Macroeconomic Dynamics, **1**, 1997, 423–451. Printed in the United States of America.

ASSET PRICING WITH BORROWING CONSTRAINTS AND EX ANTE HETEROGENEITY

Pamela Labadie
George Washington University

In answer to the question "Will borrowing constraints necessarily intensify aggregate fluctuations and aggregate cyclical variability?" it has been found that complete markets equilibrium displays aggregate fluctuations that may be dampened when borrowing constraints are introduced. Like others, I find that variability in the distribution of labor productivity shocks amplifies aggregate fluctuations. I also find that allowing agents to have different permanent incomes amplifies aggregate fluctuations, enriching the asset-pricing implications of the complete contingent claims model when demand aggregation is not possible. Although agents are able to equalize their intertemporal marginal rates of substitution (IMRS) of consumption state-by-state, the IMRS of labor is not equalized across agents, creating gains from specialization. To determine how frictions affect aggregate variability, two types of borrowing constraints are studied. In the first model, dividend payments are restricted and, in the second, nonhuman wealth is restricted to be positive. Either type of borrowing constraint can dampen aggregate fluctuations.

Keywords: Asset Pricing, Borrowing Constraints, Ex Ante Heterogeneity

1. INTRODUCTION

In their seminal paper, Scheinkman and Weiss (1986, p. 37) ask the following question: "Will borrowing constraints necessarily intensify aggregate fluctuations and aggregate cyclical variability?" They construct a model with the property that the complete contingent claims equilibrium displays no aggregate fluctuations. There are two important features of their model: Agents are identical ex ante, or equivalently have identical permanent incomes, and become differentiated over time by the history of their idiosyncratic productivity shocks; and second, the distribution of the labor productivity shocks is time and state invariant. In an example in Appendix D of their paper, they show that allowing variability in the distribution of the productivity shocks can give rise to aggregate fluctuations in the complete contingent claims equilibrium, which are dampened when borrowing constraints are imposed. I examine not only how variability in the distribution of productivity shocks affects aggregate fluctuations but I also allow agents to

I wish to thank S. Altuğ, Christian Gilles, Anthony Smith, and an anonymous referee for comments made on earlier drafts. Address correspondence to: Pamela Labadie, George Washington University, Department of Economics, 608 Funger Hall, Washington, DC 20052, USA; e-mail: labadie@gwis2.circ.gwu.edu.

 1365-1005/97 $9.00 + .10

have different permanent incomes. As in the Scheinkman and Weiss example, I find that allowing variability in the distribution of labor productivity shocks gives rise to aggregate fluctuations that are dampened when trade is restricted through borrowing constraints. I also find that allowing agents to have different permanent incomes can lead to the same result.

If agents have different permanent incomes but demand aggregation is possible, then a representative consumer can be constructed as described by Rubinstein (1974). In model studied here, agents have linear disutility of labor so that the first-order conditions for consumption and labor cannot be averaged over agents to construct a representative agent, although a composite consumer can be constructed as described by Constantinides (1982). This point is related to an issue raised by Constantinides and Duffie (1996). They study a model in which there are uninsurable, persistent, and heteroskedastic labor income shocks. Each agent initially faces an identical distribution of labor income. Over time, the sample path of the ratio of an agent's labor income to aggregate income is nonstationary. When borrowing constraints are imposed in such a model, agents are no longer able to insure against the persistent and heteroskedastic income shocks. Marginal rates of substitution in consumption are not equalized state-by-state so that a representative agent cannot be constructed. They demonstrate that incomplete consumption insurance and consumer heterogeneity greatly enrich the asset-pricing implications of their model. In particular, the mean and variance of the stochastic discount factor for asset pricing depend on the cross-sectional distribution of labor income.

The model that I examine has in common with the Constantinides and Duffie model the features that a representative agent cannot be constructed and that differences among agents persist over all sample paths. Whereas in Constantinides and Duffie, agents are identical ex ante and the ratio of an agent's income to aggregate income is nonstationary, in my model, agents have different permanent incomes and, hence are heterogeneous ex ante and the ratio of an agent's income to aggregate income is stationary. I find that allowing differences in permanent income and variability in the productivity distribution enriches the asset-pricing implications even when markets are complete because of the gains from trade and specialization. Agents will equalize their intertemporal marginal rates of substitution (IMRS) in consumption state-by-state but they do not equalize the IMRS of labor. Several examples are provided to show how these features enrich the asset-pricing implications of the model.

To determine whether borrowing constraints intensify or dampen aggregate fluctuations when agents differ in permanent income, I examine several versions of the model. If no trade is allowed—a severe form of a borrowing constraint—then per-capita output is constant even if agents have different permanent incomes. In the complete contingent claims equilibrium, if agents have different permanent incomes, there are aggregate fluctuations in production. When dividend payments are unrestricted, the complete contingent claims equilibrium can be used to determine equity prices. An agent can borrow against future income by issuing equity or claims to a dividend stream paid out of his or her labor earnings. In Section 4, the implications for the economy are examined when dividend payments are restricted, which is a form of a borrowing constraint. Dividend payments in the United States are often subject to legal restrictions, such as state laws restricting payments so that a firm is always solvent. In general, if agents have different permanent incomes, the complete contingent claims equilibrium displays greater aggregate fluctuations than the equilibrium with restricted dividend payments. When agents are identical ex ante, however, restricting dividend payments intensifies aggregate fluctuations.

In Section 5, another type of borrowing constraint is examined. The model is a discrete-time version of the Scheinkman and Weiss model. As in Scheinkman and Weiss, an important determinant of aggregate economic activity is the cross-sectional distribution of nonhuman wealth. In general, changes in this distribution amplify aggregate fluctuations.

2. DESCRIPTION OF MODEL

The model is a discrete-time version of the Scheinkman and Weiss (1986) model. There is a countable infinity of agents: a fraction $0 \leq \alpha_1 \leq 1$ is type 1 and the rest α_2 are type-2.[1] Each agent has time-additive preferences over consumption and leisure streams. An agent can produce the consumption good using labor as an input, but his or her productivity varies stochastically. Hence, in this economy, agents become differentiated from each other as a result of their histories of productivity. The production function for an agent is

$$y = \theta \ell,$$

where ℓ is the labor supply. The consumption good is nonstorable.

The realizations of the productivity shock across agent type and over time are determined by a stochastic process $\{s_t\}$. The state space of s_t is discrete, specifically, let $S \equiv \{1, 2\}$. If $s_t = 1$, then the productivity shock for a type-1 agent is $\theta_1(1) = 1$ whereas the productivity shock for a type-2 agent is $\theta_2(1) = \underline{\theta}$, where $0 \leq \underline{\theta} < 1$. When $s_t = 2$, $\theta_1(2) = \underline{\theta}$ and $\theta_2(2) = 1$. Under this specification, productivity shocks across agent types are perfectly negatively correlated. The evolution of s_t over time is independent and identically distributed over time with π_i denoting the probability that $s = i$.

Each agent has preferences over stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ of the form

$$E_0 \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - \ell_{i,t}] \right\}. \tag{1}$$

Assumption 1. Let $U: \Re^+ \to \Re^+$ be bounded, strictly concave, twice continuously differentiable, and increasing, with $\lim_{c \to 0} U'(c) = +\infty$ and $\lim_{c \to \infty} U'(c) = 0$. Let $U'(c)c$ be nondecreasing and concave in c so that relative risk aversion is nondecreasing and less than or equal to unity.

3. COMPLETE MARKETS AND HETEROGENEITY

In this section, the effects of differences in permanent income and variability in the distribution of labor productivity are studied when markets are complete. Agents may have different permanent incomes, depending on their type. Differences in permanent incomes across agent types occur whenever $\pi_1 \neq \pi_2$ so that the frequency of being highly productive varies across agent types. A type 1 agent has a probability of being productive that is different from the probability of a type 2 agent, and this difference persists along all time paths. Variability in the distribution of labor productivity occurs whenever $\alpha_1 \neq \alpha_2$, so that the proportion of agents who are highly productive varies over states.

3.1. Central Planner's Problem

The model is initially formulated as a central planning problem because the conditions for nonnegativity of labor supply are easily derived. The central planner solves

$$\max_{\{c_{i,t}\},\{\ell_{i,t}\}} E_0 \sum_{i=1}^{2} \left(\eta_i \left\{ \sum_{t=0}^{\infty} \beta^t [U(c_{i,t}) - \ell_{i,t}] \right\} \right), \tag{2}$$

subject to

$$\alpha_1\theta_1(s_t)\ell_{1,t} + \alpha_2\theta_2(s_t)\ell_{2,t} = \alpha_1 c_{1,t} + \alpha_2 c_{2,t}, \tag{3}$$

$$\ell_{i,t} \geq 0, \tag{4}$$

where $\eta_i > 0$ is the Pareto weight of the type-i agent. Only stationary solutions are examined. Let $\psi(s)$ denote the multiplier associated with (3) and $\mu_i(s)$ denote the multiplier associated with (4) in state s. The first-order conditions are

$$\begin{aligned} \eta_i U'[c_i(s)] &= \alpha_i \psi(s), \\ \eta_i &= \alpha_i\theta_i(s)\psi(s) + \mu_i(s), \end{aligned} \tag{5}$$

for $i = 1, 2$. For a type-i agent who works,

$$\psi(s) = \frac{\eta_i}{\alpha_i\theta_i(s)}$$

and

$$U'[c_i(s)] = \frac{1}{\theta_i(s)}.$$

There are two possible outcomes—$\mu_i(s) = 0$ or $\mu_i(s) > 0$—for each type of agent in each state, two states $s = 1, 2$, and two types of agents, and so, there are 2^3 possible combinations of (μ_1, μ_2). For a given s, cases in which $\mu_1(s) > 0$ and $\mu_2(s) > 0$ can be ruled out because consumption is nonstorable so that some agent must work each period. This restriction eliminates 7 of the 16 combinations. Cases in which one type of agent never works or the other works in both states can be

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

ruled out because $\eta_i > 0$ and preferences are identical, eliminating two cases. The case in which both types of agents work in both states can be ruled out by noting that the condition for both types to work in state s,

$$\frac{\eta_1}{\alpha_1\theta_1(s)} = \frac{\eta_2}{\alpha_2\theta_1(s)},$$

cannot hold simultaneously for both states, under the assumption that the productivity shocks are perfectly negatively correlated across agent types. This leaves six possible combinations of multipliers.

