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Interval oscillation criteria for self-adjoint matrix Hamiltonian systems

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By using a monotonic functional on a suitable matrix space, some new oscillation criteria for self-adjoint matrix Hamiltonian systems are obtained. They are different from most known results in the sense that the results of this paper are based on information only for a sequence of subintervals of $[t_0, \infty)$, rather than for the whole half-line. We develop new criteria for oscillations involving monotonic functionals instead of positive linear functionals or the largest eigenvalue. The results are new, even for the particular case of self-adjoint second-differential systems which can be applied to extreme cases such as $\lambda_{\max}[-\int_{t_0}^{\infty} C(s)\,\mathrm{d}s] = -\infty$.

1. Introduction

In this paper, we consider the following self-adjoint matrix Hamiltonian system

$$\left.\begin{aligned} X'(t) &= A(t)X(t) + B(t)Y(t), \\ Y'(t) &= C(t)X(t) - A^{\mathrm{T}}(t)Y(t), \end{aligned}\right\} \quad t \geqslant t_0, \tag{1.1}$$

where $X(t)$, $Y(t)$, $A(t)$, $B(t)$ and $C(t)$ are $n \times n$ real continuous matrix functions with $B(t) = B^{\mathrm{T}}(t) > 0$, $C(t) = C^{\mathrm{T}}(t)$. Here and below, the transpose of matrix M is denoted by M^{T} and its positive definiteness is denoted by $M > 0$. In what follows, we denote by $\mathcal{S}$ the subspace of all $n \times n$ symmetric matrices and by $E_n \in \mathcal{S}$ the identity matrix.

A solution $(X(t), Y(t))$ of system (1.1) is said to be non-trivial if $\det X(t) \neq 0$ for at least one $t \in [t_0, \infty)$ and a non-trivial solution $(X(t), Y(t))$ of (1.1) is said to be prepared or self-conjugate if

$$X^{\mathrm{T}}(t)Y(t) - Y^{\mathrm{T}}(t)X(t) = 0, \quad t \geqslant t_0. \tag{1.2}$$

System (1.1) is said to be oscillatory on $[t_0, \infty)$ if there is a non-trivial prepared solution $(X(t), Y(t))$ of (1.1) such that $\det X(t)$ vanishes at least once on $[T, \infty)$ for each $T \geqslant t_0$. Otherwise, it is said to be non-oscillatory. It is well known [14, theorem 8.1, p. 303] that if system (1.1) is oscillatory, then every non-trivial prepared solution $(\bar{X}(t), \bar{Y}(t))$ of (1.1) has the property that $\det \bar{X}(t)$ vanishes at least once on $[T, \infty)$ for every $T > t_0$.

The oscillatory properties of (1.1) are important in the optimization of certain functionals associated with (1.1). Therefore, such properties have been studied quite extensively. In particular, letting $A(t) \equiv 0$, $P(t) = B^{-1}(t)$ and $Q(t) = -C(t)$, the system (1.1) reduces to the self-adjoint system

$$(P(t)X'(t))' + Q(t)X(t) = 0, \tag{1.3}$$

and to

$$X''(t) + Q(t)X(t) = 0 \tag{1.4}$$

for $P(t) = E_n$. In these cases, for the matrix system (1.1) and less general system (1.3), as well as (1.4), oscillation has been the subject of study by several authors for many years, as seen in [1–6, 9, 10, 12–18] and references therein. Some results have concentrated on showing that (1.1) is oscillatory by applying a positive functional (see [10, 16–18]). Several others have involved the eigenvalues of the integrals of $A(t)$, $B(t)$ and $C(t)$ (see [1, 2, 4, 6, 13, 15] and references therein).

For any $K \in \mathcal{S}$, we assume its eigenvalues $\lambda_i[K]$, $i = 1, 2, \dots, n$, are ordered such that $\lambda_{\min}[K] = \lambda_n[K] \leqslant \cdots \leqslant \lambda_2[K] \leqslant \lambda_1[K] = \lambda_{\max}[K]$. It was conjectured by Hinton and Lewis [6] that (1.4) is oscillatory if

$$\lim_{t\to\infty} \lambda_{\max}\biggl[\int_0^t Q(s)\,\mathrm{d}s\biggr] = \infty.$$

This conjecture was partly proved by several authors and finally settled by Byers *et al*. [2]. The result has a more general form that is applicable to the system (1.4), (1.3) and (1.1) (see [1–4, 9, 10, 12, 13, 15]).

In 1987, Butler *et al*. [1] showed that the system (1.4) is oscillatory when

$$\limsup_{t\to\infty} \frac{1}{t}\int_a^t \lambda_{\max}[Q(s)]\,\mathrm{d}s = \infty,$$

provided that

$$\liminf_{t\to\infty} \frac{1}{t}\int_a^t \biggl(\operatorname{tr}\int_a^\tau Q(s)\,\mathrm{d}s\biggr)\mathrm{d}\tau > -\infty.$$

In 1993, Erbe *et al*. [4] proved that system (1.4) is oscillatory if

$$\limsup_{t\to\infty} \frac{1}{t^m}\lambda_{\max}\biggl[\int_a^t (t-s)^m Q(s)\,\mathrm{d}s\biggr]\mathrm{d}t = \infty \tag{1.5}$$

for some $m > 1$.

In 2003, by using a monotonic subhomogeneous functional q of degree μ on $\mathcal{S}$, Yang *et al*. [16] and Meng *et al*. [12] obtained oscillation theorems for (1.1) which extend those of Erbe *et al*. [4] and Kamenev [7], respectively. Other oscillation results based on the Wintner-type criterion for (1.1) and the special system (1.3) can also be found in the recent paper [18] and the references therein.

There are many other oscillation criteria. System (1.1) is obviously an extension of (1.3). On the one hand, we notice that, in the criteria given by [1–4, 6, 10, 13, 18], only the largest eigenvalue or the positive linear functional of some integrals is involved. It is not clear from these how oscillation is affected by the other eigenvalues and the other functionals. On the other hand, many papers involve $P(t)$ and the integral of $Q(t)$ and, hence, require the information integral of $Q(t)$ on the entire half-line $[t_0, \infty)$. But, from the Sturm separation theorem, if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0, \infty)$, as $a_i \to \infty$, such that for each i there exists a solution of the scalar equation (1.3) that has at least two zeros in $[a_i, b_i]$, then every solution of the scalar equation (1.3) is oscillatory, no matter how 'bad' the scalar equation (1.3) is (or P and Q are) on the remaining part of $[t_0, \infty)$. In [3,8,9,15,17], however, some interval criteria were established for the oscillation of (1.3) and for second-order scalar differential equations. However, these oscillation criteria in the literature are unable to be used to study the matrix Hamiltonian system (1.1). Results in [3, 8, 9, 15, 17] are substantially different from the existing results for oscillation. Moreover, the conditions of all theorems in Meng *et al.* [12] involve a fundamental matrix solution $\Phi(t)$ of the system $x' = A(t)x$, which cannot be solved in a closed form except for in a few special cases. In view of these facts, it is therefore of interest to find new oscillation criteria for (1.1).

This paper establishes some new oscillation criteria for the self-adjoint Hamiltonian matrix system (1.1) which improve the previous results, even for second-order ordinary differential equations. We approach our goal by means of a somewhat general monotonic functional q, as well as a new matrix Riccati-type transformation containing commonly used generalized Riccati transformations $W(t) = \{v(YX^{-1} + fB^{-1})\}(t)$ and the other generalized Riccati transformation $W(t) = (\mu Y X^{-1})(t)$, where $v(t) = \exp\{\int^t f(s)\,\mathrm{d}s\}$. Our results will reveal that not only the positive linear functional but also the nonlinear functional may be used to determine the oscillation. In particular, the theorems obtained in this paper further replace the conditions in the theorems of [12] by conditions independent of any fundamental matrix solution, i.e. involving only the coefficient matrices A, B and C. Moreover, by choosing appropriate functionals q and averaging functions, we can present a series of explicit oscillation criteria. Thus, the results of this paper extend, improve and unify a number of existing results. For more details, see remarks 2.4, 2.9, 2.10 and 2.11.

This paper is organized as follows: § 2 states the main results, and their proofs are given in § 3.

To state some of our theorems we need the following definitions and a lemma.

Definition 1.1 (Meng and Mingarelli [12]; Yang and Cheng [16]). A functional q such that $\mathcal{S} \to \mathbb{R}$ is said to be monotonic (or non-decreasing) if $J - K \geqslant 0$ implies that $q[J] \geqslant q[K]$ for J, $K \in \mathcal{S}$.

Definition 1.2. A functional $q : \mathcal{S} \to \mathbb{R}$ is said to be subhomogeneous if $q[\lambda K] \leqslant \lambda q[K]$ whenever $K \in \mathcal{S}$ and $\lambda \geqslant 0$. Furthermore, a functional $q : \mathcal{S} \to \mathbb{R}$ is said to be subhomogeneous of degree μ if there exists a $\mu \in \mathbb{R}$ such that, for any $K \in \mathcal{S}$, any $\lambda \geqslant 1$, $q[\lambda K] \leqslant \lambda^\mu q[K]$.

The first part of definition 1.2 is found in [5] and the second in [12].

Definition 1.3 (Hartman [5]). A linear functional $\mathsf{L} : \mathcal{S} \to \mathbb{R}$ satisfying

$$\mathsf{L}[K + J] = \mathsf{L}[K] + \mathsf{L}[J], \qquad \mathsf{L}[\lambda K] = \lambda \mathsf{L}[K]$$

for K, $J \in \mathcal{S}$, $\lambda \in \mathbb{R}$, is said to be 'positive' if $\mathsf{L}[K] > 0$ whenever $K \in \mathcal{S}$ and $K > 0$.

