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Distributed model predictive coverage control for decoupled mobile robots

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SUMMARY
A distributed coverage control scheme based on the state space model predictive control, which is known as receding horizon control (RHC) for decoupled systems, is presented. An optimal control problem is formulated for a set of decoupled robotic systems where a cost function couples the dynamical behavior of the robots. The coupling is described through a connected graph using a Voronoi diagram, where each robot is a node and the cost and constraints of the optimization problem associated with each robot are a function of its state and of the states of its neighbors. The complexity of the problem is addressed by breaking a centralized receding horizon controller into distinct RHC controllers of smaller sizes. Each RHC controller is associated with a different node and it computes the local control inputs based only on the position of the robot and that of its neighbors. The stability of the distributed scheme is analyzed and its properties compared with the linear quadratic regulator (LQR) design which has been proposed in the literature. Moreover, the proposed coverage algorithm is also applied to deploy a group of mobile robots in a desired formation pattern. The simulation results are used to illustrate the good performance of the proposed coverage control scheme.

KEYWORDS: Coverage control; Distributed control; Optimization; Receding horizon control; Stability.

1. Introduction

The application of multi-agent systems to control a group of robots has gained significant attention in recent years due to considerable advancements in computer technology. In a multi-agent system, several autonomous agents are simultaneously coordinated and controlled in order to achieve a common system objective. The underlying assumption is that in multi-agent systems, the agents are distributed in a predetermined fashion and each agent will act autonomously while exchanging local information with neighboring agents.[1,2] Furthermore, it has been established that the distributed control approach among autonomous agents provides better scalability and improved tractability than centralized approaches.

With the progresses made in real-time optimization-based control, some researchers have suggested new distributed control algorithms in an attempt to manipulate constraints in real time.[3,4] One major factor for consideration in developing reliable distributed control algorithms is the location of nodes for the robot network in the mission space. This is referred to as the coverage control or active sensing problem.[5,6]

The deployment location of the mobile robot must provide for maximum information retrieval, satisfactory communication level, and effective energy consumption.[7] Similar to challenges in facility location optimization such as static problems, an offline scheme can be implemented to determine coverage control by deploying the robots in an optimal location that will not require mobility. As an alternative, in a coordinated-movement dynamic scheme, the mobile robots can be deployed into a geographical area with the highest information density. However, due to similarities between facility

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location and coverage control optimization, issues regarding robot deployment have been studied using facility location optimization.[5]

It has been determined that the level of sensitivity and domain of coverage of mobile robots in their deployment location are essential to the overall efficiency of the system network. This involves a comprehensive coverage metric encompassing an optimized sensing performance and placement of mobile robots.[5,8] Researchers have used the Voronoi partitioning of the region model to reduce the challenges of locational optimization.[9] The focus of the original algorithm, for optimal mobile robot placement, was on the coordination and control of mobile robots, leading to the development of more enhanced formulations and coordination algorithms by other scholars.[5,8] As such, during recent years, formulating a cooperative control design among distributed agents assigned to a specific task that can navigate autonomously without collision has received significant attention.[1,10] Consequently, the concept of coordination and control algorithms for networked dynamic systems have become a central focus for researchers in the systems and control arena, drawing overwhelming attention.[5,11] For example, Dunbar and colleagues suggested a design for a formation pattern in a multi-agent system based on RHC.[12] In our work, we propose a distributed coverage control scheme based on RHC for the coordination of multiple mobile robots. Meguerdchian and colleagues purported the centroidal Voronoi configuration as a solution to problems associated with area of coverage in a way that clarifies the issue of coverage control. They presented their algorithms in a centralized manner as a practical approach and as having a possibility for application.[13] Since a centralized approach does not suit the distributed communication and computation structure of sensor networks, we propose our algorithm in a distributed manner, which provides better scalability and improved tractability than centralized approaches. Cortes and associates in ref. [5] suggested a decentralized coverage control algorithm for multi-robots with special dynamics in an area in a way that the mission space is partitioned in Voronoi cells. From this perspective, which is considered in this paper, they discussed the sensory control issue, which in fact is the problem of locational optimization for sensors. By contrast, one of the main advantages of our approach is its generality. Our algorithm can be extended to be applied to robots with any dynamics and it can handle any constraint imposed on the system. In addition, since our algorithm determines the coverage graph at each update instance, it is adaptive, which means it can address any possible change in the network topology such as robot failure or departure.

One of the advantages of using the RHC approach in this paper is that we have a straightforward formulation based on well-understood concepts and we can explicitly handle the constraints. It is easy to understand this property from Section III where we define the coverage problem in the RHC framework. Moreover, compared with other approaches, development time in RHC is much shorter than that for competing advanced control methods. In addition, one of the other main advantages is that RHC is easier to maintain. Therefore, changing models or specs does not require complete redesign. We verify this property in Section 7, where the proposed distributed receding horizon coverage control (DRHCC) algorithm is applied to deploy a group of mobile robots in desired formation patterns. Therefore, just by redefining the density function of the event, we can use the coverage control scheme as a formation scheme. To show the advantage of using RHC instead of a LQR controller, both approaches are applied to the same system. The Experimental result, which is presented in Section 8, shows the unstable response to a certain offset in the position of robots for the LQR controller and the larger region of attraction of RHC compared with the LQR controller.

The concepts exploited for the theoretical framework include locational optimization, RHC, distributed coverage control, and centroidal Voronoi partitions, which are briefly discussed in the next section. The centralized receding horizon coverage control (CRHCC) approach is presented in Section 3. By using the results of this section, the DRHCC algorithm is given in Section 4. In Section 5, the stability analysis of the closed-loop system is studied and it is proven that our distributed coverage scheme can optimally stabilize robots at the centroidal Voronoi configuration, which is the most optimal configuration to cover an event. In Section 6, the control scheme is tested in three simulation environments to illustrate its good performance in environments with any probability density function of events as well as its ability to generalize to much larger groups of mobile robots. In Section 7, the proposed DRHCC algorithm is applied to deploy a group of mobile robots in desired formation patterns. To show the advantage of using RHC instead of the LQR controller, both approaches are applied to the same system. The experimental result is presented in Section 8. In Section 9, we draw conclusions and discuss future work.

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2. Preliminaries

2.1. Locational optimization

This section presents some facts regarding the method used to describe coverage control for the mobile sensing network in ref. [5] and in the framework of locational optimization presented in ref. [9], which underpins the coverage algorithms depicted in the Voronoi diagram.

Assume that S is a convex space in $\Re^2$ and $P = (p_1, ..., p_n)$ is the location of n mobile robots, i.e. $p_i \in S$ denotes the ith robot's position. Furthermore, assume that the movement of each robot is confined in S and $W = \{W_1, \ldots, W_n\}$ is a tessellation of S such that $I(W_i) \cap I(W_j) = \emptyset$. $I(.)$ denotes the interior space of each W_i and $\bigcup_{i=1}^{n} W_i = S$. Therefore, it is supposed that each agent i is only responsible for covering its domain W_i. To obtain the probability of an event occurring at a point in S, the mapping $\phi : S \to \Re^+$ is defined. Note that in this sense, ϕ is the distribution density function. As robot i moves further away from any given point s inside the mission space S, its sensing performance at point s taken from the ith sensor located at $p_i \in W_i$ reduces with the distance $d(s, p_i) = \|s - p_i\|$ because of noise and the loss of resolution. This reduction is defined by function $g : \Re^+ \to \Re^+$. As a measurement of the system's performance, the coverage cost function is described as follows:

$$J(P, W) = \sum_{i=1}^{n} J(p_i, W_i) = \sum_{i=1}^{n} \int_{W_i} g(d(s, p_i))\phi(s)ds \tag{1}$$

where J is a differentiable function. Note that, the cost function J must be minimized in regards to the location of robots and space partition.

2.2. Centroidal Voronoi configuration

A collection of points $P = \{p_1, \ldots, p_n\}$ generates the Voronoi diagram, which is defined as $V = \{V_1, \ldots, V_n\}$ and V_i. This is commonly referred to as the Voronoi domain or Voronoi cell associated with point p_i and is defined by:

$$V_i = \left\{s \in S : d(s, p_i) \le d(s, p_j), \forall j \ne i\right\}$$

The above definition is commonly used to describe the Voronoi partition based on refs. [5], [9] Voronoi partitioning is one of the most important tools in localization optimization theory.

Definition 1[14]: For robot i, all neighboring Voronoi robots (meaning N_i) are described as a collection of robots with a shared Voronoi cell border. Based on definition of Voronoi partitioning, we have

$$\min_{i\in 1,\ldots,n} g(d(s, p_i)) = g(d(s, p_i))$$

for each $s \in V_j$; accordingly,

$$J(P, V(P)) = \int_S \min_{i\in 1,\ldots,n} g(d(s, p_i))\phi(s)ds \tag{2}$$

Proposition 1[5]**:** One of the necessary conditions to minimize (1) is that W partitioning must be equal to Voronoi configuration $V(P)$.