For the remaining cases, solve (5) for μ_i to obtain the restriction that $\eta_i \geq \alpha_i\theta_i(s)\psi(s)$ if $\mu_i \geq 0$. This restriction can be used to rule out three cases for which the values of (η_1, η_2) fail to satisfy the condition

$$\frac{1}{\underline{\theta}} \geq \left(\frac{\alpha_2}{\alpha_1}\frac{\eta_1}{\eta_2}\right) \geq \underline{\theta}, \tag{6}$$

and hence violate the nonnegativity of μ. There are three possible solutions:

(1) Agent 1 works in both states and agent 2 works only in state 2 when

$$\frac{1}{\underline{\theta}} = \frac{\alpha_2}{\alpha_1}\frac{\eta_1}{\eta_2}.$$

(2) Agent 1 works only in state 1 and agent 2 works in both states when

$$\frac{\alpha_2}{\alpha_1}\frac{\eta_1}{\eta_2} = \underline{\theta}.$$

(3) Equation (6) holds as a strict inequality in both directions and each agent specializes by working only in the period in which he or she is most productive.

To proceed further, it is useful to examine the associated contingent claims equilibrium because the Pareto weights (η_1, η_2) can be conveniently linked to the model's parameters.

3.2. Contingent Claims Equilibrium

At time zero, all agents trade in the market for claims to consumption and labor supply contingent on state s at time t. Only stationary equilibria are considered. Define $p(t, s)$ as the price of a right to delivery of one unit of consumption in state s at time t for all and $s \in S$. In a complete markets equilibrium, a representative type-i consumer maximizes (1) subject to the single budget constraint

$$\sum_{t=0}^{\infty}\left\{\sum_{s=1}^{2} p(t, s)(c_{i,t} - \theta_i(s)\ell_{i,t})\right\} \leq 0 \tag{7}$$

and nonnegativity constraint

$$\ell_{i,t} \geq 0. \tag{8}$$

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

Let λ_i denote the Lagrange multiplier associated with the budget constraint (7) for a type-i agent and let $\phi_{i,t}$ denote the Lagrange multiplier associated with the nonnegativity constraint (8). The first-order conditions with respect to $c_{i,t}$ and $\ell_{i,t}$ for an agent of type i are

$$\beta^t \pi_s U'(c_{i,t}) = \lambda_i p(t,s), \tag{9}$$

$$\beta^t \pi_s = \lambda_i \theta_i(s) p(t,s) + \beta^t \pi_s \phi_{i,t}. \tag{10}$$

Notice that λ_i does not vary over time or over s. The contingent claims equilibrium and the solution to the central planning problem are equivalent if

$$\eta_1 = (\alpha_1/\lambda_1),$$

$$\eta_2 = (\alpha_2/\lambda_2).$$

For the type-i agent that works in state s, (10) implies that

$$\frac{p(t,s)}{\beta^t} = \frac{\pi_s}{\lambda_i \theta_i(s)}$$

and

$$U'[c_i(s)] = \frac{1}{\theta_i(s)}.$$

Denote g as the function

$$g(x) = (U')^{-1}(x),$$

which is well defined because utility is twice continuously differentiable and strictly concave. Denote $x \equiv \lambda_1/\lambda_2$. The market-clearing conditions for $s = 1, 2$ are

$$\alpha_1 \ell_1(1) + \alpha_2 \underline{\theta} \ell_2(1) = \alpha_1 g(1) + \alpha_2 g(1/x), \tag{11}$$

$$\alpha_1 \underline{\theta} \ell_1(2) + \alpha_2 \ell_2(2) = \alpha_1 g(x) + \alpha_2 g(1). \tag{12}$$

The expected present value of lifetime earnings of a type-1 agent is

$$\sum_{t=0}^{\infty} \left[\sum_s p(t,s)\theta_1(s)\ell_1(s) \right] = \sum_{t=0}^{\infty} \beta^t \left[\frac{\pi_1 \ell_1(1)}{\lambda_1} + \frac{\pi_2 \underline{\theta} \ell_1(2)}{\lambda_2} \right]$$
$$= \frac{1}{1-\beta} \frac{1}{\lambda_1} [\pi_1 \ell_1(1) + x \pi_2 \ell_1(2) \underline{\theta}], \tag{13}$$

and the expected present value of his or her consumption expenditures is

$$\sum_{t=0}^{\infty} \sum_s p(t,s) c_1(s) = \frac{1}{1-\beta} \frac{1}{\lambda_1} [\pi_1 g(1) + \pi_2 x g(x)]. \tag{14}$$

Equate (14) to the expected present value of type 1's earnings (13), solve for $\ell_1(1)$, substitute into market-clearing condition (11), and simplify to obtain

$$(\pi_2/\pi_1)x[g(x) - \underline{\theta}\ell_1(2)] = (\alpha_2/\alpha_1)[g(1/x) - \underline{\theta}\ell_2(1)]. \tag{15}$$

As described in the central planning problem, there are three possible cases.

(1) In the first case, the type-1 agent works in both states, the type-2 agent works only in state 2 and $x = \underline{\theta}$. In (15), set $x = \underline{\theta}$, $\ell_2(1) = 0$, and solve for $\ell_1(2)$ to obtain

$$\underline{\theta}\ell_1(2) = g(\underline{\theta}) - (\alpha_2/\alpha_1)(\pi_1/\pi_2)(1/\underline{\theta})g(1/\underline{\theta}). \tag{16}$$

Recall that $0 < \underline{\theta} < 1$ so that $g(\underline{\theta}) > g(1/\underline{\theta})$ because $g' > 0$. For $\ell_1(2) > 0$ to obtain, the following condition must hold:

$$\underline{\theta}g(\underline{\theta}) > \left(\frac{\alpha_2}{\alpha_1}\frac{\pi_1}{\pi_2}\right)g\left(\frac{1}{\underline{\theta}}\right). \tag{17}$$

(2) In the second case, the type-2 agent works in both states, the type-1 agent works only in state 1, and $x = 1/\underline{\theta}$. Set $\ell_1(2) = 0$ in (15) and solve for $\ell_2(1)$ to establish a nonnegativity constraint on $\ell_2(1)$ analogous to (17) of the form

$$\frac{1}{\underline{\theta}}g\left(\frac{1}{\underline{\theta}}\right) > \left(\frac{\alpha_1}{\alpha_2}\frac{\pi_2}{\pi_1}\right)g(\underline{\theta}). \tag{18}$$

(3) The final case, in which each agent specializes, can be determined as follows. Set $\ell_1(2) = \ell_2(1) = 0$ in (15) to obtain

$$\left(\frac{\alpha_1}{\alpha_2}\frac{\pi_2}{\pi_1}\right)xg(x) = g\left(\frac{1}{x}\right). \tag{19}$$

Under Assumption 1, the left side of (19) is decreasing in x and the right is increasing so that there is a unique solution $\hat{x}[(\alpha_1/\alpha_2)(\pi_2/\pi_1)]$.[2] For this to be the solution to the problem, $\hat{x}$ must satisfy

$$\underline{\theta} < \hat{x}\left(\frac{\alpha_1}{\alpha_2}\frac{\pi_2}{\pi_1}\right) < \frac{1}{\underline{\theta}}. \tag{20}$$

Output is

$$\alpha_1\ell_1(1) = \alpha_1 g(1) + \alpha_2 g(1/x),$$
$$\alpha_2\ell_2(2) = \alpha_1 g(\hat{x}) + \alpha_2 g(1).$$

It is straightforward to show that $\hat{x}(a)$ is increasing in a, so that x is increasing in α_1 (decreasing in α_2) and increasing in π_2 (decreasing in α_1). This implies that $\alpha_1\ell_1(1)$ is increasing in x and $\alpha_2\ell_2(2)$ is decreasing in x.

For any utility function satisfying Assumption 1 and set of parameters (α_1, α_2, β, π_1, π_2), only one of the three conditions will be satisfied.

The same set of conditions for specialization in labor supply can be derived using the expected present value of consumption and earnings for a type-2 agent and the market-clearing condition for $s = 2$. For a type-2 agent, the expected present value of labor income is

$$\sum_{t=0}^{\infty}\left[\sum_{s} p(t,s)\theta_2(s)\ell_2(s)\right] = \frac{1}{1-\beta}\frac{1}{\lambda_2}\left[\frac{\pi_1\underline{\theta}\ell_2(1)}{x} + \pi_2\ell_2(2)\right] \tag{21}$$

and the expected present value of his or her consumption is

$$\sum_{t=0}^{\infty}\sum_{s} p(t,s)c_2(s) = \frac{1}{1-\beta}\frac{1}{\lambda_2}\left[\pi_1\frac{1}{x}g\left(\frac{1}{x}\right) + \pi_2 g(1)\right]. \tag{22}$$

Equate (21) to (22), solve for $\ell_2(2)$, substitute into (12), and simplify to obtain an equation that will result in the identical set of conditions that depends on the value of the multipliers ϕ_i.

In the examples studied below, the parameter values are chosen so that the last case, in which each agent works only when he is most productive, is the solution to the model.

Example 1: Scheinkman and Weiss economy

Suppose that $\pi_1 = \frac{1}{2}$ and that $\alpha_1 = \frac{1}{2}$. Each period, one-half of the agents is productive so that the distribution of labor productivity is invariant over states and each type of agent expects to be productive with the same probability as any other, so that permanent incomes are identical across agents. Under these assumptions, a stationary solution is $\lambda_1 = \lambda_2$ because (15) reduces to $g(x) = xg(x^{-1})$. In this case, all agents consume a constant amount, equal to $g(1)$, at all dates and in all states, regardless of type. Output is constant and equal to $2g(1)$. Prices are also constant and the real interest rate r satisfies

$$E_t\left(\frac{p_{t+1}}{p_t}\right) = \frac{1}{1+r} = \beta.$$

This is the case of complete insurance in which the opportunities to pool risk enable all agents to consume a fixed amount regardless of the particular time path of their earnings stream. Agents have identical permanent incomes and each specializes in labor supply, working only in those states in which he or she is most productive. Each agent's consumption is perfectly correlated with aggregate output. This is referred to below as the "benchmark case."

Example 2: Differences in permanent income

Suppose now that $\pi_1 = \frac{2}{3}$ but retain the assumption that $\alpha_1 = \frac{1}{2}$, so that the distribution of labor productivity is state invariant. Although one-half of the agents is productive at each point in time, just as before, notice that the expected present value of the lifetime earnings for a type-1 agent is greater than that of a type-2

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

agent. Equation (15) now becomes

$$\tfrac{1}{2}g(x)x = g(1/x).$$

Suppose utility displays constant relative risk aversion so that $U'(c) = c^{-\gamma}$. Although this utility function violates Assumption 1, a unique, positive, and finite real solution can be found as long as $\gamma \neq 2$. Solve (15) to obtain $x = (\frac{1}{2})^{\gamma/(2-\gamma)}$. When $s = 1$, type-1 agents consume $g(1)$ and type-2 agents consume $(\frac{1}{2})^{1/(2-\gamma)}$. When $s = 2$, type-2 agents consume $g(1)$ and type-1 agents consume $2^{1/(2-\gamma)}$. Per-capita output in state 1 and state 2 is

$$\ell_1(1) = 1 + \left(\tfrac{1}{2}\right)^{1/(2-\gamma)},$$
$$\ell_2(2) = 1 + 2^{1/(2-\gamma)}.$$

The real interest rate r_1 when type-1 agents are productive ($s = 1$) is

$$\left(\frac{1}{1+r_1}\right) = \beta\left[\pi_1 + \pi_2(2)^{-\frac{\gamma}{2-\gamma}}\right],$$

and the real interest rate r_2 when type-2 agents are productive ($s = 2$) is

$$\left(\frac{1}{1+r_2}\right) = \beta\left[\pi_1(2)^{\frac{\gamma}{2-\gamma}} + \pi_2\right].$$

Hence, both agents experience fluctuations in consumption over time, depending on the realization of the state variable. Each agent works only in the state in which he or she is most productive, so that the labor participation profile over states is identical to that of the benchmark economy, although the amount of labor supplied varies. The economy experiences aggregate fluctuations in output, prices, and real interest rates because agents are no longer identical in expected present value of lifetime earnings. There is no market incompleteness here and all risks are pooled.

