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A UNIFIED METHOD FOR MAXIMAL TRUNCATED CALDERÓN–ZYGMUND OPERATORS IN GENERAL FUNCTION SPACES BY SPARSE DOMINATION

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Abstract In this note we give simple proofs of several results involving maximal truncated Calderón–Zygmund operators in the general setting of rearrangement-invariant quasi-Banach function spaces by sparse domination. Our techniques allow us to track the dependence of the constants in weighted norm inequalities; additionally, our results hold in $\mathbb{R}^n$ as well as in many spaces of homogeneous type.

Keywords: sparse domination; weighted inequalities; Banach function spaces; maximal truncations

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1. Introduction

Sparse domination has recently been an extremely active area of research in harmonic analysis. This technique dates back to Andrei Lerner from his alternative, simple proof of the A_2 theorem [**20**, **21**], proved originally by Hytönen [**14**]. Lerner is able to bound all *Calderón–Zygmund operators* by a supremum of a special collection of dyadic, positive operators called *sparse operators*. This bound led almost instantly to a proof of the sharp dependence of the constant in related weighted norm inequalities, the A_2 theorem, a problem that had been actively worked on for over a decade.

There have been many improvements to Lerner's techniques as well as extensions of his ideas to a wide range of spaces and operators. These results are too numerous to list fully, though we mention a few of them in our references; we refer the interested reader to the many recent papers and monographs involving sparse domination for more references and background. We could have made use of some of these improvements, such as [**6**, **8**, **19**, **22**], but since we are looking at weighted norm inequalities, Lerner's original technology also works.

In this paper, we concentrate on several of the results in the paper [9] involving the maximal truncated Calderón–Zygmund operator. Specifically, we study the behaviour of the maximal truncated Calderón–Zygmund operator on rearrangement-invariant Banach function spaces (RIBFSs) and rearrangement-invariant quasi-Banach function spaces (RIQBFS); we also show some modular inequalities. To bring our results into context, we recall a few definitions.

Let T be a Calderón–Zygmund operator in $\mathbb{R}^n$ with standard kernel K satisfying the following size and smoothness conditions.

(1) $|K(x,y)| \le c/(|x-y|^n)$, where $x \ne y$.

(2) There exists $0 < \delta \le 1$ such that

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \le c\frac{|x-x'|^\delta}{|x-y|^{n+\delta}},$$

where $|x-x'| \le |x-y|/2$ and c is some absolute constant.

(3) T is bounded on L^2.

Given a Calderón–Zygmund operator T, define its maximal truncated operator by

$$T^{**}f(x) = \sup_{0<\varepsilon_1<\varepsilon_2} \left| \int_{\varepsilon_1<|x-y|<\varepsilon_2} K(x,y)f(y)\,\mathrm{d}y \right|.$$

We say that a weight w belongs to the Muckenhoupt class $A_p, 1 < p < \infty$, if for every cube $Q \subset \mathbb{R}^n$,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,\mathrm{d}x\right)\left(\frac{1}{|Q|}\int_Q w(x)^{1-p'}\,\mathrm{d}x\right)^{p-1} \le [w]_{A_p}.$$

When $p = 1$, w belongs to A_1 if $Mw(x) \le [w]_{A_1} w(x)$ almost everywhere (a.e.). Moreover, we denote $A_\infty = \bigcup_{p\ge 1} A_p$. In this paper we use the Fujii–Wilson definition of the Muckenhoupt class A_∞. Namely, the weight w belongs to the class A_∞ if and only if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)}\int_Q M(w\chi_Q) < \infty,$$

where the supremum is taken with respect to all cubes in $\mathbb{R}^n$ whose sides are parallel to the axes.

In this language, the A_2 theorem states that for $w \in A_p$, $p > 1$,

$$\|T\|_{L^p(w)\mapsto L^p(w)} \le C(T,p)[w]_{A_p}^{\max\{1,1/(p-1)\}}$$

and the exponent is sharp. We refer the reader to the books [10, 11] for more information.

By using sparse domination, we show that under certain conditions, the following statements hold, with explicit dependence of the constant C on the weight w (which is detailed and discussed in the body of this paper):

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(i)

$$\|T^{**}f\|_{\mathbb{X}(w)} \le C\|f\|_{\mathbb{X}(w)},$$

where $\mathbb{X}$ is some RIBFS or RIQBFS (see Theorems 3.5 and 3.7 for a precise statement);

(ii)

$$\int_{\mathbb{R}^n} \phi(T^{**}f(x))w(x)\,\mathrm{d}x \le C\int_{\mathbb{R}^n} \phi(f(x))w(x)\,\mathrm{d}x,$$

where ϕ is an N-function and w is some Muckenhoupt weight.

As stated, additionally we track the dependence of the constants on the weight characteristic and provide some commentary. In particular, the dependence on the constant that we obtain improves on that in [**16**] in certain cases, even on the space $L^2(w)$ (see remarks following Theorem 3.7).

Our approach simplifies the original proof, which is done by using extrapolation [**9**]. Moreover, by taking the advantage of the sparse domination, we can track the constant C and study its dependence with respect to w. Finally this technique is general enough to hold in many spaces of homogeneous type (SHT). These are doubling measure spaces equipped with a quasimetric—more references and a precise definition are contained in [**2**]. For simplicity, we structure our results in $\mathbb{R}^n$, and we indicate throughout the note where additional steps are needed for SHT and what they are; we mention any restrictions on the space when they arise.

The structure of the paper is as follows: § 2 provides background, especially concerning RIQBFS; and § 3 includes our main results, proofs and remarks.

Throughout this paper, for $a, b \in \mathbb{R}$, $a \lesssim b$ ($a \gtrsim b$, respectively) means there exists a positive number C, which is independent of a and b, such that $a \le Cb$ ($a \ge Cb$, respectively).

2. Preliminaries

In this section, we collect several basic facts for RIBFSs, RIQBFSs and modular inequalities.

2.1. RIBFSs and RIQBFSs

Denote by $\mathcal{M}$ the set of measurable functions on $(\mathbb{R}^n, \mathrm{d}x)$ and by $\mathcal{M}^+$ the non-negative ones. A *rearrangement-invariant Banach norm* is a mapping $\rho : \mathcal{M}^+ \mapsto [0,\infty]$ such that the following properties hold.

(a) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e.; $\rho(f+g) \le \rho(f) + \rho(g)$; $\rho(af) = a\rho(f)$ for $a \ge 0$.

(b) If $0 \le f \le g$ a.e., then $\rho(f) \le \rho(g)$.

(c) If $f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$.

(d) If E is a measurable set such that $|E| < \infty$, then $\rho(\chi_E) < \infty$, and $\int_E f\,\mathrm{d}x \le C_E\rho(f)$, for some constant $0 < C_E < \infty$, depending on E and ρ, but independent of f.

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(e) $\rho(f)=\rho(g)$ if f and g are equimeasurable, that is, $d_f(\lambda)=d_g(\lambda), \lambda\geq 0$, where d_f (d_g, respectively) denotes the distribution function of f (g, respectively).