According to ref. [5]

$$\frac{\partial J_V(P)}{\partial p_i} = \frac{\partial J(p_i, V_i)}{\partial p_i} = \int_{V_i} \frac{\partial}{\partial p_i} g(d(s, p_i))\phi(s)ds$$

Hence, the partial derivative of J_V with respect to the ith robot is only associated with the position of the robot itself and that of its neighbors. The (generalized) mass and first moment (not normalized)

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and center of the Voronoi cell are defined as:

$$M_{V_i} = \int_{V_i} \phi(s)ds, \quad L_{V_i} = \int_{V_i} s\phi(s)ds, \quad C_{V_i} = \frac{L_{V_i}}{M_{V_i}} \tag{3}$$

By using the above definition and proposition and letting $g = \frac{1}{2}\|s - p_i\|$, we have

$$\frac{\partial J_V(P)}{\partial p_i} = \int_{V_i} \frac{\partial}{\partial p_i} g(d(s, p_i))\phi(s)ds = M_{V_i}(p_i - C_{V_i}) \tag{4}$$

Thus, the local minimum points for the locational optimization cost function J_V are the centroids of Voronoi cells. In other words, to minimize J_V, each robot must not only be a generator point of its own Voronoi cell but also be at the center of the cell. Accordingly, the critical partitions and points for J are called centroidal Voronoi partitions. We refer to a robots configuration as a centroidal Voronoi configuration if it gives rise to a centroidal Voronoi partition.[5]

2.3. RHC approach

RHC is an optimization approach that can be used for systems, even if some constraints on states and inputs exist. In RHC, the current control law is obtained by solving a finite horizon optimal problem at each sampling instance. Each optimization generates an open-loop optimal control trajectory, and the first portion of this optimal control trajectory is applied to the system until the next sampling time.[15,16]

In what follows, first, we suggest that CRHCC drives a group of n mobile robots at the centroidal Voronoi configuration, and then this centralized approach will be extended to a distributed approach.

3. CRHCC Approach

The CRHCC approach for multiple mobile robots is proposed in this section. The objective is to asymptotically force a group of n mobile robots toward the centroidal Voronoi configuration in a cooperative manner using RHC. To do so, let $P(t) = (p_1, \ldots, p_n)$ be an n-vector whose elements are the robots' positions, i.e. $p_i(t) = (x_i(t), y_i(t))$, and $C_V = (C_{V_1}, \ldots, C_{V_n})$ be a vector of the Voronoi cells' centroids. The overall system dynamic can be described as:

$$\dot{P}(t) = u(t), \ t \geq 0, \tag{5}$$

where $P(0)$ is known and $P(t) \in \Re^{2n}$ and $u(t) \in \Re^{2n}$ are the state and input vectors, respectively. It is assumed that there exist some constraints on state and input, i.e. $P(t) \in \aleph^n$, $u(t) \in \boldsymbol{u}^n$, and $\aleph^n$ and $\boldsymbol{u}^n$ are the state and input constraints sets, respectively.

Assumption 1: $\boldsymbol{u}$ is a compact and a connected set that contains the origin in its interior and each robot can measure all of its states.

The coverage algorithm proposed in this paper is based on the Voronoi diagram. In ref. [14], Aurenhammer showed that the dual of the Voronoi diagram is the Delaunay triangulation, which lies under the graph theory concept. To proceed, the coverage problem is investigated by using graph theory.

Lemma 1[14]: Two points of P in the Voronoi diagram are connected with a Delaunay edge if their corresponding Voronoi cells are adjacent.

These two points (or robots) are called neighbors.

By drawing the robots' Voronoi diagram and its corresponding Delaunay graph, the set of the robots' positions can be shown with a graph where its vertexes are the robots' positions and its edges are connecting segments between any two neighboring robots. We denote the coverage graph topology by $G = (V, E), V = \{1, \ldots, n\}$, $E \subset V \times V$. Each edge in the graph is illustrated with an ordered pair $(i, j) \in E$, where i, j are any two neighboring robots. Our coverage graph is assumed to be undirected. Hence, $(i, j) \equiv (j, i)$. Robots i, j are called neighbors if they are in the coverage graph $(i, j) \in E$. The set of neighbors of the ith robot is denoted by $N_i \subset V$.

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Each element of E is denoted by e_i. Accordingly, $E = \{e_1, \ldots, e_M\}$, where M is the number of Delaunay edges.

To proceed, we need to define the proposed notion of *"coverage vector" and "coverage matrix"*. Before that, we define the desired connecting vector between any two neighbors in a coverage graph, denoted by $d_{ij} \in \Re^2$ as follows:

$$d_{ij} = C_{V_j} - C_{V_i} \tag{6}$$

This vector has the following property:

$$d_{ij} = -d_{ji} \tag{7}$$

Definition 2: *"coverage vector" and "coverage matrix"*: the *"coverage vector"* denoted by COV is defined as $\text{COV} = (\text{cov}_1, \ldots, \text{cov}_l, \ldots, \text{cov}_M, \text{cov}_{M+1}, \ldots, \text{cov}_{M+n}) \in \Re^{2(M+n)}$, where

$$cov_l = p_i - p_j + d_{ij}, \quad l = 1, \ldots, M, \quad (i, j) \in E \tag{8}$$

and

$$cov_{M+k} = p_k - C_{V_k} \quad k = 1, \ldots, n \tag{9}$$

The robots will be in the centroidal Voronoi configuration, namely $P = C_V$, when $\text{COV} \equiv 0$. Hence, we can write the linear mapping from P to COV as follows:

$$COV = TP + \bar{d}, \tag{10}$$

where $\bar{d} = (\ldots, d_{ij}, \ldots, -C_{V_k}, \ldots), k = 1, \ldots, n,$ for all $(i, j) \in E$.

We call T the " coverage matrix".

From the definition of the coverage vector, we know that $\text{COV} = TP + \bar{d} \rightarrow$ if $P = C_V$ then $\text{COV} \equiv 0$ Therefore,

$$TC_V + \bar{d} = 0 \Rightarrow \bar{d} = -TC_V \tag{11}$$

The substitution of (11) into (10) yields

$$COV = TP - TC_V = T(P - C_V) \tag{12}$$

The coverage matrix T used in (10) has full rank and it is equal to $\dim(P) = 2n$.

Definition 3: The CRHCC cost function is defined as follows:

$$\begin{aligned} H(P(t), u(.), h_p) = \int_t^{t+h_p} \Bigg[& \sum_{(i,j)\in E} \xi \| p_i(\tau) - p_j(\tau) + d_{ij} \|^2 \\ & + \xi \| P(\tau) - C_V \|^2 + \eta \| u(\tau) \|^2 \Bigg] d\tau \\ & + \sigma \| P(t + h_p) - C_V \|^2, \end{aligned} \tag{13}$$

where ξ, η, σ are positive weighting constants.

As proved in ref. [14], a Delaunay graph is connected. Furthermore, as stated before, the coverage graph considered in this paper is Delaunay and thus it is a connected graph. If the coverage graph was not connected, it could be separated into at least two sub-graphs. Moreover, the cost function would be separated into more than one coupled cost function. Since the coverage matrix has full rank, $T^T T$ is a positive definite matrix; hence, by using (12), one can get

$$\|COV\|^2 = ((P - C_V)^T T^T T(P - C_V)) = \|P - C_V\|^2_{T^T T} \tag{14}$$

We denote $Q = \xi T^T T$, $G = \sigma I$ and $R = \eta I$ (where I is an identity matrix). Since ξ, η, σ are positive, the matrices Q, G, and R are positive definite matrices. Considering $C_V = (C_{V_1}, \ldots, C_{V_n})$, (13) can be rewritten as follows:

$$H(P(t), u(.), h_p) = \int_t^{t+h_p} \left[\|P(\tau; P(t)) - C_V\|_Q^2 + \|u(\tau)\|_R^2\right] d\tau + \|P(t+h_p; P(t)) - C_V\|_G^2 \quad (15)$$

Now, by using the above concepts, the CRHCC problem can be stated as follows:

Problem 1: CRHCC problem:
Find

$$H^*(P(.), u^*(.), h_p) = \min_{u(.)} H(P(t), u(.), h_p),$$

with

$$H(P(.), u(.), h_p) = \int_t^{t+h_p} \left[\|P(\tau; P(t)) - C_V\|_Q^2 + \|u(\tau)\|_R^2\right] d\tau + \left\|P(t+h_p; P(t)) - C_V\right\|_G^2$$

subject to:

$$\left.\begin{array}{l} \dot{P}(\beta) = u(\beta) \\ u(\beta) \in \boldsymbol{u}^n \\ P(\beta; P(t)) \in \aleph^n \end{array}\right\} \beta \in \left[t, t+h_p\right],$$

$$P(t+h_p; P(t)) \in \Psi(\omega) := \left\{P \in \Re^{2n} : \|P - C_V\|_G^2 \le \omega, \omega \ge 0\right\} \quad (16)$$

Note that (16) represents the terminal constraint. Assume that the first segment of the optimal control problem is solved at time instance t_0, where h_c is the receding horizon update period and the closed-loop system that we wish to stabilize at C_V is

$$\dot{P}(\tau) = u^*(\tau), \tau \ge t_0, \quad (17)$$

where $u^*(\beta; P(t))$, $\beta \in [t, t+h_p]$ is the open-loop optimal solution of Problem 1. This optimal control solution is applied to the system until $t + h_c$, i.e. the applied control to the system in the time interval $\tau \in [t, t+h_c)$, $0 < h_c \le h_p$ is $u^*(\tau) = u^*(\tau; P(t))$, $\tau \in [t, t+h_c)$. The open-loop optimal state trajectory is denoted as $P^*(\tau; P(t))$.

