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Spectral properties of the Neumann–Poincaré operator and cloaking by anomalous localized resonance for the elasto-static system†

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We first investigate spectral properties of the Neumann–Poincaré (NP) operator for the Lamé system of elasto-statics. We show that the elasto-static NP operator can be symmetrized in the same way as that for the Laplace operator. We then show that even if elasto-static NP operator is not compact even on smooth domains, it is polynomially compact and its spectrum on two-dimensional smooth domains consists of eigenvalues that accumulate to two different points determined by the Lamé constants. We then derive explicitly eigenvalues and eigenfunctions on discs and ellipses. Using these resonances occurring at eigenvalues is considered. We also show on ellipses that cloaking by anomalous localized resonance takes place at accumulation points of eigenvalues.

Key words: Neumann-Poincaré operator, Lamé system, linear elasticity, spectrum, resonance, cloaking by anomalous localized resonance.

1 Introduction

The Neumann–Poincaré (NP) operator for the Laplace operator is a boundary integral operator that appears naturally when solving classical boundary value problems for the Laplace equation using layer potentials. Recently, there is rapidly growing interest in the spectral properties of the NP operator in relation to plasmonics and cloaking by anomalous localized resonance (CALR). Plasmon resonance and anomalous localized resonance occur at eigenvalues and at the accumulation point of eigenvalues, respectively (see for example [1, 17] and references therein). We emphasize that the spectral nature

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of the NP operator differs depending on smoothness of the domain on which the NP operator is defined. If the domain has a smooth boundary, $C^{1,\alpha}$ for some $\alpha > 0$ to be precise, then the NP operator is compact on L^2 or $H^{-1/2}$ space. Since the NP operator can be realized as a self-adjoint operator by introducing a new inner product (see [10,12]), its spectrum consists of eigenvalues converging to 0. If the domain has a corner, the corresponding NP operator may exhibit a continuous spectrum. For this and recent development of spectral theory of the NP operator for the Laplace operator, we refer to [9,11,21,22] and references therein.

The purpose of this paper is two-fold. We first extend the spectral theory of the NP operator for the Laplace operator to that for the Lamé system of elasto-statics, and then investigate resonance and CALR.

To describe results of this paper in a precise manner, we first introduce some notation. Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with the Lipschitz boundary, and let (λ, μ) be the Lamé constants for Ω satisfying the strong convexity condition

$$\mu > 0 \quad \text{and} \quad d\lambda + 2\mu > 0. \tag{1.1}$$

The isotropic elasticity tensor $\mathbb{C} = (C_{ijkl})_{i,j,k,l=1}^d$ and the corresponding elastostatic system $\mathcal{L}_{\lambda,\mu}$ are defined by

$$C_{ijkl} := \lambda\, \delta_{ij}\delta_{kl} + \mu\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{1.2}$$

and

$$\mathcal{L}_{\lambda,\mu}\mathbf{u} := \nabla \cdot \mathbb{C}\widehat{\nabla}\mathbf{u} = \mu\Delta\mathbf{u} + (\lambda + \mu)\nabla\nabla \cdot \mathbf{u}, \tag{1.3}$$

where $\widehat{\nabla}$ denotes the symmetric gradient, namely,

$$\widehat{\nabla}\mathbf{u} := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right).$$

Here, T indicates the transpose of a matrix. The corresponding conormal derivative on $\partial\Omega$ is defined to be

$$\partial_\nu\mathbf{u} := (\mathbb{C}\widehat{\nabla}\mathbf{u})\mathbf{n} = \lambda(\nabla \cdot \mathbf{u})\mathbf{n} + 2\mu(\widehat{\nabla}\mathbf{u})\mathbf{n} \quad \text{on } \partial\Omega, \tag{1.4}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$.

Let $\Gamma = \left(\Gamma_{ij}\right)_{i,j=1}^d$ be the Kelvin matrix of fundamental solutions to the Lamé operator $\mathcal{L}_{\lambda,\mu}$, namely,

$$\Gamma_{ij}(\mathbf{x}) = \begin{cases} -\dfrac{\alpha_1}{4\pi}\dfrac{\delta_{ij}}{|\mathbf{x}|} - \dfrac{\alpha_2}{4\pi}\dfrac{x_i x_j}{|\mathbf{x}|^3}, & \text{if } d = 3, \\ \dfrac{\alpha_1}{2\pi}\delta_{ij}\ln|\mathbf{x}| - \dfrac{\alpha_2}{2\pi}\dfrac{x_i x_j}{|\mathbf{x}|^2}, & \text{if } d = 2, \end{cases} \tag{1.5}$$

where

$$\alpha_1 = \frac{1}{2}\left(\frac{1}{\mu} + \frac{1}{2\mu + \lambda}\right) \quad \text{and} \quad \alpha_2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{2\mu + \lambda}\right). \tag{1.6}$$

The NP operator for the Lamé system is defined by

$$\mathbf{K}[\mathbf{f}](\mathbf{x}) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_\mathbf{y}}\Gamma(\mathbf{x} - \mathbf{y})\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}) \quad \text{a.e. } \mathbf{x} \in \partial\Omega. \tag{1.7}$$

Here, p.v. stands for the Cauchy principal value, and the conormal derivative $\partial_{\nu_y}\Gamma(\mathbf{x}-\mathbf{y})$ of the Kelvin matrix with respect to $\mathbf{y}$-variables is defined by

$$\partial_{\nu_y}\Gamma(\mathbf{x}-\mathbf{y})\mathbf{b} = \partial_{\nu_y}(\Gamma(\mathbf{x}-\mathbf{y})\mathbf{b}) \tag{1.8}$$

for any constant vector $\mathbf{b}$.

The NP operator $\mathbf{K}$ is connected to the Lamé system $\mathcal{L}_{\lambda,\mu}$ in the following way. The Dirichlet boundary value problem for the Lamé system

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u} = \mathbf{g} & \text{on } \partial\Omega \end{cases} \tag{1.9}$$

can be solved using the double layer potential, namely, $\mathbf{u} = \mathbf{D}[\mathbf{f}]$ where

$$\mathbf{D}[\mathbf{f}](\mathbf{x}) := \int_{\partial\Omega} \partial_{\nu_y}\Gamma(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \partial\Omega \tag{1.10}$$

for some potential $\mathbf{f}$ on $\partial\Omega$. In fact, such $\mathbf{u}$ satisfies $\mathcal{L}_{\lambda,\mu}\mathbf{u} = 0$ in Ω. So, to solve (1.9), the boundary condition should be fulfilled. It is known (see [6]) that the following jump formula holds:

$$\mathbf{D}[\mathbf{f}]|_{\pm} = \left(\mp\frac{1}{2}I + \mathbf{K}\right)[\mathbf{f}] \quad \text{a.e. on } \partial\Omega. \tag{1.11}$$

Here and afterwards, the subscripts $+$ and $-$ indicate the limits (to $\partial\Omega$) from outside and inside Ω, respectively. So, (1.9) is solved by finding the solution of the integral equation

$$\left(\frac{1}{2}I + \mathbf{K}\right)[\mathbf{f}] = \mathbf{g} \quad \text{on } \partial\Omega. \tag{1.12}$$

In this paper, we show that the NP operator $\mathbf{K}$ can be realized as a self-adjoint operator on $H^{1/2}(\partial\Omega)^d$ ($H^{1/2}$ is a Sobolev space) by introducing a new inner product in a way parallel to the case of the Laplace operator. But, there is a significant difference between NP operators for the Laplace operator and the Lamé operator. The NP operator for the Lamé operator is *not* compact even if the domain has a smooth boundary (this was observed in [6] correcting an error in [15]), which means that we cannot infer directly that the NP operator has point spectrum (eigenvalues). However, we are able to show in this paper that the elasto-static NP operator on planar domains with $C^{1,\alpha}$ boundaries has only point spectrum. In fact, we show that on such domains

$$\mathbf{K}^2 - k_0^2 I \quad \text{is compact,} \tag{1.13}$$

where

$$k_0 = \frac{\mu}{2(2\mu+\lambda)}. \tag{1.14}$$

It is worth mentioning that we are able to prove (1.13) only in two dimensions, and it is not clear if it is true in three dimensions. Probably, there is a polynomial p such that $p(\mathbf{K})$ is compact. As an immediate consequence of (1.13), we show that the spectrum of $\mathbf{K}$ consists of eigenvalues that accumulate at k_0 and $-k_0$. We then explicitly compute

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eigenvalues of $\mathbf{K}$ on discs and ellipses. It turns out that k_0 and $-k_0$ are eigenvalues of infinite multiplicities (there are two other eigenvalues of finite multiplicities) on discs, while on ellipses k_0 and $-k_0$ are accumulation points of eigenvalues, but not eigenvalues, and the rates of convergence to k_0 and $-k_0$ are exponential.

Using the spectral properties of the NP operator, we investigate resonance, especially CALR. CALR on dielectric plasmonic material was first discovered in [18]. It is shown that if we coat a dielectric material of circular shape by a plasmonic material of negative dielectric constant (with a dissipation), then huge resonance occurs and the polarizable dipole is cloaked when it is within the cloaking region. This result has been extended to general sources other than polarizable dipole sources [1, 14]. It is also shown in [3] that CALR occurs not only on the coated structure, but also on ellipses.

In this paper, we show that CALR also occurs on elastic structures. We consider an ellipse Ω embedded in $\mathbb{R}^2$, where the Lamé constants of the background are (λ, μ) and those of Ω are $(c + i\delta)(\lambda, \mu)$. Here, c is a negative constant and δ is a loss parameter that tends to 0. So, Ω represents an elastic material with negative Lamé constants. Discussion on the existence of such materials is beyond the scope of this paper. However, we refer to [16] for existence (in composites) of negative stiffness material, and to [13] for effective properties. We show that if c satisfies

$$k(c) := \frac{c+1}{2(c-1)} = k_0 \quad \text{or} \quad -k_0, \tag{1.15}$$

then CALR takes place as $\delta \to 0$. See Section 4 for the precise description of CALR with estimates. Here, we highlight a few points. In dielectric case, CALR occurs when $k(c) = 0$ or $c = -1$ since 0 is the accumulation point of eigenvalues. In the elasto-static case, (1.15) is fulfilled if (and only if)

$$c = -\frac{\lambda + 3\mu}{\lambda + \mu} \quad \text{or} \quad c = -\frac{\lambda + \mu}{\lambda + 3\mu}. \tag{1.16}$$

It turns out that the cloaking region when $k(c) = k_0$ is different from that when $k(c) = -k_0$. We also mention that since $0 < k_0 < 1/2$, (1.15) holds only if $c < 0$. The inclusion Ω is assumed to be elliptic shape since eigenvalues and eigenfunctions of the NP operator can be explicitly computed. We emphasize that anomalous localized resonance does not occur on a disc since there k_0 and $-k_0$ are (isolated) eigenvalues of the corresponding NP operator.

The rest of this paper is organized as follows. In Section 2, we show that the elasto-static NP operator can be symmetrized by introducing a proper inner product on $H^{1/2}$-space. In Section 3, we prove (1.13), and as a consequence that the NP operator on a smooth domain has eigenvalues accumulating to $\pm k_0$. We also present eigenvalues of the NP operator on discs and ellipses in Section 3, whose proofs are given in Appendix A. Section 4 is to investigate the anomalous localized resonance whose detail is given in Appendix B.

2 Spectral properties of the NP operator

2.1 Layer potentials and the NP operator

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d = 2, 3$) with the Lipschitz boundary. Let $H^{1/2}(\partial\Omega)$ denote the usual L^2-Sobolev space of order $1/2$, namely, the collection of all φ on $\partial\Omega$ satisfying

$$\int_{\partial\Omega}\int_{\partial\Omega} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^d} d\sigma(x) d\sigma(y) < \infty, \tag{2.1}$$

and $H^{-1/2}(\partial\Omega)$ its dual space. Let $\mathcal{H} := H^{1/2}(\partial\Omega)^d$ and $\mathcal{H}^* := H^{-1/2}(\partial\Omega)^d$. The duality pairing of $\mathcal{H}^*$ and $\mathcal{H}$ is denoted by $\langle \cdot, \cdot \rangle$. Let Ψ be the set of all functions $\mathbf{v} = (v_1, \ldots, v_d)^T$ such that

$$\partial_j v_i + \partial_i v_j = 0 \quad \text{in } \Omega, \quad 1 \leqslant i, j \leqslant d. \tag{2.2}$$

Observe that Ψ in two dimensions is spanned by

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} y \\ -x \end{bmatrix}, \tag{2.3}$$

and in three dimensions, it is spanned by

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}, \quad \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}, \quad \begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}. \tag{2.4}$$

It is worth mentioning that if $\mathbf{v} \in \Psi$, then $\mathbf{v}$ satisfies $\mathcal{L}_{\lambda,\mu}\mathbf{v} = 0$ in Ω and $\partial_\nu \mathbf{v} = 0$ on $\partial\Omega$, and the converse holds.

Define

$$\mathcal{H}^*_\Psi := \{\boldsymbol{\varphi} \in \mathcal{H}^* \;:\; \langle \boldsymbol{\varphi}, \mathbf{f} \rangle = 0 \text{ for all } \mathbf{f} = \mathbf{v}|_{\partial\Omega}, \quad \mathbf{v} \in \Psi\}. \tag{2.5}$$

Since Ψ contains constant functions, we have, in particular, $\int_{\partial D} \boldsymbol{\varphi} d\sigma = 0$ if $\boldsymbol{\varphi} \in \mathcal{H}^*_\Psi$. We emphasize that if $\mathcal{L}_{\lambda,\mu}\mathbf{u} = 0$ in Ω, then $\partial_\nu \mathbf{u} \in \mathcal{H}^*_\Psi$. In fact, if $\mathbf{f} = \mathbf{v}|_{\partial\Omega}$ for some $\mathbf{v} \in \Psi$, then

$$\int_{\partial\Omega} \partial_\nu \mathbf{u} \cdot \mathbf{f} d\sigma = \int_{\partial\Omega} \partial_\nu \mathbf{u} \cdot \mathbf{f} d\sigma - \int_{\partial\Omega} \mathbf{u} \cdot \partial_\nu \mathbf{v} d\sigma = 0.$$

In addition to double layer potentials in (1.10), the single layer potential on $\partial\Omega$ associated with the Lamé parameter (λ, μ) is defined by

$$\mathbf{S}[\boldsymbol{\varphi}](\mathbf{x}) := \int_{\partial\Omega} \Gamma(\mathbf{x} - \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y}) d\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^d$$

for $\boldsymbol{\varphi} \in \mathcal{H}^*$, where Γ is the Kelvin matrix defined in (1.5). Like (1.11), the single layer potential enjoys the following jump relation:

$$\partial_\nu \mathbf{S}[\boldsymbol{\varphi}]|_\pm = \left(\pm \frac{1}{2} I + \mathbf{K}^* \right) [\boldsymbol{\varphi}] \quad \text{a.e. on } \partial\Omega, \tag{2.6}$$

where $\mathbf{K}^*$ is the adjoint operator of $\mathbf{K}$ on $L^2(\partial\Omega)^d$, that is,

$$\mathbf{K}^*[\boldsymbol{\varphi}](\mathbf{x}) := \text{p.v.} \int_{\partial\Omega} \partial_{\nu_{\mathbf{x}}} \Gamma(\mathbf{x}-\mathbf{y})\boldsymbol{\varphi}(\mathbf{y})d\sigma(\mathbf{y}) \quad \text{a.e. } \mathbf{x} \in \partial\Omega. \tag{2.7}$$

The operator $\mathbf{K}^*$ is also called the (elasto-static) NP operator on $\partial\Omega$.

The following lemma collects some facts to be used in the sequel, proofs of which can be found in [5, 6, 19].

Lemma 2.1

(i) $\mathbf{K}$ *is bounded on* $\mathcal{H}$*, and* $\mathbf{K}^*$ *is on* $\mathcal{H}^*$*.*
(ii) *The spectrum of* $\mathbf{K}^*$ *on* $\mathcal{H}^*$ *lies in* $(-1/2, 1/2]$.
(iii) $1/2I - \mathbf{K}^*$ *is invertible on* $\mathcal{H}^*_\Psi$*.*
(iv) $\mathbf{S}$ *as an operator defined on* $\partial\Omega$ *is bounded from* $\mathcal{H}^*$ *into* $\mathcal{H}$*.*
(v) $\mathbf{S} : \mathcal{H}^* \to \mathcal{H}$ *is invertible in three dimensions.*

In two dimensions $\mathbf{S}$ may not be invertible. In fact, there may be a bounded domain Ω on which $\mathbf{S}[\boldsymbol{\varphi}] = 0$ on $\partial\Omega$ for some $\boldsymbol{\varphi} \neq 0$ (see the next subsection). It is worthwhile mentioning that there is such a domain for the Laplace operator [23].

Lemma 2.2 *Ψ is the eigenspace of* $\mathbf{K}$ *on* $\mathcal{H}$ *corresponding to* $1/2$.