By means of ρ, the *RIBFS* is defined as

$$\mathbb{X}=\{f\in\mathcal{M}:\|f\|_{\mathbb{X}}:=\rho(|f|)<\infty\}.$$

Moreover, the *associate space of* $\mathbb{X}$ is the Banach function $\mathbb{X}'$ defined by

$$\mathbb{X}'=\left\{f\in\mathcal{M},\|f\|_{\mathbb{X}'}=\sup\left\{\int_{\mathbb{R}^n}fg\,\mathrm{d}x:g\in\mathcal{M}^+,\rho(g)\leq 1\right\}<\infty\right\}.$$

Note that in the present setting, $\mathbb{X}$ is an RIBFS if and only if $\mathbb{X}'$ is an RIBFS (see, for example, [**3**, Chapter 2 and Corollary 4.4]). For SHT, we require that the underlying space be *resonant* (that is, a σ-finite space that is completely non-atomic, or is atomic with all atoms having equal measure).

An important feature for these spaces is the Lorentz–Luxemburg theorem, which asserts that $\mathbb{X}=\mathbb{X}''$ and hence we have

$$\|f\|_{\mathbb{X}}=\sup\left\{\left|\int_{\mathbb{R}^n}fg\,\mathrm{d}x\right|:g\in\mathbb{X}',\|g\|_{\mathbb{X}'}\leq 1\right\}.$$

Recall that the decreasing rearrangement of f is the function f^* on $[0,\infty)$ defined by

$$f^*(t)=\inf\{\lambda\geq 0:d_f(\lambda)\leq t\},\quad t\geq 0.$$

It is well known that f^* is equimeasurable with f and hence by Luxemburg's representation theorem, there exists an RIBFS $\overline{\mathbb{X}}$ over $(\mathbb{R}^+,\mathrm{d}x)$, such that $f\in\mathbb{X}$ if and only if $f^*\in\overline{\mathbb{X}}$ with $\|f\|_{\mathbb{X}}=\|f^*\|_{\overline{\mathbb{X}}}$, that is, the mapping $f\mapsto f^*$ is an isometry. Furthermore, for associate space, we have $\overline{\mathbb{X}}'=\overline{\mathbb{X}'}$ and $\|f\|_{\mathbb{X}'}=\|f^*\|_{\overline{\mathbb{X}}'}$. We refer the reader to the book [**3**] for a detailed introduction to RIBFSs.

Let $w\in A_\infty$, $\mathbb{X}$ an RIBFS and $\overline{\mathbb{X}}$ as its corresponding space in $(\mathbb{R}^+,\mathrm{d}x)$. We consider the weighted version of the space $\mathbb{X}$ as follows:

$$\mathbb{X}(w)=\{f\in\mathcal{M}:\|f\|_{\mathbb{X}(w)}:=\|f_w^*\|_{\overline{\mathbb{X}}}<\infty\},$$

where $f_w^*(t)=\inf\{\lambda\geq 0:w_f(\lambda)\leq t\}$, $t\geq 0$, is the decreasing rearrangement induced by w_f, the distribution function of f with respect to the measure $w\,\mathrm{d}x$ (note that we need a resonant space to apply the representation theorem). It is known that $\mathbb{X}'(w)=\mathbb{X}(w)'$ (see [**9**]).

Next, we recall the *Boyd indices* of an RIBFS, which are closely related to some interpolation properties (see [**3**, Boyd's theorem]). Consider the dilation operator

$$D_tf(s)=f\left(\frac{s}{t}\right),\quad 0<t<\infty,\ \ f\in\overline{\mathbb{X}},$$

with norm

$$h_{\mathbb{X}}(t)=\|D_t\|_{\overline{\mathbb{X}}\mapsto\overline{\mathbb{X}}},\quad 0<t<\infty.$$

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Then the *lower and upper Boyd indices* are defined, respectively, by

$$p_{\mathbb{X}} = \lim_{t\to\infty} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \sup_{1<t<\infty} \frac{\log t}{\log h_{\mathbb{X}}(t)}, \quad q_{\mathbb{X}} = \lim_{t\to 0^+} \frac{\log t}{\log h_{\mathbb{X}}(t)} = \inf_{0<t<1} \frac{\log t}{\log h_{\mathbb{X}}(t)}.$$

We have that $1 \le p_{\mathbb{X}} \le q_{\mathbb{X}} \le \infty$, which follows from the fact that $h_{\mathbb{X}}(t)$ is submultiplicative, that is, $h_{\mathbb{X}}(ts) \le h_{\mathbb{X}}(t)h_{\mathbb{X}}(s)$, for all $s,t>0$. The relationship between the Boyd indices of $\mathbb{X}$ and $\mathbb{X}'$ is the following: $p_{\mathbb{X}'} = (q_{\mathbb{X}})'$ and $q_{\mathbb{X}'} = (p_{\mathbb{X}})'$, where p and p' are conjugate exponents (see, for example, [3, 23]).

For each $0<r<\infty$ and $\mathbb{X}$ an RIBFS, we consider the r exponent of $\mathbb{X}$, namely,

$$\mathbb{X}^r = \{f \in \mathcal{M} : |f|^r \in \mathbb{X}\},$$

with norm $\|f\|_{\mathbb{X}^r} = \||f|^r\|_{\mathbb{X}}^{1/r}$. Note that the definition of Boyd indices extends to $\mathbb{X}^r$: we have $p_{\mathbb{X}^r} = p_{\mathbb{X}} \cdot r$ and $q_{\mathbb{X}^r} = q_{\mathbb{X}} \cdot r$. It is known that if $\mathbb{X}$ is an RIBFS and $r \ge 1$, then $\mathbb{X}^r$ is still an RIBFS; however, for $0<r<1$, the space $\mathbb{X}^r$ is not necessarily Banach (see, for example, [9]). Hence, it is natural to consider the quasi-Banach case.

We start with the definition of the quasi-Banach function norm. Again, let $\rho' : \mathcal{M}^+ \mapsto [0,\infty)$ be a mapping. We say that ρ' is a *rearrangement-invariant quasi-Banach function norm* if ρ satisfies the defining conditions (a), (b), (c), (e) with the triangle inequality replaced by

$$\rho'(f+g) \le C(\rho'(f)+\rho'(g)),$$

where C is an absolute constant. Then, similarly, the *RIQBFS* is defined as the collection of all measurable functions such that $\rho'(|f|)<\infty$. In addition, for the purpose of making $\mathbb{X}^r$ become an RIBFS for some large power r, where $\mathbb{X}$ is some RIQBFS, we impose the following *p-convex* condition on $\mathbb{X}$ for $p>0$ (see, for example, [12]) by requiring

$$\left\| \left(\sum_{j=1}^{N} |f_j|^p \right)^{1/p} \right\|_{\mathbb{X}} \lesssim \left(\sum_{j=1}^{N} \|f_j\|_{\mathbb{X}}^p \right)^{1/p}.$$

Clearly, the p-convexity condition is equivalent to the fact that $\mathbb{X}^{1/p}$ is an RIBFS and, again by the Lorentz–Luxemburg theorem, we have

$$\|f\|_{\mathbb{X}} \simeq \sup\left\{ \left(\int_{\mathbb{R}^n} |f(x)|^p g(x)\,\mathrm{d}x \right)^{1/p} : g \in \mathcal{M}^+, \|g\|_{\mathbb{Y}'} \le 1 \right\},$$

where $\mathbb{Y}'$ is the associate space of the RIBFS $\mathbb{Y} = \mathbb{X}^{1/p}$. In a similar fashion, by using the fact that powers commute with f^*, we can define $\mathbb{X}(w)$ for an RIQBFS $\mathbb{X}$, $w \in A_\infty$ and $0<r<\infty$, and we have $\mathbb{X}(w)^r = \mathbb{X}^r(w)$.