Denote the eigenvalues of an $n \times n$ Hermitian matrix $K \in \mathcal{S}$ by

$$\lambda_1[K], \lambda_2[K], \dots, \lambda_n[K],$$

where $\lambda_{\min}[K] = \lambda_n[K] \leqslant \cdots \leqslant \lambda_2[K] \leqslant \lambda_1[K] = \lambda_{\max}[K]$. We may note that, because of the classical characterization of the eigenvalues of matrix in $\mathcal{S}$, the functional $q[K] = \lambda_i[K]$ $(i = 1, 2, \dots, n)$ is a monotonic functional which is traditionally called the 'eigenvalue' functional. On the other hand, it is readily verified that if $P \geqslant 0$ in $\mathcal{S}$, then the nonlinear functional $q[K] = \lambda_i[K + P]$ $(i = 1, 2, \dots, n)$ is also a monotonic functional and $\lambda_i[K + P] \geqslant \lambda_i[K]$. Furthermore, it is easy to see that the nonlinear trace functional on $\mathcal{S}$ defined by $q[K] = \operatorname{tr}[K + E_n]$ is also a monotonic functional. Moreover, a positive linear functional L is also a monotonic functional and $\mathsf{L}[J] \geqslant \mathsf{L}[K]$ for $J \geqslant K$ with $J, K \in \mathcal{S}$.

Lemma 1.4. *Let $\phi(t)$ and $\theta(t)$ be positive and smooth real-valued functions on $[t_0, \infty)$. Then the system (1.1) is oscillatory if and only if*

$$\left.\begin{aligned} U'(t) &= A(t)U(t) + \frac{\phi(t)}{\theta(t)}B(t)V(t) + \frac{1}{2}\left(\frac{\phi'}{\phi} + \frac{\theta'}{\theta}\right)(t)U(t), \\ V'(t) &= C_1(t)U(t) - A^{\mathrm{T}}(t)V(t) + \frac{1}{2}\left(\frac{\phi'}{\phi} + \frac{\theta'}{\theta}\right)(t)V(t) \end{aligned}\right\} \tag{1.6}$$

is oscillatory, where

$$C_1(t) = \frac{\theta(t)}{\phi(t)}\left\{C + \frac{\psi}{\theta}(B^{-1}A + A^{\mathrm{T}}B^{-1}) + \left(\frac{\psi}{\theta}B^{-1}\right)' - \frac{\psi^2}{\theta^2}B^{-1}\right\}(t), \tag{1.7}$$

with

$$\psi(t) = \frac{\theta(t)}{2}\left(\frac{\phi'}{\phi} - \frac{\theta'}{\theta}\right)(t) = \theta(t)\left\{\frac{1}{2}\left[\ln\frac{\phi(t)}{\theta(t)}\right]\right\}'.$$

Proof. Let us make a change of unknown variables:

$$U(t) = (\phi X)(t), \qquad V(t) = (\theta Y + \alpha B^{-1}X)(t). \tag{1.8}$$

Then system (1.1) becomes (1.6). □

Remark 1.5. Note that the ratios ψ/θ and ϕ/θ appear in (1.7), and

$$\frac{\psi(t)}{\theta(t)} = \left\{\frac{1}{2}\left[\ln\frac{\phi(t)}{\theta(t)}\right]\right\}'.$$

If we let

$$\omega(t) = \frac{\theta(t)}{\phi(t)} = \exp\left\{-2\int_{t_0}^{t} g(s)\,\mathrm{d}s\right\},$$

where $g \in C^1([t_0, \infty), \mathbb{R})$, then $\psi(t)/\theta(t) = g(t)$, $\psi(t)/\phi(t) = \omega(t)g(t)$, and $C_1(t)$ in (1.7) becomes

$$C_1(t) = \omega(t)\{C + g(B^{-1}A + A^{\mathrm{T}}B^{-1}) + (gB^{-1})' - g^2B^{-1}\}(t).$$

2. Kamenev-type theorems

Throughout this paper, we always assume that functions $\mu \in C^1([t_0,\infty),(0,\infty))$, $f \in C^1([t_0,\infty),\mathbb{R})$, ϕ and $\theta \in C^1([t_0,\infty),(0,\infty))$. Let

$$v(t) = \exp\biggl\{-2\int^t f(s)\,\mathrm{d}s\biggr\}, \qquad B_1(t) = \frac{\phi(t)}{\theta(t)}B(t), \quad t \geqslant t_0. \tag{2.1}$$

Motivated by the ideas of Kong [8] and Yang [15,17], we use $H(t,s)k(s)$ instead of $H(t,s)$ and state the following concept which will be used extensively in the remainder of the paper.

Definition 2.1. Let $D = \{(t,s) : t \geqslant s \geqslant t_0\}$, $D_0 = \{(t,s) : t > s > t_0\}$. The real-valued functions $(H,k,\mu) \in \mathcal{H}$ if there exist functions k, $\mu \in C^1([t_0,\infty),(0,\infty))$, $H \in C^1(D,\mathbb{R})$, h_1 and $h_2 \in C^1(D_0,\mathbb{R})$ satisfying the following conditions:

(H1) $H(t,s) \geqslant 0$ for $t > s \geqslant t_0$, and $H(t,t) = 0$ for $t > t_0$;

(H2) $\dfrac{\partial}{\partial t}(H(t,s)k(t)) + H(t,s)k(t)\dfrac{\mu'(t)}{\mu(t)} = h_1(t,s)$, for all $(t,s) \in D_0$;

(H3) $\dfrac{\partial}{\partial s}(H(t,s)k(s)) + H(t,s)k(s)\dfrac{\mu'(s)}{\mu(s)} = -h_2(t,s)$, for all $(x,s) \in D_0$.

For the case $(H(t,s),k(s),\mu(t)) = (H(t-s),1,1) \in \mathcal{H}$, we find that $h_1(t,s) = h_2(t,s) := h(t-s)$. The subclass of $\mathcal{H}$ containing such $(H(t-s),1,1)$ is denoted by $H \in \mathcal{H}_0$.

The main theorem of this paper is the following interval criterion for the oscillation of system (1.1).

Theorem 2.2. *Let $(H,k,\mu) \in \mathcal{H}$. Suppose that, for each $T \geqslant t_0$, there exist $b,c,d \in \mathbb{R}$ with $T \leqslant b < c < d$ such that one of the following two conditions holds.*

(I) *The following inequality is satisfied:*

$$\begin{aligned}\frac{1}{H(c,b)}\int_b^c H(s,b)k(s)T(s)\,\mathrm{d}s + \frac{1}{H(d,c)}\int_c^d H(d,s)k(s)T(s)\,\mathrm{d}s \\ > \frac{1}{H(c,b)}\int_b^c [-(J_1^{01}+J_1^{02})(s,b)]\,\mathrm{d}s + \frac{1}{H(d,c)}\int_c^d [-(J_2^{01}+J_2^{02})(d,s)]\,\mathrm{d}s.\end{aligned} \tag{2.2}$$

(II) *There exists a monotonic functional q on $\mathcal{S}$ such that either*

$$\begin{aligned}q\biggl[\frac{1}{H(c,b)}\int_b^c H(s,b)k(s)T(s)\,\mathrm{d}s + \frac{1}{H(d,c)}\int_c^d H(d,s)k(s)T(s)\,\mathrm{d}s\biggr] \\ > q\biggl[\frac{1}{H(c,b)}\int_b^c [-(J_1^{01}+J_1^{02})(s,b)]\,\mathrm{d}s + \frac{1}{H(d,c)}\int_c^d [-(J_2^{01}+J_2^{02})(d,s)]\,\mathrm{d}s\biggr]\end{aligned} \tag{2.3}$$

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or

$$
\begin{aligned}
q\biggl[&\frac{1}{H(c,b)}\int_b^c \{H(s,b)k(s)T(s)+J_1^{01}(s,b)\}\,\mathrm{d}s \\
&\quad+\frac{1}{H(d,c)}\int_c^d \{H(d,s)k(s)T(s)+J_2^{01}(d,s)\}\,\mathrm{d}s\biggr] \\
&>q\biggl[\frac{1}{H(c,b)}\int_b^c [-J_1^{02}(s,b)]\,\mathrm{d}s+\frac{1}{H(d,c)}\int_c^d [-J_2^{02}(d,s)]\,\mathrm{d}s\biggr], \qquad (2.4)
\end{aligned}
$$

where $T(t)=\mu(t)v(t)(-C_1-f[A+A^{\mathrm{T}}]+f^2B_1-f'E_n)(t)$ *and* $C_1(t)$ *is defined as in (1.7),*

$$
\begin{aligned}
J_1^{01}(s,t)&=H(s,t)k(s)v(s)\mu(s)[f(A+A^{\mathrm{T}})-A^{\mathrm{T}}B_1^{-1}A](s) \\
&\qquad+v(s)\mu(s)[\tfrac{1}{2}h_1(s,t)+H(s,t)k(s)f(s)][A^{\mathrm{T}}B_1^{-1}+B_1^{-1}A](s), \\
J_1^{02}(s,t)&=-v(s)\mu(s)[\{\tfrac{1}{2}(H(s,t)k(s))^{-1/2}h_1(s,t)+(H(s,t)k(s))^{1/2}f(s)\} \\
&\qquad\times B_1^{-1/2}(s)+(H(s,t)k(s))^{1/2}f(s)B_1^{1/2}(s)]^2 \qquad (2.5)
\end{aligned}
$$

and

$$
\begin{aligned}
J_2^{01}(t,s)&=H(t,s)k(s)v(s)\mu(s)[f(A+A^{\mathrm{T}})-A^{\mathrm{T}}B_1^{-1}A](s) \\
&\qquad-v(s)\mu(s)[\tfrac{1}{2}h_2(t,s)+H(t,s)k(s)f(s)][A^{\mathrm{T}}B_1^{-1}+B_1^{-1}A](s), \\
J_2^{02}(t,s)&=-v(s)\mu(s)[\{\tfrac{1}{2}(H(t,s)k(s))^{-1/2}h_2(t,s)+(H(2t,s)k(s))^{1/2}f(s)\} \\
&\qquad\times B_1^{-1/2}(s)-(H(t,s)k(s))^{1/2}f(s)B_1^{1/2}(s)]^2. \qquad (2.6)
\end{aligned}
$$

Then system (1.1) is oscillatory.

By applying theorem 2.2 to the subclass $\mathcal{H}_0$ of $\mathcal{H}$, we get the following results.