Proof Let $\mathbf{f} \in \Psi$. Then, $\mathbf{f} = \mathbf{v}|_{\partial\Omega}$ where $\mathbf{v}$ satisfies $\mathcal{L}_{\lambda,\mu}\mathbf{v} = 0$ in Ω and $\partial_\nu \mathbf{v} = 0$ on $\partial\Omega$. So, we have for $\mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}$

$$\begin{aligned}\mathbf{D}[\mathbf{f}](\mathbf{x}) &= \int_{\partial\Omega} \partial_{\nu_{\mathbf{y}}} \Gamma(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}) \\ &= \int_{\partial\Omega} \left[\partial_{\nu_{\mathbf{y}}} \Gamma(\mathbf{x}-\mathbf{y})\mathbf{v}(\mathbf{y}) - \Gamma(\mathbf{x}-\mathbf{y})\partial_\nu \mathbf{v}(\mathbf{y})\right] d\sigma(\mathbf{y}) = 0.\end{aligned}$$

So we infer from (1.11) that

$$\mathbf{K}[\mathbf{f}] = \frac{1}{2}\mathbf{f}. \tag{2.8}$$

Conversely, if (2.8) holds, then we have from (1.11) that $\mathbf{D}[\mathbf{f}]|_- = \mathbf{f}$ and $\mathbf{D}[\mathbf{f}](\mathbf{x}) = 0$ for $\mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}$. So $\partial_\nu \mathbf{D}[\mathbf{f}]|_- = \partial_\nu \mathbf{D}[\mathbf{f}]|_+ = 0$. It implies that $\mathbf{f} \in \Psi$. This completes the proof. □

Let $N_d := \frac{d(d+1)}{2}$, which is the dimension of Ψ. Let $\{\mathbf{f}^{(j)}\}_{j=1}^{N_d}$ be a basis of Ψ such that

$$\langle \mathbf{f}^{(i)}, \mathbf{f}^{(j)} \rangle = \delta_{ij}, \tag{2.9}$$

where δ_{ij} is Kronecker's delta. Since $\partial_\nu \mathbf{S}[\mathbf{f}^{(j)}]|_- \in \mathcal{H}^*_\Psi$ and $1/2I - \mathbf{K}^*$ is invertible on $\mathcal{H}^*_\Psi$, there is a unique $\widetilde{\boldsymbol{\varphi}}^{(j)} \in \mathcal{H}^*_\Psi$ such that

$$\left(\frac{1}{2}I - \mathbf{K}^*\right)[\widetilde{\boldsymbol{\varphi}}^{(j)}] = \partial_\nu \mathbf{S}[\mathbf{f}^{(j)}]|_- = \left(-\frac{1}{2}I + \mathbf{K}^*\right)[\mathbf{f}^{(j)}].$$

Define $\boldsymbol{\varphi}^{(j)} := \widetilde{\boldsymbol{\varphi}}^{(j)} + \mathbf{f}^{(j)}$. Then, we have

$$\mathbf{K}^*[\boldsymbol{\varphi}^{(j)}] = \frac{1}{2}\boldsymbol{\varphi}^{(j)}. \tag{2.10}$$

Moreover, we have

$$\langle \boldsymbol{\varphi}^{(j)}, \mathbf{f}^{(i)} \rangle = \langle \widetilde{\boldsymbol{\varphi}}^{(j)}, \mathbf{f}^{(i)} \rangle + \langle \mathbf{f}^{(j)}, \mathbf{f}^{(i)} \rangle = \delta_{ij}, \tag{2.11}$$

which, in particular, implies that $\boldsymbol{\varphi}^{(j)}$'s are linearly independent.

Let

$$W := \text{span}\left\{\boldsymbol{\varphi}^{(1)}, \ldots, \boldsymbol{\varphi}^{(N_d)}\right\}, \tag{2.12}$$

and let

$$\mathcal{H}_W := \{\mathbf{f} \in \mathcal{H} \ : \ \langle \boldsymbol{\varphi}, \mathbf{f} \rangle = 0 \text{ for all } \boldsymbol{\varphi} \in W\}. \tag{2.13}$$

Lemma 2.3 *The following hold.*

(i) *Each $\boldsymbol{\varphi} \in \mathcal{H}^*$ is uniquely decomposed as*

$$\boldsymbol{\varphi} = \boldsymbol{\varphi}' + \boldsymbol{\varphi}'' := \boldsymbol{\varphi}' + \sum_{j=1}^{N_d} \langle \boldsymbol{\varphi}, \mathbf{f}^{(j)} \rangle \boldsymbol{\varphi}^{(j)}, \tag{2.14}$$

and $\boldsymbol{\varphi}' \in \mathcal{H}^*_\Psi$.

(ii) *Each $\mathbf{f} \in \mathcal{H}$ is uniquely decomposed as*

$$\mathbf{f} = \mathbf{f}' + \mathbf{f}'' := \mathbf{f}' + \sum_{j=1}^{N_d} \langle \boldsymbol{\varphi}^{(j)}, \mathbf{f} \rangle \mathbf{f}^{(j)}, \tag{2.15}$$

and $\mathbf{f}' \in \mathcal{H}_W$.

(iii) $\mathbf{S}$ *maps W into Ψ, and $\mathcal{H}^*_\Psi$ into $\mathcal{H}_W$.*

(iv) *W is the eigenspace of $\mathbf{K}^*$ corresponding to the eigenvalue* $1/2$.

Proof For $\boldsymbol{\varphi} \in \mathcal{H}^*$, and let $\boldsymbol{\varphi}''$ be as in (2.14). Then, one can immediately see from (2.11) that $\langle \boldsymbol{\varphi}', \mathbf{f}^{(j)} \rangle = 0$ for all j, and hence $\boldsymbol{\varphi}' \in \mathcal{H}^*_\Psi$. Uniqueness of the decomposition can be proved easily. (ii) can be proved similarly.

Thanks to (2.10), we have $\partial_\nu \mathbf{S}[\boldsymbol{\varphi}^{(j)}]|_- = 0$, and so $\mathbf{S}[\boldsymbol{\varphi}^{(j)}]|_{\partial\Omega} \in \Psi$. If $\boldsymbol{\varphi} \in \mathcal{H}^*_\Psi$, then

$$\langle \boldsymbol{\varphi}^{(j)}, \mathbf{S}[\boldsymbol{\varphi}] \rangle = \langle \boldsymbol{\varphi}, \mathbf{S}[\boldsymbol{\varphi}^{(j)}] \rangle = 0$$

for all j. So, $\mathbf{S}$ maps $\mathcal{H}^*_\Psi$ into $\mathcal{H}_W$. This proves (iii).

Suppose that $\mathbf{K}^*[\boldsymbol{\varphi}] = 1/2\boldsymbol{\varphi}$ and that $\boldsymbol{\varphi}$ admits the decomposition (2.14). Then, $\mathbf{K}^*[\boldsymbol{\varphi}'] = 1/2\boldsymbol{\varphi}'$. So we have from (iii) that $\mathbf{S}[\boldsymbol{\varphi}'] \in \Psi$, and hence $\langle \boldsymbol{\varphi}', \mathbf{S}[\boldsymbol{\varphi}'] \rangle = 0$. Since $\int_{\partial\Omega} \boldsymbol{\varphi}' d\sigma = 0$, we have from (2.6)

$$\begin{aligned} -\langle \boldsymbol{\varphi}', \mathbf{S}[\boldsymbol{\varphi}'] \rangle &= \langle \partial_\nu \mathbf{S}[\boldsymbol{\varphi}']|_-, \mathbf{S}[\boldsymbol{\varphi}'] \rangle - \langle \partial_\nu \mathbf{S}[\boldsymbol{\varphi}']|_+, \mathbf{S}[\boldsymbol{\varphi}'] \rangle \\ &= \|\nabla \mathbf{S}[\boldsymbol{\varphi}']\|^2_{L^2(\Omega)} + \|\nabla \mathbf{S}[\boldsymbol{\varphi}']\|^2_{L^2(\mathbb{R}^d \setminus \Omega)}. \end{aligned} \tag{2.16}$$

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So $\mathbf{S}[\varphi'] = \text{const.}$ in $\mathbb{R}^d$. Thus, we have $\varphi' = \partial_\nu \mathbf{S}[\varphi']|_+ - \partial_\nu \mathbf{S}[\varphi']|_- = 0$, and hence $\varphi \in W$. Thus, (iv) is proved. This completes the proof. □

2.2 Symmetrization of the NP operator

In this section, we introduce a new inner product on $\mathcal{H}^*$ (and $\mathcal{H}$) that makes the NP operator $\mathbf{K}^*$ self-adjoint.

In three dimensions, $\mathbf{S}[\varphi](\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$. Using this fact, one can show that $-\mathbf{S}$ is positive-definite. In fact, similarly to (2.16), we obtain

$$-\langle \varphi, \mathbf{S}[\varphi] \rangle \gtrsim \|\nabla \mathbf{S}[\varphi]\|^2_{L^2(\Omega)} + \|\nabla \mathbf{S}[\varphi]\|^2_{L^2(\mathbb{R}^d \setminus \Omega)} \geqslant 0. \tag{2.17}$$

If $\langle \varphi, \mathbf{S}[\varphi] \rangle = 0$, then $\mathbf{S}[\varphi]$ is constant in $\mathbb{R}^3$. Thus, we have $\varphi = \partial_\nu \mathbf{S}[\varphi]|_+ - \partial_\nu \mathbf{S}[\varphi]|_- = 0$. So, if we define

$$(\varphi, \psi)_* := -\langle \varphi, \mathbf{S}[\varphi] \rangle, \tag{2.18}$$

it is an inner product on $\mathcal{H}^*$.

In two dimensions, the same argument shows that $-\mathbf{S}$ is positive-definite on $\mathcal{H}^*_\Psi$. In fact, if $\varphi \in \mathcal{H}^*_\Psi$, then $\mathbf{S}[\varphi](\mathbf{x}) = O(|\mathbf{x}|^{-1})$ as $|\mathbf{x}| \to \infty$, and hence we can apply the same argument as in three dimensions. However, $-\mathbf{S}$ may fail to be positive on W: if Ω is the disc of radius r (centred at 0), then we have

$$\mathbf{S}[\mathbf{c}](\mathbf{x}) = \left[\alpha_1 r \ln r - \frac{\alpha_2 r}{2} \right] \mathbf{c} \quad \text{for } \mathbf{x} \in \Omega \tag{2.19}$$

for any constant vector $\mathbf{c} = (c_1, c_2)^T$. It shows that $-\mathbf{S}$ can be positive or negative depending on r. To see (2.19), we note that

$$\begin{aligned} \mathbf{S}[\mathbf{c}]_i(\mathbf{x}) &= \frac{\alpha_1 c_i}{2\pi} \int_{\partial\Omega} \ln|\mathbf{x} - \mathbf{y}| d\sigma(\mathbf{y}) - \frac{\alpha_2}{2\pi} \sum_{j=1}^{2} c_j \int_{\partial\Omega} \frac{(\mathbf{x} - \mathbf{y})_i (\mathbf{x} - \mathbf{y})_j}{|\mathbf{x} - \mathbf{y}|^2} d\sigma(\mathbf{y}) \\ &= \alpha_1 c_i \mathcal{S}[1](\mathbf{x}) - \alpha_2 \Big(x_i \mathbf{c} \cdot \nabla \mathcal{S}[1](\mathbf{x}) - \mathbf{c} \cdot \nabla \mathcal{S}[y_i](\mathbf{x}) \Big), \end{aligned}$$

where $\mathcal{S}$ is the electro-static single layer potential, namely,

$$\mathcal{S}[f](\mathbf{x}) = \frac{1}{2\pi} \int_{\partial\Omega} \ln|\mathbf{x} - \mathbf{y}| f(\mathbf{y})\, d\sigma(\mathbf{y}). \tag{2.20}$$

It is known (see [1]) that $\mathcal{S}[1](\mathbf{x}) = r \ln r$ and $\mathcal{S}[y_i](\mathbf{x}) = -\frac{r x_i}{2}$ for $\mathbf{x} \in \Omega$. So we have (2.19).

We introduce a variance of $\mathbf{S}$ in two dimensions. For $\varphi \in \mathcal{H}^*$, define using the decomposition (2.14)

$$\widetilde{\mathbf{S}}[\varphi] := \mathbf{S}[\varphi'] - \sum_{j=1}^{3} \langle \varphi, \mathbf{f}^{(j)} \rangle \mathbf{f}^{(j)}. \tag{2.21}$$

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We emphasize that $\widetilde{\mathbf{S}}[\varphi] = \mathbf{S}[\varphi]$ for all $\varphi \in \mathcal{H}^*_\Psi$ and $\widetilde{\mathbf{S}}[\varphi^{(j)}] = \mathbf{f}^{(j)}$, $j = 1, 2, 3$. In view of (2.11) and Lemma 2.3 (iii), we have

$$-\langle \varphi, \widetilde{\mathbf{S}}[\varphi] \rangle = -\langle \varphi', \mathbf{S}[\varphi'] \rangle + \sum_{j=1}^{3} |\langle \varphi, \mathbf{f}^{(j)} \rangle|^2. \tag{2.22}$$

So, $-\widetilde{\mathbf{S}}$ is positive-definite on $\mathcal{H}^*$. In fact, since $-\langle \varphi', \mathbf{S}[\varphi'] \rangle \geqslant 0$, we have $-\langle \varphi, \widetilde{\mathbf{S}}[\varphi] \rangle \geqslant 0$. If $-\langle \varphi, \widetilde{\mathbf{S}}[\varphi] \rangle = 0$, then $-\langle \varphi', \mathbf{S}[\varphi'] \rangle = 0$ and $\sum_{j=1}^{3} |\langle \varphi, \mathbf{f}^{(j)} \rangle|^2 = 0$. So, $\varphi' = 0$ and $\langle \varphi, \mathbf{f}^{(j)} \rangle = 0$ for all j, and hence $\varphi = 0$.

Let us also denote $\mathbf{S}$ in three dimensions by $\widetilde{\mathbf{S}}$ for convenience. Define

$$(\varphi, \psi)_* := -\langle \varphi, \widetilde{\mathbf{S}}[\psi] \rangle, \quad \varphi, \psi \in \mathcal{H}^*. \tag{2.23}$$

Proposition 2.4 *$(\cdot,\cdot)_*$ is an inner product on $\mathcal{H}^*$. The norm induced by $(\cdot,\cdot)_*$, denoted by $\|\cdot\|_*$, is equivalent to $\|\cdot\|_{-1/2}$.*

Proof Positive-definiteness of $-\widetilde{\mathbf{S}}$ implies that $\widetilde{\mathbf{S}} : \mathcal{H}^* \to \mathcal{H}$ is bijective. So, we have

$$\|\varphi\|_{-1/2} \approx \|\widetilde{\mathbf{S}}[\varphi]\|_{1/2}.$$

Here and throughout this paper, $A \lesssim B$ means that there is a constant C such that $A \leqslant CB$, and $A \approx B$ means $A \lesssim B$ and $B \lesssim A$. It then follows from the definition (2.23) that

$$|(\varphi, \varphi)_*| \leqslant \|\varphi\|_{-1/2} \|\widetilde{\mathbf{S}}[\varphi]\|_{1/2} \lesssim \|\varphi\|^2_{-1/2}.$$

We have from the Cauchy Schwarz inequality

$$|\langle \varphi, \widetilde{\mathbf{S}}[\psi] \rangle| = |(\varphi, \psi)_*| \leqslant \|\varphi\|_* \|\psi\|_* \lesssim \|\varphi\|_* \|\widetilde{\mathbf{S}}[\psi]\|_{1/2}.$$

So we have

$$\|\varphi\|_{-1/2} = \sup_{\psi \neq 0} \frac{|\langle \varphi, \widetilde{\mathbf{S}}[\psi] \rangle|}{\|\widetilde{\mathbf{S}}[\psi]\|_{1/2}} \lesssim \|\varphi\|_*.$$

This completes the proof. □

We may define a new inner product on $\mathcal{H}$ by

$$(\mathbf{f}, \mathbf{g}) := (\widetilde{\mathbf{S}}^{-1}[\mathbf{f}], \widetilde{\mathbf{S}}^{-1}[\mathbf{g}])_* = -\langle \widetilde{\mathbf{S}}^{-1}[\mathbf{f}], \mathbf{g} \rangle, \quad \mathbf{f}, \mathbf{g} \in \mathcal{H}. \tag{2.24}$$

Proposition 2.5 *$(\cdot,\cdot)$ is an inner product on $\mathcal{H}$. The norm induced by $(\cdot,\cdot)$, denoted by $\|\cdot\|$, is equivalent to $\|\cdot\|_{1/2}$. Moreover, $\widetilde{\mathbf{S}}$ is an isometry between $\mathcal{H}^*$ and $\mathcal{H}$.*

As shown in [12], the NP operator $\mathbf{K}^*$ can be realized as a self-adjoint operator on $\mathcal{H}^*$ using Plemelj's symmetrization principle that states for the Lamé system:

$$\mathbf{S}\mathbf{K}^* = \mathbf{K}\mathbf{S}. \tag{2.25}$$

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This relation is a consequence of Green's formula. In fact, if $\mathcal{L}_{\lambda,\mu}\mathbf{u} = 0$ in Ω, then we have for $\mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}$:

$$\mathbf{S}\left[\partial_\nu \mathbf{u}|_-\right](\mathbf{x}) - \mathbf{D}[\mathbf{u}|_-](\mathbf{x}) = 0.$$

Substituting $\mathbf{u}(\mathbf{x}) = \mathbf{S}[\boldsymbol{\varphi}](\mathbf{x})$ for some $\boldsymbol{\varphi} \in \mathcal{H}^*$ into the above relation yields

$$\mathbf{S}\left(-\frac{1}{2}I + \mathbf{K}^*\right)[\boldsymbol{\varphi}](\mathbf{x}) = \mathbf{D}\mathbf{S}[\boldsymbol{\varphi}](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \setminus \overline{\Omega}.$$

Letting $\mathbf{x}$ approach to $\partial\Omega$, we have from (1.11)

$$\mathbf{S}\left(-\frac{1}{2}I + \mathbf{K}^*\right)[\boldsymbol{\varphi}](\mathbf{x}) = \left(-\frac{1}{2}I + \mathbf{K}\right)\mathbf{S}[\boldsymbol{\varphi}](\mathbf{x}), \quad \mathbf{x} \in \partial\Omega.$$

So we have (2.25).