Remark 2.1. We list some typical examples of RIBFSs and RIQBFSs here: the Lebesgue space L^p, the Lorentz space $L^{p,q}$, the Orlicz spaces L^ϕ, the Lorentz Γ-spaces $\Gamma^q(v)$ and the Marcinkiewicz spaces $\mathbb{M}_\varphi$. We refer the reader to the work [9] for a detailed introduction to these spaces, as well as their Boyd indices.

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2.2. Modular inequality

To set up our modular inequality results, we start recalling some basic properties of Young functions, as well as N-functions. Let Φ be the collection of all the functions $\phi : [0,\infty) \mapsto [0,\infty)$ satisfy the following conditions:

(1) ϕ is non-negative and increasing;

(2) $\phi(0^+) = 0$ and $\phi(\infty) = \infty$.

If $\phi \in \Phi$ is convex, then we say that ϕ is a *Young function*. Moreover, an *N-function* ϕ is a Young function such that

$$\lim_{t\to 0^+} \frac{\phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\phi(t)}{t} = \infty.$$

We say that $\phi \in \Phi$ is *quasi-convex* if there exist a convex function $\widetilde{\phi}$ and $a_1 \geq 1$ such that

$$\widetilde{\phi}(t) \leq \phi(t) \leq a_1 \widetilde{\phi}(a_1 t), \quad t \geq 0.$$

For a positive increasing function ϕ, we define the lower and upper dilation indices of ϕ, respectively, by

$$i_\phi = \lim_{t\to 0^+} \frac{\log h_\phi(t)}{\log t} = \sup_{0<t<1} \frac{\log h_\phi(t)}{\log t}, \quad I_\phi = \lim_{t\to\infty} \frac{\log h_\phi(t)}{\log t} = \inf_{1<t<\infty} \frac{\log h_\phi(t)}{\log t},$$

where

$$h_\phi(t) = \sup_{s>0} \frac{\phi(st)}{\phi(s)}, \quad t > 0.$$

Observe that $0 \leq i_\phi \leq I_\phi \leq \infty$. Moreover, as we mentioned before, the dilation indices are closely related to Boyd indices. More precisely, we have

$$p_{\mathbb{X}} = \frac{1}{I_\phi}, \quad q_{\mathbb{X}} = \frac{1}{i_\phi},$$

where $\mathbb{X}$ is the Marcinkiewicz space induced by ϕ (see [9]), while

$$p_{\mathbb{X}} = i_\phi, \quad q_{\mathbb{X}} = I_\phi,$$

where $\mathbb{X}$ is the Orlicz space induced by ϕ (see [7]).

The following Δ_2 condition is crucial. Given a function $\phi \in \Phi$, we say that ϕ satisfies the Δ_2 condition if ϕ is doubling, that is,

$$\phi(2t) \leq C\phi(t), \quad t > 0.$$

It is well known that if ϕ is quasi-convex, then $i_\phi \geq 1$, $\phi \in \Delta_2$ if and only if $I_\phi < \infty$ and $\overline{\phi} \in \Delta_2$ if and only if $i_\phi > 1$, where $\overline{\phi}(s) = \sup_{t>0}\{st - \phi(t)\}, s > 0$, is the *complementary function* of ϕ (see, for example, [25]). Here are some important properties of $\overline{\phi}$.

(1) (Young's inequality) $st \leq \phi(s) + \overline{\phi}(t)$, $s, t \geq 0$.

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(2) When ϕ is an N-function, $\overline{\phi}$ is also an N-function, and the following inequality holds:

$$t \leq \phi^{-1}(t)\overline{\phi}^{-1}(t) \leq 2t, \quad t \geq 0. \tag{2.1}$$

(3) If ϕ is an N-function, then there exists $0 < \alpha < 1$ such that ϕ^{α} is quasi-convex if and only if $\overline{\phi} \in \Delta_2$, where $\phi^{\alpha}(t) = \phi(t)^{\alpha}$.

We are now ready to define the modular inequality. Given $w \in A_\infty$ and $\phi \in \Phi$, we define the modular

$$\rho_w^{\phi}(f) = \int_{\mathbb{R}^n} \phi(|f(x)|)w(x)\,\mathrm{d}x.$$

The collection of functions

$$M_w^{\phi} = \{f : \rho_w^{\phi}(f) < \infty\}$$

is referred to as a modular space. A sublinear operator T satisfies a modular inequality on M_w^{ϕ} if there exist constants $c_1, c_2 > 0$ such that

$$\rho_w^{\phi}(Tf) \leq c_1 \rho_w^{\phi}(c_2 f),$$

and satisfies a weak modular inequality on M_w^{ϕ} if there exist $c_3, c_4 > 0$ such that

$$\sup_{\lambda} \phi(\lambda) w\{x \in \mathbb{R}^n : f(x) > \lambda\} \leq c_3 \sup_{\lambda} \phi(\lambda) w\{x \in \mathbb{R}^n : c_4 g(x) > \lambda\}.$$

Note that weighted modular estimates are not necessarily associated with Banach or quasi-Banach spaces and so duality cannot be used. Modular inequalities were originally developed as a means for providing endpoint estimates for certain operators, such as iterates of the Hardy–Littlewood maximal function [**9**].

3. Main result

We need some dyadic calculus from [**20**, **21**]. By a *dyadic grid* $\mathcal{D}$, we mean a collection of cubes with the following properties.

(i) For any $Q \in \mathcal{D}$, its sidelength ℓ_Q is of the form 2^k, $k \in \mathbb{Z}$.

(ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathcal{D}$.

(iii) The cubes of a fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

An important property for a dyadic grid is the three-lattice theorem. This asserts that there are 3^n dyadic grids $\mathcal{D}_\alpha$ such that for any cube $Q \subset R^n$ there exists a cube $Q_\alpha \in \mathcal{D}_\alpha$ such that $Q \subset Q_\alpha$ and $\ell_{Q_\alpha} \leq c_n l_Q$. Moreover, in [**5**], the author showed that the optimal number of the dyadic grids is $n+1$ (see [**5**, **17**, **20**] for a discussion).

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We say that $\mathcal{S} \subset \mathcal{D}$ is a *sparse family* of cubes if for every $Q \in \mathcal{S}$,

$$\left| \bigcup_{P\in\mathcal{S}, P\subsetneq Q} P \right| \le \frac{1}{2}|Q|.$$

Equivalently, if we define

$$E(Q) = Q \setminus \bigcup_{P\in\mathcal{S}, P\subsetneq Q} P,$$

then the sets $E(Q)$ are pairwise disjoint and $|E(Q)| \ge \frac{1}{2}|Q|$. Note that in general, the constant $\frac{1}{2}$ in the above definition can be replaced by any $\gamma \in (0, 1)$. However, we will use $\frac{1}{2}$ for simplicity. Note that the concept of dyadic grid has been well studied in SHT, as well as the analogue of the three-lattice theorem (called Mei's theorem; see, for example, [**2**, **15**, **22**]).