THEOREM 2.3. *Assume for any* $A_0 \geqslant a$, *that there exists* $H\in\mathcal{H}_0$ *such that one of the following two conditions holds.*

(I) *For* $A_0 \leqslant b < c$,

$$
\int_b^c H(s-b)\{T_0(s)+T_0(2c-s)\}\,\mathrm{d}s > \int_b^c \{-(J_{10}^{01}+J_{10}^{02})(s-b)-(J_{20}^{01}+J_{20}^{01})(2c-s)\}\,\mathrm{d}s. \qquad (2.7)
$$

(II) *There exists a monotonic functional* q *on* $\mathcal{S}$ *satisfying the condition that, for* $A_0 \leqslant b < c$, *either*

$$
\begin{aligned}
q\biggl[&\int_b^c H(s-b)\{T_0(s)+T_0(2c-s)\}\,\mathrm{d}s\biggr] \\
&>q\biggl[-\int_b^c \{(J_{10}^{01}+J_{10}^{02})(s-b)+(J_{20}^{01}+J_{20}^{01})(2c-s)\}\,\mathrm{d}s\biggr] \qquad (2.8)
\end{aligned}
$$

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or

$$q\bigg[\int_b^c \{H(s-b)[T_0(s)+T_0(2c-s)]+[J_{10}^{01}(s-b)+J_{20}^{01}(2c-s)]\}\,\mathrm{d}s\bigg] > q\bigg[-\int_b^c \{J_{10}^{02}(s-b)+J_{20}^{02}(2c-s)\}\,\mathrm{d}s\bigg], \tag{2.9}$$

where $T_0(t) = v(t)(-C_1 - f[A+A^{\mathrm{T}}] + f^2 B_1 - f' E_n)(t)$ *and* $C_1(t)$ *is defined as in (1.7),*

$$\begin{aligned} J_{10}^{01}(s-b) &= H(s-b)v(s)[f(A+A^{\mathrm{T}}) - A^{\mathrm{T}} B_1^{-1} A](s) \\ &\quad + v(s)[\tfrac{1}{2}h(s-b) + H(s-b)f(s)][A^{\mathrm{T}} B_1^{-1} + B_1^{-1} A](s), \\ J_{10}^{02}(s-b) &= -v(s)[\{\tfrac{1}{2}(5H(s-b))^{-1/2}h(s-b) + (6H(s-b))^{1/2} f(s)\} B_1^{-1/2}(s) \\ &\quad + (7H(s-b))^{1/2} f(s) B_1^{1/2}(s)]^2 \end{aligned}$$

and

$$\begin{aligned} J_{20}^{01}(2c-s) &= H(s-b)v(2c-s)[f(A+A^{\mathrm{T}}) - A^{\mathrm{T}} B_1^{-1} A](2c-s) - v(2c-s) \\ &\quad \times [\tfrac{1}{2}h(s-b) + H(s-b)f(2c-s)][A^{\mathrm{T}} B_1^{-1} + B_1^{-1} A](2c-s), \\ J_{20}^{02}(2c-s) &= -v(2c-s) \\ &\quad \times [\{\tfrac{1}{2}(H(s-b))^{-1/2}h(s-b) + (H(s-b))^{1/2} f(2c-s)\} \\ &\quad \times B_1^{-1/2}(2c-s) - (H(s-b))^{1/2} f(2c-s) B_1^{1/2}(2c-s)]^2. \end{aligned}$$

Then system (1.1) is oscillatory.

Remark 2.4. Assume that $K, P \in \mathcal{S}$ and $P > 0$. Let $q[K]$ be a monotonic functional. Take $q[K] = \lambda_i[K]$ $(i = 1, 2, \dots, n)$ or $\mathsf{L}[K]$ as in theorems 2.2(II) or 2.3, respectively, with the other conditions unchanged. Then system (1.1) is oscillatory. Also, if we take the nonlinear functional $q[K] = \lambda_i[K+P]$ $(i = 1, 2, \dots, n)$ or $\mathrm{tr}[K + E_n]$ in theorems 2.2(II) or 2.3 with the other conditions unchanged, then system (1.1) is still oscillatory.

Next consider theorem 2.2 with special functions such as $k(t)$, $\mu(t)$, $f(t)$, $H(t,s)$ and special monotonic functional such as q. Then, we apply theorem 2.2 to get the following interesting theorems.

Theorem 2.5. *Let* $(H, k, \mu) \in \mathcal{H}$. *Suppose, for each* $l \geqslant t_0$, *one of the following two conditions holds.*

(I) *The inequalities are both satisfied:*

$$\limsup_{t\to\infty} \int_l^t \{H(s,l)k(s)T(s) + (J_1^{01} + J_1^{02})(s,l)\}\,\mathrm{d}s > 0 \tag{2.10}$$

and

$$\limsup_{t\to\infty} \int_l^t \{H(t,s)k(s)T(s) + (J_2^{01} + J_2^{02})(t,s)\}\,\mathrm{d}s > 0. \tag{2.11}$$

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(II) *There exists a positive linear functional* L *satisfying*

$$\limsup_{t\to\infty}\int_l^t\{H(s,l)k(s)\mathsf{L}[T(s)]+\mathsf{L}[(J_1^{01}+J_1^{02})(s,l)]\}\,\mathrm{d}s>0 \tag{2.12}$$

and

$$\limsup_{t\to\infty}\int_l^t\{H(t,s)k(s)\mathsf{L}[T(s)]+\mathsf{L}[(J_2^{01}+J_2^{02})(t,s)]\}\,\mathrm{d}s>0, \tag{2.13}$$

where $T(t)=\mu(t)v(t)(-C_1-f[A+A^{\mathrm{T}}]+f^2B_1-f'E_n)(0t)$ *and* $C_1(1t)$ *is defined as in (1.7),* $J_1^{01}(3s,t)$ *and* $J_1^{02}(4s,t)$ *are defined by (2.5),* $J_2^{01}(t,s)$ *and* $J_2^{02}(t,s)$ *by (2.6). Then (1.1) is oscillatory.*

Now, we choose $H(t,s)=(t-s)^{\alpha}$ for $\alpha>1$ and $\mu(t)=k(t)=1$, $f(t)\equiv 0$ for $t\geqslant t_0$. Then $H\in\mathcal{H}_0$ and $h(t-s)=\alpha(t-s)^{\alpha-1}$. Based on theorem 2.2, we obtain the following result.

THEOREM 2.6. *Let* B *be a constant matrix and* $B_1^{-1}(t)\leqslant B$ *for* $t\geqslant t_0$. *Then (1.1) is oscillatory provided that, for each* $l\geqslant t_0$ *and for some* $\alpha>1$, *one of the following conditions holds.*

(I) *The following two inequalities hold:*

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha-1}}\int_l^t\{(t-s)^{\alpha}M(s)+\tfrac{1}{2}\alpha(t-s)^{\alpha-1}N(s)\}\,\mathrm{d}s>\frac{\alpha^2}{4(\alpha-1)}B \tag{2.14}$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha-1}}\int_l^t\{(s-l)^{\alpha}M(s)-\tfrac{1}{2}\alpha(s-l)^{\alpha-1}N(s)\}\,\mathrm{d}s>\frac{\alpha^2}{4(\alpha-1)}B. \tag{2.15}$$

(II) *There exists a monotonic functional* q *such that*

$$\limsup_{t\to\infty}q\biggl[\frac{1}{t^{\alpha-1}}\int_l^t\{(t-s)^{\alpha}M(s)+\tfrac{1}{2}\alpha(t-s)^{\alpha-1}N(s)\}\,\mathrm{d}s\biggr]>q\biggl[\frac{\alpha^2}{4(\alpha-1)}B\biggr] \tag{2.16}$$

and

$$\limsup_{x\to\infty}q\biggl[\frac{1}{t^{\alpha-1}}\int_l^t\{(s-l)^{\alpha}M(s)-\tfrac{1}{2}\alpha(s-l)^{\alpha-1}N(s)\}\,\mathrm{d}s\biggr]>q\biggl[\frac{\alpha^2}{4(\alpha-1)}B\biggr], \tag{2.17}$$

where $M(s)=-(C_1+A^{\mathrm{T}}B_1^{-1}A)(s)$, $N(s)=-(A^{\mathrm{T}}B_1^{-1}+B_1^{-1}A)(s)$ *and* $C_1(t)$ *defined as in (1.7).*

Define

$$R(t)=\int_l^t\frac{\theta(s)}{\phi(s)}\,\mathrm{d}s,\quad t\geqslant l\geqslant t_0,$$

and let

$$H(t,s)=[R(t)-R(s)]^{\alpha},\quad t\geqslant t_0,$$

where $\alpha>1$ is a constant.

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THEOREM 2.7. *Suppose that $\lim_{t\to\infty} R(t) = \infty$ holds. Let B_0 be a constant matrix and $B^{-1}(t) \leqslant B_0$ for $t \geqslant t_0$. Then system (1.1) is oscillatory provided that, for each $l \geqslant t_0$ and for some $\alpha > 1$, one of the following two conditions holds.*

(I) *The following two inequalities are satisfied:*

$$\limsup_{t\to\infty} \frac{1}{R^{\alpha-1}(t)} \int_l^t \{[R(t)-R(s)]^\alpha M(s) + \tfrac{1}{2}\alpha[R(t)-R(s)]^{\alpha-1} N_0(s)\}\,\mathrm{d}s > \frac{\alpha^2}{4(\alpha-1)} B_0 \quad (2.18)$$

and

$$\limsup_{t\to\infty} \frac{1}{R^{\alpha-1}(t)} \int_l^t \{[R(s)-R(l)]^\alpha M(s) - \tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1} N_0(s)\}\,\mathrm{d}s > \frac{\alpha^2}{4(\alpha-1)} B_0. \quad (2.19)$$

(II) *There exists a positive linear functional* L *such that*

$$\limsup_{t\to\infty} \frac{1}{R^{\alpha-1}(t)} \int_l^t \{[R(t)-R(s)]^\alpha \mathsf{L}[M(s)] + \tfrac{1}{2}\alpha[R(t)-R(s)]^{\alpha-1} \mathsf{L}[N_0(s)]\}\,\mathrm{d}s > \frac{\alpha^2}{4(\alpha-1)} \mathsf{L}[B_0] \quad (2.20)$$

and

$$\limsup_{x\to\infty} \frac{1}{R^{\alpha-1}(t)} \int_l^t \{[R(s)-R(l)]^\alpha \mathsf{L}[M(s)] - \tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1} \mathsf{L}[N_0(s)]\}\,\mathrm{d}s > \frac{\alpha^2}{4(\alpha-1)} \mathsf{L}[B_0], \quad (2.21)$$

where $M(s) = -(C_1 + A^{\mathrm{T}} B_1^{-1} A)(s)$, $N_0(s) = -(A^{\mathrm{T}} B^{-1} + B^{-1} A)(s)$ and $C_1(t)$ defined as in (1.7).