The relation (2.25) holds with $\mathbf{S}$ replaced by $\widetilde{\mathbf{S}}$, namely,

$$\widetilde{\mathbf{S}}\mathbf{K}^* = \mathbf{K}\widetilde{\mathbf{S}}. \tag{2.26}$$

In fact, if $\boldsymbol{\varphi} \in W$, then $\mathbf{K}^*[\boldsymbol{\varphi}] = 1/2\boldsymbol{\varphi}$ and $\widetilde{\mathbf{S}}[\boldsymbol{\varphi}] \in \Psi$. So, we have

$$\widetilde{\mathbf{S}}\mathbf{K}^*[\boldsymbol{\varphi}] = \mathbf{K}\widetilde{\mathbf{S}}[\boldsymbol{\varphi}].$$

This proves (2.26).

Proposition 2.6 *The NP operators* $\mathbf{K}^*$ *and* $\mathbf{K}$ *are self-adjoint with respect to* $(\cdot,\cdot)_*$ *and* $(\cdot,\cdot)$, *respectively.*

Proof According to (2.26), we have

$$\begin{aligned}(\boldsymbol{\varphi}, \mathbf{K}^*[\boldsymbol{\psi}])_* &= -\langle \boldsymbol{\varphi}, \widetilde{\mathbf{S}}\mathbf{K}^*[\boldsymbol{\psi}]\rangle = -\langle \boldsymbol{\varphi}, \mathbf{K}\widetilde{\mathbf{S}}[\boldsymbol{\psi}]\rangle \\ &= -\langle \mathbf{K}^*[\boldsymbol{\varphi}], \widetilde{\mathbf{S}}[\boldsymbol{\psi}]\rangle = (\mathbf{K}^*[\boldsymbol{\varphi}], \boldsymbol{\psi})_*.\end{aligned}$$

So $\mathbf{K}^*$ is self-adjoint. That $\mathbf{K}$ is self-adjoint can be proved similarly. □

3 Spectrum of NP operators on smooth planar domains

In this section, we prove (1.13) when $\partial\Omega$ is $C^{1,\alpha}$ for some $\alpha > 0$. For that purpose, we look into $\mathbf{K}$ in more explicit form. The definition (1.8) and straightforward computations show that

$$\partial_{\nu_\mathbf{y}}\Gamma(\mathbf{x}-\mathbf{y}) = \frac{\mu}{2\mu+\lambda}\mathbf{K}_1(\mathbf{x},\mathbf{y}) - \mathbf{K}_2(\mathbf{x},\mathbf{y}), \tag{3.1}$$

where

$$\mathbf{K}_1(\mathbf{x},\mathbf{y}) = \frac{\mathbf{n}_\mathbf{y}(\mathbf{x}-\mathbf{y})^T - (\mathbf{x}-\mathbf{y})\mathbf{n}_\mathbf{y}^T}{\omega_d|\mathbf{x}-\mathbf{y}|^d}, \tag{3.2}$$

$$\mathbf{K}_2(\mathbf{x},\mathbf{y}) = \frac{\mu}{2\mu+\lambda}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_\mathbf{y}}{\omega_d|\mathbf{x}-\mathbf{y}|^d}\mathbf{I} + \frac{2(\mu+\lambda)}{2\mu+\lambda}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_\mathbf{y}}{\omega_d|\mathbf{x}-\mathbf{y}|^{d+2}}(\mathbf{x}-\mathbf{y})(\mathbf{x}-\mathbf{y})^T, \tag{3.3}$$

where ω_d is 2π if $d = 2$ and 4π if $d = 3$, and $\mathbf{I}$ is the $d \times d$ identity matrix. Let

$$\mathbf{T}_j[\boldsymbol{\varphi}](\mathbf{x}) := \text{p.v.} \int_{\partial\Omega} \mathbf{K}_j(\mathbf{x}, \mathbf{y})\boldsymbol{\varphi}(\mathbf{y})\, d\sigma(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega,\ j = 1, 2, \tag{3.4}$$

so that

$$\mathbf{K} = \frac{\mu}{2\mu + \lambda}\mathbf{T}_1 - \mathbf{T}_2. \tag{3.5}$$

Note that each term of $\mathbf{K}_2$ has the term $(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{y}}$. Since $\partial\Omega$ is $C^{1,\alpha}$, we have

$$|(\mathbf{x} - \mathbf{y}) \cdot \mathbf{n}_{\mathbf{y}}| \leqslant C|\mathbf{x} - \mathbf{y}|^{1+\alpha}$$

for some constant C because of orthogonality of $\mathbf{x} - \mathbf{y}$ and $\mathbf{n}_{\mathbf{y}}$. So we have

$$|\mathbf{K}_2(\mathbf{x}, \mathbf{y})| \leqslant C|\mathbf{x} - \mathbf{y}|^{-d+1+\alpha}.$$

So $\mathbf{T}_2$ is compact on $\mathcal{H}$ (see, for example, [8]), and $\mathbf{T}_1$ is responsible for non-compactness of $\mathbf{K}$.

3.1 Compactness of $\mathbf{K}^2 - k_0^2 I$ and spectrum

Proposition 3.1 *Let Ω be a bounded $C^{1,\alpha}$ domain in $\mathbb{R}^2$ for some $\alpha > 0$. Then, $\mathbf{K}^2 - k_0^2 I$ is compact on $\mathcal{H}$.*

Proof In view of (3.5), it suffices to show that $\mathbf{T}_1^2 - \frac{1}{4}I$ is compact.

In two dimensions, we have

$$\mathbf{K}_1(\mathbf{x}, \mathbf{y}) = \frac{1}{2\pi|\mathbf{x} - \mathbf{y}|^2} \begin{bmatrix} 0 & K(\mathbf{x}, \mathbf{y}) \\ -K(\mathbf{x}, \mathbf{y}) & 0 \end{bmatrix},$$

where

$$K(\mathbf{x}, \mathbf{y}) := -n_2(\mathbf{y})(x_1 - y_1) + n_1(\mathbf{y})(x_2 - y_2).$$

Let

$$\mathcal{R}[\varphi](\mathbf{x}) = \frac{1}{2\pi}\text{p.v.} \int_{\partial\Omega} \frac{K(\mathbf{x}, \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^2}\varphi(\mathbf{y})\, d\sigma(\mathbf{y}). \tag{3.6}$$

Then, we have

$$\mathbf{T}_1[\boldsymbol{\varphi}] = \begin{bmatrix} \mathcal{R}[\varphi_2] \\ -\mathcal{R}[\varphi_1] \end{bmatrix}. \tag{3.7}$$

For $\mathbf{x} \in \partial\Omega$, set $\Omega_\epsilon := \Omega \setminus B_\epsilon(\mathbf{x})$ where $B_\epsilon(\mathbf{x})$ is the disc of radius ϵ centred at $\mathbf{x}$. For $\varphi \in H^{1/2}(\partial\Omega)$, let u be the solution to $\Delta u = 0$ in Ω with $u = \varphi$ on $\partial\Omega$. Since

$$\text{rot}\frac{\mathbf{x} - \mathbf{y}}{|\mathbf{x} - \mathbf{y}|^2} = 0, \quad \mathbf{x} \neq \mathbf{y},$$

we obtain from Stokes' formula

$$\begin{aligned}\mathcal{R}[\varphi](\mathbf{x}) &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\partial\Omega_\epsilon}\frac{-(x_1-y_1)n_2(\mathbf{y})+(x_2-y_2)n_1(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\varphi(\mathbf{y})\,d\sigma(\mathbf{y})\\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\Omega_\epsilon}\frac{-(x_1-y_1)\partial_2 u(\mathbf{y})+(x_2-y_2)\partial_1 u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\,d\mathbf{y}.\end{aligned}$$

Let v be a harmonic conjugate of u in Ω and $\psi := v|_{\partial\Omega}$ so that

$$\psi = \mathcal{T}[\varphi], \tag{3.8}$$

where $\mathcal{T}$ is the Hilbert transformation on $\partial\Omega$. Then, we have from divergence theorem

$$\begin{aligned}\mathcal{R}[\varphi](\mathbf{x}) &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\Omega_\epsilon}\frac{(x_1-y_1)\partial_1 v(\mathbf{y})+(x_2-y_2)\partial_2 v(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^2}\,d\mathbf{y}\\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\partial\Omega_\epsilon}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\psi(\mathbf{y})\,d\sigma(\mathbf{y}).\end{aligned}$$

Observe that

$$\frac{1}{2\pi}\int_{\partial\Omega_\epsilon}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\psi(\mathbf{y})\,d\sigma(\mathbf{y})$$

is the electro-static double layer potential of ψ, and $\mathbf{x}\notin\Omega_\epsilon$. So by the jump formula of the double layer potential (see [8]), we have

$$\lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\partial\Omega_\epsilon}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\psi(\mathbf{y})\,d\sigma(\mathbf{y}) = -\frac{1}{2}\psi(\mathbf{x})+\mathcal{K}[\psi](\mathbf{x}),$$

where

$$\mathcal{K}[\psi](\mathbf{x}) := \frac{1}{2\pi}\int_{\partial\Omega}\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}_\mathbf{y}}{|\mathbf{x}-\mathbf{y}|^2}\psi(\mathbf{y})\,d\sigma(\mathbf{y}),\quad \mathbf{x}\in\partial\Omega. \tag{3.9}$$

It is worth to mention that $\mathcal{K}$ is the electro-static NP operator.

So far we have shown that

$$\mathcal{R}[\varphi] = -\frac{1}{2}\mathcal{T}[\varphi]+\mathcal{K}\mathcal{T}[\varphi]. \tag{3.10}$$

Since $\mathcal{T}$ is bounded and $\mathcal{K}$ is compact on $H^{1/2}(\partial\Omega)$, we have

$$\mathcal{R} = -\frac{1}{2}\mathcal{T} + \text{compact operator}. \tag{3.11}$$

Since $\mathcal{T}^2 = -I$, we infer that $\mathcal{R}^2+\frac{1}{4}I$ is compact, and so is $\mathbf{T}_1^2-\frac{1}{4}I$. This completes the proof. □

Since $\mathbf{K}^2 - k_0^2 I$ is compact and self-adjoint, it has eigenvalues converging to 0. The proof of Proposition 3.1 shows that neither $\mathbf{K}-k_0 I$ nor $\mathbf{K}+k_0 I$ is compact, so we obtain the following theorem.

Theorem 3.2 *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{1,\alpha}$ boundary for some $\alpha > 0$.*

(i) *The spectrum of* $\mathbf{K}$ *on* $\mathcal{H}$ *consists of eigenvalues accumulating at* k_0 *and* $-k_0$, *and their multiplicities are finite if they are not equal to* k_0 *or* $-k_0$.

(ii) *The spectrum of* $\mathbf{K}^*$ *on* $\mathcal{H}^*$ *is the same as that of* $\mathbf{K}$ *on* $\mathcal{H}$.

(iii) *The set of linearly independent eigenfunctions of* $\mathbf{K}$ *makes a complete orthogonal system of* $\mathcal{H}$.

(iv) $\boldsymbol{\varphi}$ *is an eigenfunction of* $\mathbf{K}^*$ *on* $\mathcal{H}^*$ *if and only if* $\widetilde{\mathbf{S}}[\boldsymbol{\varphi}]$ *is an eigenfunction of* $\mathbf{K}$ *on* $\mathcal{H}$.

3.2 Spectral expansion of the fundamental solution

Let $\{\boldsymbol{\psi}_j\}$ be a complete orthonormal (with respect to the inner product $(\cdot,\cdot)_*$) system of $\mathcal{H}^*_\Psi$ consisting of eigenfunctions of $\mathbf{K}^*$ on $\partial\Omega$ in two dimensions. Then, they, together with $\boldsymbol{\varphi}^{(j)}$, $j = 1, 2, 3$, defined in Section 2, make an orthonormal system of $\mathcal{H}^*$. Then, by Theorem 3.2 (iv) $\{\mathbf{S}[\boldsymbol{\psi}_j]\}$ together with $\mathbf{f}^{(i)}$ is a complete orthonormal system of $\mathcal{H}$ with respect to the inner product $(\cdot,\cdot)$.

Let $\Gamma(\mathbf{x}-\mathbf{y})$ be the Kelvin matrix defined in (1.5). If $\mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega}$ and $\mathbf{y} \in \partial\Omega$, then there are (column) vector-valued functions $\mathbf{a}_j$ and $\mathbf{b}_i$ such that

$$\Gamma(\mathbf{x}-\mathbf{y}) = \sum_{j=1}^{\infty} \mathbf{a}_j(\mathbf{x})\mathbf{S}[\boldsymbol{\psi}_j](\mathbf{y})^T + \sum_{i=1}^{3} \mathbf{b}_i(\mathbf{x})\mathbf{f}^{(i)}(\mathbf{y})^T. \tag{3.12}$$

It then follows that

$$\begin{aligned}\int_{\partial\Omega} \Gamma(\mathbf{x}-\mathbf{y})\boldsymbol{\psi}_l(\mathbf{y})\,d\sigma(\mathbf{y}) &= \sum_{j=1}^{\infty} \mathbf{a}_j(\mathbf{x})\langle\boldsymbol{\psi}_l, \mathbf{S}[\boldsymbol{\psi}_j]\rangle + \sum_{i=1}^{3} \mathbf{b}_i(\mathbf{x})\langle\boldsymbol{\psi}_l, \mathbf{f}^{(i)}\rangle \\ &= -\sum_{j=1}^{\infty} \mathbf{a}_j(\mathbf{x})(\boldsymbol{\psi}_j, \boldsymbol{\psi}_l)_* + \sum_{i=1}^{3} \mathbf{b}_i(\mathbf{x})(\boldsymbol{\varphi}^{(i)}, \boldsymbol{\psi}_l)_* = -\mathbf{a}_l(\mathbf{x}).\end{aligned}$$

In other words, we obtain $\mathbf{a}_l(\mathbf{x}) = -\mathbf{S}[\boldsymbol{\psi}_l](\mathbf{x})$. Likewise, one can show $\mathbf{b}_i(\mathbf{x}) = \widetilde{\mathbf{S}}[\boldsymbol{\varphi}^{(i)}](\mathbf{x})$. So, we obtain

$$\Gamma(\mathbf{x}-\mathbf{y}) = -\sum_{j=1}^{\infty} \mathbf{S}[\boldsymbol{\psi}_j](\mathbf{x})\mathbf{S}[\boldsymbol{\psi}_j](\mathbf{y})^T + \sum_{i=1}^{3} \widetilde{\mathbf{S}}[\boldsymbol{\varphi}^{(i)}](\mathbf{x})\mathbf{f}^{(i)}(\mathbf{y})^T, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega},\ \mathbf{y} \in \partial\Omega.$$

Since both sides of above are solutions of the Lamé equation in $\mathbf{y}$ for a fixed $\mathbf{x}$, we obtain the following theorem from the uniqueness of the solution to the Dirichlet boundary value problem.

Theorem 3.3 (expansion in 2D) *Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{1,\alpha}$ boundary for some $\alpha > 0$ and let $\{\boldsymbol{\psi}_j\}$ be a complete orthonormal system of $\mathcal{H}^*_\Psi$ consisting of eigenfunctions of $\mathbf{K}^*$. Let $\Gamma(\mathbf{x}-\mathbf{y})$ be the Kelvin matrix of the fundamental solution to the Lamé*

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system. It holds that

$$\Gamma(\mathbf{x}-\mathbf{y}) = -\sum_{j=1}^{\infty} \mathbf{S}[\psi_j](\mathbf{x})\mathbf{S}[\psi_j](\mathbf{y})^T + \sum_{i=1}^{3} \widetilde{\mathbf{S}}[\varphi^{(i)}](\mathbf{x})\mathbf{f}^{(i)}(\mathbf{y})^T, \quad \mathbf{x} \in \mathbb{R}^2 \setminus \overline{\Omega},\ \mathbf{y} \in \Omega. \quad (3.13)$$

In three dimensions, one can prove the following theorem similarly. We emphasize that it has not been proved that the NP operator on smooth domain has a discrete spectrum.