The fluctuations in output in this example reflect the gains from trade when agents are heterogeneous. To see this, consider the behavior of output when agents are prohibited from borrowing or writing contingent claims contracts. The type-i agent will maximize (1) subject to the constraint

$$\theta_i(s)\ell_{i,t} \geq c_{i,t}.$$

The solution is

$$U'(c_{i,t}) = \frac{1}{\theta_i(s_t)},$$
$$\ell_i(s) = \frac{1}{\theta_i(s)}g\left[\frac{1}{\theta_i(s)}\right],$$

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

for all states. Per-capita output each period is

$$\alpha_1 g\left[\frac{1}{\theta_1(s)}\right] + \alpha_2 g\left[\frac{1}{\theta_2(s)}\right],$$

and, as long as $\alpha_1 = \alpha_2$, per capita output is constant across states even if $\pi_1 \neq 0.5$. Both type of agents will work in both states. The severe borrowing constraint prevents an agent from substituting labor effort intertemporally, and has the effect of smoothing an agent's labor market participation over time relative to the complete markets equilibrium. When borrowing is allowed, the relative price of leisure (the disutility of labor) falls when agents are able to specialize in production.

Example 3: Variability in the distribution of labor productivity

Suppose that $\pi_1 = \frac{1}{2}$ but that $\alpha_1 = \frac{2}{3}$, so that the distribution of labor productivity varies over states, with $\frac{2}{3}$ of the agents highly productive when $s = 1$ and only $\frac{1}{3}$ of the agents highly productive when $s = 2$. In the case of constant relative risk aversion, $x = (\frac{1}{2})^{\gamma/(\gamma-2)}$. When type-1 agents are productive, type-2 agents consume $2^{1/(2-\gamma)}$ whereas type-1 agents consume $g(1)$. When type-2 agents are productive, they consume $g(1)$ and type-1 agents consume $2^{1/(\gamma-2)}$. Not surprisingly, the time path of consumption, labor, and interest rates vary because the fraction of agents that are productive is stochastic. This result is similar to the result of Scheinkman and Weiss (1986, Example 1, Appendix D).

This model provides a convenient framework to show how permanent income effects and variability in the distribution of the productivity shock are important when the conditions for aggregation do not hold, whether or not markets are complete. Conditions for demand aggregation are described by Rubinstein (1974). Demand aggregation fails in this model because of the linear disutility of labor. The pair of first-order conditions cannot be averaged over agents to construct a representative agent whose first-order conditions for utility maximization results in the same contingent claims prices as the model under study. To see this, multiply the first-order condition for labor by α_i for agent i and sum over both agents to obtain

$$1 = \frac{p(s,t)}{\beta^t \pi_s}[\alpha_1\lambda_1\theta_1(s) + \alpha_2\lambda_2\theta_2(s)] + \sum_{i=1}^{2}\alpha_i\phi_i.$$

Under the conditions on x such that each agent works only in the period in which he or she is most productive, this condition reduces to

$$\frac{p(1,t)}{\beta^t \pi_1} = \lambda_1^{-1} \quad \text{in } s = 1$$

and

$$\frac{p(2,t)}{\beta^t \pi_2} = \lambda_2^{-1} \quad \text{in } s = 2.$$

Rewrite the first-order conditions for consumption by raising each side of the equation to the $(1/\gamma)$ power, multiply by α_i, sum over i, and rewrite to obtain

$$\alpha_1 c_1(s) + \alpha_2 c_2(s) = \left(\alpha_1 \lambda_1^{-\frac{1}{\gamma}} + \alpha_2 \lambda_2^{-\frac{1}{\gamma}}\right)\left[\frac{p(t,s)}{\beta^t \pi_s}\right]^{-\frac{1}{\gamma}}.$$

The left side is per-capita or average consumption. For $s = 1$ and substituting for the contingent claims price, the equation above simplifies to

$$\left(\frac{\alpha_1 + \alpha_2 x^{-\frac{1}{\gamma}}}{\alpha_1 + \alpha_2 x^{\frac{1}{\gamma}}}\right)^{-\gamma} = 1.$$

Notice that this equation does not hold in general unless $x = 1$, which is the benchmark case. Hence, demand aggregation is not possible in this model, except for this special case. Although agents will equalize their equal IMRS state-by-state, the IMRS is not equal to that of a representative agent.

The consumption of a representative agent equals per-capita output $y(1) = \alpha_1 + \alpha_2 g(1/x)$ in state 1 and $y_2 = \alpha_1 g(x) + \alpha_2$ in state 2. The expected IMRS for the representative agent in each state is

$$M_r(1) = \beta\left[\pi_1 + \pi_2 (y_2/y_1)^{-\gamma}\right],$$
$$M_r(2) = \beta\left[\pi_1 (y_1/y_2)^{-\gamma} + \pi_2\right].$$

A composite consumer can be constructed by maximizing the weighted sum of individual utilities subject to aggregate feasibility conditions, where the weight for a type-i agent is equal to λ_i^{-1}; see Constantinides (1982) for a discussion.

In Table 1, the effect of differences in permanent income on the mean and standard deviation of output and the IMRS in consumption is reported for various values of the parameter γ when utility is assumed to display constant relative risk aversion. These results should be compared to the benchmark case in which output and the IMRS are constant across states and agents.

In Panel A of Table 1, the distribution of productivity shocks over the population is fixed, with one-half productive each period. Type-1 agents have a higher permanent income than type-2 agents because $\pi_1 = 0.6$, so that type 1 have a greater frequency of being more productive. Output and the expected IMRS fluctuate because of the gains from specialization when agents have different permanent incomes. In Panel B, variability in the distribution of productivity shocks gives rise to aggregate fluctuations, as in Scheinkman and Weiss (1986). In Table 1, the labor participation profile of an agent remains the same as the case with identical permanent incomes in that each agent works only in the period in which he or she is most productive. Notice that the mean and the standard deviation of output and the IMRS are not monotonic in the risk aversion parameter γ. The consumption by each type of agent continues to be perfectly correlated with aggregate output.

TABLE 1. Complete markets equilibrium ($\beta = 0.99$)

	γ				
	0.5	0.75	1.5	3.0	4.0
			Panel A: $\pi_1 = 0.6, \alpha_1 = 0.5$		
Mean of output	1.045	1.059	1.264	1.010	0.9899
SD[a] of output	0.1340	0.1617	0.4422	0.2040	0.1000
Mean of IMRS	0.994	1.004	1.387	1.387	1.155
SD of IMRS	0.0648	0.1162	0.6655	0.8276	0.4715
			Panel B: $\pi_1 = 0.5, \alpha_1 = 0.6$		
Mean of output	1.045	1.059	1.000	0.9898	0.9897
SD of output	0.1404	0.1703	0.2020	0.1000	0.1000
Mean of IMRS	0.9945	1.004	1.404	1.162	1.155
SD of IMRS	0.0671	0.1215	0.7620	0.4469	0.4715

[a]SD = Standard deviation.

3.2.1. Pricing equity shares. A complete contingent claims equilibrium can be used to price any security and, in this section, the focus is on pricing equity shares. The model is solved below for several parameter values by allowing the technology shock to be serially correlated but retaining the assumption that the shock is perfectly negatively correlated across agents at a point in time. Let π_{ij} denote the conditional probability of moving from state i to state j in one step.

Assume that each agent acts as a firm and issues claims to a dividend stream, paying $d_i(s)$ in dividends when the agent is productive and nothing otherwise. There is one equity share outstanding for each agent. This trading mechanism allows each agent to borrow against future income. The representative type-i agent maximizes (1) subject to

$$\theta_i(s)\ell_i(s) + \sum_{j=1}^{2} z_{i,t}^j[Q_j(s) + d_j(s)] \geq c_{i,t} + \sum_j z_{i,t+1}^j Q_j(s) + \frac{D_i(s)}{\alpha_i}, \quad \textbf{(23)}$$

where $Q_j(s)$ denotes the price of a type-j share when the current state is s, $z_{i,t}^j$ denotes the shares of type j equities held by a type-i agent at time t, and $D_i(s)$ is treated as exogenous by the type-i agent. Let $\hat{\psi}$ denote the Lagrange multiplier for (23) and let $\hat{\mu}$ denote the Lagrange multiplier on the nonnegativity constraint for ℓ. The first-order conditions are

$$U'(c_{i,t}) = \hat{\psi}_{i,t}, \quad \textbf{(24)}$$

$$1 = \hat{\psi}_{i,t}\theta_i(s) + \hat{\mu}_{i,t}, \quad \textbf{(25)}$$

$$\hat{\psi}_{i,t} Q_{j,t} = \beta E_s[\hat{\psi}_{i,t+1}(Q_{j,t+1} + d_{j,t+1})]. \quad \textbf{(26)}$$

TABLE 2. Results of model for pricing equity shares ($\beta = 0.99$)

	$\alpha_1 = 0.5$	$\alpha_1 = 0.45$	
	$\pi_{11} = 0.6, \pi_{22} = 0.5$	$\pi_{11} = 0.5, \pi_{22} = 0.5$	$\pi_{11} = 0.6, \pi_{22} = 0.5$
		Panel A: $\gamma = 0.75$	
SD[a] of output	0.0892	0.0799	0.1688
SD of equity return	0.0605	0.0286	0.1130
Equity premium	0.0037	0.0037	0.0165
SD of risk-free rate	0.0607	0.0302	0.1130
SD of consumption for 1	0.0871	0.0493	0.2006
SD of consumption for 2	0.0741	0.0449	0.1429
		Panel B: $\gamma = 3.00$	
SD of output	0.1118	0.1020	0.2214
SD of equity return	0.3394	0.3235	0.8025
Equity premium	0.1191	0.0975	0.4541
SD of risk-free rate	0.3005	0.2951	0.5363
SD of consumption for 1	0.0994	0.0909	0.1716
SD of consumption for 2	0.1242	0.1111	0.2622

[a] SD = Standard deviation.