From the above oscillation criteria, we can obtain different sufficient conditions for the oscillation of all prepared solutions of system (1.1) by choosing $H(t,s) = (t-s)^\alpha$ for $\alpha > 1$.

THEOREM 2.8. *Let B_0 be a constant matrix and $B_1^{-1}(t) \leqslant B$ for $t \geqslant t_0$. Then system (1.1) is oscillatory provided that, for each $l \geqslant t_0$ and for some $\alpha > 1$, one of the following two conditions holds.*

(I) *The following inequality is satisfied:*

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t [(s-l)^\alpha (M(s) + M(2t-s)) + \tfrac{1}{2}\alpha (s-l)^{\alpha-1} (N(s) - N(2t-s))]\,\mathrm{d}s > \frac{\alpha^2}{2(\alpha-1)} B. \quad (2.22)$$

(II) *There exists either a monotonic subhomogeneous functional q or a positive linear functional q on $\mathcal{S}$ such that*

$$\limsup_{t\to\infty}\frac{1}{t^{\alpha-1}}q\biggl\{\int_l^t[(s-l)^\alpha(M(s)+M(2t-s)) + \tfrac{1}{2}\alpha(s-l)^{\lambda-1}(N(s)-N(2t-s))]\,\mathrm{d}s\biggr\} > \frac{\alpha^2}{2(\alpha-1)}q[B], \tag{2.23}$$

where $M(s) = -(C_1 + A^{\mathrm{T}}B_1^{-1}A)(s)$, $N(s) = -(A^{\mathrm{T}}B_1^{-1} + B_1^{-1}A)(s)$ and $C_1(t)$ defined as in (1.7).

Remark 2.9. Our results improve and generalize earlier results by Kumari and Umamaheswaram [10, theorems 2.3–2.9], Meng and Mingarelli [12], Yang and Cheng [16] for (1.1) and Kong [8,9], Meng *et al*. [13], Erbe *et al*. [4] for (1.3) or its special cases, and many existing results for the scalar system $x''(t) + q(t)x(t) = 0$.

Remark 2.10. Theorems 2.6–2.8 give an improvement of the Kamenev criterion [7] to the Hamiltonian system (1.1). If we compare theorems 2.6–2.8 with the Kamenev-type condition (1.5), we can see that the former is an essential improvement of the latter. In particular, taking $n = 1$, $A(t) = 0$, $B(t) = 1$ and $C(t) = -\gamma/t^2$, from (1.1), it then follows the Euler equation

$$x''(t) + \frac{\gamma}{t^2}x(t) = 0. \tag{2.24}$$

It is well known that (2.24) is oscillatory for $\gamma > \frac{1}{4}$ and non-oscillatory for $\gamma \leqslant \frac{1}{4}$. Applying theorem 2.6 to (2.24), we see that (2.24) is oscillatory for $\gamma > \frac{1}{4}$ (see example 2.14, below). This implies that our results are sharper.

Remark 2.11. Although the conditions in theorems 2.2, 2.3 and 2.5–2.8 seem to be more complicated than the previous ones, with appropriate use of the functions H, k, f, μ, ϕ, θ and functional q and from theorems 2.2, 2.3 and 2.5–2.8 we can derive a number of oscillation criteria. For example, we choose $\mu(t) = k(t) = 1$ for $t \geqslant t_0$. Then theorems 2.2, 2.3 and 2.5 can reduce to many interesting conclusions. Furthermore, let $\phi(t) = \theta(t) = 1$ for $t \geqslant t_0$. Then $C_1(t) = C(t)$ for $t \geqslant t_0$. Therefore, from theorems 2.2, 2.3 and 2.5–2.8, many other corollaries follow. Moreover, by applying theorems 2.2, 2.3 and 2.5–2.8 in the special cases considered, one can derive many new criteria for the oscillation of the system (1.1) by varying the positive linear functional and the nonlinear functionals as considered. The details are omitted here.

The following example illustrates theorem 2.3. It is easy to see that [10, theorem 2.2] is not applicable in our example. Furthermore, the results in [1–9,13,15,17] are also not suitable since $A(t) \not\equiv 0$ or $n \not\equiv 1$.

Example 2.12. Consider the four-dimensional Hamiltonian system (1.1), where

$$A(t) = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad C(t) = -\begin{bmatrix} q_1(t)-2 & 0 \\ 0 & q_2(t)-1 \end{bmatrix}, \quad t \geqslant 0, \tag{2.25}$$

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and X and Y are $n\times 2$ matrix functions of t on $[0,\infty)$, and $q_i(t)$ $(i=1,2)$ are given as follows:

$$q_i(t)=\begin{cases}5(t-3n), & 3n\leqslant t\leqslant 3n+1,\\ 5(-t+3n+2), & 3n+1<t\leqslant 3n+2,\\ \xi_i(t), & 3n+2<t\leqslant 3n+3,\end{cases}$$

$n\in N_0=\{0,1,2,3,\dots\}$, $\xi_i(t)$ $(i=1,2)$ are arbitrary functions such that $q_i(t)$ is continuous.

Nevertheless, $\xi_i(t)$ can be selected as a 'bad' term $C(t)$ such that either the integral $\int_0^\infty q_i(s)\,\mathrm{d}s$ does not exist, or

$$\lambda_{\max}\biggl[-\int_0^\infty C(s)\,\mathrm{d}s\biggr]=-\infty.$$

So the results of [1–18] cannot apply to the system containing the coefficient (2.25). In fact, the system containing the coefficient (2.25) is oscillatory by theorem 2.3. We now give a proof.

Proof. Choose $\phi(t)=\theta(t)=1$, $f(t)=0$, $\mu(t)=k(t)=1$ and $H(t-s)=(t-s)^2$. Then $v(t)=1$, $B_1^{-1}(t)=B(t)$, $C_1(t)=C(t)$ and $h(t,s)=2(t-s)$. After a simple computation, it follows that

$$(C+A^{\mathrm{T}}B^{-1}A)(s)=\operatorname{diag}(q_1(s)-2,q_2(s)-1),\qquad (A^{\mathrm{T}}B^{-1}+B^{-1}A)(s)=0,$$
$$H(s-b)T_0(s)+J_{01}^{10}(s-b)=\operatorname{diag}(q_1(s),q_2(s)),$$
$$H(s-b)T_0(2c-s)+J_{02}^{10}(2c-s)=\operatorname{diag}(q_1(2c-s),q_2(2c-s))$$

and

$$J_{01}^{20}(s-b)=-\operatorname{diag}(1,\tfrac{1}{2}),\qquad J_{02}^{20}(2c-s)=-\operatorname{diag}(1,\tfrac{1}{2}).$$

For any $l>0$ there exists $n\in N_0$ such that $3n>l$. Let $b=3n$, $c=3n+1$ and $q[K]=\lambda_{\max}[K]$. By some simple computations, we find that (2.9) holds. Then, by theorem 2.3(II), the system containing the coefficient (2.25) is oscillatory. □

Example 2.13. Let a be a constant. Consider the four-dimensional system (1.1), where

$$A(t)=\begin{bmatrix}0 & a\\ -a & 0\end{bmatrix},\quad B(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\quad C(t)=-\begin{bmatrix}a^2+\frac{3}{4} & 0\\ 0 & a^2-\frac{5}{4}\end{bmatrix},\quad t\geqslant 0, \tag{2.26}$$

and X and Y are 2×2 matrix functions of t on $[0,\infty)$. With theorem 2.5, the verification of oscillation for the system containing the coefficient (2.26) is valid. Let $\phi(t)\equiv\theta(t)\equiv k(t)\equiv 1$, $f(t)\equiv\frac{1}{2}$, $\mu(t)=\mathrm{e}^t$ and $H(t,s)=(t-s)^2$. Then $v(t)\equiv\mathrm{e}^{-t}$, $B_1(t)=E_n$, $C_1(t)=C(t)$, $h_1(t,s)=(t-s)(2+t-s)$, $h_2(t,s)=(t-s)(2-t+s)$ and

$$T(t)=\mu(t)v(t)(-C_1-f(A^{\mathrm{T}}+A)+f^2B_1-f'E_n)(t)=-C+\tfrac{1}{4}B,$$
$$(A^{\mathrm{T}}B^{-1}A)(s)=a^2E_2.$$

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Define $\mathsf{L}[K] = k_{11}$, where $K = (k_{ij}) \in \mathcal{S}$. By some simple computations, we obtain

$$\begin{aligned}\limsup_{t\to\infty} \int_l^t \{H(s,l)k(s)\mathsf{L}[T(s)] + \mathsf{L}[(J_1^{01} + J_1^{02})(s,l)]\}\,\mathrm{d}s \\ &= \limsup_{t\to\infty} \int_l^t \{(s-l)^2\mathsf{L}[\mathrm{diag}(1,-1)] - \tfrac{1}{4}(2+s-l)^2\mathsf{L}[\mathrm{diag}(1,1)]\}\,\mathrm{d}s \\ &= \limsup_{t\to\infty}\{\tfrac{1}{4}(t-l)^3 - \tfrac{1}{2}(t-l)^2 - (t-l)\} \\ &> 0\end{aligned}$$

and

$$\begin{aligned}\limsup_{t\to\infty} \int_l^t \{H(t,s)k(s)\mathsf{L}[T(s)] + \mathsf{L}[(J_2^{01} + J_2^{02})(t,s)]\}\,\mathrm{d}s \\ = \limsup_{t\to\infty}\{\tfrac{1}{4}(t-l)^3 + \tfrac{1}{2}(t-l)^2 - (t-l)\} > 0.\end{aligned}$$

It follows from theorem 2.5(II) that the system containing the coefficient (2.26) is oscillatory. In fact, let $a = 0$. Then

$$(X(t), Y(t)) = \left(\begin{bmatrix} \cos\frac{\sqrt{3}}{2}t & \sin\frac{\sqrt{3}}{2}t \\ \exp\{\frac{\sqrt{5}}{2}t\} & \exp\{\frac{\sqrt{5}}{2}t\}\end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{3}}{2}\sin\frac{\sqrt{3}}{2}t & \frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}}{2}t \\ \frac{\sqrt{5}}{2}\exp\{\frac{\sqrt{5}}{2}t\} & \frac{\sqrt{5}}{2}\exp\{\frac{\sqrt{5}}{2}t\}\end{bmatrix}\right)$$

is such an oscillatory solution of the system containing the coefficient (2.26). However, if $a = \pm\frac{1}{2}$ in the system containing the coefficient (2.26), then we have $\mathrm{tr}[-C(t)] = 0$.