Theorem 3.4 (expansion in 3D) *Let Ω be a bounded domain in $\mathbb{R}^3$. Suppose that the NP operator $\mathbf{K}^*$ admits eigenfunctions $\{\psi_j\}$ that is a complete orthonormal system of $\mathcal{H}^*$. It holds that*

$$\Gamma(\mathbf{x}-\mathbf{y}) = -\sum_{j=1}^{\infty} \mathbf{S}[\psi_j](\mathbf{x})\mathbf{S}[\psi_j](\mathbf{y})^T, \quad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega},\ \mathbf{y} \in \Omega. \quad (3.14)$$

The expansion formula, sometimes called an addition formula, for the fundamental solution to the Laplace operator on discs, balls, ellipses, and ellipsoids is classical and well-known. That on ellipsoids is attributed to Heine (see [7]). The formulas describe expansions of the fundamental solution to the Laplace operator in terms of spherical harmonics (balls) and ellipsoidal harmonics (ellipses). General addition formula of the fundamental solution to the Laplace operator as in Theorems 3.3 and 3.4 was found in [3]. It shows that the addition formula is a spectral expansion by eigenfunctions of the NP operator. Above theorems extend the formula to the Kelvin matrix of the fundamental solution to the Lamé system. Using explicit forms of eigenfunctions to be derived in the next subsection, one can compute the expansion formula on discs and ellipses explicitly, even though we will not write down the formulas since they are too long.

3.3 Spectrum of the NP operator on discs and ellipses

In this section, we write down spectrum of the NP operator on discs and ellipses. Detailed derivation of the spectrum is presented in Appendix A.

Suppose that Ω is a disc. The spectrum of $\mathbf{K}^*$ is as follows:

Eigenvalues:

$$\frac{1}{2}, \quad -\frac{\lambda}{2(2\mu+\lambda)}, \quad \pm k_0. \quad (3.15)$$

It is worth mentioning that the second eigenvalue above is less than $1/2$ in an absolute value because of the strong convexity condition (1.1).

Eigenfunctions:

(i) $1/2$:

$$(1,0)^T, \quad (0,1)^T, \quad (y,-x)^T, \quad (3.16)$$

(ii) $-\frac{\lambda}{2(2\mu+\lambda)}$:

$$(x,y)^T, \quad (3.17)$$

(iii) k_0:

$$\begin{bmatrix}\cos m\theta\\ \sin m\theta\end{bmatrix}, \quad \begin{bmatrix}-\sin m\theta\\ \cos m\theta\end{bmatrix}, \quad m = 2, 3, \ldots, \tag{3.18}$$

(iv) $-k_0$:

$$\begin{bmatrix}\cos m\theta\\ -\sin m\theta\end{bmatrix}, \quad \begin{bmatrix}\sin m\theta\\ \cos m\theta\end{bmatrix}, \quad m = 1, 2, \ldots. \tag{3.19}$$

We emphasize that eigenfunctions are not normalized.

We now describe eigenvalues and eigenfunctions on ellipses. Suppose that Ω is an ellipse of the form

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} < 1, \quad a \geqslant b > 0. \tag{3.20}$$

Put $R := \sqrt{a^2 - b^2}$. Then, the elliptic coordinates (ρ, ω) are defined by

$$x_1 = R\cosh\rho\cos\omega, \quad x_2 = R\sinh\rho\sin\omega, \quad \rho \geqslant 0,\ 0 \leqslant \omega \leqslant 2\pi, \tag{3.21}$$

in which the ellipse Ω is given by $\partial\Omega = \{(\rho, \omega) : \rho = \rho_0\}$, where ρ_0 is defined to be $a = R\cosh\rho_0$ and $b = R\sinh\rho_0$. Define

$$h_0(\omega) := R\sqrt{\sinh^2\rho_0 + \sin^2\omega}. \tag{3.22}$$

To make expressions short, we set

$$q := (\lambda + \mu)\sinh 2\rho_0, \tag{3.23}$$

and

$$\gamma_n^{\pm} := \sqrt{e^{4n\rho_0}\mu^2 + (\lambda + \mu)(\lambda + 3\mu) + nq(\pm 2e^{2n\rho_0}\mu + nq)}. \tag{3.24}$$

The spectrum of $\mathbf{K}^*$ is as follows:

Eigenvalues:

$$\frac{1}{2}, \quad k_{j,n},\ j = 1, \ldots, 4, \tag{3.25}$$

where

$$\begin{aligned}
k_{1,n} &= \frac{e^{-2n\rho_0}}{2(\lambda + 2\mu)}(-qn + \gamma_n^-), \quad n \geqslant 1,\\
k_{2,n} &= \frac{e^{-2n\rho_0}}{2(\lambda + 2\mu)}(qn + \gamma_n^+), \quad n \geqslant 2,\\
k_{3,n} &= \frac{e^{-2n\rho_0}}{2(\lambda + 2\mu)}(-qn - \gamma_n^-), \quad n \geqslant 1,\\
k_{4,n} &= \frac{e^{-2n\rho_0}}{2(\lambda + 2\mu)}(qn - \gamma_n^+), \quad n \geqslant 1.
\end{aligned} \tag{3.26}$$

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Eigenfunctions:

(i) 1/2:

$$\frac{1}{h_0(\omega)}\begin{bmatrix}1\\0\end{bmatrix},\quad \frac{1}{h_0(\omega)}\begin{bmatrix}0\\1\end{bmatrix},\quad \frac{1}{h_0(\omega)}\begin{bmatrix}\left((\lambda+\mu)e^{-2\rho_0}-(\lambda+3\mu)\right)\sin\omega\\ \left((\lambda+\mu)e^{-2\rho_0}+(\lambda+3\mu)\right)\cos\omega\end{bmatrix}, \tag{3.27}$$

(ii) $k_{j,n}$, $j=1,2,3,4$:

$$\begin{aligned}
\boldsymbol{\varphi}_{1,n} &= \boldsymbol{\psi}_{1,n} + \frac{p_n}{k_0+k_{1,n}}\boldsymbol{\psi}_{3,n}, \quad && n\geqslant 1,\\
\boldsymbol{\varphi}_{2,n} &= \boldsymbol{\psi}_{2,n} + \frac{p_n}{k_0+k_{2,n}}\boldsymbol{\psi}_{4,n}, && n\geqslant 2,\\
\boldsymbol{\varphi}_{3,n} &= \frac{k_0+k_{3,n}}{p_n}\boldsymbol{\psi}_{1,n} + \boldsymbol{\psi}_{3,n}, && n\geqslant 1,\\
\boldsymbol{\varphi}_{4,n} &= \frac{k_0+k_{4,n}}{p_n}\boldsymbol{\psi}_{2,n} + \boldsymbol{\psi}_{4,n}, && n\geqslant 1,
\end{aligned} \tag{3.28}$$

where

$$p_n = \left(\frac{1}{2}-k_0\right)e^{-2n\rho_0}, \tag{3.29}$$

and

$$\begin{aligned}
\boldsymbol{\psi}_{1,n}(\omega) &= \frac{1}{h_0(\omega)}\begin{bmatrix}\cos n\omega\\ \sin n\omega\end{bmatrix}, \quad \boldsymbol{\psi}_{2,n}(\omega) = \frac{1}{h_0(\omega)}\begin{bmatrix}-\sin n\omega\\ \cos n\omega\end{bmatrix},\\
\boldsymbol{\psi}_{3,n}(\omega) &= \frac{1}{h_0(\omega)}\begin{bmatrix}\cos n\omega\\ -\sin n\omega\end{bmatrix}, \quad \boldsymbol{\psi}_{4,n}(\omega) = \frac{1}{h_0(\omega)}\begin{bmatrix}\sin n\omega\\ \cos n\omega\end{bmatrix}.
\end{aligned} \tag{3.30}$$

A remark on $k_{2,1}$ in (3.26) is in order. It is given by

$$k_{2,1} = \frac{e^{-2\rho_0}}{2(\lambda+2\mu)}(q+\gamma_1^+),$$

where

$$\gamma_1^+ := \sqrt{e^{4\rho_0}\mu^2+(\lambda+\mu)(\lambda+3\mu)+q(2e^{2\rho_0}\mu+q)}.$$

Since

$$\mu^2e^{4\rho_0}+(\lambda+\mu)(\lambda+3\mu)+q(2e^{2\rho_0}\mu+q) = \frac{1}{4}\left[(\lambda+3\mu)e^{2\rho_0}+(\lambda+\mu)e^{-2\rho_0}\right]^2,$$

we have $k_{2,1}=\frac{1}{2}$ and the corresponding eigenfunction is

$$\boldsymbol{\varphi}_{2,1} = h_0^{-1}(\omega)\begin{bmatrix}\left((\lambda+\mu)e^{-2\rho_0}-(\lambda+3\mu)\right)\sin\omega\\ \left((\lambda+\mu)e^{-2\rho_0}+(\lambda+3\mu)\right)\cos\omega\end{bmatrix}.$$

So it is listed as an eigenfunction for 1/2.

Let us now look into the asymptotic behaviour of eigenvalues as $n \to \infty$. One can easily see from the definition (3.24) that

$$\gamma_n^{\pm} = \mu e^{2n\rho_0} \pm qn \mp \frac{(\lambda+\mu)(\lambda+3\mu)q}{2\mu^2} n e^{-2n\rho_0} + e^{-2n\rho_0} O(1),$$

where $O(1)$ indicates constants bounded independently of n. So one infer from (3.26) that

$$\begin{aligned} k_{1,n} &= k_0 - \frac{q}{\lambda+2\mu} n e^{-2n\rho_0} + n^2 e^{-4n\rho_0} O(1), \\ k_{2,n} &= k_0 + \frac{q}{\lambda+2\mu} n e^{-2n\rho_0} + n^2 e^{-4n\rho_0} O(1), \\ k_{3,n} &= -k_0 - \frac{(\lambda+\mu)(\lambda+3\mu)q}{4\mu^2(\lambda+2\mu)} n e^{-4n\rho_0} + e^{-4n\rho_0} O(1), \\ k_{4,n} &= -k_0 + \frac{(\lambda+\mu)(\lambda+3\mu)q}{4\mu^2(\lambda+2\mu)} n e^{-4n\rho_0} + e^{-4n\rho_0} O(1), \end{aligned} \tag{3.31}$$

as $n \to \infty$. In particular, we see that $k_{1,n}$ and $k_{2,n}$ converge to k_0, while $k_{3,n}$ and $k_{4,n}$ to $-k_0$ as $n \to \infty$. We emphasize that the convergence rates are exponential.

4 Anomalous localized resonance and cloaking

4.1 Resonance estimates

Let Ω be a bounded domain in $\mathbb{R}^2$ with $C^{1,\alpha}$ boundary. Let (λ,μ) be the Lamé constants of $\mathbb{R}^2 \setminus \Omega$ satisfying the strong convexity condition (1.1). Let $(\widetilde{\lambda},\widetilde{\mu})$ be Lamé constants of Ω. We assume that $(\widetilde{\lambda},\widetilde{\mu})$ is of the form

$$(\widetilde{\lambda},\widetilde{\mu}) := (c+i\delta)(\lambda,\mu), \tag{4.1}$$

where $c < 0$ and $\delta > 0$. Let $\widetilde{\mathbb{C}}$ be the isotropic elasticity tensor corresponding to $(\widetilde{\lambda},\widetilde{\mu})$, namely, $\widetilde{\mathbb{C}} = (\widetilde{C}_{ijkl})_{i,j,k,l=1}^2$ where

$$\widetilde{C}_{ijkl} := \widetilde{\lambda}\,\delta_{ij}\delta_{kl} + \widetilde{\mu}\,(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{4.2}$$

and let $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}}$ and $\partial_{\widetilde{\nu}}$ be corresponding Lamé operator and conormal derivative, respectively. Then, we have $\mathcal{L}_{\widetilde{\lambda},\widetilde{\mu}} = (c+i\delta)\mathcal{L}_{\lambda,\mu}$ and $\partial_{\widetilde{\nu}} = (c+i\delta)\partial_\nu$.

Let $\mathbb{C}_\Omega$ be the elasticity tensor in presence of inclusion Ω so that

$$\mathbb{C}_\Omega = \widetilde{\mathbb{C}}\chi_\Omega + \mathbb{C}\chi_{\mathbb{R}^2\setminus\overline{\Omega}},$$

where χ denotes the characteristic function. We consider the following transmission problem:

$$\begin{cases} \nabla\cdot\mathbb{C}_\Omega\widehat{\nabla}\mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^2, \\ \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-1}) & \text{as } |\mathbf{x}| \to \infty, \end{cases} \tag{4.3}$$

where $\mathbf{f}$ is a function compactly supported in $\mathbb{R}^2 \setminus \overline{\Omega}$ and satisfies

$$\int_{\mathbb{R}^2} \mathbf{f}\,d\mathbf{x} = 0. \tag{4.4}$$

This condition is necessary for a solution to (4.3) to exist. The problem (4.3) can be rephrased as

$$\begin{cases} \mathcal{L}_{\lambda,\mu}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathcal{L}_{\lambda,\mu}\mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^2 \setminus \overline{\Omega}, \\ \mathbf{u}|_- = \mathbf{u}|_+ & \text{on } \partial\Omega, \\ (c + i\delta)\partial_\nu \mathbf{u}|_- = \partial_\nu \mathbf{u}|_+ & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Let

$$\mathbf{F}(\mathbf{x}) := \int_{\mathbb{R}^2} \Gamma(\mathbf{x}-\mathbf{y})\mathbf{f}(\mathbf{y})d\mathbf{y}, \quad \mathbf{x} \in \mathbb{R}^2. \tag{4.6}$$

We seek the solution $\mathbf{u}_\delta$ to (4.5) in the form

$$\mathbf{u}_\delta(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \mathbf{S}[\boldsymbol{\varphi}_\delta](\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^2, \tag{4.7}$$

where $\boldsymbol{\varphi}_\delta$ is to be determined. Since $\mathbf{S}[\boldsymbol{\varphi}_\delta](\mathbf{x})$ is continuous across $\partial\Omega$, the continuity of displacement (the third condition in (4.5)) is automatically satisfied. The continuity of the traction (the fourth condition in (4.5)) leads us to

$$(c + i\delta)\left(\partial_\nu \mathbf{F} + \partial_\nu \mathbf{S}[\boldsymbol{\varphi}_\delta]|_-\right) = \partial_\nu \mathbf{F} + \partial_\nu \mathbf{S}[\boldsymbol{\varphi}_\delta]|_+ \quad \text{on } \partial\Omega.$$

Therefore, using the jump formula (2.6), we have

$$\left(k_\delta(c)I - \mathbf{K}^*\right)[\boldsymbol{\varphi}_\delta] = \partial_\nu \mathbf{F}, \tag{4.8}$$

where

$$k_\delta(c) := \frac{c + 1 + i\delta}{2(c - 1 + i\delta)}. \tag{4.9}$$

Let k_j, $j = 1, 2, \ldots$, be the eigenvalues (other than $1/2$) of $\mathbf{K}^*$ counting multiplicities, and let $\{\boldsymbol{\psi}_j\}$ be corresponding normalized eigenfunctions. Then, $\{\boldsymbol{\psi}_j, \boldsymbol{\varphi}^{(1)}, \boldsymbol{\varphi}^{(2)}, \boldsymbol{\varphi}^{(3)}\}$ is an orthonormal system of $\mathcal{H}^*$ and the solution to (4.8) is given by

$$\boldsymbol{\varphi}_\delta = \sum_{j=1}^{\infty} \frac{(\boldsymbol{\psi}_j, \partial_\nu \mathbf{F})_*}{k_\delta(c) - k_j}\boldsymbol{\psi}_j + \sum_{i=1}^{3} \frac{(\boldsymbol{\varphi}^{(i)}, \partial_\nu \mathbf{F})_*}{k_\delta(c) - 1/2}\boldsymbol{\varphi}^{(i)}.$$

Since $\mathbf{f}$ is supported in $\mathbb{R}^2 \setminus \overline{\Omega}$, $\mathcal{L}_{\lambda,\mu}\mathbf{F} = 0$ in Ω, and hence $\partial_\nu \mathbf{F} \in \mathcal{H}^*_\Psi$. So we have

$$\boldsymbol{\varphi}_\delta = \sum_{j=1}^{\infty} \frac{(\boldsymbol{\psi}_j, \partial_\nu \mathbf{F})_*}{k_\delta(c) - k_j}\boldsymbol{\psi}_j. \tag{4.10}$$

Let

$$E(\mathbf{u}) := \int_\Omega \widehat{\nabla}\mathbf{u} : \mathbb{C}\widehat{\nabla}\mathbf{u}\, d\mathbf{x}. \tag{4.11}$$

Here, $\mathbf{A} : \mathbf{B} = \sum_{i,j} a_{ij}b_{ij}$ for two matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$. We are particularly interested in estimating $\delta E(\mathbf{u}_\delta)$ since CALR is characterized by the condition $\delta E(\mathbf{u}_\delta) \to \infty$ as $\delta \to 0$. It is worth mentioning that $\delta E(\mathbf{u}_\delta)$ is the imaginary part of $\int_{\mathbb{R}^2} \widehat{\nabla}\mathbf{u}_\delta : \mathbb{C}_\Omega \widehat{\nabla}\mathbf{u}_\delta\, d\mathbf{x}$, which represents the elastic energy of the solution.

