A BOOTSTRAP ESTIMATE OF THE PREDICTIVE DISTRIBUTION OF OUTSTANDING CLAIMS FOR THE SCHNIEPER MODEL

BY

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ABSTRACT

This paper considers the bootstrapping approach for measuring reserve uncertainty when applying the model of Schnieper for reserves which separate Incurred But Not Reported (IBNR) and Incurred But Not Enough Reserved (IBNER) claims. The Schnieper method has been explored in Liu and Verrall (2009), and the Mean Square Errors of Prediction (MSEP) derived. This paper takes this further by deriving the full predictive distribution, using bootstrapping. Numerical examples are provided and the MSEP from the bootstrapping approach are compared with those obtained analytically.

KEYWORDS

Bootstrapping; Claims Reserving; Schnieper model; IBNR and IBNER claims.

1. INTRODUCTION

The model of Schnieper (1991) separates out IBNR and IBNER claims, with the intention of providing better estimates of outstanding liabilities in cases when the over claims data are inherently volatile. Although Mack (1993) used some of the ideas from Schnieper, there has not been much attention paid to the original paper since it was published. However, Liu and Verrall (2009) have derived approximations to the Mean Square Errors of Prediction (MSEP) of the reserves and we believe that the method has the potential to be useful in practice. In this paper, we continue with the development of the statistical background for the original method by showing how the complete predictive distribution can be approximated using bootstrapping methods. This is a very important additional step to the theory derived in Liu and Verrall (2009), since the MSEP is of only limited value in the context of risk assessment and capital modelling. For a proper assessment of risk, and to use the model in the modern solvency setting, it is far better to use the predictive distribution. Also, a simulation approach is often used in this context, and bootstrapping has been found to be very convenient for this.

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Section 2 gives a brief outline of the model of Schnieper. For more details, see Schnieper (1991) and Liu and Verrall (2009). In Section 3 of this paper, we show how to construct an appropriate resampling procedure for the Schnieper method, within a Generalised Linear Models (GLM) framework. Note that the bootstrapping is a general method, which can be applied to any fully defined model in order to obtain the sampling distribution for any statistic of interest. As was shown in England and Verrall (1999) and England (2002), it is straightforward to extend the bootstrapping procedure to obtain an approximation to the predictive distribution. This requires a final step to be added to the resampling method, which simulates a future observation from the appropriate process distribution. A more complete discussion of bootstrapping methods can be found in England and Verrall (2006), which also contains a fuller review of the literature on bootstrapping for claims reserving in general. Note that the Schnieper method is a recursive method for claims reserving, and the appropriate background for this can be found in England and Verrall (2006). The paper by England and Verrall (1999), which first considered bootstrapping for the chain-ladder technique, was based on the over-dispersed Poisson model which is non-recursive. For ease of implementation, the detailed algorithm which can be used to obtain the bootstrap approximation to the predictive distribution for the Schnieper method is given in the Appendix. In Section 4, we apply the bootstrap method to the data from Schnieper (1991) and show that the results are very close to the results for the analytical estimation error derived in Liu and Verrall (2009). This section also shows the full predictive distribution. Section 5 contains the conclusion.

2. The Schnieper Model

The idea behind the model of Schnieper (1991) is to separate a triangle of potentially volatile claims data into two separate triangles: a triangle of the IBNER claims and a triangle of the real IBNR claims. In this way, the hope is that the separate triangles will prove easier to deal with and will provide better estimates of outstanding claims, and a better idea of the forces driving these. It is assumed that the data in the two triangles are independent, and we briefly describe the models used for each of these. For more details of these models, and of the estimation of the parameters and forecasts, see Schnieper (1991) and Liu and Verrall (2009).