Given a dyadic grid $\mathcal{D}$ and a sparse family $\mathcal{S} \subset \mathcal{D}$, we define the *dyadic positive operator* $\mathcal{A}$ by

$$\mathcal{A}f(x) = \mathcal{A}_{\mathcal{D},\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} f_Q \chi_Q(x),$$

where $f_Q = 1/|Q| \int_Q f$. Moreover, given a measurable function f on $\mathbb{R}^n$ and a cube Q, we define the *median value* of f over Q by

$$m_f(Q) := \sup\left\{\lambda : \max\left\{\left|\left\{x \in Q : f(x) > \lambda\right\}\right|, \left|\left\{x \in \mathbb{Q} : f(x) < \lambda\right\}\right|\right\} \le |Q|/2\right\}.$$

An important property of this quantity is the following: if $f \in L^1$, then $|m_Q(T^{**}f)| \to 0$ as $|Q| \to \infty$. Indeed, by the proof of [**2**, Lemma 5.1], we see that

$$|m_Q(T^{**}f)| \le \frac{\|T^{**}f\|_{L^{1,\infty}(Q)}}{|Q|} \le \|T^{**}\|_{1,\infty} \frac{\|f\|_{L^1}}{|Q|},$$

where it is well known that $\|T^{**}\|_{1,\infty} < \infty$ (see, for example, [**11**, Theorem 4.2.4]). In SHT this is true as well, as long as $\mu(X) = \infty$, using the weak bound for T^{**} from [**13**] (note that they impose the Hörmander condition on their operator).

Finally, given any $a > 0$ and Q a cube, we denote by aQ the cube with the same centre of Q and sidelength $a\ell_Q$.

The following theorem is crucial.

Theorem 3.1. *Let T be a Calderón–Zygmund operator in $\mathbb{R}^n$ with standard kernel K (see the introduction) and $\mathcal{D}$ a dyadic grid. Then the following assertions hold.*

(1) *Let f be any measurable function on $\mathbb{R}^n$. For any $Q_0 \in \mathcal{D}$, there exists a sparse family $\mathcal{S} \subset \mathcal{D}$ such that for almost every $x \in Q_0$,*

$$|T^{**}f(x) - m_{Q_0}(T^{**}f)| \lesssim Mf(x) + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \mathcal{T}_{\mathcal{S},m}|f|(x), \tag{3.1}$$

where M is the Hardy–Littlewood maximal operator and

$$\mathcal{T}_{\mathcal{S},m}f(x) = \sum_{Q\in\mathcal{S}} f_{2^mQ}\chi_Q(x), \quad m \in \mathbb{N}.$$

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(2) *Let X be a Banach function space, that is, the very last condition (e) for RIBFSs is not required. Then*

$$\|Mf\|_{\mathbb{X}} \lesssim \sup_{\mathcal{D},\mathcal{S}} \|A_{\mathcal{D},\mathcal{S}} f\|_{\mathbb{X}}, \quad f \geq 0, \tag{3.2}$$

and for any $m \in \mathbb{N}$,

$$\sup_{\mathcal{S}\in\mathcal{D}} \|\mathcal{T}_{\mathcal{S},m} f\|_{\mathbb{X}} \lesssim m \sup_{\mathcal{D},\mathcal{S}} \|\mathcal{A}_{\mathcal{D},\mathcal{S}} f\|_{\mathbb{X}}, \quad f \geq 0. \tag{3.3}$$

In particular, we have for any Banach function space $\mathbb{X}$,

$$\|T^{**} f\|_{\mathbb{X}} \lesssim \sup_{\mathcal{D},\mathcal{S}} \|\mathcal{A}_{\mathcal{D},\mathcal{S}} |f|\|_{\mathbb{X}}. \tag{3.4}$$

(See [**20**].) This result holds in SHT by following the proofs in [**2**, **20**] for T, but substituting the sublinearity of T^{**} for the linearity of T.

3.1. Maximal truncated Calderón–Zygmund operator on RIBFSs and RIQBFSs

We start by considering the behaviour of T^{**} on RIBFSs and RIQBFSs.

Lemma 3.2. *Let $1 < p < \infty$ and let $w \in A_p$. Then $w \in A_{p-\varepsilon}$, where*

$$\varepsilon = \frac{p-1}{1 + 2^{n+1}[\sigma]_{A_\infty}}$$

in which $\sigma = w^{1-p'}$ is the dual weight. Furthermore

$$[w]_{A_{p-\varepsilon}} \leq 2^{p-1}[w]_{A_p}$$

*(see [**24**, Corollary 1.1.1 and Lemma 1.1.3]).*

Note that a version of this lemma is true in SHT; see [**18**].

Lemma 3.3. *Let $\mathbb{X}$ be an RIQBFS which is p-convex for some $0 < p \leq 1$. Then if $1 < p_{\mathbb{X}} \leq \infty$, then M is bounded on $\mathbb{X}(w)$ for all $w \in A_{p_{\mathbb{X}}}$. Moreover, when $1 < p_{\mathbb{X}} < \infty$, we have*

$$\|M\|_{\mathbb{X}(w)\mapsto\mathbb{X}(w)} \leq C[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}},$$

where C is an absolute constant only depending on $p_{\mathbb{X}}$ and n.

Proof. The proof of this lemma is contained in the proof of [**9**, Theorem 2.3]. Moreover, the upper bound of $\|M\|_{\mathbb{X}(w)\mapsto\mathbb{X}(w)}$ comes from tracking the constant by using Lemma 3.2. □

We also need the weighted dyadic Hardy–Littlewood maximal operator $M_w^{\mathcal{D}}$, given by

$$M_w^{\mathcal{D}} f(x) = \sup_{x\in Q, Q\in\mathcal{D}} \frac{1}{w(Q)} \int_Q |f(y)| w(y)\,\mathrm{d}y, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where $w \in A_\infty$ and $\mathcal{D}$ is the given dyadic grid. It is well known that $M_w^{\mathcal{D}}$ maps $L^p(w)$ strongly to $L^p(w)$ for $1 < p \leq \infty$ and $L^1(w)$ weakly to $L^{1,\infty}(w)$ (see [**10**, Theorem 7.1.9] or

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[22]). We have the following result for the dyadic maximal function that can be obtained in a similar manner to [9, Theorem 3.2]. Note that this result is independent of the weight characteristic since we are using the dyadic maximal function.

Lemma 3.4. *Let $\mathbb{X}$ be an RIQBFS which is p-convex for some $0 < p \le 1$. If $p_{\mathbb{X}} > 1$ and $w \in A_\infty$, then $M_w^{\mathcal{D}}$ is bounded on $\mathbb{X}(w)$. More precisely, we have $\|M_w^{\mathcal{D}}\| \le C$, where the absolute constant C only depends on $p_{\mathbb{X}}$ and n.*

We first deal with the case where $\mathbb{X}$ is an RIBFS.

Theorem 3.5. *Let T be a Calderón–Zygmund operator with standard kernel K. Furthermore, let $\mathbb{X}$ be an RIBFS and $w \in A_{p_{\mathbb{X}}}$. Then if $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, then*

$$\|T^{**}f\|_{\mathbb{X}(w)} \le C[w]_{A_\infty}[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\|f\|_{\mathbb{X}(w)},$$

where C is an absolute constant only depending on $p_{\mathbb{X}}$ and n.

Proof. First we note that $p_{\mathbb{X}'} = (q_{\mathbb{X}})' = q_{\mathbb{X}}/(q_{\mathbb{X}} - 1) > 1$, which follows from the fact that $1 < q_{\mathbb{X}} < \infty$.