To present results of this section, let us introduce a notation. For two quantities A_δ and B_δ depending on δ, $A_\delta \sim B_\delta$ means that there are constants C_1 and C_2 independent of $\delta \leqslant \delta_0$ for some δ_0 such that

$$C_1 \leqslant \frac{A_\delta}{B_\delta} \leqslant C_2.$$

Proposition 4.1 *Let* $\mathbf{u}_\delta$ *be the solution to (4.5). It holds that*

$$E(\mathbf{u}_\delta - \mathbf{F}) \sim \sum_{j=1}^{\infty} \frac{|(\boldsymbol{\psi}_j, \partial_\nu \mathbf{F})_*|^2}{|k_\delta(c) - k_j|^2}. \tag{4.12}$$

Proof Using Green's formula for Lamé operator and the jump formula (2.6), we have

$$\begin{aligned} E(\mathbf{u}_\delta - \mathbf{F}) = E(\mathbf{S}[\boldsymbol{\varphi}_\delta]) &= \int_\Omega \widehat{\nabla}\mathbf{S}[\boldsymbol{\varphi}_\delta] : \mathbb{C}\widehat{\nabla}\mathbf{S}[\boldsymbol{\varphi}_\delta] \, d\mathbf{x} \\ &= \int_{\partial\Omega} \mathbf{S}[\boldsymbol{\varphi}_\delta] \cdot \partial_\nu \mathbf{S}[\boldsymbol{\varphi}_\delta] \, d\sigma \\ &= \left\langle (-\frac{1}{2}I + \mathbf{K}^*)[\boldsymbol{\varphi}_\delta], \mathbf{S}[\boldsymbol{\varphi}_\delta] \right\rangle = \left((\frac{1}{2}I - \mathbf{K}^*)[\boldsymbol{\varphi}_\delta], \boldsymbol{\varphi}_\delta \right)_* . \end{aligned}$$

It then follows from (4.10) that

$$E(\mathbf{u}_\delta - \mathbf{F}) = \sum_{j=1}^{\infty} \frac{(1/2 - k_j)|(\boldsymbol{\psi}_j, \partial_\nu \mathbf{F})_*|^2}{|k_\delta(c) - k_j|^2}.$$

Since $k_{j,n}$ accumulates at k_0 or $-k_0$ and $-1/2 < k_{j,n} < 1/2$, there is a constant $C > 0$ such that

$$C \leqslant |\frac{1}{2} - k_j| < 1 \tag{4.13}$$

for all j. So we have (4.12). □

Note that $k_\delta(c) \to k(c)$ as $\delta \to 0$, where $k(c)$ be the number defined in (1.15). More precisely, we have

$$|k_\delta(c) - k(c)| \sim \delta. \tag{4.14}$$

So we obtain the following theorem from Proposition 4.1, which shows that resonance occurs at the eigenvalue as $\delta \to 0$ at the rate of δ^{-2}.

Theorem 4.2 *Let* c *be such that* $k(c) = k_j$ *for some* j *and suppose that* $(\boldsymbol{\psi}_j, \partial_\nu \mathbf{F})_* \neq 0$. *Then, we have*

$$E(\mathbf{u}_\delta) \sim \delta^{-2}, \tag{4.15}$$

as $\delta \to 0$.

Proof We have from (4.12)

$$E(\mathbf{u}_\delta - \mathbf{F}) \gtrsim \delta^{-2}.$$

Since $E(\mathbf{F})$ is finite, we obtain (4.15). □

4.2 Anomalous localized resonance on ellipses

Anomalous localized resonance occurs at the accumulation points of eigenvalues of the NP operator (if the accumulation points are not eigenvalues). So, we assume

$$k(c) = k_0 \quad \text{or} \quad -k_0. \tag{4.16}$$

For the analysis of anomalous localized resonance, we need the explicit form of eigenvalues and eigenfunctions. So we assume Ω is an ellipse of the form (3.20) so that $\Omega = \{\rho < \rho_0\}$ in elliptic coordinates. We emphasize that anomalous localized resonance does not occur on discs since $\pm k_0$ are eigenvalues.

We further assume that the source function $\mathbf{f}$ is a polarizable dipole, namely,

$$\mathbf{f} = \mathbf{A}\nabla\delta_{\mathbf{z}} = \begin{bmatrix} \mathbf{a}_1 \cdot \nabla\delta_{\mathbf{z}} \\ \mathbf{a}_2 \cdot \nabla\delta_{\mathbf{z}} \end{bmatrix},$$

where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the constant vectors, $\mathbf{A} = (\mathbf{a}_1\ \mathbf{a}_2)^T$, and $\mathbf{z} \in \mathbb{R}^2 \setminus \overline{\Omega}$. In this case, the function $\mathbf{F}$ defined by (4.6) is given by

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}_{\mathbf{z}}(\mathbf{x}) = \left((\mathbf{A}\nabla_{\mathbf{x}})^T \Gamma(\mathbf{x} - \mathbf{z})\right)^T. \tag{4.17}$$

Let $(\rho_{\mathbf{z}}, \omega_{\mathbf{z}})$ be the elliptic coordinates of $\mathbf{z}$, and let $\mathbf{U}(\theta)$ be the rotation by the angle θ, namely,

$$\mathbf{U}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{4.18}$$

We obtain the following theorems, when $k(c) = k_0$.

Theorem 4.3 *Assume $k(c) = k_0$. Let $\mathbf{u}_\delta$ be the solution to (4.5). If $\mathbf{a}_1 \neq \mathbf{U}(-\pi/2)\mathbf{a}_2$, then we have*

$$E(\mathbf{u}_\delta) \sim \begin{cases} |\log\delta|\, \delta^{-3+\rho_{\mathbf{z}}/\rho_0} & \text{if } \rho_0 < \rho_{\mathbf{z}} \leqslant 3\rho_0, \\ 1 & \text{if } \rho_{\mathbf{z}} > 3\rho_0, \end{cases} \tag{4.19}$$

as $\delta \to 0$.

Theorem 4.4 *Assume $k(c) = k_0$. Let $\mathbf{x} = (\rho, \omega)$ in the elliptic coordinates. Then, it holds for all $\mathbf{x}$ satisfying $\rho + \rho_{\mathbf{z}} - 4\rho_0 > 0$ that*

$$|\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_{\mathbf{z}}(\mathbf{x})| \lesssim \sum_{n=1}^{\infty} \frac{e^{-n(\rho+\rho_{\mathbf{z}}-4\rho_0)}}{n}. \tag{4.20}$$

In particular, for any $\overline{\rho} > 4\rho_0 - \rho_{\mathbf{z}}$ there exists some $C = C_{\overline{\rho}} > 0$ such that

$$\sup_{\rho \geqslant \overline{\rho}} |\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_{\mathbf{z}}(\mathbf{x})| < C. \tag{4.21}$$

We also obtain the following theorems when $k(c) = -k_0$.

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Theorem 4.5 *Assume that* $k(c) = -k_0$. *If* $\mathbf{a}_1 \neq \mathrm{U}(-\pi/2)\mathbf{a}_2$, *then we have*

$$E(\mathbf{u}_\delta) \sim \begin{cases} |\log\delta|^3 \delta^{-5/2+\rho_{\mathbf{z}}/2\rho_0} & \text{if } \rho_0 < \rho_{\mathbf{z}} \leqslant 5\rho_0, \\ 1 & \text{if } \rho_{\mathbf{z}} > 5\rho_0, \end{cases} \tag{4.22}$$

as $\delta \to 0$.

Theorem 4.6 *Assume* $k(c) = -k_0$. *Let* $\mathbf{x} = (\rho, \omega) \in \mathbb{R}^2$ *in the elliptic coordinates. Then, it holds for all* $\mathbf{x}$ *satisfying* $\rho + \rho_{\mathbf{z}} - 6\rho_0 > 0$ *that*

$$|\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_{\mathbf{z}}(\mathbf{x})| \lesssim \sum_{n=1}^{\infty} n e^{-n(\rho+\rho_{\mathbf{z}}-6\rho_0)}. \tag{4.23}$$

In particular, for any $\overline{\rho} > 6\rho_0 - \rho_{\mathbf{z}}$ *there exists some* $C = C_{\overline{\rho}} > 0$ *such that*

$$\sup_{\rho \geqslant \overline{\rho}} |\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_{\mathbf{z}}(\mathbf{x})| < C. \tag{4.24}$$

Theorems 4.3 and 4.4 show that CALR occurs when $k(c) = k_0$. In fact, (4.19) shows that $\delta E(\mathbf{u}_\delta) \to \infty$ if $\rho_{\mathbf{z}} \leqslant 2\rho_0$ and $\delta E(\mathbf{u}_\delta) \to 0$ if $\rho_{\mathbf{z}} > 2\rho_0$. On the other hand, (4.21) shows that $\mathbf{u}_\delta$ bounded if $\rho > 4\rho_0 - \rho_{\mathbf{z}}$. So, if we normalize the solution by $\mathbf{v}_\delta := (\delta E(\mathbf{u}_\delta))^{-1/2}\mathbf{u}_\delta$ so that $\delta E(\mathbf{v}_\delta) = 1$, then $\mathbf{v}_\delta \to 0$ in $\rho > 4\rho_0 - \rho_{\mathbf{z}}$ provided that $\rho_{\mathbf{z}} \leqslant 2\rho_0$. So, CALR occurs and the cloaking region is $\rho_0 < \rho_{\mathbf{z}} \leqslant 2\rho_0$. It is worth mentioning that this cloaking region coincides with that for the dielectric case (Laplace equation) obtained in [3].

Theorems 4.5 and 4.6 show that CALR occurs when $k(c) = -k_0$, and in this case the cloaking region is $\rho_0 < \rho_{\mathbf{z}} \leqslant 3\rho_0$. It is interesting to observe that the cloaking region is different from that for the case $k(c) = k_0$.

Proofs of above theorems are given in Appendix B.

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Appendix A Derivation of spectrum on discs and ellipses

The purpose of this section is to derive spectrum of the NP operator on discs and ellipses presented in Section 3.3. We use complex representations of the displacement vector and traction. So, we identify $\psi = \psi_1 + i\psi_2 \in \mathbb{C}$ with the vector $\boldsymbol{\psi} = (\psi_1, \ \psi_2)^T$, and denote

$$\mathbf{S}[\psi] = (\mathbf{S}[\boldsymbol{\psi}])_1 + i(\mathbf{S}[\boldsymbol{\psi}])_2.$$

Suppose that Ω is simply connected domain in $\mathbb{R}^2$ (bounded or unbounded), and let $\psi = \psi_1 + i\psi_2 \in H^{-1/2}(\partial\Omega)$. It is known (see [2,20]) that there are holomorphic functions

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f and g in Ω (or in $\mathbb{C} \setminus \overline{\Omega}$) such that

$$2\mu\mathbf{S}[\psi](z) = \kappa f(z) - z\overline{f'(z)} - \overline{g(z)}, \quad \kappa = \frac{\lambda + 3\mu}{\lambda + \mu} \tag{A 1}$$

in Ω (or in $\mathbb{C} \setminus \overline{\Omega}$), and the conormal derivative $\partial_\nu \mathbf{u}$ is represented as

$$\partial_\nu \mathbf{S}[\psi]|_+ d\sigma = \big((\partial_\nu \mathbf{S}[\psi])_1 + i(\partial_\nu \mathbf{S}[\psi])_2\big)|_+ d\sigma = -id\Big[f(z) + z\overline{f'(z)} + \overline{g(z)}\Big], \tag{A 2}$$

where $d\sigma$ is the line element of $\partial\Omega$ and d is the exterior derivative, namely, $d = (\partial/\partial z)dz + (\partial/\partial\bar{z})d\bar{z}$. Here, $\partial\Omega$ is positively oriented and $f'(z) = \partial f(z)/\partial z$. Moreover, it is shown that f and g are obtained by

$$f(z) = \frac{\mu\alpha_2}{2\pi}\int_{\partial\Omega} \ln(z - \zeta)\psi(\zeta)d\sigma(\zeta), \tag{A 3}$$

$$g(z) = -\frac{\mu\alpha_1}{2\pi}\int_{\partial\Omega} \ln(z - \zeta)\overline{\psi(\zeta)}d\sigma(\zeta) - \frac{\mu\alpha_2}{2\pi}\int_{\partial\Omega} \frac{\overline{\zeta}\psi(\zeta)}{z - \zeta}d\sigma(\zeta). \tag{A 4}$$

It is worth mentioning that above integrals are well defined for ψ satisfying $\int_{\partial\Omega}\psi d\sigma = 0$. If ψ is constant, then we take a proper branch cut of $\log(z - \zeta)$ for $z \in \mathbb{C} \setminus \overline{\Omega}$.

Let

$$\mathcal{L}[\psi](z) := \frac{1}{2\pi}\int_{\partial\Omega} \ln(z - \zeta)\psi(\zeta)d\sigma(\zeta) \tag{A 5}$$

so that

$$f(z) = \mu\alpha_2\mathcal{L}[\psi](z), \quad g(z) = -\mu\alpha_1\mathcal{L}[\overline{\psi}](z) - \mu\alpha_2\mathcal{L}[\overline{\zeta}\psi]'(z). \tag{A 6}$$

So, (A 1) can be rewritten as

$$2\mathbf{S}[\psi](z) = \kappa\alpha_2\mathcal{L}[\psi](z) - \alpha_2 z\overline{\mathcal{L}[\psi]'(z)} + \alpha_1\overline{\mathcal{L}[\overline{\psi}](z)} + \alpha_2\overline{\mathcal{L}[\overline{\zeta}\psi]'(z)}. \tag{A 7}$$

Observe that

$$d\Big[f(z) + z\overline{f'(z)} + \overline{g(z)}\Big] = (f'(z) + \overline{f'(z)})dz + (z\overline{f''(z)} + \overline{g'(z)})d\bar{z}.$$

So we define

$$\mathcal{C}[\psi](z) := \mathcal{L}[\psi]'(z) = \frac{1}{2\pi}\int_{\partial\Omega} \frac{\psi(\zeta)}{z - \zeta}d\sigma(\zeta), \tag{A 8}$$

then we have

$$\partial_\nu \mathbf{S}[\psi]|_+ d\sigma = -i\mu\alpha_2\left[\mathcal{C}[\psi] + \overline{\mathcal{C}[\psi]}\right]dz + i\left[\mu\alpha_1\overline{\mathcal{C}[\overline{\psi}]} - \mu\alpha_2\left(z\overline{\mathcal{C}[\psi]'} - \overline{\mathcal{C}[\overline{\zeta}\psi]'}\right)\right]d\bar{z}. \tag{A 9}$$

We shall compute $\mathbf{S}[\psi]$ and $\partial_\nu \mathbf{S}[\psi]|_+$ for proper basis functions ψ.

A.1 Discs

Since the spectrum of the NP operator is invariant under translation and scaling (and rotation), we may assume that Ω be a unit disc. Let $\psi = (\psi_1, \psi_2)^T$ be one of the following

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functions:

$$\begin{bmatrix} \cos n\theta \\ \pm \sin n\theta \end{bmatrix}, \quad \begin{bmatrix} \sin n\theta \\ \pm \cos n\theta \end{bmatrix}, \quad n = 0, 1, 2, \dots, \tag{A 10}$$

or equivalently (after identifying with $\psi = \psi_1 + i\psi_2$)

$$\beta e^{in\theta}, \quad n = 0, \pm 1, \pm 2, \dots, \tag{A 11}$$

where β is either 1 or i.

One can see from (2.19) and (2.6) that

$$\mathbf{K}^*[\mathbf{c}] = \frac{1}{2}\mathbf{c} \tag{A 12}$$

for any constant vector $\mathbf{c}$.

If $\psi = \beta e^{in\theta}$ with $n \neq 0$, then one can easily see that for $|z| > 1$

$$\mathcal{C}[\psi](z) = \begin{cases} 0 & \text{for } n \geqslant 1, \\ \beta z^{n-1} & \text{for } n \leqslant -1. \end{cases} \tag{A 13}$$

Since $\overline{\zeta^n} = \zeta^{-n}$ for $|\zeta| = 1$, we have

$$\mathcal{C}[\overline{\psi}](z) = \begin{cases} \overline{\beta} z^{-n-1} & \text{for } n \geqslant 1, \\ 0 & \text{for } n \leqslant -1. \end{cases} \tag{A 14}$$

One can also see that

$$\mathcal{C}[\overline{\zeta}\psi](z) = \begin{cases} 0 & \text{for } n \geqslant 2, \\ \beta z^{n-2} & \text{for } n = 1 \quad \text{or} \quad n \leqslant -1. \end{cases} \tag{A 15}$$

Using (A 13)–(A 15), we can show that the NP eigenvalues on the disc are given by (3.15) and corresponding eigenfunctions by (3.16)–(3.19). In fact, one can see from (A 9) and (A 13)–(A 15) that

$$\big((\partial_\nu \mathbf{S}[\beta\psi_n])_1 + i\,(\partial_\nu \mathbf{S}[\beta\psi_n])_2\big) = \begin{cases} \mu\alpha_1\beta\psi_n & \text{if } n \geqslant 2, \\ \mu\alpha_1\beta\psi_1 - \mu\alpha_2\overline{\beta}\psi_1 & \text{if } n = 1, \\ \mu\alpha_2\beta\psi_n & \text{if } n \leqslant -1, \end{cases}$$

where $\psi_n(\zeta) = \zeta^n$. Therefore, we have

$$\mathbf{K}^*[\beta\psi_n] = \begin{cases} (\mu\alpha_1 - 1/2)\beta\psi_n & \text{if } n \geqslant 2, \\ (\mu\alpha_1 - 1/2)\beta\psi_1 - \mu\alpha_2\overline{\beta}\psi_1 & \text{if } n = 1, \\ (\mu\alpha_2 - 1/2)\beta\psi_n & \text{if } n \leqslant -1. \end{cases} \tag{A 16}$$

Since $\mu\alpha_1 - 1/2 = k_0$ and $\mu\alpha_2 - 1/2 = -k_0$, (A 16) shows that the spectrum is as presented in Section 3.3. It is helpful to mention that $\beta = 1$ and $n = 1$ in (A 16) yield the second eigenvalue in (3.15) and corresponding eigenfunction.

