approximately. When $\theta \approx 294°$, it is approximately $(0.41)\dfrac{\pi R^3}{3}$.

In conclusion, if we want to maximise the total volume of the two cones then we cut out a sector with central angle approximately 115.2° from the given metal sheet and roll the two sectors into right circular cones.

Acknowledgement

I would like to sincerely thank the referee for valuable suggestions which improved the presentation of the content of the Note.

10.1017/mag.2022.21

PRITHWIJIT DE
Homi Bhabha Centre for and Science Education,
Tata Institute of Fundamental Research, Mumbai - 400088, INDIA
e-mails: *de.prithwijit@gmail.com* / *prithwijit@hbcse.tifr.res.in*

106.07 A function-based proof of the harmonic mean – geometric mean – arithmetic mean inequalities

For $a, b \in \mathbb{R}$, with $0 < b \leqslant a$, the harmonic, geometric and arithmetic means of a and b are respectively defined by

$$H(a, b) = \frac{2}{\frac{1}{a} + \frac{1}{b}} = \frac{2ab}{a + b}, \; G(a, b) = \sqrt{ab} \text{ and } A(a, b) = \frac{a + b}{2}.$$

Theorem: For $0 < b \leqslant a$, $H(a, b) \leqslant G(a, b) \leqslant A(a, b)$, that is

$$\frac{2ab}{a + b} \leqslant \sqrt{ab} \leqslant \frac{a + b}{2}.$$

Proof: If $x = \dfrac{b}{a}$, then $x \in (0, 1]$ and the inequalities to prove are

$$\frac{2x}{1 + x} \leqslant \sqrt{x} \leqslant \frac{1 + x}{2}.$$

There are easy purely algebraic proofs for these inequalities [1]. Here, instead, we propose an elementary approach based on the graph of some functions to prove them.

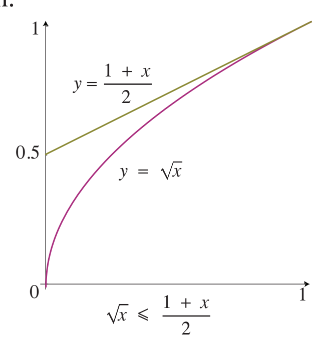

FIGURE 1: $G \leqslant A$

From now on, we use the notation $H = H(a, b)$, $G = G(a, b)$ and $A = A(a, b)$.

Step 1: $G \leqslant A$ and $G = A$ if, and only if, $a = b$, that is $\sqrt{x} \leqslant \dfrac{1 + x}{2}$ for $x \in (0, 1]$, with equality only for $x = 1$.

If $f(x) = \frac{1}{2}(1 + x)$, and $g(x) = \sqrt{x}$, then $f(0) = \frac{1}{2}$, $g(0) = 0$, $f(1) = g(1) = 1$. In addition, $f'(x) = \frac{1}{2}$, while $g'(x) = 1/(2\sqrt{x})$. Since $g'(x) > \frac{1}{2}$, for $x \in (0, 1)$, with $g'(1) = \frac{1}{2}$, it follows that $g(x) < f(x)$ for $x \in (0, 1)$, and the proof is complete.

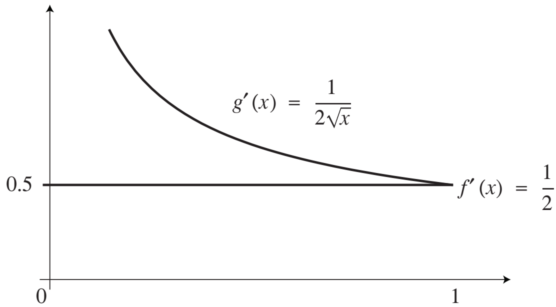

FIGURE 2: $g'(x) = \dfrac{1}{2\sqrt{x}} \geqslant f'(x) = \dfrac{1}{2}$

Step 2: Note that $G(A, H) = G$, because $\sqrt{x} = \sqrt{\dfrac{2x}{1 + x} \cdot \dfrac{1 + x}{2}}$. Since $G \leqslant A$, then $H \leqslant G \leqslant A$, as the following figure shows:

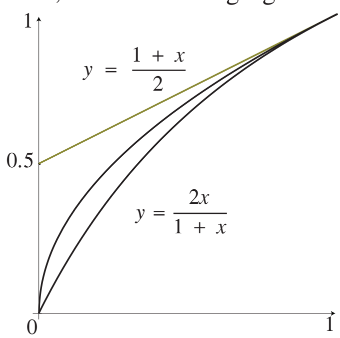

FIGURE 3: $G(A, H) = G$. Since $G \leqslant A$, then $H \leqslant G \leqslant A$

Reference

1. P. S. Bullen, *Handbook of means and their inequalities* (Springer) 2003.

10.1017/mag.2022.22
Published by Cambridge University Press on behalf of The Mathematical Association

ÁNGEL PLAZA
Universidad de Las Palmas de Gran Canaria, Spain
e-mail: *angel.plaza@ulpgc.es*