The nonnegativity of $\hat{\mu}$ restricts $\hat{\psi}$ such that

$$\hat{\psi}_i \leq \frac{1}{\theta_i(s)}.$$

The equilibrium derived in the complete contingent claims model can be replicated by setting $d_1(2) = d_2(1) = 0$, $D_i(i) = d_i(i)$, $z_j^j = 0$, $z_i^j = 1/\alpha_i$ for $i \neq j$, and

$$d_1(1) = \alpha_2 g(1/x),$$

$$d_2(2) = \alpha_1 g(x).$$

The stationary equilibrium equity prices satisfy

$$Q_1(1) = \frac{\beta}{U'\left[g\left(\frac{1}{x}\right)\right]}\{\pi_{11}U'[g(1/x)][d_1 + Q_1(1)] + \pi_{12}U'[g(1)]Q_1(2)\},$$

$$Q_2(2) = \frac{\beta}{U'[g(x)]}\{\pi_{21}U'[g(1)]Q_2(1) + \pi_{12}U'[g(x)][d_2 + Q_2(2)]\}.$$

Table 2 reports the results of solving the model for a variety of parameter values.

Recall that in the benchmark case, where $\alpha_1 = 0.5$ and $\pi_{11} = \pi_{22} = 0.5$, there are no fluctuations in endogenous variables and consumption is constant

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

over time and equal across agents. The case in which $\alpha_1 = 0.5$, reported in column 2, highlights how differences in permanent income can give rise to aggregate fluctuations (the permanent income effect) and illustrates some of the implications for asset prices. Notice that, as γ increases, the standard deviations of output and asset returns increase but that the standard deviation of asset returns increase proportionately more. Columns 3 and 4, in which $\alpha_1 = 0.45$, is the case in which the distribution of labor productivity varies over states. When $\pi_{11} = \pi_{22}$, each agent expects to be as productive as often as any other, but the fluctuation in the fraction of productive agents affects each agent's trading opportunities and, hence, his or her ability to smooth consumption intertemporally. The last column combines both the permanent income effect and variability in the distribution of labor productivity.[3] The conclusion is that differences in permanent income and variability in the distribution of labor productivity can give rise to richer asset-pricing dynamics, even in a complete markets setting.

4. RESTRICTIONS ON DIVIDEND PAYMENTS

Dividend payments are often subject to legal restrictions. For example, if a firm enters into a debt contract by issuing a bond, the debt contract may restrict dividends to protect the assets available to service the debt contract. There are also state laws in the United States restricting dividends if the payments would make the firm insolvent. Restricting dividend payments in this model limits the ability of agents to borrow against future income and, in this sense, is a form of a borrowing constraint.

The effects of restricting dividend payments on equilibrium output and prices can be studied by maximizing (1) subject to the budget constraint (23), resulting in the set of first-order conditions (24)–(26). As described in the central planning problem, cases can be ruled out in which a particular agent works in neither state or where neither agent works in a particular state.

In the discussion below, I examine only the steady state by assuming that $z_j^j = 0$ and $z_j^i = 1/\alpha_j$ for $i \neq j$. A general proof of existence and uniqueness of a stationary equilibrium would require that equilibrium prices and allocations be determined for any feasible allocation of equity claims, or any (z_1^j, z_2^j) such that $z_i^j \geq 0$ and $\alpha_1^{-1} z_1^j + \alpha_2^{-1} z_2^j = 1$ for $j = 1, 2$. To show that an outcome in which both types of agents work in both states cannot be an equilibrium, examine the first-order conditions for equity shares. For a type-1 agent, the first-order condition for holding a type-j equity in state 1 is

$$Q_j(1) = \beta\left\{\pi_1[Q_j(1) + d_1(1)] + \frac{\pi_2}{\underline{\theta}}[Q_j(2) + d_j(2)]\right\}, \tag{27}$$

and the first-order condition for a type-2 agent is

$$Q_j(1) = \beta\{\pi_1[Q_j(1) + d_1(1)] + \pi_2\underline{\theta}[Q_j(2) + d_j(2)]\}. \tag{28}$$

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

Under the assumption that $0 < \underline{\theta} < 1$, the first-order conditions for the two agents cannot hold simultaneously; a similar argument can be constructed for $s = 2$; hence, an equilibrium in which each agent type works in both states can be ruled out. Similar arguments can be constructed as in the central planning problem to eliminate all but the following three cases.

(1) For the case where agent 1 works in both states and agent 2 works only in state 2 to be an equilibrium, the restrictions on the equilibrium dividend payments can be derived as follows. For a type-1 agent to work in $s = 2$, the dividend payment received by type 1 in state 2 must satisfy

$$\frac{1}{\underline{\theta}} < U'\left[\frac{d_2(2)}{\alpha_1}\right]; \tag{29}$$

otherwise, the type 1 agent will not work when $s = 2$. Agent 2 will not work in $s = 1$ as long as

$$U'\left[\frac{d_1(1)}{\alpha_2}\right] < \frac{1}{\underline{\theta}}. \tag{30}$$

The first-order condition for equity shares of a type-1 agent satisfies (27) and, for agent 2, the condition is

$$Q_j(1) = \beta\left\{\pi_1[Q_j(1) + d_j(1)] + \frac{\pi_2}{U'\left[\alpha_2^{-1}d_1(1)\right]}[Q_j(2) + d_j(2)]\right\}. \tag{31}$$

For the first-order conditions to hold simultaneously,

$$U'\left[\frac{d_1(1)}{\alpha_2}\right] = \underline{\theta}. \tag{32}$$

(2) For the case where agent 2 works in both states and agent 1 works only in state 2, the dividend payment to agent 1 in $s = 2$ must satisfy

$$U'\left[\frac{d_2(2)}{\alpha_1}\right] < \frac{1}{\underline{\theta}}; \tag{33}$$

otherwise, agent 1 will work. For agent 2 to work in $s = 1$,

$$U'\left[\frac{d_1(1)}{\alpha_2}\right] > \frac{1}{\underline{\theta}}. \tag{34}$$

Examining the first-order conditions for equity share prices for the two types of agents restricts dividends such that

$$U'\left[\frac{d_2(2)}{\alpha_2}\right] = \underline{\theta}, \tag{35}$$

for the conditions to hold simultaneously.

(3) For the final case, in which each agent works only in the period in which he is most productive, the equilibrium dividends must satisfy (30), (33),

$$\underline{\theta} < U'\left[\frac{d_1(1)}{\alpha_2}\right],$$

$$\underline{\theta} < U'\left[\frac{d_2(2)}{\alpha_1}\right],$$

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

and

$$\frac{1}{U'\left[\frac{d_2(2)}{\alpha_1}\right]} = U'\left[\frac{d_1(1)}{\alpha_2}\right]. \qquad \textbf{(36)}$$

Suppose that $\pi_1 = 0.5$ and $\alpha_1 = 0.5$, parameter values that correspond to the benchmark case, and set $\underline{\theta} = 0.5$. Dividend payments can be restricted by picking $d_1(1)$ to satisfy

$$g\left(\frac{1}{\underline{\theta}}\right) < \frac{d_1(1)}{\alpha_2} < g(1) < g(\underline{\theta}).$$

Under this restriction on dividends paid by type-1 agents to their shareholders, type-2 agents will not work. The equity prices will satisfy the first-order condition (31) for type-1 agents and the first-order condition for type-2 agents is

$$Q_j(1) = \beta\left\{\pi_1[Q_j(1) + d_j(1)] + \pi_2 U'\left[\alpha_1^{-1} d_2(2)\right][Q_j(2) + d_j(2)]\right\}. \qquad \textbf{(37)}$$

For both first-order conditions to hold simultaneously, the dividend payments must satisfy

$$\frac{d_2(2)}{\alpha_1} = g\left\{\frac{1}{U'\left[\alpha_2^{-1} d_1(1)\right]}\right\}.$$

The effects of restricting dividend payments are reported in Table 3.

Panel A of Table 3 shows output and the IMRS when $\pi_1 = \pi_2$ and $\alpha_1 = \alpha_2$. When markets are complete, this is the benchmark case in which there are no aggregate fluctuations. Clearly restricting dividend payments gives rise to aggregate fluctuations. Panel B of Table 3 should be compared to Panel A of Table 1, which reports the same set of statistics for complete markets. For values of the risk aversion parameter $\gamma < 1$, output is more variable when dividends are restricted,

TABLE 3. Restricted dividend payments ($\beta = 0.99$)

	γ				
	0.5	0.75	1.5	3.0	4.0
		Panel A:	$\alpha_1 = 0.5$, $\pi_1 = 0.5$		
Mean of output	1.027	1.012	1.003	1.000	1.002
SD[a] of output	0.1653	0.1092	0.0543	0.0271	0.0203
Mean of IMRS	0.9917	0.9937	1.005	1.050	1.098
SD of IMRS	0.0403	0.0605	0.1219	0.2510	0.3441
		Panel B:	$\alpha_1 = 0.5$, $\pi_1 = 0.6$		
Mean of output	1.060	1.034	1.014	1.006	1.005
SD of output	0.1620	0.1070	0.0532	0.0266	0.0199
Mean of IMRS	0.9916	0.9935	1.004	1.048	1.094
SD of IMRS	0.0398	0.0600	0.1223	0.2577	0.3592

[a] SD = Standard deviation.

but the IMRS is less so than the complete markets case. As γ increases, the variability of output and the IMRS in the complete markets case rise by more than the restricted payments model. This demonstrates that borrowing constraints can dampen aggregate fluctuations that would otherwise occur in a model of complete markets and ex ante heterogeneity.

5. BORROWING CONSTRAINTS

Under the complete markets specification, if there is variability in the distribution of productivity shocks or if agents have different permanent incomes, then the economy will display aggregate fluctuations. The effects of borrowing constraints on equilibrium allocations and prices are now studied in a version of the Scheinkman and Weiss model. In the discussion below, set $\underline{\theta} = 0$ so that, each period, one type of agent is productive while the other is not.

Suppose that there is a durable and nondepletable asset that is fixed at one unit in per-capita supply. The asset is bought and sold at a real price q_t at time t. Let $x_{i,t}$ denote the asset holdings of the average type i at the beginning of period t. Because the supply of the asset is fixed at unity, market clearing requires that

$$\alpha_1 x_{1,t} + \alpha_2 x_{2,t} = 1. \tag{38}$$

From (38), x_2 can be determined once x_1 is known and conversely. Define $\hat{x}_1(x_2)$ as the value of x_1 satisfying (38) given x_2 and define $\hat{x}_2(x_1)$ analogously. Hence, the state of the system is summarized by the pair (s, x_1) or (s, x_2). The state variables for an individual agent consist of the system state variables and his or her initial asset holdings z_i.