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A.2 Ellipses

Suppose that $\Omega = \{(\rho, \omega) : \rho < \rho_0\}$ in elliptic coordinates as in Section 3.3. Let (ρ, η) be the elliptic coordinate of $z \in \mathbb{C} \setminus \overline{\Omega}$ and (ρ_0, ω) that of $\zeta \in \partial\Omega$ so that

$$z = R\cosh(\rho + i\eta), \quad \zeta = R\cosh(\rho_0 + i\omega), \tag{A 17}$$

where $R = \sqrt{a^2 - b^2}$. It is known (see, for example, [3]) that

$$\begin{aligned}\frac{1}{2\pi}\ln|z-\zeta| = &-\sum_{m=1}^{\infty}\frac{1}{m\pi}(\cosh m\rho_0 \cos m\omega e^{-m\rho}\cos m\eta + \sinh m\rho_0 \sin m\omega e^{-m\rho}\sin m\eta)\\ &+\frac{1}{2\pi}\left(\rho + \ln\left(\frac{R}{2}\right)\right), \qquad \text{for } \rho_0 < \rho.\end{aligned}$$

It is convenient to write $\xi = \rho + i\eta$ so that

$$\frac{1}{2\pi}\ln(z-\zeta) = -\sum_{m=1}^{\infty}\frac{1}{m\pi}\cosh m(\rho_0 + i\omega)e^{-m\xi} + \frac{1}{2\pi}(\xi + C) \tag{A 18}$$

for some constant C whose real part is $\ln(R/2)$.

Let $\boldsymbol{\psi}_{j,n}$ be functions defined in (3.30). After complexification, they can be written as

$$\psi = \beta h_0(\omega)^{-1}\psi_n(\omega),$$

where β is either 1 or i, and $\psi_n(\omega) = e^{in\omega}$. In fact, we have

$$\begin{cases}\psi = \boldsymbol{\psi}_{1,n} \text{ and } \overline{\psi} = \boldsymbol{\psi}_{3,n} & \text{if } \beta = 1 \text{ and } n \geqslant 1,\\ \psi = \boldsymbol{\psi}_{3,-n} \text{ and } \overline{\psi} = \boldsymbol{\psi}_{1,-n} & \text{if } \beta = 1 \text{ and } n \leqslant -1,\\ \psi = \boldsymbol{\psi}_{2,n} \text{ and } \overline{\psi} = -\boldsymbol{\psi}_{4,n} & \text{if } \beta = i \text{ and } n \geqslant 1,\\ \psi = \boldsymbol{\psi}_{4,-n} \text{ and } \overline{\psi} = -\boldsymbol{\psi}_{2,-n} & \text{if } \beta = i \text{ and } n \leqslant -1.\end{cases} \tag{A 19}$$

Let us compute $\mathcal{L}[h_0^{-1}\psi_n]$. Since $d\sigma(\omega) = h_0(\omega)d\omega$ on $\partial\Omega$, we have

$$\begin{aligned}\mathcal{L}[h_0^{-1}\psi_n](z) &= \frac{1}{2\pi}\int_0^{2\pi}\ln(z-\zeta)\psi_n(\omega)d\omega\\ &= -\sum_{m=1}^{\infty}\frac{e^{-m\xi}}{m\pi}\int_0^{2\pi}\cosh m(\rho_0 + i\omega)e^{in\omega}d\omega + \frac{1}{2\pi}(\xi + C)\int_0^{2\pi}e^{in\omega}d\omega.\end{aligned}$$

Thus, we have for $n \neq 0$

$$\mathcal{L}[h_0^{-1}\psi_n](z) = -\frac{e^{-|n|\xi - n\rho_0}}{|n|}. \tag{A 20}$$

We also obtain

$$\mathcal{L}[h_0^{-1}\overline{\psi_n}](z) = -\frac{e^{-|n|\xi + n\rho_0}}{|n|}. \tag{A 21}$$

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Since $\partial z/\partial \xi = R\sinh\xi$, we have

$$\mathcal{C}[h_0^{-1}\psi_n](z) = \mathcal{L}[h_0^{-1}\psi_n]'(z) = \frac{e^{-|n|\xi - n\rho_0}}{R\sinh\xi}, \tag{A 22}$$

and

$$\mathcal{C}[h_0^{-1}\overline{\psi_n}](z) = \mathcal{L}[h_0^{-1}\overline{\psi_n}]'(z) = \frac{e^{-|n|\xi + n\rho_0}}{R\sinh\xi}. \tag{A 23}$$

Since $\zeta = R\cosh(\rho_0 + i\omega)$ on $\partial\Omega$, we have

$$\overline{\zeta}\psi_n(\zeta) = \frac{R}{2}\left[e^{\rho_0}\psi_{n-1}(\omega) + e^{-\rho_0}\psi_{n+1}(\omega)\right],$$

and hence

$$\mathcal{L}[\overline{\zeta}h_0^{-1}\psi_n](z) = \frac{R}{2}e^{\rho_0}\mathcal{L}[h_0^{-1}\psi_{n-1}](z) + \frac{R}{2}e^{-\rho_0}\mathcal{L}[h_0^{-1}\psi_{n+1}](z).$$

It then follows from (A 22) and (A 23) that

$$\begin{aligned}\mathcal{C}[\overline{\zeta}h_0^{-1}\psi_n](z) = \mathcal{L}[\overline{\zeta}h_0^{-1}\psi_n]'(z) &= \frac{e^{-|n-1|\xi-(n-2)\rho_0} + e^{-|n+1|\xi-(n+2)\rho_0}}{2\sinh\xi} \\ &= \begin{cases} \dfrac{e^{-n(\xi+\rho_0)}\cosh(\xi+2\rho_0)}{\sinh\xi} & \text{if } n \geqslant 1, \\ \dfrac{e^{n(\xi-\rho_0)}\cosh(\xi-2\rho_0)}{\sinh\xi} & \text{if } n \leqslant -1. \end{cases}\end{aligned} \tag{A 24}$$

Let $\psi = \beta h_0^{-1}\psi_n$ where $n \neq 0$ and $\beta = 1, i$. According to (A 7), we have

$$2\mathbf{S}[\psi](z) = \beta\kappa\alpha_2\mathcal{L}[h_0^{-1}\psi_n](z) + \overline{\beta}\alpha_2\left(\overline{\mathcal{L}[\overline{\zeta}h_0^{-1}\psi_n]'(z)} - z\overline{\mathcal{L}[h_0^{-1}\psi_n]'(z)}\right) + \beta\alpha_1\overline{\mathcal{L}[h_0^{-1}\overline{\psi_n}](z)}.$$

In view of (A 22) and (A 24), we have

$$\overline{\mathcal{L}[\overline{\zeta}h_0^{-1}\psi_n]'(z)} - z\overline{\mathcal{L}[h_0^{-1}\psi_n]'(z)} = \begin{cases} \dfrac{e^{-n(\overline{\xi}+\rho_0)}[\cosh(\overline{\xi}+2\rho_0) - \cosh\xi]}{\sinh\overline{\xi}}, & n \geqslant 1, \\ \dfrac{e^{n(\overline{\xi}-\rho_0)}[\cosh(\overline{\xi}-2\rho_0) - \cosh\xi]}{\sinh\overline{\xi}}, & n \leqslant -1. \end{cases}$$

Thus, we obtain

$$2\mathbf{S}[\psi](z) = \begin{cases} -\dfrac{\beta(\kappa\alpha_2 e^{-n(\xi+\rho_0)} + \alpha_1 e^{-n(\overline{\xi}-\rho_0)})}{n} \\ \qquad + \dfrac{\overline{\beta}\alpha_2 e^{-n(\overline{\xi}+\rho_0)}[\cosh(\overline{\xi}+2\rho_0) - \cosh\xi]}{\sinh\overline{\xi}}, & n \geqslant 1, \\ \dfrac{\beta(\kappa\alpha_2 e^{n(\xi-\rho_0)} + \alpha_1 e^{n(\overline{\xi}+\rho_0)})}{n} \\ \qquad + \dfrac{\overline{\beta}\alpha_2 e^{n(\overline{\xi}-\rho_0)}[\cosh(\overline{\xi}-2\rho_0) - \cosh\xi]}{\sinh\overline{\xi}}, & n \leqslant -1. \end{cases} \tag{A 25}$$

Let

$$h_\rho(\omega) = h(\rho,\omega) := R\sqrt{\sinh^2\rho + \sin^2\omega}. \tag{A 26}$$

Using the identity

$$\sinh(\rho - i\eta)\sinh(\rho + i\eta) = \sinh^2\rho + \sin^2\eta = \frac{h_\rho^2}{R^2},$$

we obtain the following lemma.

Lemma A.1 *The single layer potentials outside the ellipse, $\rho \geqslant \rho_0$, are computed as follows: for $\beta = 1$ or i and $\psi_n(\omega) = e^{in\omega}$, $n = 1, 2, \ldots$,*

$$\begin{aligned}
\mathbf{S}[\beta h_0^{-1}\psi_n](z) &= -\frac{\beta}{2n}\left[\kappa\alpha_2 e^{-n(\rho+\rho_0)}e^{-in\eta} + \alpha_1 e^{-n(\rho-\rho_0)}e^{in\eta}\right] \\
&\quad + \frac{\overline{\beta}\alpha_2 R^2}{4h_\rho^2}e^{-n(\rho+\rho_0)}\left[e^{in\eta}\sinh 2(\rho+\rho_0) + \frac{e^{-2\rho_0} - e^{2\rho}}{2}e^{i(n+2)\eta} + \frac{e^{-2\rho} - e^{2\rho_0}}{2}e^{i(n-2)\eta}\right], \\
\mathbf{S}[\beta h_0^{-1}\psi_{-n}](z) &= -\frac{\beta}{2n}\left[\kappa\alpha_2 e^{-n(\rho-\rho_0)}e^{-in\eta} + \alpha_1 e^{-n(\rho+\rho_0)}e^{in\eta}\right] \\
&\quad + \frac{\overline{\beta}\alpha_2 R^2}{4h_\rho^2}e^{-n(\rho-\rho_0)}\left[e^{in\eta}\sinh 2(\rho-\rho_0) + \frac{e^{2\rho_0} - e^{2\rho}}{2}e^{i(n+2)\eta} + \frac{e^{-2\rho} - e^{-2\rho_0}}{2}e^{i(n-2)\eta}\right],
\end{aligned}$$

where $z = R\cosh(\rho + i\eta)$.

As an immediate consequence of above lemma, we see that there is a constant C such that

$$\left|\mathbf{S}[\boldsymbol{\psi}_{j,n}](z)\right| \leqslant C\frac{e^{-n(\rho-\rho_0)}}{n}, \quad j = 1, 2, \tag{A 27}$$

and

$$\left|\mathbf{S}[\boldsymbol{\psi}_{j,n}](z)\right| \leqslant Ce^{-n(\rho-\rho_0)}, \quad j = 3, 4, \tag{A 28}$$

for all n.

If $z \in \partial\Omega$, namely, $\rho = \rho_0$, then we have

$$\cosh\xi = \cosh(\overline{\xi} - 2\rho_0).$$

Thus, we obtain for $z \in \partial\Omega$

$$2\mathbf{S}[\psi](z) = \begin{cases} -\beta\dfrac{\kappa\alpha_2 e^{-2n\rho_0}}{n}e^{-in\eta} - \beta\dfrac{\alpha_1}{n}e^{in\eta} + \overline{\beta}2\alpha_2\sinh 2\rho_0 e^{-2n\rho_0}e^{in\eta} & \text{if } n \geqslant 1, \\ \beta\dfrac{\kappa\alpha_2}{n}e^{in\eta} + \beta\dfrac{\alpha_1 e^{2n\rho_0}}{n}e^{-in\eta} & \text{if } n \leqslant -1, \end{cases}$$

which can be rephrased as

$$2h_0^{-1}\mathbf{S}[\psi] = \begin{cases} -\dfrac{\beta}{\overline{\beta}}\dfrac{\kappa\alpha_2 e^{-2n\rho_0}}{n}\overline{\psi} - \left(\dfrac{\alpha_1}{n} - \dfrac{\overline{\beta}}{\beta}2\alpha_2\sinh 2\rho_0 e^{-2n\rho_0}\right)\psi & \text{if } n \geqslant 1, \\ \dfrac{\kappa\alpha_2}{n}\psi + \dfrac{\beta}{\overline{\beta}}\dfrac{\alpha_1 e^{2n\rho_0}}{n}\overline{\psi} & \text{if } n \leqslant -1. \end{cases} \tag{A 29}$$

So we obtain the following lemma from (A 19).

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Lemma A.2 *It holds that*

$$h_0^{-1}(\omega)\mathbf{S}[\boldsymbol{\psi}_{1,n}](\omega) = -\left(\frac{\alpha_1}{2n} - \alpha_2 \sinh 2\rho_0 e^{-2n\rho_0}\right)\boldsymbol{\psi}_{1,n} - \frac{\kappa\alpha_2 e^{-2n\rho_0}}{2n}\boldsymbol{\psi}_{3,n},$$
$$h_0^{-1}(\omega)\mathbf{S}[\boldsymbol{\psi}_{2,n}](\omega) = -\left(\frac{\alpha_1}{2n} + \alpha_2 \sinh 2\rho_0 e^{-2n\rho_0}\right)\boldsymbol{\psi}_{2,n} - \frac{\kappa\alpha_2 e^{-2n\rho_0}}{2n}\boldsymbol{\psi}_{4,n},$$
$$h_0^{-1}(\omega)\mathbf{S}[\boldsymbol{\psi}_{3,n}](\omega) = -\frac{\alpha_1 e^{-2n\rho_0}}{2n}\boldsymbol{\psi}_{1,n} - \frac{\kappa\alpha_2}{2n}\boldsymbol{\psi}_{3,n},$$
$$h_0^{-1}(\omega)\mathbf{S}[\boldsymbol{\psi}_{4,n}](\omega) = -\frac{\alpha_1 e^{-2n\rho_0}}{2n}\boldsymbol{\psi}_{2,n} - \frac{\kappa\alpha_2}{2n}\boldsymbol{\psi}_{4,n}.$$

Now we compute $\partial_\nu \mathbf{S}[\psi]|_+(\omega)$ using (A 9). We obtain from (A 22)

$$\mathcal{C}[h_0^{-1}\psi_n]'(z) = -\frac{e^{-|n|\xi - n\rho_0}(|n|\sinh\xi + \cosh\xi)}{R^2\sinh^3\xi}. \tag{A 30}$$

We also obtain from (A 24)

$$\mathcal{C}[\overline{\zeta}h_0^{-1}\psi_n]'(z) = \begin{cases} -\dfrac{e^{-n(\xi+\rho_0)}[n\cosh(\xi+2\rho_0)\sinh\xi + \cosh 2\rho_0]}{R\sinh^3\xi}, & n \geqslant 1, \\ -\dfrac{e^{n(\xi-\rho_0)}[-n\cosh(\xi-2\rho_0)\sinh\xi + \cosh 2\rho_0]}{R\sinh^3\xi}, & n \leqslant -1. \end{cases} \tag{A 31}$$

Let $\psi = \beta h_0^{-1}\psi_n$. Since $d\sigma(z) = h_0(\eta)d\eta$, $dz = iR\sinh\xi d\eta$, and $d\overline{z} = -iR\sinh\overline{\xi}d\eta$ on $\partial\Omega$, we have from (A 9) that

$$h_0(\eta)\partial_\nu\mathbf{S}[\psi]|_+ = R\mu\alpha_2\sinh\xi\left[\mathcal{C}[\psi] + \overline{\mathcal{C}[\psi]}\right] + R\sinh\overline{\xi}\left[\mu\alpha_1\overline{\mathcal{C}[\overline{\psi}]} - \mu\alpha_2\left(z\overline{\mathcal{C}[\psi]'} - \overline{\mathcal{C}[\overline{\zeta}\psi]'}\right)\right].$$

We then obtain from (A 22), (A 23), (A 30), and (A 31) (after tedious computations which we omit) that

$$\partial_\nu\mathbf{S}[\psi]|_+(\eta) = \left[\mu\alpha_1\beta - 2n\mu\alpha_2\overline{\beta}\sinh 2\rho_0 e^{-2n\rho_0}\right]h_0^{-1}\psi_n(\eta) + \mu\alpha_2\beta e^{-2n\rho_0}h_0^{-1}\psi_{-n}(\eta),$$

for $n \geqslant 1$. It is helpful to mention that the following identities are used:

$$\cosh\xi\cosh\overline{\xi} - \cosh 2\rho_0 = -\sinh\xi\sinh\overline{\xi}, \quad \cosh\xi = \cosh(\overline{\xi} - 2\rho_0).$$