By (3.4), it suffices to show that for any $\mathcal{D}$ a dyadic grid and $\mathcal{S} \in \mathcal{D}$ a sparse family, we have

$$\|\mathcal{A}_{\mathcal{D},\mathcal{S}}|f|\|_{\mathbb{X}(w)} \lesssim [w]_{A_\infty}[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\|f\|_{\mathbb{X}(w)}.$$

Indeed, for any $\|h\|_{\mathbb{X}'(w)} \le 1$ and Q a dyadic cube, put

$$h_{Q,w} = \frac{1}{w(Q)}\int_Q h(x)w(x)\,\mathrm{d}x$$

and then, by Lemmas 3.3 and 3.4, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} \mathcal{A}_{\mathcal{D},\mathcal{S}}|f|(x)h(x)w(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\Bigg(\sum_{Q\in\mathcal{S}}|f|_Q\chi_Q(x)\Bigg)h(x)w(x)\,\mathrm{d}x \\
&= \sum_{Q\in\mathcal{S}} f_Q \cdot h_{Q,w}\cdot w(Q) \\
&\le \sum_{Q\in\mathcal{S}}\Bigg(\frac{1}{w(Q)}\int_Q (Mf(x))^{1/2}(M_w^{\mathcal{D}}h(x))^{1/2}w(x)\,\mathrm{d}x\Bigg)^2 w(Q) \\
&\le 8[w]_{A_\infty}\int_{\mathbb{R}^n} Mf(x)M_w^{\mathcal{D}}h(x)w(x)\,\mathrm{d}x \\
&\le 8[w]_{A_\infty}\|Mf\|_{\mathbb{X}(w)}\|M_w^{\mathcal{D}}h\|_{\mathbb{X}'(w)} \\
&\le [w]_{A_\infty}[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\|f\|_{\mathbb{X}(w)}\|h\|_{\mathbb{X}'(w)} \\
&\le 8[w]_{A_\infty}[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\|f\|_{\mathbb{X}(w)},
\end{aligned}$$

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where in the second inequality we apply the Carleson embedding theorem by noting that the Carleson condition

$$\sum_{Q\subseteq R} w(Q) \le 2[w]_{A_\infty} w(R) \tag{3.5}$$

holds for any dyadic cube $R \in \mathcal{D}$ (see [16, Lemma 4.1]).

The desired result follows by taking supremum over all dyadic grids $\mathcal{D}$ and all their sparse families $\mathcal{S}$. □

The following corollary is straightforward.

Corollary 3.6. *Let T be a Calderón–Zygmund operator with standard kernel K. Furthermore, let $\mathbb{X}$ be an RIQBFS, which is p-convex for some $p > 0$, and $w \in A_{p_{\mathbb{X}}/p}$. Then if $p < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, then*

$$\|(T^{**}f)^{1/p}\|_{\mathbb{X}(w)} \le C[w]_{A_\infty}^{1/p}[w]_{A_{p_{\mathbb{X}}/p}}^{1/p_{\mathbb{X}}}\|f^{1/p}\|_{\mathbb{X}(w)},$$

where C is a constant only depending on p, $p_{\mathbb{X}}$ and n.

Proof. This is because $\mathbb{X}^{1/p}$ is an RIBFS and $p_{\mathbb{X}^{1/p}} = p_{\mathbb{X}}/p$. □

Next, we deal with the case where $\mathbb{X}$ is an RIQBFS, which is proved in a different way.

Theorem 3.7. *Let T be a Calderón–Zygmund operator with standard kernel K. Furthermore, let $\mathbb{X}$ be an RIQBFS, which is p-convex for some $0 < p \le 1$, and $w \in A_{p_{\mathbb{X}}}$. Then if $1 < p_{\mathbb{X}} \le q_{\mathbb{X}} < \infty$, then*

$$\|T^{**}f\|_{\mathbb{X}(w)} \le C[w]_{A_\infty}^{1/p}[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\|f\|_{\mathbb{X}(w)},$$

where C is an absolute constant only depending on $p_{\mathbb{X}}$ and n.

Proof. Since $\mathbb{X}$ is p-convex, we have that $\mathbb{Y} = \mathbb{X}^{1/p}$ is an RIBFS. Take and fix any $h \in \mathbb{Y}'(w)$ with $\|h\|_{\mathbb{Y}'(w)} \le 1$. We have the following claim: for any dyadic grid $\mathcal{D}$ and $\mathcal{S} \in \mathcal{D}$ a sparse family,

$$I := \int_{\mathbb{R}^n}\left(Mf(x) + \sum_{m=0}^{\infty}\frac{1}{2^{m\delta}}\mathcal{T}_{\mathcal{S},m}|f|(x)\right)^p h(x)w(x)\,\mathrm{d}x \le C[w]_{A_\infty}\left([w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\right)^p\|f^p\|_{\mathbb{Y}(w)}$$
$$= C[w]_{A_\infty}\left([w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}\right)^p\|f\|_{\mathbb{X}(w)}^p.$$

Indeed, we have

$$I \le \int_{\mathbb{R}^n} Mf(x)^p h(x)w(x)\,\mathrm{d}x + \sum_{m=0}^{\infty}\int_{\mathbb{R}^n}\frac{1}{2^{m\delta p}}[\mathcal{T}_{\mathcal{S},m}|f|(x)]^p h(x)w(x)\,\mathrm{d}x$$
$$:= I_1 + \sum_{m=0}^{\infty}\frac{I_{2,m}}{2^{m\delta p}},$$

where $I_{2,m} := \int_{\mathbb{R}^n}[\mathcal{T}_{\mathcal{S},m}|f|(x)]^p h(x)w(x)\,\mathrm{d}x$.

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Estimation of I_1. By Lemma 3.3 and duality, we have

$$I_1 \leq \|(Mf)^p\|_{\mathbb{Y}(w)} = \|Mf\|^p_{\mathbb{X}(w)} \leq \left(C[w]^{1/p_{\mathbb{X}}}_{A_{p_{\mathbb{X}}}}\right)^p \|f\|^p_{\mathbb{X}(w)}.$$

Estimation of $I_{2,m}, m \in \mathbb{N}$. Using the fact that $0 < p \leq 1$, the above estimation on I_1 and duality, we see that for each $m \in \mathbb{N}$,

$$\begin{aligned}
I_{2,m} &= \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{S}} f_{2^m Q}\chi_Q(x)\right)^p h(x)w(x)\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^n} \left(\sum_{Q\in\mathcal{S}} f^p_{2^m Q}\chi_Q(x)\right) h(x)w(x)\,\mathrm{d}x \\
&= \sum_{Q\in\mathcal{S}} f^p_{2^m Q}\int_Q h(x)w(x)\,\mathrm{d}x = \sum_{Q\in\mathcal{S}} f^p_{2^m Q} h_{Q,w}\cdot w(Q) \\
&\leq \sum_{Q\in\mathcal{S}} \left(\frac{1}{w(Q)}\int_Q (Mf(x))^{p/2}(M^{\mathcal{D}}_w h(x))^{1/2} w(x)\,\mathrm{d}x\right)^2 w(Q) \\
&\lesssim [w]_{A_\infty}\int_{\mathbb{R}^n} (Mf(x))^p M^{\mathcal{D}}_w h(x)w(x)\,\mathrm{d}x \\
&\quad \text{(by Carleson embedding theorem)} \\
&\leq [w]_{A_\infty}\|(Mf)^p\|_{\mathbb{Y}(w)}\|M^{\mathcal{D}}_w h\|_{\mathbb{Y}'(w)} \\
&\leq C[w]_{A_\infty}\left([w]^{1/p_{\mathbb{X}}}_{A_{p_{\mathbb{X}}}}\right)^p \|f\|^p_{\mathbb{X}(w)},
\end{aligned}$$

where in the last inequality we use the fact that $\|M^{\mathcal{D}}_w h\|_{\mathbb{Y}'(w)} \lesssim \|h\|_{\mathbb{Y}'(w)} \leq 1$.