Based on the CRHCC results obtained in this section, a DRHCC algorithm is proposed in the next section.

4. DRHCC Approach

In the DRHCC approach, the objective is to force a group of n robots to the centroidal Voronoi configuration in a distributed manner using RHC.

Let $p_i \in \Re^2$ and $u_i \in \Re^2$ be the state and control input of the ith robot, where $i = 1, \ldots, n$. It is assumed that the robots' dynamics are decoupled from each other and hence their dynamics can be written as follows:

$$\dot{p}_i(t) = u_i(t), t \ge 0, \quad p_i(0) \text{ given} \quad (18)$$

To achieve the desired cost function, the coupling inherent with the centralized approach is eliminated by defining n different costs, one for each robot, and only the connections between any given robot and its neighbors are present. To facilitate the results, the terminal constraint and terminal cost are assumed to be decoupled, i.e. $G = diag(G_1, \ldots, G_n)$. Furthermore, in addition to previous

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constraints, a compatibility constraint is added to ensure that each robot does not move away too far from the trajectory expected by its neighbors. This will be explained later. It is also assumed that h_p, h_c are identical for all robots. By considering (5) and defining $P(t) = (p_1, \ldots, p_n)$ and $u = (u_1, \ldots, u_n)$, the overall system dynamic can be decomposed into n sub-systems having the dynamics given by (18). Accordingly, the objective is to design a DRHCC algorithm for each robot that drives the robot to the centroid of its own cell in the centroidal Voronoi configuration, while cooperating with its neighbors.

Definition 4: The DRHCC cost function for each robot with the objective of reaching its cell's centroid in the centroidal Voronoi configuration in a cooperative way with its neighbors is defined as follows:

$$H_i(p_i, p_j, u_i, h_p) = \int_t^{t+h_p} \left[\sum_{j \in \mathrm{N}_i} \frac{\xi}{2} \left\| p_i(\tau) - p_j(\tau) + d_{ij} \right\|^2 + \xi \left\| p_i(\tau) - C_{V_i} \right\|^2 + \eta \left\| u_i(\tau) \right\|^2 \right] d\tau + \left\| p_i(t + h_p, p_i(t)) - C_{V_i} \right\|^2_{G_i} \tag{19}$$

In the newly considered system, the state and control constraints are separated for each robot, i.e. $p_i(t) \in \aleph \subseteq \Re^2$ and $u_i(t) \in \boldsymbol{u} \subseteq \Re^2$.

Given $R = diag(R_1, \ldots, R_n)$, the control cost can be rewritten as $\|u\|^2_R = \sum_{i=1}^n \|u_i\|^2_{R_i}$, where each $R_i = \sigma I$ is a positive definite matrix. To proceed further, the notions of the "*distributed coverage vector*" and "*distributed coverage matrix*" are needed. Before that, some notations must be defined.

As stated before, N_i is the set that contains the neighbors of the ith robot's neighbors. Therefore, there exists a Delaunay edge between the robot and each of its neighbors. Let $p_{-i} = (\ldots, p_j, \ldots)$ and $C_{V_{-i}} = (\ldots, C_{V_j}, \ldots)$, where $j \in N_i$ denotes the state and centroid vectors of the neighbors of the ith robot, respectively. Now, for each robot, we define the following vector:

$$\overline{cov}^i = \left(\ldots, \overline{cov}^i_l, \ldots, \overline{cov}^i_{|N_i|+1} \right),$$

where

$$\overline{cov}^i_l = p_i - p_j + d_{ij} \quad l = 1, \ldots, |N_i| \, \forall j \in N_i \tag{20}$$

$$\overline{cov}^i_{|N_i|+1} = p_i - C_{V_i} \tag{21}$$

and $|N_i|$ is the number of elements in N_i. Let the linear mapping from $P^i = (p_i, p_{-i})$ to $\overline{cov}^i$ be written as follows:

$$\overline{cov}^i = \bar{T}^i P^i + \bar{d}^i, \tag{22}$$

where

$$\bar{d}^i = \left(\ldots, d_{ij}, \ldots, -C_{V_i} \right), j \in N_i \tag{23}$$

We can now state the following definitions:

Definition 5: *"Distributed coverage vector" and "distributed coverage matrix"*:
For each ith robot, the "*distributed coverage vector*" is defined as follows:

$$cov^i = (\ldots, cov^i_l, \ldots, cov^i_{|N_i|+1}),$$

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where

$$cov_l^i = \frac{1}{2}\overline{cov}_l^i, l = 1, \ldots, |N_i| \tag{24}$$

$$cov_{|N_i|+1}^i = \overline{cov}_{|N_i|+1}^i \tag{25}$$

The "*distributed coverage matrix*" is defined as matrix T^i in the following equation:

$$cov^i = T^i P^i + d^i, \tag{26}$$

where $d^i = \left(\ldots, \frac{1}{2}d_{ij}, \ldots, C_{V_i}\right) \forall j \in N_i$ and $P^i = (p_i, p_{-i})$.

Since $Q \neq diag(Q_1, \ldots, Q_n)$, the term $\frac{1}{2}$is added into (24) in order to satisfy the following equation:

$$\|P - C_V\|_Q^2 = \sum_{i=1}^n \left\| \begin{matrix} p_i - C_{V_i} \\ p_{-i} - C_{V_{-i}} \end{matrix} \right\|_{Q_i}^2 = \sum_{i=1}^n \|P^i - C_V^i\|_{Q_i}^2, \quad C_V^i = \left(C_{V_i}, C_{V_{-i}}\right)$$

Note that, if robot i and its neighbors are located at their centroids in the centroidal Voronoi configuration, i.e. $P^i = C_V^i$, then (20) and (21) and therefore (24) and (25) will be equal to zero. Hence,

$$T^i C_V^i + d^i = 0 \Rightarrow d^i = -T^i C_V^i \Rightarrow cov^i = T^i (P^i - C_V^i) \tag{27}$$

Similar to the centralized case, it can be proven that the distributed coverage matrix T^i in (27) has full rank and it is equal to $\dim(p) = 2$.

Proposition 2: The cost function given by (19) can be rewritten as follows:

$$\begin{aligned} H_i(P^i(t), u(.), h_p) = &\int_t^{t+h_p} \left[\left\|p^i(\tau) - C_V^i\right\|_{Q_i}^2 + \|u_i(\tau)\|_{R_i}^2\right] d\tau \\ &+ \left\|p_i(t + h_p; p(t)) - C_{V_i}\right\|_{G_i}^2 \end{aligned} \tag{28}$$

and $\sum_{i=1}^n H^i(P^i(t), u_i(.), h_p) = H(P(t), u(.).h_p)$, where H is the CRHCC cost function.

Proof: By using (27), it can be seen that

$$\left\|cov^i\right\|^2 = \left\|P^i - C_V^i\right\|_{T^i T^{iT}}^2.$$

Since $R = diag(R_1, \ldots, R_n)$ and $G = diag(G_1, \ldots, G_n)$, then by defining $Q_i = \xi T^i T^{i^T}$ one can rewrite (19) as follows:

$$\begin{aligned} H_i(P^i(t), u(.), h_p) = &\int_t^{t+h_p} \left(\left\|p^i(\tau) - C_V^i\right\|_{Q_i}^2 + \|u_i(\tau)\|_{R_i}^2\right) d\tau \\ &+ \left\|p_i(t + h_p; p(t)) - C_{V_i}\right\|_{G_i}^2 \end{aligned}$$

This is indeed (28), which is useful in stability analysis. Now, according to definitions 2–5, it is concluded that $\sum_{i=1}^n H^i(P^i(t), u_i(.), h_p) = H(P(t), u(.), h_p)$, i.e. the sum of n distributed cost functions is equivalent to the centralized cost function. ■

Now suppose that n DRHCC optimal problems, one corresponding to each robot, are all solved at a common time instance called "update time", denoted by $t_k = t_0 + h_c k$, $k \in \{0, 1, \ldots\}$. As stated in

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(19) and (28), for each cost function, a term contains the connection between the corresponding robot and its neighbors. Therefore, at every update time, when local optimal problems are solved, each robot must know the state trajectories of all its neighbors over the time interval $[t_k, t_k + h_p]$. However, such information does not exist at instance t_k. Therefore, each robot must estimate the state trajectories of its neighbors at $[t_k, t_k + h_p]$ and then solve its optimal control problem. The trajectories that each robot estimates for its neighbors are called the *estimated trajectories*. Since each robot is assumed to have information about the dynamics of its neighbors, an estimated control (defined shortly) is obtained from which the state trajectories are derived. To ensure compatibility between the actual and estimated trajectories, an additional constraint called the "compatibility constraint" is added to the DRHCC problem of each robot.