The following example uses theorem 2.6.

Example 2.14. Let γ_i ($1 \leqslant i \leqslant n$) be constants and $\gamma_1 > \frac{1}{4}$. Consider the $2n$-dimensional system (1.1), where

$$A(t) = [0]_{n\times n}, \qquad B(t) \geqslant E_n, \qquad C(t) = -\,\mathrm{diag}\left(\frac{\gamma_1}{t^2}, \frac{\gamma_2}{t^2}, \dots, \frac{\gamma_n}{t^2}\right), \tag{2.27}$$

and X and Y are $n \times n$ matrix functions in t on $[1,\infty)$. The previous results in [1–16, 18] are relatively difficult to calculate for the case (2.27). However, with theorem 2.6, the verification of oscillation for the system containing the coefficient (2.27) is valid. Choose $\phi(t) \equiv \theta(t) \equiv 1$. Then $B_1(t) \equiv B(t)$ and $C_1(t) = C(t)$. Furthermore, it is easy to see $0 < B^{-1}(t) \leqslant E_n$, $M(t) = -C(t)$ and $N(t) = 0$.

Define $q[K] = \mathsf{L}[K] = k_{11}$, where $K = (k_{ij}) \in \mathcal{S}$. By theorem 2.6(II), the system containing the coefficient (2.27) is oscillatory.

For $\alpha > 1$, we may get theorem 2.6(II) if

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t (s-l)^\alpha \mathsf{L}[-C(s)]\,\mathrm{d}s > \frac{\alpha^2}{4(\alpha-1)}\mathsf{L}[E_n] \tag{2.28}$$

and

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t (t-s)^\alpha \mathsf{L}[-C(s)]\,\mathrm{d}s > \frac{\alpha^2}{4(\alpha-1)}\mathsf{L}[E_n]. \tag{2.29}$$

Indeed, (2.28) and (2.29) are equivalent to

$$\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t (s-l)^\alpha \frac{\gamma_1}{s^2}\,\mathrm{d}s = \frac{\gamma_1}{\alpha-1} > \frac{\alpha^2}{4(\alpha-1)} \quad \text{or} \quad \gamma_1 > \tfrac{1}{4}\alpha^2 > \tfrac{1}{4}.$$

3. Proofs of the main results

Before proving theorem 2.2, we need the following lemmas. We believe that the following lemmas, which will be used in establishing oscillation criteria for (1.1), are of independent interest.

Lemma 3.1. *Let $(U(t), V(t))$ be a prepared solution of (1.6) such that $\det U(t) \neq 0$ on $[t_0, \infty)$. Then for all $\mu \in C^1([t_0,\infty),(0,\infty))$ and $f \in C^1([t_0,\infty),\mathbb{R})$, the matrix function $W(x)$ on $[t_0,\infty)$ defined by*

$$W(t) = \mu(t)v(t)[V(t)U^{-1}(t) + f(t)E_n] = \mu(t)Z(t) \tag{3.1}$$

satisfies the Riccati system

$$W'(t) = \left(\frac{\mu'}{\mu}W\right)(t) - \left[\frac{1}{\mu v}WB_1W + A^{\mathrm{T}}W + WA - f(WB_1 + B_1W - 2W)\right](t) - T(t), \tag{3.2}$$

where $T(t) = \mu(t)v(t)(-C_1 - f[A + A^{\mathrm{T}}] + f^2B_1 - f'E_n)(t)$ and $C_1(t)$ is defined as in (1.7).

Proof. From (1.2), (1.8) and (3.1) it follows that

$$\begin{aligned} W^{\mathrm{T}}(t) &= \mu(t)v(t)\{[U^{-1}]^{\mathrm{T}}V^{\mathrm{T}} + fE_n\}(t) \\ &= \mu(t)v(t)\{[\phi^{-1}X^{-1}]^{\mathrm{T}}[\theta Y^{\mathrm{T}} + \alpha X^{\mathrm{T}}B^{-1}] + fE_n\}(t) \\ &= \mu(t)v(t)\{\phi^{-1}\theta[X^{-1}]^{\mathrm{T}}X^{\mathrm{T}}YX^{-1} + \phi^{-1}\alpha[X^{-1}]^{\mathrm{T}}X^{\mathrm{T}}B^{-1} + fE_n\}(t) \\ &= \mu(t)v(t)\{(\theta Y + \alpha B^{-1}X)(\phi^{-1}X^{-1}) + fE_n\}(t) \\ &= \mu(t)v(t)\{VU^{-1} + fE_n\}(t) = W(t), \end{aligned}$$

which implies that W is Hermitian. Differentiating the Hermitian matrix (3.1) and invoking (1.6), we obtain

$$\begin{aligned} W'(t) &= \mu'(t)Z(t) + \mu(t)Z'(t) \\ &= \left(\frac{\mu'}{\mu}W\right)(t) \\ &\quad + (\mu v)(t)\left\{-\left[A^{\mathrm{T}}Z + ZA + \frac{1}{v}ZB_1Z\right](t) \right. \\ &\qquad\qquad \left. + (f[ZB_1 + B_1Z - 2Z])(t) - \left(\frac{1}{\mu}T\right)(t)\right\} \\ &= \left(\frac{\mu'}{\mu}W\right)(t) - \left[\frac{1}{\mu v}WB_1W + A^{\mathrm{T}}W + WA \right. \\ &\qquad\qquad \left. - f(WB_1 + B_1W - 2W)\right](t) - T(t) \end{aligned}$$

for $t \in [t_0, \infty)$. □

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Lemma 3.2. *Let $(U(t),V(t))$ be a prepared solution of (1.6) such that* $\det U(t)\neq 0$ *on $[c,d)\subset[t_0,\infty)$. For all $\mu\in C^1([t_0,\infty),(0,\infty))$, $f\in C^1([t_0,\infty),\mathbb{R})$, let*

$$W(t)=\mu(t)v(t)[V(t)U^{-1}(t)+f(t)E_n]=\mu(t)Z(t),\quad t\in[c,d).$$

Then, for any $(H,k,\mu)\in\mathcal{H}$,

$$\int_c^d H(d,s)k(s)T(s)\,\mathrm{d}s\leqslant H(d,c)k(c)W(c)+\int_c^d[-(J_2^{01}+J_2^{02})(d,s)]\,\mathrm{d}s,\tag{3.3}$$

where

$$T(t)=\mu(t)v(t)(-C_1-f[A+A^{\mathrm{T}}]+f^2B_1-f'E_n)(t),$$

$C_1(t)$ is defined as in (1.7), $J_2^{01}(d,s)$ and $J_2^{02}(d,s)$ are defined as in (2.6).

Proof. By lemma 3.1, $W(x)$ satisfies the Riccati system (3.2). Replacing t by s in (3.2) and then multiplying the subsequent equation by $H(t,s)k(s)$ and integrating with respect to s from c to t ($d>t\geqslant c$), and then after some simple computation, we have

$$\begin{aligned}\int_c^t &H(t,s)k(s)T(s)\,\mathrm{d}s\\ &=-\int_c^t H(t,s)k(s)W'(s)\,\mathrm{d}s+\int_c^t H(t,s)k(s)\frac{\mu'(s)}{\mu(s)}W(s)\,\mathrm{d}s\\ &\quad-\int_c^t H(t,s)k(s)\left[\frac{1}{\mu v}WB_1W+A^{\mathrm{T}}W+WA-f(WB_1+B_1W-2W)\right](s)\,\mathrm{d}s\\ &=H(t,c)k(c)W(c)-\int_c^t\left[-\frac{\partial}{\partial s}[H(t,s)k(s)]-H(t,s)k(s)\frac{\mu'(s)}{\mu(s)}\right]W(s)\,\mathrm{d}s\\ &\quad-\int_c^t H(t,s)k(s)\left[\frac{1}{\mu v}WB_1W+A^{\mathrm{T}}W+WA-f(WB_1+B_1W-2W)\right](s)\,\mathrm{d}s\\ &=H(t,c)k(c)W(c)-\int_a^x S^{-1}(s)G(t,s)S^{-1}(s)\,\mathrm{d}s,\end{aligned}\tag{3.4}$$

where

$$S(s)=\left[\frac{1}{\mu(s)v(s)}B_1(s)\right]^{1/2}$$

and

$$\begin{aligned}G(t,s)=H(t,s)k(s)\{[SWS][SWS]\}(s)+[h_2(t,s)+2H(t,s)k(s)f(s)](SWS)(s)\\ +H(t,s)k(s)\{S[A^{\mathrm{T}}W+WA-f(WB_1+B_1W)]S\}(s).\end{aligned}$$

Let

$$\begin{aligned}Q(t,s)=[H(t,s)k(s)]^{1/2}[SW-f\mu vS+S^{-1}A](s)\\ +\{\tfrac{1}{2}[H(t,s)k(s)]^{-1/2}h_2(t,s)+[H(t,s)k(s)]^{1/2}f(s)\}S^{-1}(s).\end{aligned}$$