It then follows from (2.6) that

$$\mathbf{K}^*[\psi](\eta) = \left[\left(\mu\alpha_1 - \frac{1}{2}\right)\beta - 2n\mu\alpha_2\overline{\beta}\sinh 2\rho_0 e^{-2n\rho_0}\right]h_0^{-1}\psi_n(\eta) + \mu\alpha_2\beta e^{-2n\rho_0}h_0^{-1}\psi_{-n}(\eta). \tag{A 32}$$

Similarly, one can see for $n \leqslant -1$ that

$$\partial_\nu\mathbf{S}[\beta h_0^{-1}\psi_n]|_+(\eta) = \mu\alpha_1\beta e^{2n\rho_0}h_0^{-1}\psi_{-n}(\eta) + \mu\alpha_2\beta h_0^{-1}\psi_n(\eta),$$

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and hence

$$\mathbf{K}^*[\beta h_0^{-1}\psi_n](\eta) = \mu\alpha_1\beta e^{-2n\rho_0}h_0^{-1}\psi_{-n}(\eta) + \left(\mu\alpha_2 - \frac{1}{2}\right)\beta h_0^{-1}\psi_n(\eta).$$

Note that $\mu\alpha_1 - \frac{1}{2} = k_0$. We then obtain from (A 19) that

$$\begin{aligned} \mathbf{K}^*[\boldsymbol{\psi}_{1,n}](\omega) &= \left(k_0 - 2n\mu\alpha_2 \sinh 2\rho_0 e^{-2n\rho_0}\right)\boldsymbol{\psi}_{1,n} + \mu\alpha_2 e^{-2n\rho_0}\boldsymbol{\psi}_{3,n}, \\ \mathbf{K}^*[\boldsymbol{\psi}_{3,n}](\omega) &= \mu\alpha_1 e^{-2n\rho_0}\boldsymbol{\psi}_{1,n} - k_0\boldsymbol{\psi}_{3,n}, \end{aligned} \tag{A 33}$$

with $\beta = 1$, and

$$\begin{aligned} \mathbf{K}^*[\boldsymbol{\psi}_{2,n}](\omega) &= \left[k_0 + 2n\mu\alpha_2 \sinh 2\rho_0 e^{-2n\rho_0}\right]\boldsymbol{\psi}_{2,n} + \mu\alpha_2 e^{-2n\rho_0}\boldsymbol{\psi}_{4,n}, \\ \mathbf{K}^*[\boldsymbol{\psi}_{4,n}](\omega) &= \mu\alpha_1 e^{-2n\rho_0}\boldsymbol{\psi}_{2,n} - k_0\boldsymbol{\psi}_{4,n}, \end{aligned} \tag{A 34}$$

with $\beta = i$.

We see from (A 33) that for each n $\mathbf{K}^*$ acts on the space spanned by $\boldsymbol{\psi}_{1,n}$ and $\boldsymbol{\psi}_{3,n}$ like the matrix

$$\begin{bmatrix} k_0 - 2n\mu\alpha_2 \sinh 2\rho_0 e^{-2n\rho_0} & \mu\alpha_2 e^{-2n\rho_0} \\ \mu\alpha_1 e^{-2n\rho_0} & -k_0 \end{bmatrix}.$$

So by finding the eigenvalues and eigenvectors of this matrix, one can see that $k_{1,n}$ and $k_{3,n}$ in (3.26) are eigenvalues and $\boldsymbol{\varphi}_{1,n}$ and $\boldsymbol{\varphi}_{3,n}$ in (3.28) are corresponding eigenfunctions. One can also see from (A 34) that $k_{2,n}$ and $k_{4,n}$ are eigenvalues and $\boldsymbol{\varphi}_{2,n}$ and $\boldsymbol{\varphi}_{4,n}$ are corresponding eigenfunctions.

Appendix B Proofs of CALR

Let $k_{j,n}$ ($j = 1, \ldots, 4$, $n = 1, \ldots$) be eigenvalues of the NP operator given in (3.26) and $\boldsymbol{\varphi}_{j,n}$ be corresponding eigenfunctions given in (3.28). Put $\boldsymbol{\phi}_{j,n} := \boldsymbol{\varphi}_{j,n}/\|\boldsymbol{\varphi}_{j,n}\|_*$ and

$$\alpha_{j,n}(\mathbf{z}) := (\boldsymbol{\phi}_{j,n}, \partial_\nu \mathbf{F}_\mathbf{z})_*.$$

Then, we have from (4.7) and (4.10) that

$$\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_\mathbf{z}(\mathbf{x}) = \sum_{j=1}^{4}\sum_n \frac{\alpha_{j,n}(\mathbf{z})}{k_\delta(c) - k_{j,n}}\mathbf{S}[\boldsymbol{\phi}_{j,n}](\mathbf{x}), \tag{B 1}$$

and from (4.12) that

$$E(\mathbf{u}_\delta - \mathbf{F}_z) \approx \sum_{j=1}^{4}\sum_n \frac{|\alpha_{j,n}(\mathbf{z})|^2}{|k_\delta(c) - k_{j,n}|^2}. \tag{B 2}$$

We obtain from Green's formula and the jump relation (2.6) that

$$\alpha_{j,n}(\mathbf{z}) = -\langle \boldsymbol{\phi}_{j,n}, \mathbf{S}[\partial_\nu \mathbf{F}_\mathbf{z}]\rangle = -\langle \partial_\nu \mathbf{S}[\boldsymbol{\phi}_{j,n}]_-, \mathbf{F}_\mathbf{z}\rangle = \left(-k_{j,n} + \frac{1}{2}\right)\langle \boldsymbol{\phi}_{j,n}, \mathbf{F}_\mathbf{z}\rangle.$$

So we have from (4.17) that

$$\alpha_{j,n}(\mathbf{z}) = -\left(k_{j,n} - \frac{1}{2}\right)(\mathbf{A}\nabla)^T\, \mathbf{S}[\boldsymbol{\phi}_{j,n}](\mathbf{z}).$$

Thanks to (4.13), we have

$$|\alpha_{j,n}(\mathbf{z})| \approx \frac{\left|(\mathbf{A}\nabla)^T\, \mathbf{S}[\boldsymbol{\varphi}_{j,n}](\mathbf{z})\right|}{\|\boldsymbol{\varphi}_{j,n}\|_*}. \tag{B 3}$$

We now estimate $(\mathbf{A}\nabla)^T\, \mathbf{S}[\boldsymbol{\varphi}_{j,n}](\mathbf{z})$. Let us compute $\left\|\boldsymbol{\varphi}_{j,n}\right\|_*$ first. From (3.30) and Lemma A.2, one can easily see that

$$\left\|\boldsymbol{\psi}_{1,n}\right\|_*^2 = -\left\langle \boldsymbol{\psi}_{1,n}, \mathbf{S}[\boldsymbol{\psi}_{1,n}]\right\rangle = \pi\left(\frac{\alpha_1}{n} - 2\alpha_2 \sinh 2\rho_0 e^{-2n\rho_0}\right).$$

We can also see that

$$\left\|\boldsymbol{\psi}_{2,n}\right\|_*^2 = \pi\left(\frac{\alpha_1}{n} + 2\alpha_2 e^{-2n\rho_0}\sinh 2\rho_0\right),$$
$$\left\|\boldsymbol{\psi}_{3,n}\right\|_*^2 = \left\|\boldsymbol{\psi}_{4,n}\right\|_*^2 = \frac{\pi\kappa\alpha_2}{n}.$$

In addition, we have

$$\left(\boldsymbol{\psi}_{1,n}, \boldsymbol{\psi}_{3,n}\right)_* = \left(\boldsymbol{\psi}_{2,n}, \boldsymbol{\psi}_{4,n}\right)_* = \frac{\pi\alpha_1 e^{-2n\rho_0}}{n}.$$

It then follows from (3.28) and (3.29) that

$$\begin{aligned}\left\|\boldsymbol{\varphi}_{1,n}\right\|_*^2 &= -\left\langle \boldsymbol{\psi}_{1,n} + \frac{p_n}{k_0 + k_{1,n}}\boldsymbol{\psi}_{3,n}, \mathbf{S}[\boldsymbol{\psi}_{1,n}] + \frac{p_n}{k_0 + k_{1,n}}\mathbf{S}[\boldsymbol{\psi}_{3,n}]\right\rangle \\ &= \left\|\boldsymbol{\psi}_{1,n}\right\|_*^2 + \left(\frac{p_n}{k_0 + k_{1,n}}\right)^2 \left\|\boldsymbol{\psi}_{3,n}\right\|_*^2 + \frac{2p_n}{k_0 + k_{1,n}}\left(\boldsymbol{\psi}_{1,n}, \boldsymbol{\psi}_{3,n}\right)_* \\ &= \frac{\pi\alpha_1}{n} + e^{-2n\rho_0}O(1).\end{aligned} \tag{B 4}$$

In the same way, we also obtain

$$\left\|\boldsymbol{\varphi}_{2,n}\right\|_*^2 = \frac{\pi\alpha_1}{n} + e^{-2n\rho_0}O(1), \tag{B 5}$$
$$\left\|\boldsymbol{\varphi}_{3,n}\right\|_*^2 = \frac{\pi\kappa\alpha_2}{n} + ne^{-4n\rho_0}O(1), \tag{B 6}$$
$$\left\|\boldsymbol{\varphi}_{4,n}\right\|_*^2 = \frac{\pi\kappa\alpha_2}{n} + ne^{-4n\rho_0}O(1). \tag{B 7}$$

Let us introduce two notations to make expressions short. Let (ρ, ω) be the elliptic coordinates of $\mathbf{z}$ and let

$$\mathbf{b}(\mathbf{z}) := \begin{bmatrix}\cos\omega \sinh\rho \\ \sin\omega \cosh\rho\end{bmatrix}, \tag{B 8}$$

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and

$$\widetilde{\mathbf{U}}(\mathbf{z}) = (e^{2(\rho-\rho_0)} - e^{-2(\rho-\rho_0)})\mathbf{I} + (e^{2\rho_0} - e^{2\rho})\mathbf{U}(-2\omega) + (e^{-2\rho} - e^{-2\rho_0})\mathbf{U}(2\omega), \quad \text{(B 9)}$$

where $\mathbf{U}(\theta)$ is the rotation by the angle θ.

Lemma B.1 *The matrix $\widetilde{\mathbf{U}}(\mathbf{z})$ is non-singular for any $0 \leqslant \omega \leqslant \pi/2$ and $\rho > \rho_0$.*

Proof Put $\eta := \rho - \rho_0$. Then, we have

$$\widetilde{\mathbf{U}}(\rho,\omega) = \left(e^{2\eta} - e^{-2\eta}\right)\mathbf{I} - e^{2\rho_0}\left(e^{2\eta}-1\right)\mathbf{U}(-2\omega) - e^{-2\rho_0}\left(1-e^{-2\eta}\right)\mathbf{U}(2\omega).$$

Assume that $0 < \omega < \pi/2$. Since $\eta > 0$, we have

$$e^{2\rho_0}\left(e^{2\eta}-1\right) > e^{-2\rho_0}\left(1-e^{-2\eta}\right) > 0.$$

It means that

$$e^{2\rho_0}\left(e^{2\eta}-1\right)\mathbf{U}(-2\omega)\mathbf{a} + e^{-2\rho_0}\left(1-e^{-2\eta}\right)\mathbf{U}(2\omega)\mathbf{a} \neq c\mathbf{a}$$

for any real number c and constant vector $\mathbf{a}$, which implies that $\widetilde{\mathbf{U}}(\rho,\omega)\mathbf{a} \neq 0$, and hence $\widetilde{\mathbf{U}}(\rho,\omega)$ is non-singular.

If $\omega = 0$, one can easily show that

$$e^{2\eta} - e^{-2\eta} - e^{2\rho_0}\left(e^{2\eta}-1\right) - e^{-2\rho_0}(1-e^{-2\eta}) \neq 0$$

for any $\eta, \rho_0 > 0$, and hence $\widetilde{\mathbf{U}}(\rho,0)$ is non-singular. Similarly one can see that $\widetilde{\mathbf{U}}(\rho,\pi/2)$ is non-singular. □

Through long but straightforward computations, which will be presented at the end of this subsection, we see that

$$(\mathbf{A}\nabla)^T\mathbf{S}[\boldsymbol{\psi}_{1,n}](\mathbf{z}) = \frac{R\alpha_1 e^{-n(\rho-\rho_0)}}{2h(\rho,\omega)^2}\left(\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2\right)\cdot\mathbf{U}(n\omega)\mathbf{b}(\mathbf{z}), \quad \text{(B 10)}$$

$$(\mathbf{A}\nabla)^T\mathbf{S}[\boldsymbol{\psi}_{2,n}](\mathbf{z}) = \frac{R\alpha_1 e^{-n(\rho-\rho_0)}}{2h(\rho,\omega)^2}\left(\mathbf{U}(-\pi/2)\mathbf{a}_1 + \mathbf{a}_2\right)\cdot\mathbf{U}(n\omega)\mathbf{b}(\mathbf{z}), \quad \text{(B 11)}$$

modulo $ne^{-n(\rho+\rho_0)}O(1)$, and

$$(\mathbf{A}\nabla)^T\mathbf{S}[\boldsymbol{\psi}_{3,n}](\mathbf{z}) = -\frac{n\alpha_2 R^3 e^{-n(\rho-\rho_0)}}{8h(\rho,\omega)^4}\left[\widetilde{\mathbf{U}}(\rho,\omega)(\mathbf{a}_1 + \mathbf{U}\left(\pi/2\right)\mathbf{a}_2)\right]\cdot\mathbf{U}(n\omega)\mathbf{b}(\mathbf{z}), \quad \text{(B 12)}$$

$$(\mathbf{A}\nabla)^T\mathbf{S}[\boldsymbol{\psi}_{4,n}](\mathbf{z}) = \frac{n\alpha_2 R^3 e^{-n(\rho-\rho_0)}}{8h(\rho,\omega)^4}\left[\widetilde{\mathbf{U}}(\rho,\omega)\left(\mathbf{U}(-\pi/2)\mathbf{a}_1 + \mathbf{a}_2\right)\right]\cdot\mathbf{U}(n\omega)\mathbf{b}(\mathbf{z}), \quad \text{(B 13)}$$

modulo $e^{-n(\rho-\rho_0)}O(1)$.

We have from (3.28) that

$$(\mathbf{A}\nabla)^T\,\mathbf{S}[\boldsymbol{\varphi}_{1,n}](\mathbf{z}) = (\mathbf{A}\nabla)^T\,\mathbf{S}[\boldsymbol{\psi}_{1,n}](\mathbf{z}) + \frac{p_n}{k_0 + k_{1,n}}(\mathbf{A}\nabla)^T\,\mathbf{S}[\boldsymbol{\psi}_{3,n}](\mathbf{z}).$$

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One can see from (3.29) and (B 12) that

$$\left| \frac{p_n}{k_0 + k_{1,n}} (\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\psi}_{3,n}](\mathbf{z}) \right| \lesssim n e^{-n(\rho+\rho_0)}.$$

So, we have from (B 10) that

$$(\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\varphi}_{1,n}](\mathbf{z}) = \frac{R\alpha_1 e^{-n(\rho-\rho_0)}}{2h(\rho,\omega)^2} \left(\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2\right) \cdot \mathbf{U}(n\omega)\mathbf{b}(\mathbf{z}) \tag{B 14}$$

modulo $ne^{-n(\rho+\rho_0)}O(1)$. Similarly, one can show using (B 11) and (B 13) that

$$(\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\varphi}_{2,n}](\mathbf{z}) = \frac{R\alpha_1 e^{-n(\rho-\rho_0)}}{2h(\rho,\omega)^2} \left(\mathbf{U}(-\pi/2)\mathbf{a}_1 + \mathbf{a}_2\right) \cdot \mathbf{U}(n\omega)\mathbf{b}(\mathbf{z}), \tag{B 15}$$

modulo $ne^{-n(\rho+\rho_0)}O(1)$.