Definition 6: Estimated control

At every time interval $\tau \in [t_k, t_k + h_p]$, the estimated control for each robot is defined as follows:

$$\begin{cases} \hat{u}_i(\tau; p_i(t_k)) = 0; & \text{if } P^i(t_k) = C_V^i \\ \hat{u}_i(\tau; p_i(t_k)) = u_i^*(\tau; p_i(t_{k-1})); & \text{Otherwise} \end{cases}$$

The actual and estimated state trajectories are denoted by $p_i(.; p_i(t_k))$ and $\hat{p}_i(.; p_i(t_k))$, respectively. Note that, $\hat{p}_i(t_k; p_i(t_k)) = p_i(t_k; p_i(t_k)) = p_i(t_k)$, $i = 1, \ldots, n$.

The DRHCC problem can now be stated as follows.

Problem 2: With a given fixed update period $h_c \in (0, h_p)$ and an optimization period h_p, for every $i = 1, \ldots, n$ and at any sampling time t_k and with given $\hat{u}_{-i}(\beta; p_{-i}(t_k))$, $p_i(t_k)$, $p_{-i}(t_k)$, $\hat{u}_i(\beta; p_i(t_k))$ at $\beta \in [t_k, t_k + h_p]$, we find $H_i^*(p_i(t_k), p_{-i}(t_k), h_p) = \min_{u(.)} H_i(p^i(t_k), u_i(.; p_i(t_k), h_p)$, where

$$H_i(p^i(t), u_i(.), h_p) = \int_{t_k}^{t_k+h_p} \left[\sum_{j \in N_i} \frac{\xi}{2} \left\| p_i(\tau) - p_j(\tau) + d_{ij} \right\|^2 + \xi \left\| p_i(\tau) - C_{V_i} \right\|^2 + \eta \left\| u_i(\tau) \right\|^2 \right] dt + \left\| p_i(t + h_p, p_i(t)) - C_{V_i} \right\|_{G_i}^2$$

subject to the following:

$\dot{p}_i(\beta; p_i(t_k)) = u_i(\beta)$, $\dot{\hat{p}}_j(\beta; p_j(t_k)) = \hat{u}_j(\beta)$, $j \in N_i$, $u_i(\beta; p_i(t_k)) \in \boldsymbol{u}$, $p_i(\beta; p_i(t_k)) \in \aleph$,

$$\| p_i(\beta; p_i(t_k)) - \hat{p}_i(\beta; p_i(t_k)) \| \le h_c^2 \kappa, \kappa \in (0, \infty), \tag{29}$$

$p_i(t_k + h_p; p_i(t_k)) \in \Psi_i(\varepsilon_i)$, where $\beta \in [t_k, t_k + h_p]$, and

$$\Psi_i(\in_i) =: \left\{ p_i \in R^2 : \left\| p_i - C_{V_i} \right\|_{G_i}^2 \le \varepsilon_i, \varepsilon_i \ge 0 \right\} \tag{30}$$

and where $h_c^2 \kappa \approx 0$.

Equation (29) is called the compatibility constraint and (30) is the target or terminal set. The optimal solution for each DRHCC problem is denoted by $u_i^*(\tau; p_i(t_k))$, $\tau \in [t_k, t_k + h_p]$ and the closed-loop system where we wish to stabilize is

$$\dot{P}(\tau) = u^*(\tau)), \quad \tau \ge 0, \tag{31}$$

where $u^*(\tau; P(t_k)) = (u_1^*(\tau; p_1(t_k)), \ldots, u_n^*(\tau; p_n(t_k)))$. The optimal state trajectory for the ith robot is denoted by $p_i^*(\tau; p_i(t_k))$, $\tau \in [t_k, t_k + h_p]$.

The augmented optimal state trajectory for $\tau \in [t_k, t_k + h_p]$ is $P^*(\tau; P(t_k)) = (p_1^*(\tau; p_1(t_k)), \ldots, p_n^*(\tau; p_n(t_k)))$. Note that the ith robot's optimal control, i.e. $u_i^*(\tau; p_i(t_k))$,

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Table I. DRHCC algorithm.

Name: DRHCC algorithm **Goal:** Asymptotically drive a group of n mobile robots toward the centroidal Voronoi configuration.
At instance $t_0 - h_c$ every robot: A1- senses its position and transmits information about its position to its neighbors and receives its neighbors' positions A2- computes its Voronoi region $V_i(t_0 - h_c)$ and the centroid of that according to (3). A3- follows the initial setting procedure given in Algorithm 1 **At every update instance t_k, $k \in N$ each robot:** B1- senses its own and its neighbors' positions (or receives its neighbors' positions). B2- computes its Voronoi region $V_i(t_k)$ B3- computes the centroid of its cell according to (3) B4- transmits the information about its Voronoi center to each of its neighbors in the system and retrieves the same information from its neighbors B5- computes its own and its neighbors' estimated trajectories by using (18) B6- computes its distributed optimal control trajectory B7- $u_i^*(\tau; p_i(t_k))$ over the interval $\tau \in [t_k, t_k + h_p]$ by using Problem 2 **Over every interval $[t_{k-1}, t_k)$, each robot:** C1- applies the distributed optimal control trajectory that has been obtained at t_{k-1} C2- computes its estimated control for $[t_k, t_k + h_p]$ according to Definition 6 C3- transmits its estimated control that was computed in C2 to all neighbors and receives their estimated control

is dependent on its initial state, $p_i(t_k)$, and the initial states of its neighbors. In the DRHCC problem, initialization is more difficult. As stated before, each robot has an estimated trajectory at t_0. To solve the optimal problem corresponding to a robot, the estimated control information of its neighbors is needed. Since the estimated control of each step is assumed to be the optimal control obtained in the previous step, and since prior to t_0 no optimal problem has been solved, one must define an initialization method to obtain the estimated control. The time instance that this initialization occurs is denoted by $t_0 - h_c$.

Algorithm 1- (*Initial setting method*): at time instance $t_0 - h_c$, we solve Problem 2 with initial state $p^i(t_0 - h_c)$ and $\hat{u}_i(\tau; p_i(t_0 - h_c)) = 0$ for all $\tau \in [t_0 - h_c, t_0 - h_c + h_p]$ and $\kappa = +\infty$.

The optimal control obtained by solving this problem with the above conditions is the estimated control for the time interval $[t_0, t_0 + h_p]$.$\kappa = +\infty$ implies that the compatibility constraint is not important prior to t_0. The state and control trajectories obtained at $t_0 - h_c$ over interval $\tau \in [t_0 - h_c, t_0 - h_c + h_p]$ are denoted by $p_i^*(\tau; p_i(t_0 - h_c))$ and $u_i^*(\tau; p_i(t_0 - h_c))$. This optimal control is applied to the ith robot over $[t_0 - h_c, t_0)$. The proposed DRHCC algorithm is given in Table I.

5. Stability Analysis

The stabilization of the closed-loop system (31) is investigated in this section. As stated in Section IV, the overall optimal cost function for the system is denoted as follows:

$$H^*(P(t_k), h_p) = \sum_{i=1}^{n} H_i^*(p_i^*(t_k), p_{-i}(t_k), h_p) \tag{32}$$

Proposition 3: C_V in the centroidal Voronoi configuration is an equilibrium point of the closed-loop system (31).

Proof: If each robot is located at C_V at time $t_0 - h_c$, i.e. $P(t_0 - h_c) = C_V$, then the optimal solution for Problem 2 over the time interval $\tau \in [t_0 - h_c, t_0 - h_c + h_p]$ is $u^*(\tau, C_V) = 0$. On the other hand, by using the systems dynamics given by (5), one can write $\dot{P}(t) = 0$. Furthermore, since every optimal control is applied to the system until the next update time, the estimated control is equal

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to zero at t_0, i.e. $\hat{u}_i(.; p_i(t_0)) = 0$ and hence $P(t_0) = C_V$. Therefore, over interval $\tau \in [t_k, t_k + h_p]$ we have $u^*(\tau, P(t_k)) = 0$ and hence C_V is an equilibrium point for the closed-loop system (31).■

Theorem 1: Based on the DRHCC algorithm given in Table I, the closed-loop system (31) converges to the centroidal Voronoi configuration and C_V is an asymptotically stable equilibrium point for the closed-loop system (31), with $\aleph^n$ as its region of attraction.

Proof: Since the mission space is assumed to be convex, each Voronoi cell is also convex and it contains its centroid in its interior. Consequently, each robot always moves inside its cell and therefore never leaves the mission space S. Assuming Problem 2 is feasible at initialization, $u_i(\tau; p_i(t_0 - h_c))$ is an admissible control over $\tau \in [t_0 - h_c, t_0 - h_c + h_p]$. Presuming that all robots are located inside S at $t_0 - h_c$, then at times $t_0 - h_c + h_c \leq \tau \leq t_0 - h_c + h_p$ the state and control trajectories are admissible and as a result for all time instances after initialization, they are admissible as well. Therefore, $\aleph^n$ is a positively invariant set, meaning that the closed-loop state trajectory for all $t \geq t_0 - h_c$ is contained in the interior of $\aleph^n$.