Then

$$Q^{\mathrm{T}}(t,s) = [H(t,s)k(s)]^{1/2}[WS - f\mu v S + A^{\mathrm{T}}S^{-1}](s) + \{\tfrac{1}{2}[H(t,s)k(s)]^{-1/2}h_2(t,s) + [H(t,s)k(s)]^{1/2}f(s)\}S^{-1}(s).$$

Note that

$$\begin{aligned}
&[h_2(t,s) + 2f(s)H(t,s)k(s)]W(s) \\
&\quad + H(t,s)k(s)\bigg[\frac{1}{\mu v}WB_1W + A^{\mathrm{T}}W + WA - f(WB_1 + B_1W)\bigg](s) \\
&\quad = S^{-1}(s)G(t,s)S^{-1}(s) \\
&\quad = (Q^{\mathrm{T}}Q)(t,s) + H(t,s)k(s)v(s)\mu(s)[f(A + A^{\mathrm{T}}) - A^{\mathrm{T}}B_1^{-1}A](s) \\
&\qquad - v(s)\mu(s)\{\tfrac{1}{2}h_2(t,s) + H(t,s)k(s)f(s)\}[A^{\mathrm{T}}B_1^{-1} + B_1^{-1}A](s) \\
&\qquad - v(s)\mu(s)\{[\tfrac{1}{2}(H(t,s)k(s))^{-1/2}h_2(t,s) + (H(t,s)k(s))^{1/2}f(s)]B_1^{-1/2}(s) \\
&\qquad\qquad - (H(t,s)k(s))^{1/2}f(s)B_1^{1/2}(s)\}^2 \\
&\quad = (Q^{\mathrm{T}}Q)(t,s) + (J_2^{01} + J_2^{02})(t,s).
\end{aligned} \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$\int_c^t H(t,s)k(s)T(s)\,\mathrm{d}s = H(t,c)k(c)W(c) - \int_c^t \{(Q^{\mathrm{T}}Q)(t,s) + (J_2^{01} + J_2^{02})(t,s)\}\,\mathrm{d}s,$$

which implies that

$$\begin{aligned}
&\int_c^t H(t,s)k(s)T(s)\,\mathrm{d}s \\
&\qquad = H(t,c)k(c)W(c) + \int_c^t [-(J_2^{01} + J_2^{02})(t,s)]\,\mathrm{d}s - \int_c^t (Q^{\mathrm{T}}Q)(t,s)\,\mathrm{d}s \\
&\qquad \leqslant H(t,c)k(c)W(c) + \int_c^t [-(J_2^{01} + J_2^{02})(t,s)]\,\mathrm{d}s, \quad t \in [c,d).
\end{aligned} \tag{3.6}$$

Letting $t \to d^-$ in (3.6), we obtain (3.3). This completes the proof of lemma 3.2. □

LEMMA 3.3. *Let $(U(t),V(t))$ be a prepared solution of (1.6) such that* $\det U(t) \neq 0$ *on $[b,c) \subset [t_0,\infty)$. For all $\mu \in C^1([t_0,\infty),(0,\infty))$, $f \in C^1([t_0,\infty),\mathbb{R})$, let*

$$W(t) = \mu(t)v(t)[V(t)U^{-1}(t) + f(t)E_n] = \mu(t)Z(t), \quad t \in (b,c].$$

Then, for any $(H,k,\mu) \in \mathcal{H}$,

$$\int_b^c H(s,b)k(s)T(s)\,\mathrm{d}s \leqslant -H(c,b)k(c)W(c) + \int_b^c [-(J_1^{01} + J_1^{02})(s,b)]\,\mathrm{d}s, \tag{3.7}$$

where

$$T(t) = \mu(t)v(t)(-C_1 - f[A + A^{\mathrm{T}}] + f^2B_1 - f'E_n)(t),$$

$C_1(t)$ is defined as in (1.7), and $J_1^{01}(d,s)$ and $J_1^{02}(d,s)$ are defined as in (2.5).

Proof. Now go back to (3.2), replace t by s, multiply by $H(s,t)k(s)$, and integrate the result with respect to s from t to c for $t \in (b,c]$, to obtain

$$\begin{aligned}
&\int_t^c H(s,t)k(s)T(s)\,\mathrm{d}s \\
&= -\int_t^c H(s,t)k(s)W'(s)\,\mathrm{d}s + \int_t^c H(s,t)k(s)\frac{\mu'(s)}{\mu(s)}W(s)\,\mathrm{d}s \\
&\quad - \int_t^c H(s,t)k(s)\Big\{\frac{1}{\mu v}WB_1W + [A^{\mathrm{T}}W + WA - f(WB_1 + B_1W) - 2W]\Big\}(s)\,\mathrm{d}s \\
&= -H(c,t)k(c)W(c) + \int_t^c [h_1(s,t) + 2f(s)H(s,t)k(s)]W(s)\,\mathrm{d}s \\
&\quad - \int_t^c H(s,t)k(s)\Big\{\frac{1}{\mu v}WB_1W + [A^{\mathrm{T}}W + WA - f(WB_1 + B_1W)]\Big\}(s)\,\mathrm{d}s \\
&= -H(c,t)k(c)W(c) - \int_t^c \{(Q_1^{\mathrm{T}}Q_1)(s,t) + (J_1^{01} + J_1^{02})(s,t)\}\,\mathrm{d}s, \qquad (3.8)
\end{aligned}$$

where

$$R(s) = \left(\frac{1}{\mu(s)v(s)}B_1(s)\right)^{1/2}$$

and

$$\begin{aligned}
Q_1(s,t) &= [H(s,t)k(s)]^{1/2}[SW - f\mu vS + S^{-1}A](s) \\
&\quad - [\tfrac{1}{2}[H(s,t)k(s)]^{-1/2}h_1(s,t) + [H(s,t)k(s)]^{1/2}f(s)]S^{-1}(s).
\end{aligned}$$

Note that

$$\begin{aligned}
(Q_1^{\mathrm{T}}Q_1)(s,t) &= H(s,t)k(s)\Big[\frac{1}{\mu v}WB_1W + A^{\mathrm{T}}W + WA - f(WB_1 + B_1W)\Big](s) \\
&\quad - [h_1(s,t) + 2f(s)H(s,t)k(s)]W(s) - J_1(s,t),
\end{aligned}$$

where $J_1(s,t)$ is defined as in (2.5). It follows from (3.8) that

$$\begin{aligned}
\int_t^c H(s,t)k(s)T(s)\,\mathrm{d}s &= -H(c,t)k(c)W(c) \\
&\quad + \int_t^c [-(J_1^{01} + J_1^{02})(s,t)]\,\mathrm{d}s - \int_t^c (Q_1^{\mathrm{T}}Q_1)(s,t)\,\mathrm{d}s \\
&\leqslant -H(c,t)k(c)W(c) + \int_t^c [-(J_1^{01} + J_1^{02})(s,t)]\,\mathrm{d}s.
\end{aligned}$$

Letting $t \to b^+$ in the above inequality, we obtain (3.7). The proof of lemma 3.3 is complete. □

LEMMA 3.4. *Let $(H,k,\mu) \in \mathcal{H}$. For any prepared solution $(U(t),V(t))$ of (1.6), $\det U(t) \neq 0$ in the open interval (b,d). Then, for any constant $c \in (b,d) \subset [t_0,\infty)$,*

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$$\frac{1}{H(c,b)}\int_b^c H(s,b)k(s)T(s)\,\mathrm{d}s+\frac{1}{H(d,c)}\int_c^d H(d,s)k(s)T(s)\,\mathrm{d}s$$
$$\leqslant \frac{1}{H(c,b)}\int_b^c[-(J_1^{01}+J_1^{02})(s,b)]\,\mathrm{d}s+\frac{1}{H(d,c)}\int_c^d[-(J_2^{01}+J_2^{02})(d,s)]\,\mathrm{d}s, \tag{3.9}$$

or there exists a monotonic functional q on $\mathcal{S}$ such that

$$q\biggl[\frac{1}{H(c,b)}\int_b^c H(s,b)k(s)T(s)\,\mathrm{d}s+\frac{1}{H(d,c)}\int_c^d H(d,s)k(s)T(s)\,\mathrm{d}s\biggr]$$
$$\leqslant q\biggl[\frac{1}{H(c,b)}\int_b^c[-(J_1^{01}+J_1^{02})(s,b)]\,\mathrm{d}s+\frac{1}{H(d,c)}\int_c^d[-(J_2^{01}+J_2^{02})(d,s)]\,\mathrm{d}s\biggr], \tag{3.10}$$

where $T(t)=\mu(t)v(t)(-C_1-f[A+A^{\mathrm{T}}]+f^2B_1-f'E_n)(t)$, $C_1(t)$ is defined as in (1.7), and $J_1(s,t)$ and $J_2(t,s)$ are defined as in (2.5) and (2.6), respectively.

Proof. From lemma 3.2 and dividing both sides of (3.3) by $H(b,c)$, it follows that

$$\frac{1}{H(c,b)}\int_b^c H(s,b)k(s)T(s)\,\mathrm{d}s\leqslant -k(c)W(c)+\frac{1}{H(c,b)}\int_b^c[-(J_1^{01}+J_1^{02})(s,b)]\,\mathrm{d}s. \tag{3.11}$$

By lemma 3.3 and a similar argument to that above, we obtain

$$\frac{1}{H(d,c)}\int_c^d H(d,s)k(s)T(s)\,\mathrm{d}s\leqslant k(c)W(c)+\frac{1}{H(d,c)}\int_c^d[-(J_2^{01}+J_2^{02})(d,s)]\,\mathrm{d}s. \tag{3.12}$$

Then, adding (3.11) and (3.12), it follows that (3.9) holds.