Indeed, by Lemma 3.4, it suffices to show that $p_{\mathbb{Y}'} > 1$. A simple calculation shows that

$$p_{\mathbb{Y}'} = (q_{\mathbb{Y}})' = \frac{q_{\mathbb{Y}}}{q_{\mathbb{Y}} - 1} = \frac{q_{\mathbb{X}^{1/p}}}{q_{\mathbb{X}^{1/p}} - 1} = \frac{q_{\mathbb{X}}}{q_{\mathbb{X}} - p} > 1$$

for $1 < q_{\mathbb{X}} < \infty$. Thus, combining the estimation of both I_1 and $I_{2,m}$, we get

$$\begin{aligned}
I &\leq C[w]_{A_\infty}\left([w]^{1/p_{\mathbb{X}}}_{A_{p_{\mathbb{X}}}}\right)^p \left(\|f\|^p_{\mathbb{X}(w)} + \sum_{m=0}^{\infty}\frac{\|f\|^p_{\mathbb{X}(w)}}{2^{m\delta p}}\right) \\
&\leq C[w]_{A_\infty}\left([w]^{1/p_{\mathbb{X}}}_{A_{p_{\mathbb{X}}}}\right)^p \|f\|^p_{\mathbb{X}(w)}. \qquad (3.6)
\end{aligned}$$

Recall that $|m_Q(T^{**}f)| \to 0$ as $|Q| \to \infty$. The desired result follows from (3.1), (3.6) and Lebesgue's domination theorem. □

We make several remarks on the above results.

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Remark 3.8.

(1) It is clear that Theorem 3.5 is a particular case of Theorem 3.7.

(2) There is another approach for proving Theorem 3.5 by using extrapolation. We sketch the proof here. First, we have for any $w \in A_p$, where $1 < p < \infty$,

$$\|T^{**}f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)}$$

(see [**10**, Theorem 7.4.6]). Second, we apply the extrapolation theory of RIBFSs (see [**7**, Theorem 4.10]) to the above inequality to get the desired result. However, by doing so, it is not clear how the operator norm of T^{**} depends on the weight w.

(3) The modifications for SHT include assuming that the Lebesgue differentiation theorem holds. For a thorough reference on this property and SHT in general, see [**1**].

(4) The dependence on the weight of $[w]_{A_\infty}[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}$ (see Theorem 3.5) in the constant is indicative of the method of proof—the term $[w]_{A_\infty}$ comes from a Coifman–Fefferman-style argument using Carleson embedding, while the term $[w]_{A_{p_{\mathbb{X}}}}^{1/p_{\mathbb{X}}}$ comes from the bound for the maximal function. By observing the proof and Buckley's proof of the sharp bound for the maximal function, we see that our bound should be sharp in terms of the characteristics. Therefore, we expect that the dependence on the constants is sharp.

Remark 3.9. We note that even when considering the space $L^2(w)$, in certain cases, our constant dependence in Theorem 3.5 improves on the dependence in the work [**16**] (note that the result in [**16**] is for the standard Calderón–Zygmund operator, ours is for the maximal truncated Calderón–Zygmund operator). In particular, our bound is

$$[w]_{A_2}^{1/2}[w]_{A_\infty},$$

while Hytönen and Pérez obtain a bound of

$$[w]_{A_2}^{1/2}([w]_{A_\infty} + [w^{-1}]_{A_\infty})^{1/2}.$$

Let $n = 1$. For the case of power weights $w(x) = |x|^a$ with $0 < a < 1$, we have that $[w]_{A_2} \approxeq 1/(1+a) \cdot 1/(1-a)$, $[w]_{A_\infty} \approxeq 1/(1+a)$ and $[w^{-1}]_{A_\infty} \approxeq 1/(1-a)$. Therefore,

$$[w]_{A_\infty} \approxeq \frac{1}{1+a} \lesssim (2[w]_{A_\infty}[w^{-1}]_{A_\infty})^{1/2} = ([w]_{A_\infty} + [w^{-1}]_{A_\infty})^{1/2}.$$

Hence in these cases, our bound is smaller (see [**16**] for computations and more details).

To close this subsection, we prove the following bound for the median (which holds in SHT), which can be substituted in the above proofs for $\mathbb{X}(w) = L^p(w)$. This bound shows the dependence of the constant on the weight characteristic and allows us to consider SHT of finite measure for the case of the RIBFS $L^p(w)$.

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Proposition 3.10. *We have that* $\|m_Q(T^{**})\|_{L^p(w)} \leq C_{T,n}[w]_{A_p}^{1/p}\|f\|_{L^p(w)}$.

Proof. By [2, Lemma 3.15] and Hölder's inequality, we have

$$\begin{aligned}\|m_Q(T^{**})\|_{L^p(w)} &\leq \left(\int_Q \frac{\|T^{**}\|_{1,\infty}^p\|f\|_{L^1(Q)}^p}{|Q|^p} w(x)\,\mathrm{d}x\right)^{1/p} \\ &\leq \frac{\|T^{**}\|_{1,\infty}}{|Q|}\left(\int_Q f^p w\right)^{1/p}\left(\int_Q w^{-p'/p}\right)^{1/p'}\left(\int_Q w(x)\,\mathrm{d}x\right)^{1/p} \\ &\leq \|T^{**}\|_{1,\infty}\|f\|_{L^p(w)}[w]_{A_p}^{1/p}. \qquad \square\end{aligned}$$

This bound for the median mirrors the Buckley bound for the maximal function [4].

3.2. Maximal truncated Calderón–Zygmund operator of modular inequality type

We need the following lemma, which can be regarded as a modular inequality version of Lemma 3.3.

Lemma 3.11. *Let* $\phi \in \Phi$ *be such that* ϕ *is quasi-convex. Furthermore, let* $w \in A_{i_\phi}$. *If* $1 < i_\phi < \infty$, *then*

$$\int_{\mathbb{R}^n} \phi(Mf(x))w(x)\,\mathrm{d}x \leq C_0 \int_{\mathbb{R}^n} \phi\Big(C_0[w]_{A_{i_\phi}}^{1/i_\phi}|f(x)|\Big)w(x)\,\mathrm{d}x,$$

where C_0 *is an absolute constant that only depends on* ϕ *and* α.

Proof. The proof of this lemma is contained in the proof of [9, Theorem 3.7]. Moreover, the constant follows from Lemma 3.2. □

Similarly, as what we did for the RIBFS and RIQBFS case, we need the following version of the modular inequality for the weighted Hardy–Littlewood maximal function.

Lemma 3.12. *Let* $w \in A_\infty$ *and* $\phi \in \Phi$ *be such that there exists* $0 < \alpha < 1$ *for which* ϕ^α *is a quasi-convex function. Then there exists some constant* $a_2 > 1$, *depending on* ϕ *and* w, *such that*

$$\int_{\mathbb{R}^n} \phi(M_w^{\mathcal{D}}f(x))w(x)\,\mathrm{d}x \leq a_2 \int_{\mathbb{R}^n} \phi(a_2|f(x)|)w(x)\,\mathrm{d}x,$$

where the constant a_2 *only depends on* ϕ *and* α, *and is independent of* w.

(See [9, Propostion 5.1], where it is stated that the constant depends on w, but by their proof one sees that it is in fact independent of w.)