Now, let $V = H^*(P(t_k), h_p)$ where V is a Lyapunov function. From (28) and (32) and given the fact that $G_i \geq 0$, one can deduce that $H^*(P(t_k), h_p)$ is a non-negative function. If $H^*(P(t_k), h_p) = 0$, then based on (28) and (32) for every robot $\left\| p_i^*(t_k + h_p, p_i(t_k)) - C_{V_i} \right\|_{G_i}^2 = 0$ and

$$\int_{t_k}^{t_k+h_p} \left[\frac{\xi}{2} \sum_{j \in N_i} \left\| p_i^*(\tau, p_i(t_k)) - \hat{p}_j(\tau, p_j(t_k)) + d_{ij} \right\|^2 + \xi \left\| p_i^*(\tau, p_i(t_k)) - C_{V_i} \right\|^2 + \eta \left\| u_i^*(\tau, p_i(t_k)) \right\|^2 \right] d\tau = 0$$

Since the function under the integral in (28) is piecewise continuous and non-negative over the time interval $[t_k, t_k + h_p]$, one can write

$$\sum_{j \in N_i} \frac{\xi}{2} \left\| p_i^*(\tau, p_i(t_k)) - \hat{p}_j(\tau, p_j(t_k)) + d_{ij} \right\|^2 + \xi \left\| p_i^*(\tau, p_i(t_k)) - C_{V_i} \right\|^2 + \eta \left\| u_i^*(\tau, p_i(t_k)) \right\|^2 = 0, \forall \tau \in \left[t_k, t_k + h_p\right]$$

Hence, $u^*(\tau, P(t_k)) = 0$ over the interval $\tau \in [t_k, t_k + h_p]$, and given the fact that G_i is a positive definite matrix, then $p_i^*(t_k + h_p, p_i(t_k)) = C_{V_i}$. Note that the system dynamics given by (5) are time invariant; therefore, with initial state $P(t_k + h_p) = C_V$ and control $u(\beta) = 0, \quad \beta \in [t_k + h_p, t_k]$, one can write $P(\beta) = C_V$. Therefore, over the interval $\tau \in [t_k, t_k + h_p]$, the optimal closed-loop state is $P^*(\tau; P(t_k)) = C_V$. Furthermore, with $P(t_k) = C_V$, by using Definition 6 over the time interval $\tau \in [t_k, t_k + h_p]$, the estimated control is $\hat{u}_i(\tau, p_i(t_k)) = 0$, and hence $\hat{p}_i(\tau; p_i(t_k)) = C_{V_i}$. Since, the DRHCC optimal cost function is denoted as

$$\int_{t_k}^{t_k+h_p} \left[\sum_{j \in N_i} \frac{\xi}{2} \left\| p_i^*(\tau, p_i(t_k)) - \hat{p}_j(\tau, p_j(t_k)) + d_{ij} \right\|^2 + \xi \left\| p_i^*(\tau, p_i(t_k)) - C_{V_i} \right\|^2 + \eta \left\| u_i^*(\tau, p_i(t_k)) \right\|^2 \right] d\tau + \sigma \left\| p_i^*(t_k + h_p, p_i(t_k)) - C_{V_i} \right\|^2$$

and because $p_i(t_k) = C_{V_i}$, the optimal solutions to Problem 2, over the time interval $\tau \in [t_k, t_k + h_p]$, and for every robot, are $u_i^*(\tau; p_i(t_k)) = 0$ and $p_i^*(\tau; p_i(t_k)) = C_{V_i}$. Hence, since for every robot $H_i^* = 0$, based on Proposition 2, the total cost is $H^*(P(t_k), h_p) = 0$. Consequently, $H^*(P(t_k), h_p) > 0$ everywhere except at $P(t_k) = C_V$, where $H^*(P(t_k), h_p) = 0$. Therefore, H^* is positive definite. Since

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H^* satisfies the following:

$$H^*(P(t_k), h_p) = \sum_{i=1}^{n} \int_{t_k}^{t_k+h_p} \left[\frac{\xi}{2} \sum_{j\in N_i} \left\| p_i^*(\tau, p_i(t_k)) - \hat{p}_j(\tau, p_j(t_k)) + d_{ij} \right\|^2 \right.$$
$$\left. + \, \xi \left\| p_i^*(\tau, p_i(t_k)) - C_{V_i} \right\|^2 + \eta \left\| u_i^*(\tau, p_i(t_k)) \right\|^2 \right] d\tau$$
$$+ \left\| p_i^*(t_k + h_p, p_i(t_k)) - C_{V_i} \right\|_{G_i}^2 \tag{33}$$

and because $u_i^*(\tau, p_i(t_{k+1}))$ is an optimal control that minimizes $H_i(P(t_k), h_p)$, $G = diag(G_1, \ldots, G_n)$, $\hat{P} = (\hat{p}_1, \ldots, \hat{p}_n)$. Hence, by using Definition 6 and $Q^* = diag(Q_i)$, where $Q_i = \xi T^{i^T} T^i$ one can write as follows:

$$\Delta V = V(t_{k+1}) - V(t_k) = H^*(P(t_{k+1}), h_p) - H^*(P(t_k), h_p)$$
$$\leq - \left[\sum_{i=1}^{n} \int_{t_k}^{t_{k+1}} \frac{\xi}{2} \sum_{j\in N_i} \left\| p_i^*(\tau, p_i(t_k)) - \hat{p}_j(\tau, p_j(t_k)) + d_{ij} \right\|^2 \right.$$
$$\left. + \, \xi \left\| p_i^*(\tau, p_i(t_k)) - C_{V_i} \right\|^2 d\tau \right]$$
$$+ \sum_{i=1}^{n} \int_{t_{k+1}}^{t_k+h_p} \left\| \begin{bmatrix} p_i^*(\tau, p_i(t_k)) - C_{V_i} \\ p_{-i}^*(\tau, p_i(t_k)) - C_{V_{-i}} \end{bmatrix} \right\|_{Q_i} - \left\| \begin{bmatrix} p_i^*(\tau, p_i(t_k)) - C_{V_i} \\ \hat{p}_{-i}(\tau, p_{-i}(t_k)) - C_{V_{-i}} \end{bmatrix} \right\|_{Q_i}$$

Now, by using the definition of matrix Q_i, one can verify that it can be written as follows:

$$Q_i = \begin{bmatrix} Q_{i1} & Q_{i2} \\ Q_{i2}^T & Q_{i3} \end{bmatrix}, \quad Q_{i1} \in \Re^{2\times 2}, \quad Q_{i2} \in \Re^{2\times 2|N_i|}, \quad Q_{i3} \in \Re^{2|N_i|\times 2|N_i|} \text{ where } |N_i| = N_i - 1.$$

According to the fact that for a given vector $v = (v_1, v_2)$, $\|v\| \leq \|v_1\| + \|v_2\|$; moreover, according to the fact that in the Voronoi partition $|p_i| \leq \rho_{\max}$ and using the compatibility constraint, the above inequality can be written as follows:

$$H^*(p(t_{k+1}), h_p) - H^*(p(t_k), h_p) \leq - \left[\sum_{i=1}^{n} \int_{t_k}^{t_{k+1}} \frac{\xi}{2} \sum_{j\in N_i} \left\| p_i^*(\tau, p_i(t_k)) - \hat{p}_j(\tau, p_j(t_k)) + d_{ij} \right\|^2 \right.$$
$$\left. + \, \xi \left\| p_i^*(\tau, p_i(t_k)) - C_{V_i} \right\|^2 d\tau \right]$$
$$+ 2\lambda_{\max}(Q)\kappa\rho_{\max} h_p h_c^2 \sum_{i=1}^{n} N_i(N_i - 1) \tag{34}$$

At time $\tau = t_k$, $p_i^*(t_k; p_i(t_k)) = p_i(t_k)$ and $\hat{p}_{-i}(t_k; p_{-i}(t_k)) = p_{-i}(t_k)$; according to the definition of the coverage matrix, one can write as follows:

$$\sum_{i=1}^{n} \left[\sum_{j=1}^{N_i} \left\| p_i(\tau) - p_j(\tau) + d_{ij} \right\|^2 + \xi \left\| p_i(\tau) - C_{V_i} \right\|^2 \right] = \| p(t_k) - C_V \|_Q^2$$

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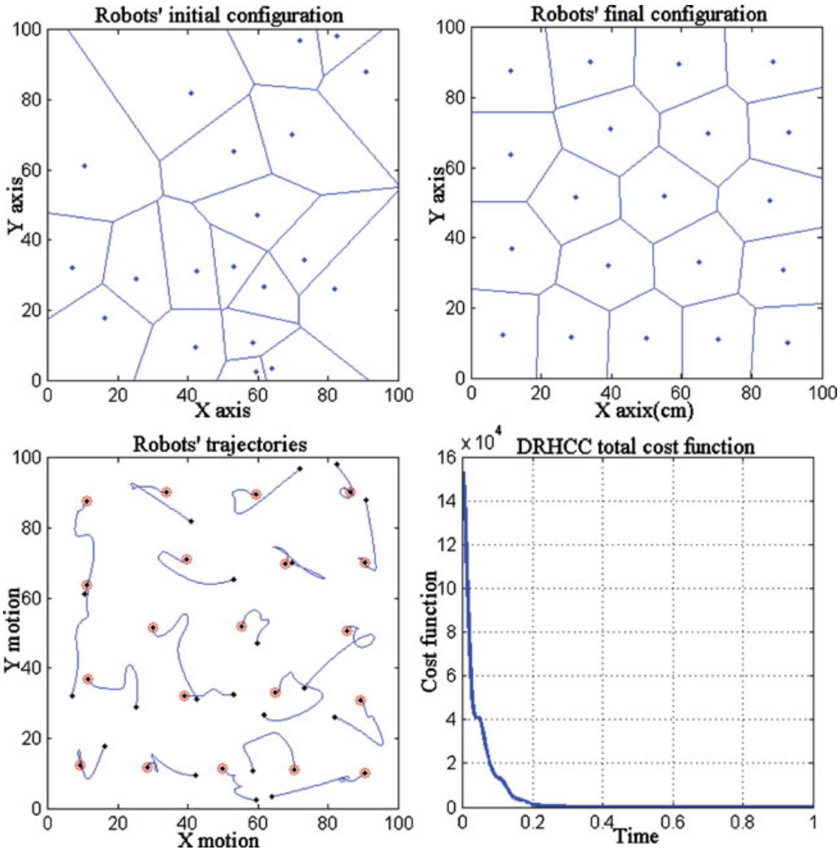

Fig. 1. (a) Initial Voronoi configuration of 20 mobile robots in an environment with uniform density of events. (b) Final configuration. (c) Robots' trajectories. (d) DRHCC total cost function.