The representative type-i agent, for $i = 1, 2$, chooses stochastic sequences $\{c_{i,t}, \ell_{i,t}\}$ to maximize (1) subject to the set of constraints

$$z_{i,t+1} - z_{i,t} = \frac{[\theta_i(s_t)\ell_{i,t} - c_{i,t}]}{q_t}, \tag{39}$$

and the nonnegativity constraints

$$z_{i,t+1} \geq 0, \qquad \ell_{i,t} \geq 0. \tag{40}$$

There are two features worth noting about this problem. First, it rules out complete insurance of idiosyncratic risk by ruling out the existence of prices to consumption contingent on any possible history. One possible reason for idiosyncratic risk to be uninsurable is that the shocks to an individual's productivity are not publicly observable. Second, the above problem assumes that individuals in this economy face borrowing constraints. The borrowing constraints are introduced through the constraint that the asset holdings of the consumer must be nonnegative at all dates and all states, or $z_{i,t+1} \geq 0$.

The consumer's problem is now studied as a dynamic programming problem. The average type-i agent who begins the period with asset holdings z solves

$$V_i(z, x, s) = \max_{\{c, \ell, z'\}} [U(c) - \ell + \beta E_s V_i(z', x', s')], \tag{41}$$

subject to the constraints (39) and (40), the initial conditions, and the law of motion for x. Let $\omega_{i,t}$ denote the multiplier on the nonnegativity constraint for $z_{i,t+1}$ in (40), let μ_i denote the Lagrange multiplier on the nonnegativity constraint for labor, and let $\xi_{i,t}$ denote the Lagrange multiplier for the budget constraint. The first-order conditions for the representative type-i agent are

$$U'(c_{i,t}) = \xi_{i,t}/q_t, \tag{42}$$

$$1 = \frac{\theta_i(s_t)\xi_{i,t}}{q_t} + \mu_{i,t}, \tag{43}$$

$$\xi_{i,t} = \beta E_s \left(\frac{\partial V_i}{\partial z_{i,t+1}} \right) + \omega_{i,t}. \tag{44}$$

The envelope condition is

$$\frac{\partial V_i}{\partial z_{i,t}} = U'(c_{i,t}) q_t. \tag{45}$$

Assume that $z \in \mathcal{Z} = [0, \bar{z}]$ where $1 < \bar{z} < \infty$ and $\ell \in [0, L]$, where $L < \infty$. Under Assumption 1, the utility function is bounded. Also assume that $x_i \in \mathcal{Z}$ for $i = 1, 2$. Define $\mathcal{S} \equiv \mathcal{Z} \times S$. Let Q be the set of functions $q_i \colon \mathcal{S} \to \Re^+$, $i = 1, 2$, such that $\{q_i \colon 0 < q_i(x, s) < \infty, (x, s) \in \mathcal{S}\}$. Notice that if $q_i(x, s)$ is strictly positive, then the set of values $\{c, \ell, z'\}$ satisfying (39) and (40) can be denoted $\phi(z, x, s)$; this set is compact and convexed valued. If q is continuous, then under Assumption 1, ϕ is continuous in s. Let $\mathcal{V}$ be the space of bounded, continuous, real-valued functions $V_i(z, x, s)$ on $\mathcal{Z} \times \mathcal{S}$ with the norm $||V_i|| = \sup|V_i(z, x, s)|$. Given any continuous, strictly positive price, it is straightforward to show that there exists a unique value function satisfying (41). This summarizes the information needed for the individual agent i.

The next step is to construct an equilibrium. A formal definition of the equilibrium is contained in the Appendix. If $\theta_i(s) = 0$, then $\mu_{i,t} = 1$, otherwise if $\theta_i(s) > 0$, then

$$\frac{\xi_{i,t}}{q_t} = \frac{1}{\theta_i(s_t)} = 1$$

and $U'(c_{i,t}) = 1$. For the unproductive agent, consumption satisfies

$$U'(c_{i,t}) = \frac{\xi_{i,t}}{q_t}.$$

Recall that if we know x_1, then we can determine x_2 from the market-clearing condition (38). Without any loss of information, define $\xi_i(x_i, s)$ as the equilibrium multiplier on constraint (39) for the representative type-i agent when the average holdings of the durable asset by type-i agents at the beginning of the period is x_i. Each period, one type of agent is productive and from the first-order conditions define

$$q(x_1, 1) = \xi_1(x_1, 1)$$

when $s = 1$ and

$$q(x_2, 2) = \xi_2(x_2, 2)$$

when $s = 2$, so that the asset price is a function of the current state and the asset holdings of the productive agent in that state.

Recall that g is a function satisfying $U'[g(k)] = k$ such that $g'(k) = (U'')^{-1} < 0$ and $g''(k) = -U'''/(U'')^3 > 0$. Market-clearing conditions for states 1 and 2 are

$$\alpha_1\theta_1(1)l_1(x_1, 1) = \alpha_1 g(1) + \alpha_2 g\left[\frac{\xi(x_2, 1)}{\xi_1(x_1, 1)}\right], \tag{46}$$

$$\alpha_2\theta_2(2)\ell_2(x_2, 2) = \alpha_1 g\left[\frac{\xi_1(x_1, 2)}{\xi_2(x_2, 2)}\right] + \alpha_2 g(1). \tag{47}$$

In equilibrium, an agent who is productive is never constrained in borrowing so that if $s = 1, \omega_{1,t} = 0$ and, if $s = 2, \omega_{2,t} = 0$. For the unproductive agent who is constrained in borrowing, consumption satisfies $c_{i,t} = x_{i,t}q_t$. For convenience, define

$$\hat{\xi}_1(x_2, s) \equiv \xi_1[\hat{x}_1(x_2), s],$$
$$\hat{\xi}_2(x_1, s) \equiv \xi_2[\hat{x}_2(x_1), s].$$

The equilibrium multiplier for the unproductive agent, who is type 1 in state 2 and type 2 in state 1, satisfies

$$\xi_1(x_1, 2) = \max\{U'[x_1\hat{\xi}_2(x_1, 2)]\hat{\xi}_2(x_1, 2), \beta E\xi_1(x_1', s')\}, \tag{48}$$

$$\xi_2(x_2, 1) = \max\{U'[x_2\hat{\xi}_1(x_2, 1)]\hat{\xi}_1(x_2, 1), \beta E\xi_2(x_2', s')\}. \tag{49}$$

For $i = 1, 2$, the solution (c_i, ℓ_i) to equations (42) and (43) can be used in the budget constraints (39) to solve for the average asset holdings next period x_i' of type-i agents.

A proof of existence and uniqueness of the equilibrium is contained in the Appendix for a more general version of the model. The unique equilibrium $(\xi_1^\star, \xi_2^\star)$ satisfies

$$\xi_1^\star(x_1, s) =$$
$$\begin{cases} \beta E\xi_1^\star\left\{x_1 + \dfrac{\alpha_2}{\xi_1^\star(x_1, 1)} g\left[\dfrac{\hat{\xi}_2^\star(x_1, 1)}{\xi_1^\star(x_1, 1)}, s'\right]\right\} & \text{if } s = 1 \\ \max\left(U'\left[x_1\hat{\xi}_2^\star(x_1, 2)\right]\hat{\xi}_2^\star(x_1, 2), \beta E\xi_1^\star\left\{x_1 - \dfrac{\alpha_1}{\alpha_2\hat{\xi}_2^\star(x_1, 2)} g\left[\dfrac{\xi_1^\star(x_1, 2)}{\hat{\xi}_2^\star(x_1, 2)}\right], s'\right\}\right) & \\ & \text{if } s = 2, \end{cases}$$

$$\xi_2^\star(x_2, s) =$$

$$\begin{cases} \beta E \xi_2^\star \left\{ x_2 + \dfrac{\alpha_1}{\alpha_2 \xi_2^\star(x_2, 2)} g\left[\dfrac{\hat{\xi}_1^\star(x_2, 2)}{\xi_2^\star(x_2, 2)}\right], s' \right\} & \text{if } s = 2 \\ \max\left(U'\left[x_2 \hat{\xi}_1^\star(x_2, 1)\right] \hat{\xi}_1^\star(x_2, 1), \beta E \xi_2^\star \left\{ x_2 - \dfrac{\alpha_2}{\alpha_1 \hat{\xi}_1^\star(x_2, 1)} g\left[\dfrac{\xi_1^\star(x_1, 2)}{\hat{\xi}_2^\star(x_1, 2)}\right], s' \right\} \right) & \\ & \text{if } s = 2. \end{cases}$$

In the Appendix, I show that the function $\xi_i^\star$ is nonincreasing and concave in its first argument. This property and the market-clearing conditions can be used to derive the following comparative dynamic results.

- An increase in the equilibrium asset holdings of the productive agent decreases output, raises asset prices, and lowers the consumption of the unproductive agent. As in Scheinkman and Weiss, the cross-sectional distribution of asset holdings is an important determinant of aggregate variability.
- An increase in π_1, the probability that $s = 1$, which causes permanent income to differ across agents, will lower production when $s = 1$. This suggests that borrowing constraints dampen aggregate fluctuations by limiting the gains from trade.

6. CONCLUSION

The standard representative agent model in which there are no frictions and information is complete has been unable to explain a variety of empirical regularities in asset prices; see Kocherlakota (1996) for a discussion and review of recent literature. This type of model provides complete consumption insurance and agents equalize their IMRS state-by-state. One promising departure from this framework is to assume that some idiosyncratic risks are uninsurable so agents are unable to smooth consumption or equalize their IMRS across states. There is a substantial and growing literature that considers the effects of borrowing constraints and other frictions on asset price behavior. Examples include the papers by Aiyagari and Gertler (1991), Heaton and Lucas (1992), and Telmer (1993). In a model with no aggregate uncertainty and with i.i.d. shocks across agents, Aiyagari and Gertler (1991) find that borrowing constraints fail to generate enough volatility in asset returns. This result is similar to the findings of Heaton and Lucas (1992) who study the effects of incorporate transactions costs, short-sales constraints, and borrowing constraints. Related papers are by Danthine et al. (1992), Heaton and Lucas (1995), and Telmer (1993).

Constantinides and Duffie (1996) have observed that, in many of these models, the idiosyncratic labor income shocks are i.i.d. so that the permanent income of agents is nearly equal, despite imperfect risk sharing. Hence, the opportunities afforded by a risk-free bond to smooth consumption are almost enough to allow agents to equalize their IMRS over states. To construct a model with asset-pricing dynamics rich enough to explain certain empirical regularities, they argue that idiosyncratic shocks must display persistence and heteroskedasticity to prevent agents from smoothing consumption.

I show that when demand aggregation is not possible and agents are heterogeneous, either because they have different permanent incomes or else there is variability in the distribution of labor productivity shocks, the asset-pricing dynamics are much richer, whether or not markets are complete. This is demonstrated by examining several versions of the model. Adding borrowing constraints or other market frictions in this model dampens aggregate fluctuations and, in some cases, reduces the variability of the IMRS, which is related to Appendix D of Scheinkman and Weiss (1986). To determine how borrowing constraints affect aggregate fluctuations, two types of borrowing-constraint models are presented. In the first model, dividend payments are restricted. Agents equalize the IMRS of consumption across states, but the gains from trade and specialization are mitigated, dampening aggregate fluctuations. In the second model, which is a discrete-time version of the Scheinkman and Weiss model, a borrowing constraint is imposed such that agents are no longer able to equalize the IMRS of consumption state-by-state. I find that the cross-sectional distribution of the asset is an important determinant of production and that the borrowing constraint may dampen output fluctuations relative to a complete markets version of the model in which agents have different permanent income.