Furthermore, by using (3.9) and the monotonic functional q on $\mathcal{S}$, we obtain (3.10). The proof is complete. □

COROLLARY 3.5. *Let $(H,k,\mu)\in\mathcal{H}$. Suppose also that, for sufficiently large $A_0\geqslant t_0$, there exist b, c, $d\in\mathbb{R}$ and $A_0\leqslant b<c<d$ satisfying either case (I) or case (II) of theorem 2.2. Then, for any prepared solution $(U(t),V(t))$ of (1.6), $\det U(t)$ has at least one zero in (b,d).*

Proof of Corollary 3.5. Assume that there is a prepared solution $(U(t),V(t))$ of the system (1.6) such that $\det U(t)\neq 0$ for $t\in(b,d)$. This allows us to make a transformation (3.1), that is,

$$W(t)=\mu(t)v(t)[V(t)U^{-1}(t)+f(t)E_n]=\mu(t)Z(t).$$

From lemma 3.4, it follows that (3.9) holds. We then reach a contradiction to (2.2). Similarly, we can obtain (3.10), which contradicts (2.3). □

Proof of theorem 2.2. The proof is by contradiction. First, we prove the case (2.2).

Without loss of generality, one can assume that there is a prepared solution $(X(t),Y(t))$ of the system (1.1) such that $\det X(t)\neq 0$ for $t\in[T,\infty)$ with $T\geqslant t_0$. Furthermore, it follows that there is a prepared solution $(U(t),V(t))$ of the system (1.6) such that $\det U(t) \neq 0$ for $t \in [T, \infty)$ with $T \geqslant t_0$. Pick a sequence $\{T_i\} \subset [T, \infty)$ such that $T_i \to \infty$ as $i \to \infty$. From (2.2), we know that, for each $i \in N$, there exist $b_i, c_i, d_i \in \mathbb{R}$ such that $T_i \leqslant b_i < c_i < d_i$, and (2.2) holds where b, c and d are replaced by b_i, c_i and d_i, respectively. From corollary 3.5, it follows that $\det U(t)$ has at least one zero t_i, $t_i \in (b_i, d_i)$ $(i = 1, 2, \dots)$. Noting that $t_i \geqslant b_i \geqslant T_i$, $i \in N$, we see that $\det U(t)$ has arbitrarily large zeros. Therefore, for the case (2.2), (1.6) is oscillatory. Moreover, by lemma 1.4, (1.1) is oscillatory.

In a manner similar to the above argument, for the case (2.3), we can show that (1.1) is also oscillatory. □

Proof of theorem 2.3. (I) Without loss of generality, one can assume that there is a prepared solution $(X(t), Y(t))$ of the system (1.1) such that $\det X(t) \neq 0$ for $t \in [T, \infty)$ with $T \geqslant t_0$. Furthermore, there is a prepared solution $(U(t), V(t))$ of the system (1.6) such that $\det U(t) \neq 0$ for $t \geqslant T \geqslant t_0$. By lemma 3.4, we find that if $H \in \mathcal{H}_0$ and $\det U(t) \neq 0$ in the open interval (b, d), then for any constant $c \in (b, d) \subset [t_0, \infty)$,

$$\begin{aligned} \frac{1}{H(c-b)} \int_b^c H(s-b)k(s)T(s)\,\mathrm{d}s + \frac{1}{H(d-c)} \int_c^d H(d-s)k(s)T(s)\,\mathrm{d}s \\ \leqslant \frac{1}{H(c-b)} \int_b^c [-(J_1^{01} + J_1^{02})(s,b)]\,\mathrm{d}s + \frac{1}{H(d-c)} \int_c^d [-(J_2^{01} + J_2^{02})(d,s)]\,\mathrm{d}s. \end{aligned} \tag{3.13}$$

In particular, let $d = 2c - b$. Then $H(d-c) = H(c-b) = H(\frac{1}{2}(d-b))$, and for any $w \in L[b, d]$, we have

$$\left.\begin{aligned} \int_c^d H(d-s)w(s)\,\mathrm{d}s &= \int_b^c H(s-b)w(2c-s)\,\mathrm{d}s, \\ \int_c^d h^2(d-s)w(s)\,\mathrm{d}s &= \int_b^c h^2(s-b)w(2c-s)\,\mathrm{d}s. \end{aligned}\right\} \tag{3.14}$$

Hence, (3.13) and (3.14) imply that

$$\int_b^c H(s-b)\{T_0(s) + T_0(2c-s)\}\,\mathrm{d}s \leqslant \int_b^c \{-J_{10}(s-b) - J_{20}(2c-s)\}\,\mathrm{d}s, \tag{3.15}$$

which contradicts (2.7). Thus, $\det U(t)$ has at least one zero in the open interval $(b, 2c-b)$. Hence, the system (1.6) is oscillatory. It follows from lemma 1.4 that system (1.1) is oscillatory.

(II) In a manner similar to the proof of case (I), we can obtain (3.15). Applying the monotonic functional q to both sides of the inequality (3.15), it follows that

$$q\biggl[\int_b^c H(s-b)\{T_0(s) + T_0(2c-s)\}\,\mathrm{d}s\biggr] \leqslant q\biggl[\int_b^c \{-J_{10}(s-b) - J_{20}(2c-s)\}\,\mathrm{d}s\biggr], \tag{3.16}$$

which contradicts (2.8). Therefore, (1.1) is also oscillatory. □

Proof of theorem 2.5. (I) For any $T \geqslant t_0$, we set $b = T$. On choosing $l = b$ in (2.10), there exists $c > b$ such that

$$\int_b^c \{H(s,b)k(s)T(s) + (J_1^{01} + J_1^{02})(s,b)\}\,\mathrm{d}s > 0. \tag{3.17}$$

We choose $l = c$ in (2.11). There then exists $d > c$ such that

$$\int_c^d [H(d,s)k(s)T(s) + (J_2^{01} + J_2^{02})(d,s)]\,\mathrm{d}s > 0. \tag{3.18}$$

From (3.17) and (3.18), we obtain (2.2). By using theorem 2.2, system (1.1) is oscillatory.

(II) For any $T \geqslant t_0$, we set $b = T$. We choose $l = b$ in (2.12). Then by definition 1.3 there exists $c > b$ such that

$$\mathsf{L}\biggl\{\int_b^c [H(s,b)k(s)T(s) + (J_1^{01} + J_1^{02})(s,b)]\,\mathrm{d}s\biggr\} > 0. \tag{3.19}$$

Let $l = c$ in (2.13). Then there exists $b > c$ such that

$$\mathsf{L}\biggl\{\int_c^d [H(d,s)k(s)T(s) + (J_2^{01} + J_2^{02})(d,s)]\,\mathrm{d}s\biggr\} > 0. \tag{3.20}$$

It follows from (3.19) and (3.20) that (2.3) for the case $q = \mathsf{L}$ of theorem 2.2 holds. Based on theorem 2.2, we can conclude the proof. □

Proof of theorem 2.6. Choose $k(t) \equiv \mu(t) \equiv 1$ and $f(t) \equiv 0$. Then $v(t) \equiv 1$, $T(t) = -C_1(t)$,

$$\begin{aligned}
J_1^{01}(s,l) &= -[(s-l)^\alpha A^{\mathrm{T}} B_1^{-1} A + \tfrac{1}{2}\alpha(s-l)^{\alpha-1} N](s),\\
J_1^{02}(s,l) &= -\tfrac{1}{4}\alpha^2 (s-l)^{\alpha-2} B_1^{-1}(s),\\
J_2^{01}(t,s) &= -[(t-s)^\alpha A^{\mathrm{T}} B^{-1} A - \tfrac{1}{2}\alpha(t-s)^{\alpha-1} N](s),
\end{aligned}$$

and

$$J_2^{02}(t,s) = -\tfrac{1}{4}\alpha^2 (t-s)^{\alpha-2} B_1^{-1}(s).$$

Noting that

$$\lim_{t\to\infty} \frac{\alpha^2}{4t^{\alpha-1}} \int_l^t (s-l)^{\alpha-2}\,\mathrm{d}s = \frac{\alpha^2}{4(\alpha-1)}, \tag{3.21}$$

$$\lim_{t\to\infty} \frac{\alpha^2}{4t^{\alpha-1}} \int_l^t (t-s)^{\alpha-2}\,\mathrm{d}s = \frac{\alpha^2}{4(\alpha-1)}, \tag{3.22}$$

from (3.21) and (3.22), it follows that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{H(t,s)k(s)T(s) + (J_2^{01} + J_2^{02})(t,s)\}\,\mathrm{d}s\\
&= \limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{(t-s)^\alpha M(s) + \tfrac{1}{2}\alpha(t-s)^{\alpha-1} N(s) - \tfrac{1}{4}\alpha^2 (t-s)^{\alpha-2} B_1^{-1}(s)\}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{(t-s)^\alpha M(s) + \tfrac{1}{2}\alpha(t-s)^{\alpha-1}N(s) - \tfrac{1}{4}\alpha^2(t-s)^{\alpha-2}B\}\,\mathrm{d}s \\
&= \limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{(t-s)^\alpha M(s) + \tfrac{1}{2}\alpha(t-s)^{\alpha-1}N(s)\}\,\mathrm{d}s + \frac{\alpha^2}{4(\alpha-1)}B \\
&> 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{H(s,l)k(s)T(s) + (J_1^{01}+J_1^{02})(s,l)\}\,\mathrm{d}s \\
&= \limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{(s-l)^\alpha M(s) - \tfrac{1}{2}\alpha(s-l)^{\alpha-1}N(s) - \tfrac{1}{4}\alpha^2(s-l)^{\alpha-2}B^{-1}(s)\}\,\mathrm{d}s \\
&\geqslant \limsup_{t\to\infty} \frac{1}{t^{\alpha-1}} \int_l^t \{(s-l)^\alpha M(s) - \tfrac{1}{2}\alpha(s-l)^{\alpha-1}N(s)\}\,\mathrm{d}s - \frac{\alpha^2}{4(\alpha-1)}B \\
&> 0.
\end{aligned}
$$

Hence,

$$
\limsup_{t\to\infty} \int_l^t \{H(s,l)k(s)T(s) + (J_1^{01}+J_1^{02})(s,l)\}\,\mathrm{d}s > 0
$$

and

$$
\limsup_{t\to\infty} \int_l^t \{H(t,s)k(s)T(s) + (J_2^{01}+J_2^{02})(t,s)\}\,\mathrm{d}s > 0
$$

imply that (2.10) and (2.11) hold. From theorem 2.5(I), it follows that the system (1.1) is oscillatory.