Thus, at any $k \in N$

$$H^*(p(s), h_p) - H^*(p(t_k), h_p) \leq -\int_{t_k}^{s} \left\| p^*(\tau; p(t_k)) - C_V \right\|_Q^2 d\tau, \;\; s \in [t_k, t_{k+1}]$$

Applying this recursively gives the result for any $t \geq t_0$ and any $t' \in (t, \infty]$, where

$$H^*(p(t'), h_p) - H^*(p(t), h_p) \leq -\int_{t}^{t'} \left\| p^*(\tau) - C_V \right\|_Q^2 d\tau,$$

for any times t, t' with $t_0 \leq t < t' \leq \infty$.

Given $\wp > 0$, choose $r \in (0, \alpha]$ such that the closed ball $B(C_V, r) = \left\{ P \in \Re^{2n} | \, \|P - C_V\| \leq r \right\}$ is a small region around C_V. For the facility, $V(P)$ is denoted by $H^*(P, h_p)$. Since $V(P)$ is continuous at $P = C_V$ and $V(P) > 0$ for all $P \neq C_V$, there exists $\alpha \in (0, \infty)$ such that $\alpha < \min_{\|P - C_V\| = r} V(P)$ Define the level set of $V(P)$. By contradiction, it can be shown that $Z_\alpha = \{P \in B(C_V, r) | V(P) \leq \alpha\}$, which is a subset contained in the interior of $B(C_V, r)$. Because $V(P)$ is monotonic, $V(P(t)) \leq V(P(t_0)) \leq \beta, \quad \forall t \geq t_0$ Therefore, Z_α is a positively invariant set for the closed-loop system (31). Since $V(C_V) = 0$ and $V(P)$ is continuous at $P = C_V$, there exists a constant $\eta \in (0, r)$, which $\|P(t_0) - C_V\| < \eta \Rightarrow V(P(t_0)) < \alpha \Rightarrow V(P(t)) < \alpha \Rightarrow \|P(t) - C_V\| < \wp$ Thus, C_V is a stable equilibrium point of the closed-loop system (31). Since $\alpha \geq V(P(t_0))$ and $0 \geq -V(P(\infty))$ by

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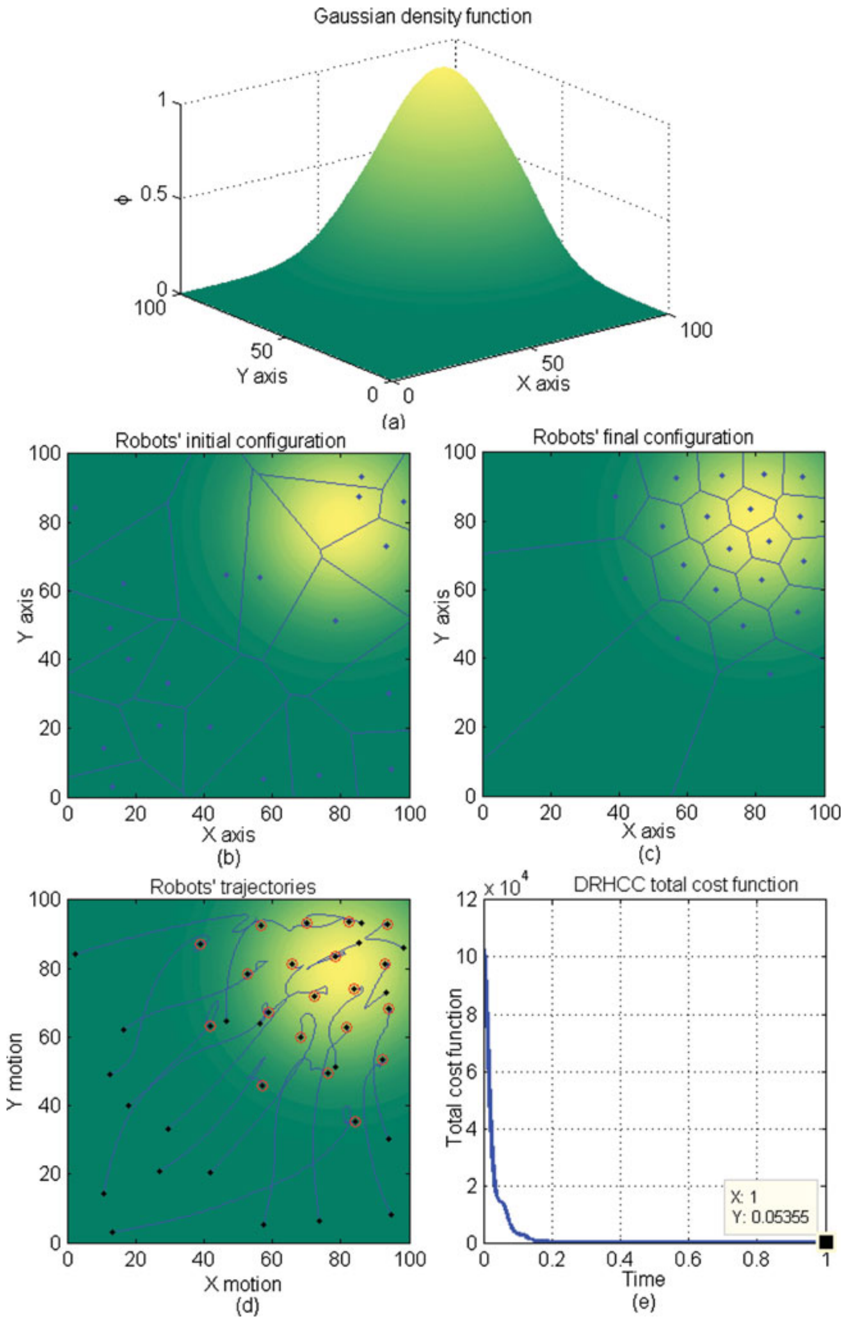

Fig. 2. (a) Gaussian density function. (b) Initial configuration of 20 linear mobile robots in an environment with Gaussian density of events. (c) Final configuration (d). Robots' trajectories. (e) DRHCC total cost function.

induction, it can be shown that

$$V(P(t_0)) - V(P(\infty)) \geq \int_{t_0}^{\infty} \left\| P^*(\tau) - C_V \right\|_Q^2 d\tau \Rightarrow \alpha \geq \int_{t_0}^{\infty} \left\| P^*(\tau) - C_V \right\|_Q^2$$

Therefore, the infinite integral above exists and is bounded. Let $\wp_1 < \wp$ be such that $P(t)$ belongs to the compact set $\{\| P(t) - C_V \| \leq \wp_1\}$ for all $t \in [t_0, \infty)$. This is known to exist because of the strict inequality bound by $\wp$ shown above. Since $u^*(t)$ is in the compact set $\boldsymbol{u}^n$ for all $t \in [t_0, \infty)$ and from (5), it is clear that $P(t)$ is continuous in P and u. Hence, we find that $P(t)$ is bounded for all