NOTES

1. The assumption that there is a countably infinity of agents is made to avoid measurability problems that can arise when a continuum of agents on the unit interval is assumed. See Feldman and Gilles (1985) for a discussion.

2. By definition of the function g, $g' = (1/U'') < 0$. Under Assumption 1,

$$\frac{\partial x g(x)}{\partial x} = g(x) + x g'(x)$$

$$= g(x)[1 + (U'/cU'')] < 0.$$

The function $g(x^{-1})$ is increasing in x. Hence, the left side of (19) is decreasing in x while the right side is increasing so that there is a unique solution.

3. For most parameter values, the model resulted in a risk-free interest rate that is positive in one state and negative in the other, the exception being the case in which agents have identical discounted present value of income (so $\alpha_1 = 0.5$, $\pi_{11} = \pi_{22} = 0.5$) resulting in risk-free rates that are constant across states. The state in which the real rate is negative depends on whether γ is greater than or less than 1 and whether α_1 differs from 0.5.

4. The proof is based on that of Theorem 3 in Deaton and Laroque (1992).

REFERENCES

Aiyagari, S.R. & M. Gertler (1991) Asset returns with transactions costs and uninsured individual risk. *Journal of Monetary Economics* **27**, 311–332.

Constantinides, G. (1982) Intertemporal asset pricing with heterogeneous consumers and without demand aggregation. *Journal of Business* **55**, 253–267.

Constantinides, G. & D. Duffie (1996) Asset pricing with heterogeneous consumers. *Journal of Political Economy* **104**, 219–240.

Danthine, J.P., J.B. Donaldson & R. Mehra (1992) The equity premium and the allocation of income risk. *Journal of Economic Dynamics and Control* **16**, 509–532.

Deaton, A. & G. Laroque (1992) On the behavior of commodity prices. *Review of Economic Studies* **59**, 1–24.

Feldman, M. & C. Gilles (1985) An expository note on individual risk without aggregate uncertainty. *Journal of Economic Theory* **35**, 26–32.

Heaton, J. & D. Lucas (1992) Evaluating the effects of incomplete markets on risk sharing and asset pricing. *Journal of Economic Dynamics and Control* **16**, 601–620.

Heaton, J. & D. Lucas (1995).

Kahn, J. (1990) Moral hazard, imperfect risk-sharing and the behavior of asset returns. *Journal of Monetary Economics* **26**, 27–44.

Kocherlakota, N. (1996) The equity premium: It's still a puzzle. *Journal of Economic Literature* **34**, 42–71.

Lucas, D. (1993) Asset pricing with undiversifiable income risk and short sales constraints: Deepening the equity premium puzzle. *Journal of Monetary Economics* **34**, 325–342.

Mankiw, N.G. (1986) The equity premium and concentrated shocks. *Journal of Financial Economics* **17**, 211–219.

Rubinstein, M. (1974) An aggregation theorem for securities markets. *Journal of Financial Economics* **1**, 225–244.

Scheinkman, J.A. & L. Weiss (1986) Borrowing constraints and aggregate economic activity. *Econometrica* **54**, 23–46.

Telmer, C.I. (1993) Asset pricing puzzles and incomplete markets. *Journal of Finance* **48**, 1803–1832.

APPENDIX

Let current utility be

$$U(c) - W(\ell).$$

Assumption A.1. Let $W : [0, L] \to \Re^+$ be linear or quadratic with $W'' \geq 0$, and $W(0) = 0$.

Define a vector-valued function $\theta : S \to [\underline{\theta}, \bar{\theta}] \times [\underline{\theta}, \bar{\theta}]$, indexed by $i \in I$. Assume that

$$\theta_i(s) = \theta_1(s) \quad \text{if } i \text{ is type 1} \tag{A.1}$$

and $\theta_i(s) = \theta_2(s)$ otherwise. For a given (t, s), this structure allows the productivity shocks $\hat{\theta}_1$ and $\hat{\theta}_2$ to take any correlation, including the perfect negative correlation studied earlier. Productivity shocks also may be serially correlated and there is no presumption that the unconditional probabilities Π_1 and Π_2 are equal so that the expected permanent incomes of agents differ across types. Let s be a stationary first-order Markov process with stationary transition function F. Let F have the Feller property.

Let φ_i denote the partial derivative of the value function with respect to its first argument, or $\varphi_i(x_i, s) = V_i'(x_i, x, s)$. Define $q_i(x_i, s) = q_j(x_j, s)$, for all $i, j \in I$. The equilibrium

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

first-order conditions for the representative type-i agent are

$$U'[\hat{c}_i(x,s)] = \frac{\xi_i(x_i,s)}{q_i(x,s)}, \quad \textbf{(A.2)}$$

$$W'[\hat{\ell}_i(x,s)] = \frac{\theta_i(s)\xi_i(x_i,s)}{q_i(x,s)}, \quad \textbf{(A.3)}$$

$$\xi_i(x_i,s) = \beta E_s \varphi_i(x_i',s') + \omega_i(x_i,s). \quad \textbf{(A.4)}$$

The envelope condition is

$$\varphi_i(x_i,s) = U'[\hat{c}_i(x_i,s)]q_i(x_i,s). \quad \textbf{(A.5)}$$

When $\omega_i > 0$, the maximum amount that an agent can consume is $c_i = x_i q$. Hence the multiplier ξ_i obeys

$$\xi_i(x_i,s) = \max\{U'[x_i q_i(x_i,s)]q_i(x_i,s), \beta E_s \xi_i(x_i',s')\}. \quad \textbf{(A.6)}$$

To construct the equilibrium, I start by fixing the marginal valuation function for the asset, which is equal to the Lagrange multiplier on the budget constraint. The price that clears the market is then determined. The next step is to hold the market-clearing price fixed to solve then for the marginal valuation function. The method of proof in this step follows that of Deaton and Laroque (1992). I then show that the marginal valuation functions are increasing and concave in the market-clearing price. In the final step, I show that there exists a unique price function that is used to construct the marginal valuation function and that also clears the market.

DEFINITION A.1. *A stationary equilibrium is a set of functions* $q_i : \mathcal{Z} \times \mathcal{S} \to \Re^+$ *for* $i = 1, 2$, $c_i(z,x,s)$, $\ell_i(z,x,s)$, *and* $z_i(z,x,s)$, *defined on* $\mathcal{Z} \times \mathcal{S}$, *such that*

- (i) $\hat{c}_i(x_i,s) = c_i(x_i,x_i,s)$, $\hat{\ell}_i(x_i,s) = \ell_i(x_i,x_i,s)$, *and* $\hat{z}_i(x_i,s) = z_i(x_i,x_i,s)$ *solve* (41) *subject to the constraints* (39) *and* (40);
- (ii) $q_1(x_1,s) = q_2(x_2,s)$;
- (iii) *markets clear*; *i.e.,*

$$1 = \alpha_1 z_1(z_1,x,s) + \alpha_2 z_2(z_2,x,s) \quad \textbf{(A.7)}$$

and

$$\alpha_1\theta_1(s)\ell_1(z_1,x,s) + \alpha_2\theta_2(s)\ell_2(z_2,x,s) = \alpha_1 c_1(z_1,x,s) + \alpha_2 c_2(z_2,x,s),$$

and $x_1' = z_1(x_1,x,s)$ *and* $x_2' = z_2(x_2,x,s)$, *where the last pair of equations determines the law of motion for the system variables* x.

Define the function h by

$$h(k) \equiv (W')^{-1}(k),$$

for $k \geq 0$ so that $h : \Re^+ \to [0, L]$. Recall that the definition of the function g is $g \equiv (U')^{-1}$ so that $g : \Re^+ \to [0, \bar{Y}]$. Given (x,s), for fixed $\xi_i \geq U'(xq)q$ and $q > 0$, equations (A.2) and (A.3) are four equations ($i = 1, 2$) in four unknowns (c_1, c_2, ℓ_1, ℓ_2) where $\xi_i = \xi_i(x_i,s)$ and $q = q_i(x_i,s)$. The values (c_i, ℓ_i) satisfy $c_i = g(\xi_i/q)$ and $\ell_i = h[(\xi_i\theta_i)/q]$. For notational convenience, define the function H as

$$H(k,\theta) \equiv h(\theta k) - g(k).$$

PROPOSITION A.1. *Under Assumption* 1, $H_1 > 0$ *and* $H_{11} < 0$, *where* H_i *denotes the partial derivative with respect to the ith argument. Also,* $\lim_{k \to 0} H(k, \theta) \to -\infty$ *and* $\lim_{k \to \infty} H(k, \theta) = L$.

It is straightforward to verify these properties under Assumption 1. Notice that g is a function satisfying $U'[g(k)] = k$ such that $g'(k) = (U'')^{-1} < 0$ and $g''(k) = -U''' / (U'')^3 > 0$; also, $W'' \geq 0$ and $W''' = 0$. For $i = 1, 2$, the solution (c_i, ℓ_i) to equations (A.2) and (A.3) can be used in the budget constraint (39) to solve for the average asset holdings next period x_i' of type-i agents.

So far, we have established that, for fixed (x_i, s) and given $q > 0$ and $\xi_i > 0$, equations (39) and (A.2) and (A.3) form a system of six equations in six unknowns (c_i, ℓ_i, z_i'). I now fix only the functions ξ_i and determine the value of the price q such that markets clear; essentially this adds one equation and one unknown. Using the budget constraint and setting $\ell_i = \hat{\ell}_i(x_i, s)$, $c_i = \hat{c}_i(x_i, s)$ and $\theta_i = \theta_i(s)$, the market-clearing price satisfies

$$
\begin{aligned}
1 &= \alpha_1 \left(\frac{\theta_1 \ell_1 - c_1}{q} + x_1 \right) + \alpha_2 \left(\frac{\theta_2 \ell_2 - c_2}{q} + x_2 \right) \\
&= 1 + \alpha_1 \left(\frac{\theta_1 \ell_1 - c_1}{q} \right) + \alpha_2 \left(\frac{\theta_2 \ell_2 - c_2}{q} \right).
\end{aligned}
$$

Subtract 1 from both sides and substitute in the function H, taking as given the values $\xi_i = \xi_i(x_i, s)$ for fixed (x_i, s), to obtain

$$
\alpha_1 H\left(\frac{\xi_1}{q}, \theta_1\right) + \alpha_2 H\left(\frac{\xi_2}{q}, \theta_2\right) = 0. \qquad \textbf{(A.8)}
$$

THEOREM A.1. *Under Assumption* 1, *for fixed* $x_i \in \mathcal{Z}$ *and* $s \in S$ *and given* $\xi_i = \xi_i(x_i, s)$ *such that* $\xi_i > 0$, *there exists a unique solution* $\hat{q} : S \times \Re^+ \times \Re^+ \to \Re^+$ *that is strictly positive and continuous. Moreover,* q *is increasing and jointly concave in* ξ_1 *and* ξ_2.