We make some easy observations of the Δ_2 condition and N-functions before we state the following lemma. First, we note that $\phi \in \Delta_2$, that is, $\phi(2t) \leq C\phi(t)$, $t \geq 0$, if and only

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if there exists some constant C' (for example, we can take $C' = \log C/\log 2$) such that for any $\lambda \geq 2$,

$$\phi(\lambda t) \leq 2^{C'}\lambda^{C'}\phi(t), \quad t > 0. \tag{3.7}$$

The proof for this claim is straightforward from the definition, and hence we omit it here. Second, since

$$\phi^{-1}(t)\overline{\phi}^{-1}(t) \geq t, \quad t \geq 0,$$

it follows that

$$t\overline{\phi}^{-1}(\phi(t)) = \phi^{-1}(\phi(t))\overline{\phi}^{-1}(\phi(t)) \geq \phi(t), \quad t \geq 0,$$

which implies that

$$\overline{\phi}\left(\frac{\phi(t)}{t}\right) \leq \phi(t), \quad t > 0. \tag{3.8}$$

Lemma 3.13. *Let ϕ be an N-function and $\phi \in \Delta_2$, that is, $I_\phi < \infty$, and $w \in A_{i_\phi}$. If $i_\phi > 1$, for each $m \in \mathbb{N}$, any dyadic grid $\mathcal{D}$ and $\mathcal{S} \in \mathcal{D}$ a sparse family, we have*

$$\int_{\mathbb{R}^n} \phi(\mathcal{T}_{\mathcal{S},m}|f|(x))w(x)\,\mathrm{d}x \leq C''[w]_{A_\infty}^{1+\alpha C'}\int_{\mathbb{R}^n}\phi(Mf(x))w(x)\,\mathrm{d}x,$$

where C' is defined in (3.7) and C'' is an absolute constant only depending on ϕ.

Proof. Since ϕ is an N-function, it is clear that the quantity $\phi(\mathcal{T}_{\mathcal{S},m}|f|(x)) = 0$ when $\mathcal{T}_{\mathcal{S},m}|f|(x) = 0$. Hence, in the sequel, we write the function

$$\frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)},$$

which takes its actual value when $\mathcal{T}_{\mathcal{S},m}|f|(x) \neq 0$ and zero when $\mathcal{T}_{\mathcal{S},m}|f|(x) = 0$.

Moreover, since ϕ is Δ_2, it follows that there exists some $0 < \alpha \leq 1$ such that $\overline{\phi}^\alpha$ is quasi-convex, that is, there exists some convex function ψ such that

$$\psi(t) \leq \overline{\phi}^\alpha(t) \leq a_3\psi(a_3 t), \quad t > 0. \tag{3.9}$$

Note that we can always assume that $a_3 \geq 2$.

Take and fix some ε satisfying

$$0 < \varepsilon < \left(\frac{1}{16a_2[w]_{A_\infty}}\right)^\alpha \cdot \frac{1}{a_2a_3^2} = \min\left\{\frac{1}{2}, \frac{1}{a_2a_3}, \left(\frac{1}{16a_2[w]_{A_\infty}}\right)^\alpha \cdot \frac{1}{a_2a_3^2}\right\}. \tag{3.10}$$

Then, by Lemma 3.12, we have

$$\begin{aligned}\int_{\mathbb{R}^n} \phi(\mathcal{T}_{\mathcal{S},m}|f|(x))w(x)\,\mathrm{d}x &= \int_{\mathbb{R}^n}\left(\sum_{Q\in\mathcal{S}} f_{2^mQ}\chi_Q(x)\right)\cdot\frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}w(x)\,\mathrm{d}x \\ &= \sum_{Q\in\mathcal{S}} f_{2^mQ}\cdot\frac{1}{w(Q)}\int_Q \frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}w(x)\,\mathrm{d}x\cdot w(Q)\end{aligned}$$

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$$\begin{aligned}
&\leq 8[w]_{A_\infty} \int_{\mathbb{R}^n} Mf(x) M_w^{\mathcal{D}}\left(\frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}\right) w(x)\,\mathrm{d}x \\
&\quad \text{(by Carleson embedding theorem)} \\
&= 8[w]_{A_\infty} \int_{\mathbb{R}^n} \frac{Mf(x)}{\varepsilon} \cdot \varepsilon M_w^{\mathcal{D}}\left(\frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}\right) w(x)\,\mathrm{d}x \\
&\leq 8[w]_{A_\infty}\left(\int_{\mathbb{R}^n} \phi\left(\frac{Mf(x)}{\varepsilon}\right) w(x)\,\mathrm{d}x\right. \\
&\quad \left. + \int_{\mathbb{R}^n} \overline{\phi}\left(\varepsilon M_w^{\mathcal{D}}\left(\frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}\right)\right) w(x)\,\mathrm{d}x\right) \\
&\qquad \text{(by (2.1))} \\
&\leq \frac{2^{C'+3}[w]_{A_\infty}}{\varepsilon^{C'}} \int_{\mathbb{R}^n} \phi(Mf(x)) w(x)\,\mathrm{d}x \\
&\qquad + 8[w]_{A_\infty} a_2 \int_{\mathbb{R}^n} \overline{\phi}\left(\frac{a_2\varepsilon\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}\right) w(x)\,\mathrm{d}x \\
&\leq \frac{2^{C'+3}[w]_{A_\infty}}{\varepsilon^{C'}} \int_{\mathbb{R}^n} \phi(Mf(x)) w(x)\,\mathrm{d}x \\
&\qquad + 8[w]_{A_\infty} a_2 \cdot (a_3^2 a_2 \varepsilon)^{1/\alpha} \int_{\mathbb{R}^n} \overline{\phi}\left(\frac{\phi(\mathcal{T}_{\mathcal{S},m}|f|(x))}{\mathcal{T}_{\mathcal{S},m}|f|(x)}\right) w(x)\,\mathrm{d}x \\
&\qquad \text{(by (3.9), (3.10))} \\
&\leq \frac{2^{C'+3}[w]_{A_\infty}}{\varepsilon^{C'}} \int_{\mathbb{R}^n} \phi(Mf(x)) w(x)\,\mathrm{d}x \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^n} \phi(\mathcal{T}_{\mathcal{S},m}|f|(x)) w(x)\,\mathrm{d}x \\
&\qquad \text{(by (3.8) and (3.10))},
\end{aligned}$$

where we have used that fact that $\psi(\lambda t) \leq \lambda\psi(t)$ for $a_3 a_2 \varepsilon = \lambda \in (0,1)$ since $\psi(0) = 0$. Hence, we have

$$\int_{\mathbb{R}^n} \phi(\mathcal{T}_{\mathcal{S},m}|f|(x)) w(x)\,\mathrm{d}x \leq C''[w]_{A_\infty}^{1+\alpha C'} \int_{\mathbb{R}^n} \phi(Mf(x)) w(x)\,\mathrm{d}x,$$

where C'' is an absolute constant only depending on ϕ. The lemma is proved. □

Theorem 3.14. *Let T be a Calderón–Zygmund operator with standard kernel K. Furthermore, let ϕ be an N-function belonging to Δ_2, that is, $I_\phi < \infty$, and $w \in A_{i_\phi}$. If $i_\phi > 1$, we have*

$$\int_{\mathbb{R}^n} \phi(|T^{**} f(x)|) w(x)\,\mathrm{d}x \leq C(\phi, w) \int_{\mathbb{R}^n} \phi(|f(x)|) w(x)\,\mathrm{d}x, \tag{3.11}$$

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where

$$C(\phi, w) = \begin{cases} C'''[w]_{A_\infty}^{1+\alpha C'}, & C_0[w]_{A_{i_\phi}}^{1/i_\phi} < 2, \\ C''[w]_{A_\infty}^{1+\alpha C'}\left([w]_{A_{i_\phi}}^{1/i_\phi}\right)^{C'}, & C_0[w]_{A_{i_\phi}}^{1/i_\phi} \geq 2, \end{cases}$$

where C_0 is the constant defined in Lemma 3.11.