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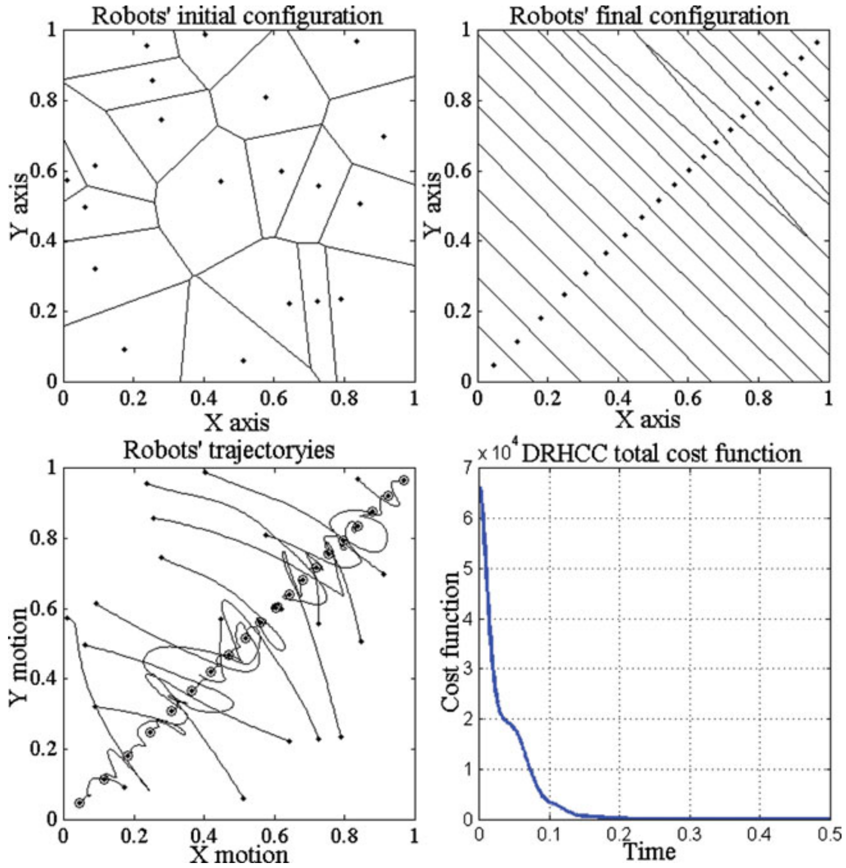

Fig. 3. A Diagonal line pattern of 20 mobile robots, by applying DRHCC algorithm.

$t \in [t_0, \infty)$. From [17], $P(t)$ is uniformly continuous in t on $[t_0, \infty)$. Since $\|P - C_V\|_Q^2$ is uniformly continuous in P on the compact set $\{\|P - C_V\| \leq \wp_1\}$, $\|P(t) - C_V\|_Q^2$ is uniformly continuous in t on $[t_0, \infty)$. Since $Q > 0$, from Barbalat's Lemma[17] it is concluded that $\|P(t) - C_V\| \to 0$ as $t \to \infty$. Thus, C_V is an asymptotically stable equilibrium point for the closed-loop system (31) with the region of attraction Z_α.

Now, for any $P(t_0) \in \aleph^n$, there exists a finite time T' such that $P(T') \in Z_\alpha$, which can be shown by contradiction as follows. Suppose $P(t) \notin Z_\alpha$ for all $t \geq t_0$. Since $V(P) > 0$ and $Q > 0$, from equation (37), for all $t \geq t_0$,

$$V(P(t + h_c)) - V(P(t)) \leq -\int_t^{t+h_c} \left\|P^*(\tau) - C_V\right\|_Q^2 d\tau \leq -h_c.\inf\left\{\|P - C_V\|_Q^2 \,|P \notin Z_\alpha\right\} \leq h_c$$

By induction, $V(P(t)) \to -\infty$ as $t \to \infty$; however, this contradicts $V(P(t)) \geq 0$. Therefore, any trajectory starting in $\aleph^n$ enters Z_α in a finite time. Finally, since $\aleph^n$ is a positively invariant set, it is a region of attraction for the closed-loop system (31). Moreover, for any $P(t) \in \aleph^n$, from the absolute continuity of $P(t')$ in $t' \geq t_0$, a small neighborhood of $P(t)$ in which the optimization problem is still feasible can always be chosen. Thus, $\aleph^n$ is open and connected. ■

6. Simulation Results

The proposed DRHCC algorithm has been numerically simulated by using two different scenarios for 20 linear mobile robots having the dynamics given by (18). In the first scenario, the event density function is assumed to be uniform, denoted as $\phi(s) = 1$. It is also assumed that the robots are initially distributed randomly in the mission space, as shown in Fig. 1(a). After 0.2 s, the robots converge to

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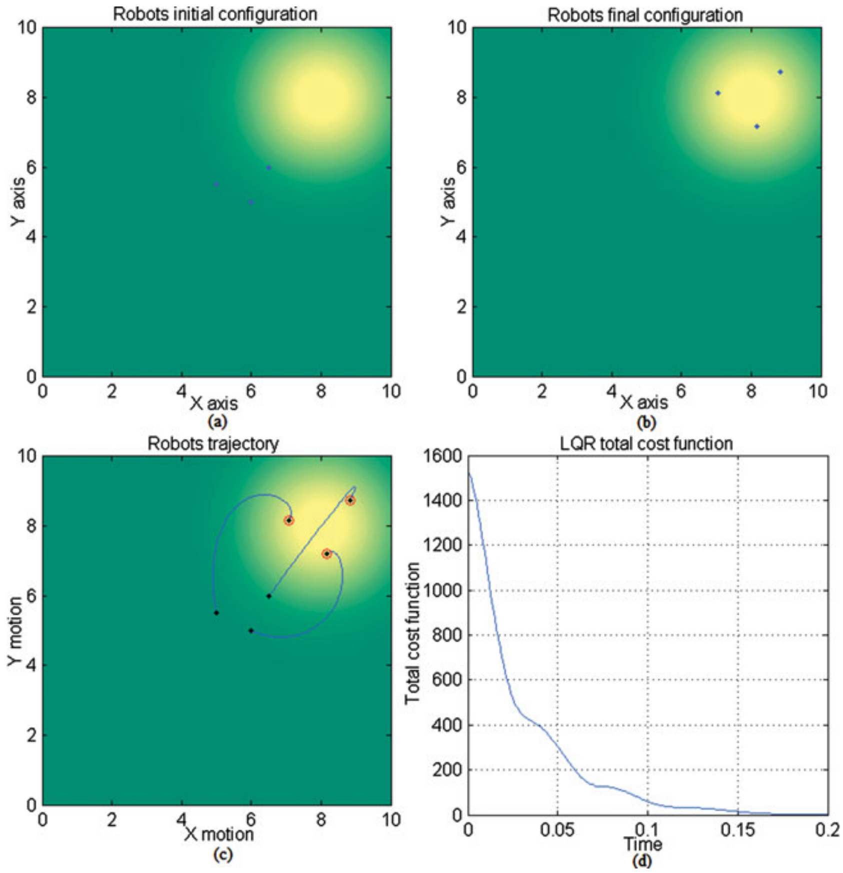

Fig. 4. (a) Initial configuration of three mobile robots in an environment with gaussian density of events. (b) Final configuration. (c) Robots' trajectories. (d) LQR total cost function.

the centroidal Voronoi configuration shown in Fig. 1(b). The robots' paths are shown in Figs. 1(c), and (d) shows the gradual reduction of $\bar{H}^*(P(t_k), h_p)$ toward zero.

In the second scenario, 20 linear mobile robots are distributed in an environment with a Gaussian events density function equal to $e^{-[(x-0.8)^2+(y-0.8)^2]}$. The simulation results for this scenario are shown in Fig. 2. The initial random distribution of robots in the mission space is shown in Fig. 1(b). As can be seen from Figs. 2(c) and (e), after 0.2 s, the robots converge to a centroidal Voronoi configuration with nearly zero total cost value. As expected, convergence to the centroidal Voronoi configuration presented in Section V is validated.

In the second scenario, 20 robots are distributed in an environment with a Gaussian events density function equal to $e^{-[(x-0.8)^2+(y-0.8)^2]}$. The simulation results for this scenario are shown in Fig. 2. Figure 2(a) shows the Gaussian density function. The initial random distribution of the robots in the mission space is shown in Fig. 2(b). The final configuration is shown in Fig. 2(c). The robots' paths are shown in Figs. 2(d), and (e) shows that the DRHCC algorithm causes the robots to converge to a centroidal Voronoi configuration with nearly zero total cost value. As expected, convergence to the centroidal Voronoi configuration presented in Section V is validated.

7. Distributed Receding Horizon Formation Control Based on the DRHCC Algorithm

The objective of this section is to asymptotically force a group of n mobile robots toward the desired formation patterns by using the DRHCC algorithm. One of the advantages of the DRHCC approach is that RHC is easier to maintain than some other approaches. Hence, changing models or specs does not require complete redesign. We verify this property in this section where the proposed DRHCC algorithm is applied to deploy a group of mobile robots in the desired formation patterns. Therefore, just by redefining the density function of the event, we can use the coverage