Proof. Under Assumption 1, the proof of Proposition A.1 can be used to show that

$$
\lim_{q \to 0} H\left(\frac{\xi}{q}, \theta\right) = L
$$

and

$$
\lim_{q \to \infty} H\left(\frac{\xi}{q}, \theta\right) = -\infty.
$$

From Proposition A.1, it also follows that the left side of A.8 is decreasing in q and the right side is increasing. As q increases, $h - g$ decreases. Hence, as an application of the implicit function theorem, there is a function $\hat{q} : S \times \Re^+ \times \Re^+ \to \Re^+$. Moreover, because H is continuous in ξ, q is continuous in ξ.

Let H_i' denote the derivative of $H[\xi_i/q, \theta_i(s)]$ with respect to its first argument. Totally differentiate $\hat{q}$ with respect to ξ_1 to obtain

$$
\alpha_1 \frac{H_1'}{q} = \left(\alpha_1 H_1' \frac{\xi_1}{q^2} + \alpha_2 H_2' \frac{\xi_2}{q^2} \right) \frac{\partial \hat{q}}{\partial \xi_1}.
$$

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

Because $H' > 0$, it follows that $\hat{q}$ is increasing in ξ_i. Let A denote the bracketed expression on the right side of the equation immediately above. The second partial derivative is

$$A\frac{\partial^2 \hat{q}}{\partial \xi_1^2} = -\left(2\frac{\partial \hat{q}}{\partial \xi_1}\right)\left[\alpha_1\left(\frac{H_1'}{q^2} + H_1''\frac{\xi}{q}\right)\left(1 - \frac{\xi}{q}\frac{\partial \hat{q}}{\partial \xi_1}\right)^2 + \left(\frac{\partial \hat{q}}{\partial \xi_1}\right)\left(\frac{\alpha_2 \xi_2^2}{q^2}\right)\left(H_2' + H_2''\frac{\xi_2}{q}\right)\right]$$
$$= \alpha_1\frac{H_1''}{q}\left(1 - \frac{\xi_1}{q}\frac{\partial \hat{q}}{\partial \xi_1}\right)^2 + \alpha_2\frac{H_2''}{q}\left(1 - \frac{\xi_2}{q}\frac{\partial \hat{q}}{\partial \xi_1}\right)^2 \leq 0.$$

A similar argument applies to ξ_2 so that $\hat{q}$ is jointly concave in (ξ_1, ξ_2). ■

Define $q_1(x_1, s) \equiv \hat{q}[s, \xi_1(x_1, s), \hat{\xi}_2(x_1, s)]$ and $q_2(x_2, s) \equiv \hat{q}[s, \hat{\xi}_1(x_2, s), \xi_2(x_2, s)]$.

Let $\mathcal{Q}(\mathcal{S})$ be the space of functions $\{q \in \mathcal{Q}(\mathcal{S}) : 0 < q(x, s) < \infty\}$ that are continuous in (x, s). Next, I hold $q \in \mathcal{Q}(\mathcal{S})$ fixed. In the discussion below, I drop the index of the agent type for convenience. Define $\mathcal{C}(\mathcal{S})$ as the space of continuous bounded functions with the sup norm that take nonnegative values defined on the state space $\mathcal{S}$ and let $\mathcal{D}(\mathcal{S}) \in \mathcal{C}(\mathcal{S})$ be the subspace of continuous functions that are nonincreasing in their first argument. The space $\mathcal{D}(\mathcal{S})$ is a Banach space.

LEMMA A.1. *Let $\xi \in \mathcal{D}(\mathcal{S})$ so that ξ is nonincreasing in its first argument. Fix $q \in \mathcal{Q}(\mathcal{S})$. Define $p \equiv q(x, s)$ and let $\psi \in \Re^+$ and let the function $G : \Re^+ \times \mathcal{S} \to \Re^+$ be defined by*

$$G(\psi, x, s) \equiv \max\left[U'(xp)p, \beta \int_S \xi\left\{x + \frac{1}{p}\left[H\left(\frac{\psi}{p}, \theta\right)\right], s'\right\} F(s, ds')\right]. \quad \textbf{(A.9)}$$

Then G is nonincreasing in ψ and x; also, $\lim_{\psi\to 0} G(\psi, x, s) = \bar{G}$ and $\lim_{\psi\to\infty} G(\psi, x, s) = \underline{G}$.

Proof. The assumptions on ξ and q ensure that G is nonincreasing in its first argument. An increase in ψ increases $H(\psi/p, \theta)$, which decreases ξ in (A.9), and hence G is nonincreasing in its second argument. Note that $\lim_{\psi\to\infty} H(\psi, \theta) = L$ and that ξ is decreasing and bounded below by 0, as is $U'(xp)p$. Hence,

$$\underline{G}(x, s) \equiv \lim_{\psi\to\infty} G(\psi, x, s)$$
$$= \max\left[U'(xp)p, \beta \int_S \xi\left\{x + \frac{1}{p}\left[H\left(\frac{\psi}{p}, \theta\right)\right], s'\right\} F(s, ds')\right] \geq 0.$$

As $\psi \to 0$, $H \to -\infty$ and ξ tends to its upper bound $\bar{\xi}$. Under Assumption 1, $U'(xp)p$ is bounded above by zero so that G is the maximum of two bounded functions and hence is bounded. ■

Let f be a solution to

$$f(x, s) = G[f(x, s), x, s]$$
$$= \max\left[U'(xp)p, \beta \int_S \xi\left(x + \frac{1}{p}\left\{H\left[\frac{f(x, s)}{p}, \theta\right]\right\}, s'\right) F(s, ds')\right]. \quad \textbf{(A.10)}$$

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

Define Υ as the operator that assigns the solution f to the function G so that

$$f = \Upsilon G.$$

LEMMA A.2. *Assume that* $\xi \in D(\mathcal{S})$ *so that* G *as defined in* (*A*.9) *satisfies the conditions of Lemma* (*A*.1). *For fixed* q *such that* $q \in \mathcal{Q}(\mathcal{S})$, *let* $f : \mathcal{S} \to \Re^{+}$ *be the solution to* (*A*.10). *Then,*

(*i*) *there exists a unique* $f^{\star} \in C(\mathcal{S})$ *satisfying* (*A*.10);
(*ii*) *the solution function satisfies* $f^{\star} \in D(\mathcal{S})$ *so that* $\Upsilon : D(\mathcal{S}) \to D(\mathcal{S})$;
(*iii*) *if* $G_1 \geq G_2$ *for all* (ψ, x, s), *then* $\Upsilon G_1 \geq \Upsilon G_2$.

Proof. Under the conditions of Lemma (A.1), $G(\psi, x, s) - \psi$ is continuously and strictly decreasing in ψ. For a fixed (x, s), $H(\psi/p, \theta)$ is increasing in ψ. For $G \in D$, let ψ satisfy

$$\begin{aligned}\psi &= G(\psi, x, s)\\ &= \max\left[U'(xp)p, \beta \int_S \xi\left\{z + \frac{1}{p}\left[H\left(\frac{\psi}{p}, \theta\right)\right], s'\right\} F(s, ds')\right]. \qquad \textbf{(A.11)}\end{aligned}$$

Clearly, the left side is increasing in ψ which, under the assumption that $G \in D$, implies that the right side is decreasing in ψ. As $\psi \to 0$, the left side tends to zero and the right side tends to a finite upper bound. As ψ increases, the left side increases and the right side tends to $\underset{\sim}{G} \geq 0$. Hence, there exists a unique ψ that satisfies (A.11).

It also follows that

$$\max\left[U'(xp)p, \beta \int_S \xi\left\{z + \frac{1}{p}\left[H\left(\frac{\psi}{p}, \theta\right)\right], s'\right\} F(s, ds')\right] - \psi$$

is continuous and strictly decreasing in ψ. Therefore, $f^{\star}$ is continuous and $f^{\star}(x, s)$ is decreasing in its first argument. Suppose that $\xi_1 \in D$ and $\xi_2 \in D$ and that $\xi_1 > \xi_2$, where

$$G_i(\psi, x, s) \equiv \max\left[U'(xp)p, \beta \int_S \xi_i\left\{x + \frac{1}{p}\left[H\left(\frac{\psi}{p}, \theta\right)\right], s'\right\} F(s, ds')\right],$$

so that $G_1(\psi, x, s) \geq G_2(\psi, x, s)$. Let ψ_i be the solution to the equation $[0 = G_i(\psi_i, x, s) - \psi_i]$. If $G_1 \geq G_2$, then $G_2(\psi_1, x, s) - \psi_1 \leq 0$ so that $\psi_1 \geq \psi_2$. Hence, $\Upsilon G_1 \geq \Upsilon G_2$. ■

For a fixed $q \in \mathcal{Q}(\mathcal{S})$ and $p = q(x, s)$, let the operator $T_q : D(\mathcal{S}) \to D(\mathcal{S})$ be defined by

$$\begin{aligned}\xi^{n+1}(x, s) &= \left(T_q \xi^n\right)(x, s)\\ &= \max\left[U'(xp)p, \beta \int_S \xi^n\left\{x + \frac{1}{p} H\left[\frac{\left(T_2\xi^n\right)(x, s)}{p}, \theta\right], s'\right\} F(s, ds')\right].\end{aligned}$$
(A.12)

THEOREM A.2. *Let* $q \in \mathcal{Q}(\mathcal{S})$ *be fixed and let* $T_q : D(\mathcal{S}) \to D(\mathcal{S})$ *be defined by* (A.12). *Under Assumption* 1, T_q *is a contraction.*

https://doi.org/10.1017/S1365100597003064 Published online by Cambridge University Press

Proof. For an initial guess, $\xi^0 \in D(\mathcal{S})$,

$$\beta \int_S \xi^0 \left\{ x + \frac{1}{p} \left[H\left(\frac{\psi}{p}, \theta \right) \right], s' \right\} F(s, ds')$$

is an element of $D(\mathcal{S})$ for fixed ψ such that $0 \leq \psi < \infty$. Under the conditions of Lemma A.2, the solution

$$f^\star(x, s) = \max\left[U'(xp)p, \beta \int_S \xi^0 \left(x + \frac{1}{p} \left\{ H\left[\frac{f^\star(x, s)}{p}, \theta \right] \right\}, s' \right) F(s, ds') \right]$$

is also an element of $D(\mathcal{S})$.