Proof. We start with the case where $i_\phi > 1$. Denote

$$2 < K_0 = 1 + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} < \infty,$$

where δ is the constant in the smoothing condition of the kernel K. Again, we prove a claim similar to that in Theorem 3.7: for any dyadic grid $\mathcal{D}$ and $\mathcal{S} \in \mathcal{D}$ a sparse family,

$$J := \int_{\mathbb{R}^n} \phi\left(Mf(x) + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \mathcal{T}_{\mathcal{S},m}|f|(x)\right) w(x)\,\mathrm{d}x \lesssim \int_{\mathbb{R}^n} \phi(|f(x)|)w(x)\,\mathrm{d}x.$$

Indeed, we have

$$\begin{aligned} J &= \int_{\mathbb{R}^n} \phi\left(K_0 \cdot \left[\frac{Mf(x)}{K_0} + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}K_0} \mathcal{T}_{\mathcal{S},m}|f|(x)\right]\right) w(x)\,\mathrm{d}x \\ &\leq C''' \int_{\mathbb{R}^n} \phi\left(\frac{Mf(x)}{K_0} + \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}K_0} \mathcal{T}_{\mathcal{S},m}|f|(x)\right) w(x)\,\mathrm{d}x \\ &\qquad \text{(by } \Delta_2 \text{ condition)} \\ &\leq \frac{C'''}{K_0} \int_{\mathbb{R}^n} \phi(Mf(x))w(x)\,\mathrm{d}x + \sum_{m=0}^{\infty} \frac{C'''}{2^{m\delta}K_0} \int_{\mathbb{R}^n} \phi(\mathcal{T}_{\mathcal{S},m}|f|(x))w(x)\,\mathrm{d}x \\ &\qquad \text{(by convexity of } \phi) \\ &\leq C'''[w]_{A_\infty}^{1+\alpha C'} \int_{\mathbb{R}^n} \phi(Mf(x))w(x)\,\mathrm{d}x \\ &\qquad \text{(by Lemma 3.13)} \\ &\leq C'''[w]_{A_\infty}^{1+\alpha C'} \int_{\mathbb{R}^n} \phi\left(C_0[w]_{A_{i_\phi}}^{1/i_\phi}|f(x)|\right) w(x)\,\mathrm{d}x \\ &\qquad \text{(by Lemma 3.11).} \end{aligned}$$

We consider two different cases.

Case I: $C_0[w]_{A_{i_\phi}}^{1/i_\phi} < 2$.
In this case, we have

$$J \leq C''[w]_{A_\infty}^{1+\alpha C'} \int_{\mathbb{R}^n} \phi(|f(x)|)w(x)\,\mathrm{d}x.$$

Case II: $C_0[w]_{A_{i_\phi}}^{1/i_\phi} \geq 2$.

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By Equation (3.7), we have

$$J \leq C''[w]_{A_\infty}^{1+\alpha C'} \left([w]_{A_{i_\phi}}^{1/i_\phi}\right)^{C'} \int_{\mathbb{R}^n} \phi(|f(x)|)w(x)\,\mathrm{d}x.$$

Finally, combining the above estimation with (3.1) and Lebesgue's domination theorem, we get the desired result. □

Remark 3.15. Our constant is not predicted to be sharp here. We conjecture that the sharp constant depends on $[w]_{A_\infty}$ linearly.

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References

1. R. Alvarado and M. Mitrea, *Hardy spaces on Ahlfors-regular quasi metric spaces*, Lecture Notes in Mathematics, Volume 2142 (Springer International Publishing, Cham, 2015).
2. T. Anderson and A. Vagharshakyan, A simple proof of the sharp weighted estimate for Calderon–Zygmund operators on homogeneous spaces, *J. Geom. Anal.* **24**(3) (2014), 1276–1297.
3. C. Bennett and R. Sharply, *Interpolation of operators* (Academic Press, New York, 1988).
4. S. M. Buckley, Estimates for operator norms on weighted spaces and reverse Jensen inequalities, *Trans. Am. Math. Soc.* **340**(1) (1993), 253–272.
5. J. Conde, A note on dyadic coverings and nondoubling Calderón–Zygmund theory, *J. Math. Anal. Appl.* **397**(2) (2013), 785–790.
6. J. M. Conde-Alonso and G. Rey, A pointwise estimate for positive dyadic shifts and some applications, *Math. Ann.* **365** (2016), 1111–1135.
7. D. Cruz-Uribe, J. M. Martell and C. Pérez, *Weights, extrapolation and the theory of Rubio de Francia*, Operator Theory: Advances and Applications, Volume 215 (Birkhäuser/Springer, Basel, 2011).
8. A. Culiuc, F. D. Plinio and Y. Ou, Uniform sparse domination of singular integrals via dyadic shifts, *Math. Res. Lett.* **25** (2018), 21–42.
9. G. Curbera, J. Cuerva, J. Martell and C. Pérez, Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals, *Adv. in Math* **203** (2006), 256–318.
10. L. Grafakos, *Classical Fourier analysis*, 3rd edn, Graduate Texts in Mathematics, Volume 249 (Springer, New York, 2014).
11. L. Grafakos, *Modern Fourier analysis*, 3rd edn, Graduate Texts in Mathematics, Volume 250 (Springer, New York, 2014).
12. L. Grafakos and N. Kalton, Some remarks on multilinear maps and interpolation, *Math. Ann.* **319** (2001), 151–180.
13. G. Hu, D. Yang and D. Yang, Boundedness of maximal singular integral operators on spaces of homogeneous type and its applications, *J. Math. Soc. Japan* **59**(2) (2007), 323–349.
14. T. Hytönen, The sharp weighted bound for general Calderón–Zygmund operators, *Ann. Math.* **175** (2012), 1473–1506.

15. T. Hytönen and A. Kairema, Systems of dyadic cubes in a doubling metric space, *Colloq. Math.* **126**(1) (2012), 1–33.
16. T. Hytönen and C. Pérez, Sharp weighted bounds involving A_∞, *Anal. PDE* **6**(4) (2013), 777–818.
17. T. Hytönen and C. Pérez, The $L(\log L)\varepsilon$ endpoint estimate for maximal singular integral operators, *J. Math. Anal. Appl.* **428** (2015), 605–626.
18. T. Hytönen, C. Pérez and E. Rela, Sharp reverse Hölder property for A_∞ weights on spaces of homogeneous type, *J. Funct. Anal.* **263**(12) (2012), 3883–3899.
19. M. T. Lacey, An elementary proof of the A2 bound, *Israel J. Math.* **217**(1) (2017), 181–195.
20. A. Lerner, A simple proof of the A_2 conjecture, *Int. Math. Res. Not.* **14** (2013), 3159–3170.
21. A. Lerner, On an estimate of Calderón–Zygmund operators by dyadic positive operators, *J. Anal. Math.* **121** (2013), 141–161.
22. A. K. Lerner and F. Nazarov, Intuitive dyadic calculus: the basics (arXiv:1508.05639, 2015).
23. L. Maligranda, *Orlicz spaces and interpolation*, Seminars in Mathematics, Volume 5 (University Estadual de Campinas, Campinas SP, Brazil, 1989).
24. C. Pérez, *Singular integrals and weights: a modern introduction*, Spring School on Function spaces and Inequalities, Paseky, June, 2013.
25. M. Rao and Z. Ren, *Theory of Orlicz spaces* (Marcel Dekker, New York, 1991).