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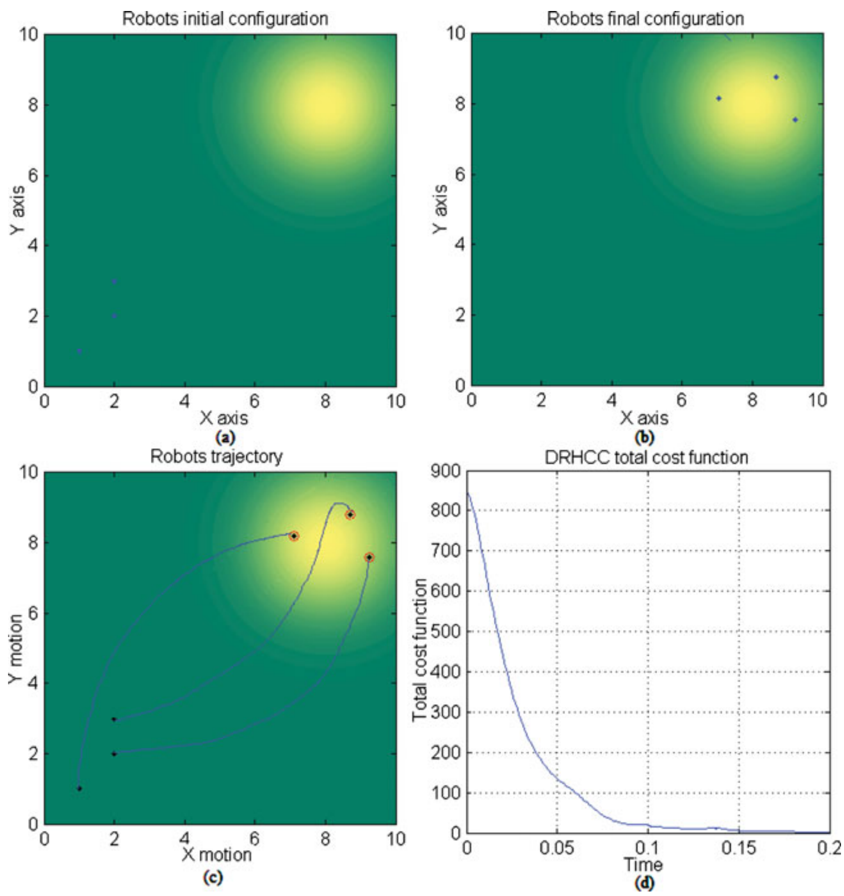

Fig. 5. (a) Initial configuration of three mobile robots with an offset position of more than seven meters in an environment with Gaussian density of events. (b) Final configuration. (c) Robots' trajectories, and (d) DRHCC total cost function.

control scheme as a formation scheme. Most of the desired patterns can be realized by using an appropriate density function of the events (20). For example, in order to line up the robots, the density function can be defined as $\phi_{line}(s) = \exp(-k(ax + by + c)^2)$. Another example is $\phi_{disk}(s) = \exp(-kSR_l(a(x - x_c)^2 + b(y - y_c)^2 - r^2))$ where $SR_l(x) = x(\arctan(lx)/\pi + (1/2))$ and $SR_l(x)$ is a smooth ramp function. This density function leads the robots to obtain a uniform distribution inside the ellipsoidal disk. We refer to refs. [18] and [19] for previous work on algorithms for geometric patterns. Now, the proposed DRHCC algorithm has been numerically simulated under a scenario for 20 mobile robots. The desired pattern is a diagonal line $y = x$, meaning that the density function is defined as $\phi = \exp(-400(x - y)^2)$ and the DRHCC algorithm of Table I is followed. The results of this scenario are shown in Fig. 3.

8. DRHCC Versus LQR

For comparison, the receding horizon coverage control approach is compared with the LQR design. Our optimal control problem can be written in the form of a standard LQR problem with the same horizon as RHC (finite horizon of 0.05 (sec)). The details about LQR may be found in any standard optimal control text.

The RHC and LQR approaches are both applied to the same system of three robots with the dynamics given by (18). Both approaches are applied to individual robots in a distributed way. We use a system of three robots for simplicity to show the differences.

In the first scenario, the robots were initially distributed in the space with an offset position less than 7 meters and the LQR approach was applied. As shown in Fig. 4, the LQR approach stabilized

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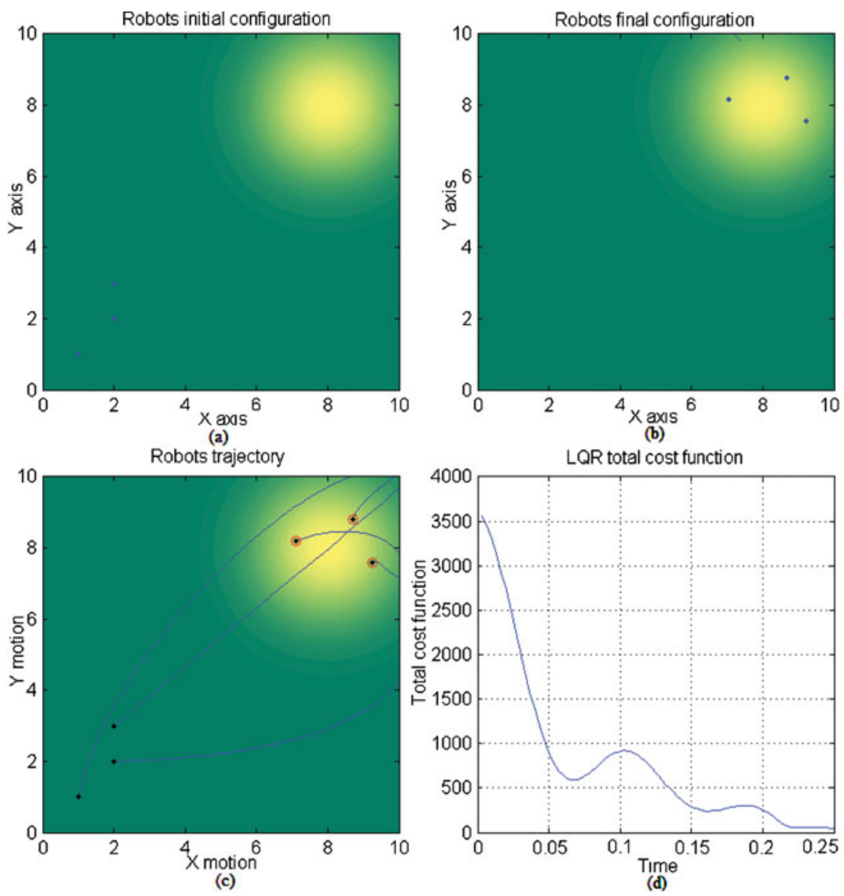

Fig. 6. (a) Initial configuration of three mobile robots with an offset position of more than seven meters in an environment with Gaussian density of events. (b) Final configuration. (c) Robots' trajectories, and (d) LQR total cost function.

the system and the robots converged to the centroidal Voronoi configuration, while the location of the robots remained in the constraint set of $x_i, y_i \in [0, 10], \forall i$.

In the second scenario, the robots were initially distributed in the space with an offset position more than 7 meters and the DRHCC approach was applied. As shown in Fig. 5, the DRHCC approach stabilized the system and the robots converged to the centroidal Voronoi configuration, while the location of the robots remained in the constraint set of $x_i, y_i \in [0, 10], \forall i$.

In the third scenario, LQR was applied to the system. As shown in Fig. 6, by applying LQR to the system, the position of the robots that had an initial offset bigger than 7 meters, exceeded the constraint set of $x_i, y_i \in [0, 10], \forall i$. In reality, this means collision with the border and even crashing. Therefore, the robots leave the mission space and then are redirected to the space. This causes an increase in the total cost function, which is related to a change in the initial condition for *P*.

In Fig. 7, the total cost functions of the DRHCC and LQR approaches applied to a system of 20 robots are shown. The red curve is the total cost function of the system when the LQR controller was used. The blue curve is the cost function while using the DRHCC approach. The region of attraction of RHC law is shown to be larger than that of the static LQR controller.

9. Conclusion

We deployed a distributed coverage control scheme based on the state space model predictive control for cooperating multiple mobile robots to cover an event. The coverage approach relies on ideas from centroidal Voronoi diagrams. The mission space was modeled by using a probability density function representing the probability of the occurrence of the events. The distributed scheme was generated by the decomposition of the single optimal coverage problem into DRHCC problems, each

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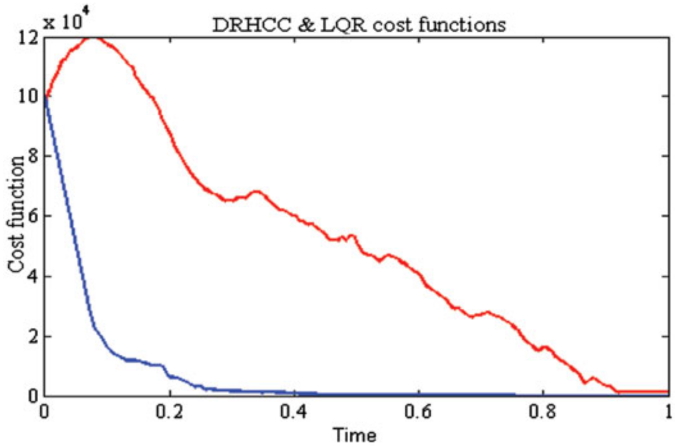

Fig. 7. Cost function of DRHCC and LQR to seven meter offset. The red curve is the cost function of LQR controller and the blue curve is the cost function of DRHCC algorithm.

associated with one robot. One aspect of the generality of our approach is that it can be extended to be applied to robots with any dynamics and can handle any constraint imposed on the system. In addition, our approach can address any possible change in the network such as robot failure or departure. Moreover, since in the DRHCC approach each robot moves toward its own center, no robot can leave its Voronoi cell. Further, assuming that the robots are small, there is no collision between them. Finally, because of the Voronoi diagram properties as well as the distributed control nature, our approach is scalable and can be generalized to large groups of mobile robots. For comparison, the DRHCC and LQR controller were both applied to a group of 20 robots. The result showed the unstable response to a certain offset in the position of the robots for the LQR controller and thus the larger region of attraction of RHC compared with the LQR controller.

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