Observe that

$$\begin{aligned} &\left|\left(\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2\right) \cdot \mathbf{U}(n\omega)\mathbf{b}(\mathbf{z})\right|^2 + \left|\left(\mathbf{U}(-\pi/2)\mathbf{a}_1 + \mathbf{a}_2\right) \cdot \mathbf{U}(n\omega)\mathbf{b}(\mathbf{z})\right|^2 \\ &= |\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2|^2 \, |\mathbf{b}(\mathbf{z})|^2. \end{aligned}$$

It then follows from (B 14) and (B 15) that

$$\left|(\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\varphi}_{1,n}](\mathbf{z})\right|^2 + \left|(\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\varphi}_{2,n}](\mathbf{z})\right|^2 = \frac{R^2\alpha_1^2 e^{-2n(\rho-\rho_0)}}{4h(\rho,\omega)^4} |\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2|^2 \, |\mathbf{b}(\mathbf{z})|^2,$$

modulo $n^2 e^{-2n(\rho+\rho_0)}O(1)$. We choose constant vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ so that

$$\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2 \neq 0. \tag{B 16}$$

Then, we have

$$\left|(\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\varphi}_{1,n}](\mathbf{z})\right|^2 + \left|(\mathbf{A}\nabla)^T \mathbf{S}[\boldsymbol{\varphi}_{2,n}](\mathbf{z})\right|^2 \approx e^{-2n(\rho-\rho_0)},$$

which, together with (B 4) and (B 5), shows

$$|\alpha_{1,n}(\mathbf{z})|^2 + |\alpha_{2,n}(\mathbf{z})|^2 \approx n e^{-2n(\rho-\rho_0)} + n^2 e^{-2n\rho} O(1). \tag{B 17}$$

Similarly, one can show that

$$|\alpha_{3,n}(\mathbf{z})|^2 + |\alpha_{4,n}(\mathbf{z})|^2 \approx n^3 e^{-2n(\rho-\rho_0)} + n^2 e^{-2n(\rho-\rho_0)} O(1). \tag{B 18}$$

Proof of Theorem 4.3. In this proof (ρ,ω) denotes the elliptic coordinates of $\mathbf{z}$. Since

$$|k_\delta(c) - k_{j,n}|^2 \approx \delta^2 + |k(c) - k_{j,n}|^2, \tag{B 19}$$

it follows from (B 2) that

$$E(\mathbf{u}_\delta - \mathbf{F}_z) \approx \sum_{j=1}^{4} \sum_n \frac{|\alpha_{j,n}(\mathbf{z})|^2}{\delta^2 + |k(c) - k_{j,n}|^2}. \tag{B 20}$$

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Since $k(c) = k_0$, we see from (3.31) that

$$|k(c) - k_{j,n}| = c_0 n e^{-2n\rho_0} + n^{3/2} e^{-3n\rho_0} O(1), \quad j = 1, 2, \tag{B 21}$$

where

$$c_0 := \frac{q}{\lambda + 2\mu}.$$

We also see from (3.31) that

$$|k(c) - k_{j,n}| \geqslant C \tag{B 22}$$

for some constant C independent of n for $j = 3, 4$. It then follows from (B 17) and (B 18) that

$$E(\mathbf{u}_\delta - \mathbf{F}_z) \approx \sum_{n=1}^{\infty} \frac{n e^{-2n(\rho-\rho_0)}}{\delta^2 + c_0^2 n^2 e^{-4n\rho_0}}. \tag{B 23}$$

For $0 < \delta \ll 1$, let $N \geqslant 1$ be the first integer such that $\delta > c_0 N e^{-2N\rho_0}$. Then, we have

$$\delta^{1/N} \sim c_0^{1/N} N^{1/N} e^{-2\rho_0} = e^{-2\rho_0} + o(1),$$

that is,

$$N \sim -\frac{1}{2\rho_0} \log \delta.$$

We then write

$$\sum_{n=1}^{\infty} \frac{n e^{-2n(\rho-\rho_0)}}{\delta^2 + c_0^2 n^2 e^{-4n\rho_0}} = \sum_{n \leqslant N} + \sum_{n > N} =: I_N + II_N. \tag{B 24}$$

For the first term, we have

$$I_N \sim \sum_{n \leqslant N} \frac{n e^{-2n(\rho-\rho_0)}}{n^2 e^{-4n\rho_0}} = \sum_{n \leqslant N} \frac{e^{2n(3\rho_0-\rho)}}{n} \sim \int_1^N e^{2(3\rho_0-\rho)s} \frac{ds}{s}.$$

If $\rho_0 < \rho < 3\rho_0$, then one sees using an integration by parts that

$$\int_1^N e^{2(3\rho_0-\rho)s} \frac{ds}{s} \sim N^{-1} e^{2N(3\rho_0-\rho)},$$

as $N \to \infty$. It is easy to see that $\int_1^N e^{2(3\rho_0-\rho)s} \frac{ds}{s} \sim \log N$ if $\rho = 3\rho_0$, and $\int_1^N e^{2(3\rho_0-\rho)s} \frac{ds}{s} \sim 1$ if $\rho > 3\rho_0$. So we obtain that

$$\sum_{n \leqslant N} \frac{n e^{2n(3\rho_0-\rho)}}{n^2 e^{-4n\rho_0}} \sim \begin{cases} |\log \delta|^{-1} \delta^{-(3-\rho/\rho_0)}, & \text{if } \rho_0 < \rho < 3\rho_0, \\ \log |\log \delta|, & \text{if } \rho = 3\rho_0, \\ 1, & \text{if } \rho > 3\rho_0. \end{cases}$$

On the other hand, we have

$$II_N \sim \frac{1}{\delta^2} \sum_{n > N} n e^{-2n(\rho-\rho_0)} \sim |\log \delta| \, \delta^{\rho/\rho_0 - 3}.$$

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So, we have from (B 23) and (B 24)

$$E(\mathbf{u}_\delta - \mathbf{F}_z) \sim \begin{cases} |\log\delta|\,\delta^{-3+\rho/\rho_0} & \text{if } \rho_0 < \rho \leqslant 3\rho_0, \\ 1 & \text{if } \rho > 3\rho_0. \end{cases}$$

Since $E(\mathbf{F}_z) < \infty$, we have (4.19), and the proof is complete. □

Proof of Theorem 4.4. Here, we denote by (ρ,ω) the elliptic coordinates of $\mathbf{x}$ and by $(\rho_\mathbf{z},\omega_\mathbf{z})$ those of $\mathbf{z}$. It follows from (B 1) and (B 19) that

$$|\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_\mathbf{z}(\mathbf{x})| \leqslant \sum_{j=1}^{4}\sum_n \frac{|\alpha_{j,n}(\mathbf{z})|}{(\delta + |k(c) - k_{j,n}|)\|\boldsymbol{\varphi}_{j,n}\|_*} \left|\mathbf{S}[\boldsymbol{\varphi}_{j,n}](\mathbf{x})\right|.$$

We then have from (B 21) and (B 22) that

$$|\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_\mathbf{z}(\mathbf{x})| \lesssim \sum_{j=1}^{2}\sum_n \frac{|\alpha_{j,n}(\mathbf{z})|}{ne^{-2n\rho_0}\|\boldsymbol{\varphi}_{j,n}\|_*} \left|\mathbf{S}[\boldsymbol{\varphi}_{j,n}](\mathbf{x})\right| + \sum_{j=3}^{4}\sum_n \frac{|\alpha_{j,n}(\mathbf{z})|}{\|\boldsymbol{\varphi}_{j,n}\|_*} \left|\mathbf{S}[\boldsymbol{\varphi}_{j,n}](\mathbf{x})\right|.$$

It then follows from (B 4)–(B 7), (A 27), (A 28), (B 17), and (B 18) that

$$\begin{aligned} |\mathbf{u}_\delta(\mathbf{x}) - \mathbf{F}_\mathbf{z}(\mathbf{x})| &\lesssim \sum_{j=1}^{2}\sum_n \frac{n^{1/2}e^{-n(\rho_\mathbf{z}-\rho_0)}}{ne^{-2n\rho_0}n^{-1/2}}\frac{e^{-n(\rho-\rho_0)}}{n} + \sum_{j=3}^{4}\sum_n \frac{n^{3/2}e^{-n(\rho_\mathbf{z}-\rho_0)}}{n^{-1/2}}e^{-n(\rho-\rho_0)} \\ &\lesssim \sum_{n=1}^{\infty} \frac{e^{-n(\rho+\rho_\mathbf{z}-4\rho_0)}}{n}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 4.5. Since $k(c) = -k_0$, we obtain from (3.31)

$$|k(c) - k_{j,n}| \approx 1, \quad j = 1, 2, \tag{B 25}$$

and

$$|k(c) - k_{j,n}| = \frac{q^2}{4\mu(\lambda+2\mu)^2}n^2e^{-4n\rho_0} + ne^{-4n\rho_0}O(1), \quad j = 3, 4. \tag{B 26}$$

The rest of the proof is similar to that of Theorem 4.3. □

Theorem 4.6 can be proved in the same way as Theorem 4.4 using (B 25) and (B 26).

Let us prove (B 10)–(B 13). We prove only (B 12) that is most involved and the rest can be proved similarly. Let

$$h_1(\rho,\omega) := \frac{e^{2(\rho-\rho_0)} - e^{-2(\rho-\rho_0)}}{h(\rho,\omega)^2}, \quad h_2(\rho,\omega) := \frac{e^{2\rho_0} - e^{2\rho}}{h(\rho,\omega)^2}, \quad h_3(\rho,\omega) := \frac{e^{-2\rho} - e^{-2\rho_0}}{h(\rho,\omega)^2},$$

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where $h(\rho, \omega)$ is defined in (A 26), and let

$$\mathbf{f}_1(\rho, \omega) = \begin{bmatrix} f_{11}(\rho, \omega) \\ f_{12}(\rho, \omega) \end{bmatrix} := h_1(\rho, \omega) e^{-n(\rho-\rho_0)} \begin{bmatrix} \cos n\omega \\ \sin n\omega \end{bmatrix},$$
$$\mathbf{f}_2(\rho, \omega) = \begin{bmatrix} f_{21}(\rho, \omega) \\ f_{22}(\rho, \omega) \end{bmatrix} := h_2(\rho, \omega) e^{-n(\rho-\rho_0)} \begin{bmatrix} \cos(n+2)\omega \\ \sin(n+2)\omega \end{bmatrix},$$
$$\mathbf{f}_3(\rho, \omega) = \begin{bmatrix} f_{31}(\rho, \omega) \\ f_{32}(\rho, \omega) \end{bmatrix} := h_3(\rho, \omega) e^{-n(\rho-\rho_0)} \begin{bmatrix} \cos(n-2)\omega \\ \sin(n-2)\omega \end{bmatrix}.$$

Then, one can see Lemma A.1 that

$$\mathbf{S}[\boldsymbol{\psi}_{3,n}](\rho, \omega) = \frac{\alpha_2 R^2}{8} [\mathbf{f}_1(\rho, \omega) + \mathbf{f}_2(\rho, \omega) + \mathbf{f}_3(\rho, \omega)] + n^{-1} e^{-n(\rho-\rho_0)} O(1). \tag{B 27}$$

Straightforward computations yield

$$\partial_\rho f_{11} = \left(\partial_\rho h_1 - n h_1\right) e^{-n(\rho-\rho_0)} \cos n\omega,$$
$$\partial_\omega f_{11} = e^{-n(\rho-\rho_0)} \left(\partial_\omega h_1 \cos n\omega - n h_1 \sin n\omega\right),$$

which can be rewritten as

$$\begin{bmatrix} \partial_\rho \\ \partial_\omega \end{bmatrix} f_{11} = e^{-n(\rho-\rho_0)} \left(\cos n\omega \begin{bmatrix} \partial_\rho \\ \partial_\omega \end{bmatrix} h_1 - n h_1 \begin{bmatrix} \cos n\omega \\ \sin n\omega \end{bmatrix} \right). \tag{B 28}$$

Recall the following chain rule:

$$\nabla = \frac{R}{h^2} \mathbf{C}(\rho, \omega) \begin{bmatrix} \partial_\rho \\ \partial_\omega \end{bmatrix},$$

where

$$\mathbf{C}(\rho, \omega) = \begin{bmatrix} \cos\omega \sinh\rho & -\sin\omega \cosh\rho \\ \sin\omega \cosh\rho & \cos\omega \sinh\rho \end{bmatrix}.$$

It then follows from (B 28) that

$$\nabla f_{11} = e^{-n(\rho-\rho_0)} \left[(\nabla h_1) \cos n\omega - \frac{R}{h^2} n h_1 \mathbf{C}(\rho, \omega) \begin{bmatrix} \cos n\omega \\ \sin n\omega \end{bmatrix} \right].$$

Observe that

$$\mathbf{C}(\rho, \omega) \begin{bmatrix} \cos n\omega \\ \sin n\omega \end{bmatrix} = \mathbf{U}(n\omega) \mathbf{b}(\rho, \omega),$$

where $\mathbf{b}(\rho, \omega)$ is defined in (B 8). So, we have

$$\nabla f_{11} = e^{-n(\rho-\rho_0)} \left[(\nabla h_1) \cos n\omega - \frac{R}{h^2} n h_1 \mathbf{U}(n\omega) \mathbf{b}(\rho, \omega) \right]. \tag{B 29}$$

Likewise, one can show that

$$\nabla f_{12} = e^{-n(\rho-\rho_0)} \left[(\nabla h_1) \sin n\omega + \frac{R}{h^2} n h_1 \mathbf{U}(\pi/2) \mathbf{U}(n\omega) \mathbf{b}(\rho, \omega) \right]. \tag{B 30}$$

It follows that

$$(\mathbf{A}\nabla)^T_{\mathbf{f}_1} = \mathbf{a}_1 \cdot \nabla(F_1)_1 + \mathbf{a}_2 \cdot \nabla(F_1)_2$$
$$= e^{-n(\rho-\rho_0)}\Big[(\mathbf{a}_1 \cdot \nabla h_1)\cos n\omega - \frac{nRh_1}{h^2}\mathbf{a}_1 \cdot \mathbf{U}(n\omega)\mathbf{b}(\rho,\omega)$$
$$+ (\mathbf{a}_2 \cdot \nabla h_1)\sin n\omega + \frac{nRh_1}{h^2}\mathbf{a}_2 \cdot \mathbf{U}(\pi/2)\mathbf{U}(n\omega)\mathbf{b}(\rho,\omega)\Big].$$

Note that $\partial^\alpha h_j$, $|\alpha| \leqslant 1$, $\alpha \in \mathbb{N}^2$, $j = 1,2,3$, is uniformly bounded as $n \to \infty$. Thus, we have

$$(\mathbf{A}\nabla)^T_{\mathbf{f}_1} = \frac{nRe^{-n(\rho-\rho_0)}h_1}{h^2}\left[-\mathbf{a}_1 \cdot \mathbf{U}(n\omega)\mathbf{b}(\rho,\omega) + \mathbf{a}_2 \cdot \mathbf{U}(\pi/2)\mathbf{U}(n\omega)\mathbf{b}(\rho,\omega)\right]$$
$$= -\frac{nRe^{-n(\rho-\rho_0)}h_1}{h^2}\left[\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2\right] \cdot \mathbf{U}(n\omega)\mathbf{b}(\rho,\omega), \tag{B 31}$$

where the equalities hold modulo $e^{-n(\rho-\rho_0)}O(1)$ terms.

Similarly, one can show that

$$\nabla f_{21} = e^{-n(\rho-\rho_0)}\Big[(\nabla h_2)\cos(n+2)\omega$$
$$- \frac{Rh_2}{h^2}\{n\mathbf{U}(2\omega)\mathbf{U}(n\omega) + 2\sin(n+2)\omega\mathbf{U}(\pi/2)\}\mathbf{b}\Big],$$
$$\nabla f_{22} = e^{-n(\rho-\rho_0)}\Big[(\nabla h_2)\sin(n+2)\omega$$
$$- \frac{Rh_2}{h^2}\mathbf{U}(-\pi/2)\{n\mathbf{U}(2\omega)\mathbf{U}(n\omega) + 2\cos(n+2)\omega\mathbf{I}\}\mathbf{b}\Big],$$

and

$$\nabla f_{31} = e^{-n(\rho-\rho_0)}\Big[(\nabla h_3)\cos(n-2)\omega$$
$$- \frac{Rh_3}{h^2}\{n\mathbf{U}(-2\omega)\mathbf{U}(n\omega) - 2\sin(n-2)\omega\mathbf{U}(\pi/2)\}\mathbf{b}\Big],$$
$$\nabla f_{32} = e^{-n(\rho-\rho_0)}\Big[(\nabla h_3)\sin(n-2)\omega$$
$$- \frac{Rh_3}{h^2}\mathbf{U}(-\pi/2)\{n\mathbf{U}(-2\omega)\mathbf{U}(n\omega) + 2\cos(n-2)\omega\mathbf{I}\}\mathbf{b}\Big].$$

So, we have

$$(\mathbf{A}\nabla)^T_{\mathbf{f}_2} = -\frac{nRe^{-n(\rho-\rho_0)}h_2}{h^2}\mathbf{U}(-2\omega)\left[\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2\right] \cdot \mathbf{U}(n\omega)\mathbf{b}(\rho,\omega), \tag{B 32}$$
$$(\mathbf{A}\nabla)^T_{\mathbf{f}_3} = -\frac{nRe^{-n(\rho-\rho_0)}h_3}{h^2}\mathbf{U}(2\omega)\left[\mathbf{a}_1 + \mathbf{U}(\pi/2)\mathbf{a}_2\right] \cdot \mathbf{U}(n\omega)\mathbf{b}(\rho,\omega). \tag{B 33}$$

Note that

$$h_1\mathbf{I} + h_2\mathbf{U}(-2\omega) + h_3\mathbf{U}(2\omega) = h^2\widetilde{\mathbf{U}}.$$

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Since

$$(\mathbf{A}\nabla)^T\mathbf{S}[\boldsymbol{\psi}_{3,n}] = \frac{\alpha_2 R^2}{8}\left((\mathbf{A}\nabla)^T_{\mathbf{f}_1} + (\mathbf{A}\nabla)^T_{\mathbf{f}_2} + (\mathbf{A}\nabla)^T_{\mathbf{f}_3}\right) + n^{-1}e^{-n(\rho-\rho_0)}O(1),$$

we obtain (B 12) from (B 31)–(B 33).

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