Let $\xi_1, \xi_2 \in D(\mathcal{S})$ and assume that $\xi_1 > \xi_2$. For fixed ψ, it follows that

$$\begin{aligned} &\max\left[U'(xp)p, \beta \int_S \xi_1 \left\{ x + \frac{1}{p} \left[H\left(\frac{\psi}{p}, \theta \right) \right], s' \right\} F(s, ds') \right] \\ &\quad \geq \max\left[U'(xp)p, \beta \int_S \xi_2 \left\{ x + \frac{1}{p} \left[H\left(\frac{\psi}{p}, \theta \right) \right], s' \right\} F(s, ds') \right]. \end{aligned}$$

Under the conditions of Lemma A.2, it follows that

$$\begin{aligned} T_q\xi_1(x, s) &= \max\left[U'(xp)p, \beta \int_S \xi_1 \left(x + \frac{1}{p} \left\{ H\left[\frac{T_q\xi_1(x, s)}{p}, \theta \right] \right\}, s' \right) F(s, ds') \right] \\ &\geq \max\left[U'(xp)p, \beta \int_S \xi_2 \left(x + \frac{1}{p} \left\{ H\left[\frac{T_q\xi_2(x, s)}{p}, \theta \right] \right\}, s' \right) F(s, ds') \right]. \end{aligned}$$

Hence, $T_q\xi_1 \geq T_q\xi_2$ so that T_q is monotone.

Let $0 < a < \infty$. To show that T_q has the discounting property, notice that

$$\begin{aligned} &\beta \int_S (\xi + a) \left(x + \frac{1}{p} \left\{ H\left[\frac{(\xi + a)(x, s)}{p}, \theta \right] \right\}, s' \right) F(s, ds') \\ &\quad \leq \beta \int_S (\xi + a) \left(x + \frac{1}{p} \left\{ H\left[\frac{\xi(x, s)}{p}, \theta \right] \right\}, s' \right) F(s, ds') \\ &\quad \leq \beta \int_S \xi \left(x + \frac{1}{p} \left\{ H\left[\frac{\xi(x, s)}{p}, \theta \right] \right\}, s' \right) F(s, ds') + \beta a, \end{aligned}$$

so that T_q has the discounting property. Hence, T_q is a contraction with unique fixed point $\xi_q^\star$. ■

The next step is to study how the unique fixed point $\xi_q^\star$ varies with q. This is accomplished by studying how the fixed point varies as p varies, where $p = q(x, s)$ as before. If one starts in the subspace of $D(S)$ consisting of increasing and concave functions in q and shows that the operator T_q maps those functions into other functions in the same subspace, because of uniqueness, then we know that the fixed point is a function that is concave in q^4.

PROPOSITION A.2. *Under Assumption* 1, $\xi^\star$ *is increasing and concave in p.*

Proof. Fix $q \in \mathcal{Q}(\mathcal{S})$. Recall that

$$\frac{\partial a(x, q, s)}{\partial q} = -\frac{1}{q^2}\left[H\left(\frac{\psi}{q}, \theta\right) + \frac{\psi}{q} H_1\left(\frac{\psi}{q}, \theta\right)\right] < 0,$$

where $a(x, p, s) = x + (1/p)H[\psi/q, \theta(s)]$. Hence, the assumption that ξ lies in the space of functions that are decreasing in the first argument is consistent with the assumption that ξ lies in the subspace of functions that are increasing in p. Suppose that $\xi \in D(\mathcal{S})$ and that ξ is concave in p. For fixed s, define the function $\hat{G}$ by

$$\hat{G}(\psi, x, s, p) = \max\left\{U'(xp)p, \beta \int_S \xi\left[x + \frac{1}{p} H\left(\frac{\psi}{p}, \theta\right), s'\right] F(s, ds')\right\}.$$

Under Assumption 1, $U'(xp)p$ is increasing and concave in p and, by assumption, ξ is increasing and concave in p. Hence, $\hat{G}$ is the maximum of two increasing and concave functions in p and is itself increasing and concave in p. Let $p_1 < p_2$ and let ψ_i be the solution to $[\hat{G}(\psi_i, x, s, p_i) - \psi_i = 0]$. Then $0 = \hat{G}(\psi_1, s, x, p_1) - \psi_1 \leq \hat{G}(\psi_1, x, s, p_2) - \psi_1$, and hence $\psi_1 \leq \psi_2$. To show that $\hat{G}$ is concave in ψ, notice that $H' > 0$ and $H'' < 0$. Hence, the first argument of ξ is increasing and concave in ψ. A property of concave functions is that, if ξ is increasing and concave and if g is concave, then $\xi \circ g$ is concave. Because $\hat{G}$ is concave in (ψ, p), the solution to $[\hat{G}(\psi, x, s, p) - \psi = 0]$ is also concave.

For $q_2 \in \mathcal{Q}(\mathcal{S})$, let $(T_2\xi)$ be the solution ψ to the equation $\hat{G}(\psi, x, s, p_2) - \psi = 0$. Let $\delta \in [0, 1]$ and let $x_1, x_2 \in X$. Define $\psi_1 = T_{q_1}\xi(x_1, s)$ and $\psi_2 = T_{q_2}\xi(x_2, s)$. Then, $\hat{G}(\psi_1, x_1, s, p_1) - \psi_1 = \hat{G}(\psi_2, x_2, s, p_2) - \psi_2 = 0$. Because $\hat{G}$ is jointly concave in (ψ, p),

$$\hat{G}[\delta T_{q_1}\xi(x_1, s) + (1-\delta)T_{q_2}\xi(x_2, s), \delta x_1 + (1-\delta)x_2, x, s, \delta p_1 + (1-\delta)p_2]$$
$$- [\delta T_{q_1}\xi(x_1, s) + (1-\delta)T_{q_2}\xi(x_2, s)] \geq 0.$$

Because $\hat{G}$ is increasing in its first argument, it follows that $T_q\xi$ is concave in q if ξ is concave in q. ■

Although we have found a fixed point $\xi_q^\star$ for a given q and determined the market clearing price $\hat{q}$ for given ξ_1 and ξ_2, we have not shown that $q = \hat{q}$. In fact, we must address the issues of whether a solution exists and, if it exists, whether it is unique. Let $\Omega_i q$ denote the fixed point of T_q for type-i agents evaluated at (x, s), where x denotes the equilibrium holdings of type-1 agents. Let ν satisfy $\nu = \hat{q}[s, \Omega_1 q(x, s), \Omega_2 q(x, s)]$, where $\hat{q}$ was defined in Theorem A.1, or

$$\alpha_1\left\{H\left[\frac{\Omega_1 q(x, s)}{\nu}, \theta_1(s)\right]\right\} + \alpha_2\left\{H\left[\frac{\Omega_2 q(x, s)}{\nu}, \theta_2(s)\right]\right\} = 0. \qquad \textbf{(A.13)}$$

PROPOSITION A.3. *The market clearing price $\hat{q}$ is an increasing and concave function of $q \in \mathcal{Q}(\mathcal{S})$.*

Proof. Under the conditions of Proposition A.2 Ω_1 and Ω_2 are increasing and concave in q for $q \in \mathcal{Q}(\mathcal{S})$. The proof of Theorem A.1 can be used to show that $\hat{q}$ is jointly increasing and concave in its second and third arguments. A property of concave functions is that, if f and g are increasing and concave, then $f \circ g$ is increasing and concave. Hence, $\hat{q}$ is increasing and concave in q. ■

Let $\mathcal{Q}(\mathcal{S})$ be the space of continuous and bounded functions $\{q \in \mathcal{Q}(\mathcal{S}) : 0 < q(x,s) \leq B\}$ and define the operator $\Psi : \mathcal{Q}(\mathcal{S}) \to \mathcal{Q}(\mathcal{S})$ by

$$q(x,s) = \Psi q(x,s) = \hat{q}[s, \Omega_1 q(x,s), \Omega_2 q(x,s)]. \tag{A.14}$$

THEOREM A.3. *Under Assumption* 1, $\Psi : \mathcal{Q}(\mathcal{S}) \to \mathcal{Q}(\mathcal{S})$ *has a unique fixed point* $q^\star$.

Proof. Notice that if

$$\alpha_1\left\{H\left[\frac{\Omega_1 q(x,s)}{q}, \theta_1(s)\right]\right\} + \alpha_2\left\{H\left[\frac{\Omega_2 q(x,s)}{q}, \theta_2(s)\right]\right\} > 0,$$

then $\nu > q$, whereas if

$$\alpha_1\left\{H\left[\frac{\Omega_1 q(x,s)}{q}, \theta_1\right]\right\} + \alpha_2\left\{H\left[\frac{\Omega_2 q(x,s)}{q}, \theta_2\right]\right\} < 0,$$

then $\nu < q$. Recall that $U'(xq)q$ is increasing in q and that $(1/q)H(\psi/q)$ is decreasing in q. Because U is increasing, concave, and bounded, $U'(c)c$ is bounded above by $\bar{U}$. For a fixed ψ such that $0 < \psi < \infty$ and fixed x, s we have

$$\lim_{q\to\infty} \Omega q = \beta \int_S \xi^\star\left\{x + \frac{1}{q}H\left[\frac{\psi}{q}, \theta(s)\right], s'\right\} F(s, ds') = \bar{G}.$$

As $q \to \infty$, $H[\bar{G}/q, \theta(s)] \to -\infty$. Then, there is some $\bar{Q}$ such that, for any $\tilde{q}(x,s) \geq \bar{Q}$,

$$\alpha_1\left\{H\left[\frac{\bar{G}}{\tilde{q}(x,s)}, \theta_1(s)\right]\right\} + \alpha_2\left\{H\left[\frac{\bar{G}}{\tilde{q}(x,s)}, \theta_2(s)\right]\right\} > 0$$

and $\Psi q \leq \bar{Q} \leq \tilde{q}$. As $q \to 0$, $H(\psi/q) \to L$ and $x' \to \infty$, so that $\xi \to \underline{G}$. As $\underline{G}/q \to \infty$, $H \to L$. Hence, there is some $\epsilon > 0$ such that

$$\alpha_1\left\{H\left[\frac{\underline{G}}{\epsilon}, \theta_1(s)\right]\right\} + \alpha_2\left\{H\left[\frac{\underline{G}}{\epsilon}, \theta_2(s)\right]\right\} > 0,$$

so that $\Psi 0 > 0$. Hence, $\Psi : [0, \bar{U}] \to [0, \bar{U}]$. Brower's Fixed-point Theorem then can be used to show that there exists a fixed point $q = \Psi q$. To establish uniqueness, $\hat{q}$ is concave in its second and third arguments. Because $\Psi(0) > 0$, $\Psi\bar{Q} < \bar{Q}$, and Ψ is concave in q, the fixed point is unique. ■

The fixed point $\Psi q(x,s)$ was constructed holding the state (x,s) fixed, so that we can define the function $q(x,s)$. This function has the properties that markets clear, and the fixed point of the marginal valuation function was constructed holding the function q fixed. This is the unique stationary equilibrium.