\cos\frac{w_2-a}{2}\bigg),\qquad d_1+d_2=2r\bigg(\cos\frac{w_1-a}{2}+\sin\frac{w_2+a}{2}\bigg).$$

If $\pi+a<w_1<w_2<2\pi-a$, then

$$s_1+s_2=2r\bigg(\sin\frac{w_2+a}{2}-\cos\frac{w_1-a}{2}\bigg),\qquad d_1+d_2=2r\bigg(\sin\frac{w_1+a}{2}-\cos\frac{w_2-a}{2}\bigg).$$

As a result,

$$\begin{aligned}
Q_s&=4\pi r^4\int_0^{\pi/2}\cos a\,\mathrm{d}a\int_{-a}^{\pi+a}\mathrm{d}w_1\int_{w_1}^{\pi+a}\bigg(\sin\frac{w_1+a}{2}+\cos\frac{w_2-a}{2}\bigg)\\
&\qquad\times\bigg(\cos\frac{w_1-a}{2}+\sin\frac{w_2+a}{2}-\sin\frac{w_1+a}{2}-\cos\frac{w_2-a}{2}\bigg)\sin\frac{w_2-w_1}{2}\,\mathrm{d}w_2\\
&\qquad+4\pi r^4\int_0^{\pi/2}\cos a\,\mathrm{d}a\int_{\pi+a}^{2\pi-a}\mathrm{d}w_1\int_{w_1}^{2\pi-a}\bigg(\sin\frac{w_2+a}{2}-\cos\frac{w_1-a}{2}\bigg)\\
&\qquad\times\bigg(\sin\frac{w_1+a}{2}-\cos\frac{w_2-a}{2}-\sin\frac{w_2+a}{2}+\cos\frac{w_1-a}{2}\bigg)\sin\frac{w_2-w_1}{2}\,\mathrm{d}w_2\\
&=\bigg(\frac{2}{3}\pi^4-2\pi^2\bigg)r^4.
\end{aligned}$$

□

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The following results directly follow from the last two lemmas and the theorems proved in Sections 3 and 4.

Theorem 6.1. *If D is a disc with radius r, then*

$$p_{46} = \frac{1}{4} - \frac{17}{\pi^2}, \quad p_{45} = \frac{29}{8\pi^2} - \frac{1}{4},$$

$$p_{44} = \frac{43}{4\pi^2} - \frac{7}{8}, \quad p_{44}^{(1)} = \frac{23}{2\pi^2} - 1, \quad p_{44}^{(2)} = \frac{1}{8} - \frac{3}{4\pi^2},$$

$$p_{43} = 1 - \frac{29}{4\pi^2}, \quad p_{43}^{(1)} = \frac{23}{4\pi^2} - \frac{1}{2}, \quad p_{43}^{(2)} = 1 - \frac{35}{4\pi^2}, \quad p_{43}^{(3)} = \frac{1}{2} - \frac{17}{4\pi^2},$$

$$p_{42} = \frac{7}{4} - \frac{121}{8\pi^2}, \quad p_{42}^{(1)} = \frac{3}{2} - \frac{13}{\pi^2}, \quad p_{42}^{(2)} = \frac{1}{4} - \frac{17}{8\pi^2},$$

$$p_{41} = \frac{29}{8\pi^2} - \frac{1}{4}, \quad p_{40} = \frac{13}{2\pi^2} - \frac{5}{8}.$$

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Competing interests

There were no competing interests to declare which arose during the preparation or publication process of this article.

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