(II) According to the linearity of the positive linear functional L and theorem 2.5(II), we can show that the system (1.1) is also oscillatory. The proof is similar to that of case (I). We omit the details. □

Proof of theorem 2.7. Set $f(t) \equiv 0$ and $k(t) \equiv \mu(t) \equiv 1$. It is easy to see that $v(t) \equiv 1$, $T(t) = -C_1(t)$,

$$
h_1(t,s) = \alpha[R(t)-R(s)]^{\alpha/2-1}\frac{\phi(t)}{\theta(t)}, \qquad h_2(t,s) = \alpha[R(t)-R(s)]^{\alpha/2-1}\frac{\phi(s)}{\theta(s)},
$$

$$
\begin{aligned}
&H(t,s)k(s)\mu(s)v(s)T(s) + (J_2^{01}+J_2^{02})(t,s) \\
&\quad = [R(t)-R(s)]^\alpha M(s) - \tfrac{1}{2}\alpha[R(t)-R(s)]^{\alpha-1}N_0(s) \\
&\qquad - \tfrac{1}{4}\alpha^2\frac{\phi(s)}{\theta(s)}[R(t)-R(s)]^{\alpha-2}B^{-1}(s)
\end{aligned}
$$

and

$$
\begin{aligned}
&H(s,l)k(s)\mu(s)v(s)T(s) + (J_1^{01}+J_1^{02})(s,l) \\
&\quad = [R(s)-R(l)]^\alpha M(s) + \tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1}N_0(s) \\
&\qquad - \tfrac{1}{4}\alpha^2\frac{\phi(s)}{\theta(s)}[R(s)-R(l)]^{\alpha-2}B^{-1}(s).
\end{aligned}
$$

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Noting that

$$\int_l^t [R(s)-R(l)]^{\alpha-2}\frac{\phi(s)}{\theta(s)}\,\mathrm{d}s = \frac{1}{\alpha-1}[R(t)-R(l)]^{\alpha-1}$$

and

$$\int_l^t [R(t)-R(s)]^{\alpha-2}\frac{\phi(s)}{\theta(s)}\,\mathrm{d}s = \frac{1}{\alpha-1}[R(t)-R(l)]^{\alpha-1},$$

in view of $\lim_{t\to\infty} R(t)=\infty$, it follows that

$$\lim_{t\to\infty}\frac{\alpha^2}{4R^{\alpha-1}(t)}\int_l^t [R(s)-R(l)]^{\alpha-2}\frac{\phi(s)}{\theta(s)}\,\mathrm{d}s = \frac{\alpha^2}{4(\alpha-1)} \tag{3.23}$$

and

$$\lim_{t\to\infty}\frac{\alpha^2}{4R^{\alpha-1}(t)}\int_l^t [R(t)-R(s)]^{\alpha-2}\frac{\phi(s)}{\theta(s)}\,\mathrm{d}s = \frac{\alpha^2}{4(\alpha-1)}. \tag{3.24}$$

(I) From (2.18) and (3.23), we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_l^t \{H(s,l)k(s)T(s)+(J_1^{01}+J_1^{02})(s,l)\}\,\mathrm{d}s\,Bs\\
&\quad=\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_l^t \biggl\{[R(s)-R(l)]^{\alpha}M(s)+\tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1}N_0(s)\\
&\qquad\qquad -\tfrac{1}{4}\alpha^2\frac{\phi(s)}{\theta(s)}[R(s)-R(l)]^{\alpha-2}B^{-1}(s)\biggr\}\,\mathrm{d}s\\
&\quad\geqslant\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_l^t \biggl\{[R(s)-R(l)]^{\alpha}M(s)+\tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1}N_0(s)\\
&\qquad\qquad -\tfrac{1}{4}\alpha^2\frac{\phi(s)}{\theta(s)}[R(s)-R(l)]^{\alpha-2}B_0\biggr\}\,\mathrm{d}s\\
&\quad=\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_l^t \{[R(s)-R(l)]^{\alpha}M(s)+\tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1}N_0(0s)\}\,\mathrm{d}s\\
&\qquad\qquad -\frac{\alpha^2}{4(1\alpha-1)}B_0\\
&\quad>0.
\end{aligned}$$

It follows that

$$\limsup_{t\to\infty}\int_l^t \{H(s,l)k(s)T(s)+(J_1^{01}+J_1^{02})(s,l)\}\,\mathrm{d}s>0$$

implies that (2.10) holds. Similarly, (2.19) implies that (2.11) holds. From case (I) of theorem 2.5, the system (1.1) is oscillatory.

(II) Analogously to case (I), from (2.20) and (3.23), it follows that

$$\begin{aligned}\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_l^t\{H(s,l)k(s)\mathsf{L}[T(s)]+\mathsf{L}[(J_1^{01}+J_1^{02})(s,l)]\}\,\mathrm{d}s\\ \geqslant\limsup_{t\to\infty}\frac{1}{R^{\alpha-1}(t)}\int_l^t\{[R(s)-R(l)]^{\alpha}\mathsf{L}[M(s)]\\ +\tfrac{1}{2}\alpha[R(s)-R(l)]^{\alpha-1}\mathsf{L}[N_0(s)]\}\,\mathrm{d}s-\frac{\alpha^2}{4(\alpha-1)}\mathsf{L}[B_0]\\ >0\end{aligned}$$

implies that

$$\limsup_{t\to\infty}\int_l^t\{H(s,l)k(s)\mathsf{L}[T(s)]+\mathsf{L}[(J_1^{01}+J_1^{02})(s,l)]\}\,\mathrm{d}s>0.$$

Therefore, (2.12) holds. Similarly, (2.21) implies that (2.13) holds. By using case (II) of theorem 2.5, the system (1.1) is also oscillatory. □

Proof of theorem 2.8. Choose $k(t)\equiv\mu(t)\equiv 1$ and $f(t)\equiv 0$. Then $v(t)\equiv 1$, $T_0(t)=-C_1(t)$,

$$\begin{aligned}J_{10}^{01}(s-b)&=-[(s-b)^{\alpha}(A^{\mathrm{T}}B_1^{-1}A)(s)-\tfrac{1}{2}\alpha(s-b)^{\alpha-1}N(s)],\\ J_{10}^{02}(s-b)&=-\tfrac{1}{4}\alpha^2(s-b)^{\alpha-2}B_1^{-1}(s),\\ J_{20}^{01}(2c-s)&=-[(s-l)^{\alpha}(A^{\mathrm{T}}B_1^{-1}A)(2c-s)+\tfrac{1}{2}\alpha(s-b)^{\alpha-1}N(2c-s)](2c-s),\end{aligned}$$

and

$$J_{20}^{02}(2c-s)=-\tfrac{1}{4}\alpha^2(s-b)^{\alpha-2}B_1^{-1}(2c-s).$$

(I) From (2.22) with (3.21), one knows that, for $l\geqslant T$,

$$\begin{aligned}\limsup_{t\to\infty}\int_l^t[(s-l)^{\alpha}(M(s)+M(2t-s))+\tfrac{1}{2}(N(s)-N(2t-s))]\,\mathrm{d}s\\ >\tfrac{1}{2}\alpha^2\int_l^t(s-l)^{\alpha-2}\,\mathrm{d}s\,B.\end{aligned}\tag{3.25}$$

Thus, for any $T\geqslant t_0$, there exist b, $c\in\mathbb{R}$ such that $T\leqslant b<c$ and

$$\begin{aligned}\int_b^c[(s-l)^{\alpha}(M(s)+M(2c-s))+\tfrac{1}{2}(N(s)-N(2c-s))]\,\mathrm{d}s\\ >\tfrac{1}{2}\alpha^2\int_b^c(s-l)^{\alpha-2}\,\mathrm{d}s\,B\\ =\int_b^c\tfrac{1}{2}\alpha^2(s-l)^{\alpha-2}B\,\mathrm{d}s\\ \geqslant\int_b^c\tfrac{1}{4}\alpha^2(s-l)^{\alpha-2}[B_1^{-1}(s)+B_1^{-1}(2c-s)]\,\mathrm{d}s,\end{aligned}$$

that is, (2.7) holds. By theorem 2.3(II), (1.1) is oscillatory.

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(II) By using (2.23) and (3.21), it follows that, for $l \geqslant T$,

$$\limsup_{t\to\infty} q\biggl\{\int_l^t [(s-l)^\alpha (M(s)+M(2t-s)) + \tfrac{1}{2}(N(s)-N(2t-s))]\,\mathrm{d}s\biggr\} > \biggl[\tfrac{1}{2}\alpha^2 \int_l^t (s-l)^{\alpha-2}\,\mathrm{d}s\biggr] q[B]. \quad (3.26)$$

Therefore, by the first part of definition 1.2 or definition 1.3, for any $T \geqslant t_0$, there exist b, $c \in \mathbb{R}$ satisfying $T \leqslant b < c$ and

$$\begin{aligned} q\biggl\{\int_b^c [(s-l)^\alpha (M(s)+M(2c-s)) + \tfrac{1}{2}(N(s)-N(2c-s))]\,\mathrm{d}s\biggr\} &> \biggl[\tfrac{1}{2}\alpha^2 \int_b^c (s-l)^{\alpha-2}\,\mathrm{d}s\biggr] q\{B\} \\ &\geqslant q\biggl\{\int_b^c \tfrac{1}{2}\alpha^2 (s-l)^{\alpha-2} B\,\mathrm{d}s\biggr\} \\ &\geqslant q\biggl\{\int_b^c \tfrac{1}{4}\alpha^2 (s-l)^{\alpha-2} [B_1^{-1}(s) + B_1^{-1}(2c-s)]\,\mathrm{d}s\biggr\}, \end{aligned}$$

that is, (2.9) holds. From theorem 2.3(II), we find that (1.1) is also oscillatory. The proof is therefore complete. □

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