Without loss of generality, we assume that the data are available in triangular form, indexed by accident year, i, and development year, j. The single triangle of data consists of the cumulative incurred claims, and are denoted by $\{X_{ij}:\ 1 \le i \le n;\ 1 \le j \le n-i+1\}$:

$$\begin{array}{llll} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & \dots & X_{2,n-1} & \\ \vdots & & & \\ X_{n1}. & & & \end{array}$$

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It is assumed that the incremental incurred claims ($X_{ij} - X_{i,j-1}$) are the sum of incremental incurred from the old claims ($-D_{ij}$) and the new claims (N_{ij}). In other words, $-D_{ij}$ represents the change in the cumulative incurred claims for claims reported in previous development periods (IBNER data), and N_{ij} is the new claims (IBNR claims) reported in development period j. Thus,

$$X_{ij} - X_{i,j-1} = D_{ij} + N_{ij}$$

and for cumulative claims:

$$X_{ij} - X_{i,j-1} - D_{ij} + N_{ij}.$$

Schnieper also assumes that a measure of the exposure, E_i, is available for each accident year i. In common with Schnieper (1991), we do not attempt to forecast beyond development year n. We refer to cumulative claims at development year n as "Ultimate Claims".

We define the information up to payment year k by H_k and the information up to development year k by F_k, where

$$H_k = \{N_{ij}, D_{ij} : 1 \leq i,\ j \leq n;\ i + j - 1 \leq k\}$$

and

$$F_k = \{N_{ij}, D_{ij} : 1 \leq i,\ j \leq n;\ j \leq k\}.$$

F_k corresponds to B_k in Mack (1993).

The general model assumptions are given as follows:

Assumption 1: There exist constants λ_j and δ_j, such that for known exposure E_i we have that,

$$E[N_{ij} \mid H_{i+j-2}] = E_i \lambda_j,\ 1 \leq i,\ j \leq n,$$

$$E[D_{ij} \mid H_{i+j-2}] = X_{i,j-1} \delta_j,\ 1 \leq i \leq n,\ 2 \leq j \leq n.$$

Assumption 2: There exist constants σ_j^2 and τ_j^2, such that

$$Var[N_{ij} \mid H_{i+j-2}] = E_i \sigma_j^2,\ 1 \leq i,\ j \leq n$$

$$Var[D_{ij} \mid H_{i+j-2}] = X_{i,j-1} \tau_j^2,\ 1 \leq i \leq n,\ 2 \leq j \leq n.$$

Assumption 3: Independence between accident years

As in Schnieper (1991), it is assumed that $\{N_{1j}, D_{1j} : 1 \leq j \leq n\} \ldots \{N_{nj}, D_{nj} : 1 \leq j \leq n\}\}$, are independent between accident years.

Assumption 4: Uncorrelatedness between development years

$\{N_{ij} \mid H_{i+j-2} : 1 \leq j \leq n\}$ and $\{D_{ij} \mid H_{i+j-2} : 1 \leq i \leq n,\ 2 \leq j \leq n\}$ and are uncorrelated.

For a discussion of these assumptions, see Liu and Verrall (2009). Based on these assumptions, estimates of the parameters may be obtained, along with predictions of the development of future claims. This is a recursive method, and full details of the derivation of these estimates may be found in Schnieper (1991) and Liu and Verrall (2009). The estimates of the parameters in the mean are given by

$$\hat{\lambda}_j = \frac{\sum_{i=1}^{n-j+1} N_{ij}}{\sum_{i=1}^{n-j+1} E_i}, \quad 1 \le j \le n.$$

and

$$\hat{\delta}_j = \frac{\sum_{i=1}^{n-j+1} D_{ij}}{\sum_{i=1}^{n-j+1} X_{i,j-1}}, \quad 2 \le j \le n.$$

Also

$$\hat{\sigma}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{1}{E_i} (N_{ij} - \hat{\lambda}_j E_i)^2, \quad 1 \le j \le n-1,$$

and

$$\hat{\tau}_j^2 = \frac{1}{n-j} \sum_{i=1}^{n-j+1} \frac{1}{X_{i,j-1}} (D_{ij} - \hat{\delta}_j X_{i,j-1})^2, \quad 2 \le j \le n-1.$$

These are the estimates that are used when the bootstrap methodology is applied. Finally, the estimate of outstanding incurred claims in the original single triangle was derived by Schnieper. Note that $\hat{E}[X_{i,j} | H_n]$ is the prediction of $X_{i,j}$, and we use the notation of $\hat{X}_{i,j}$ for this: $\hat{E}[X_{i,j} | H_n] = \hat{X}_{i,j}$. Then $\hat{X}_{i,j} = (1 - \hat{\delta}_j) \hat{X}_{i,j-1} + E_i \hat{\lambda}_j$ for $j \in \{n-i+2, n-i+3, \ldots, n\}$. Note also that $\hat{E}[X_{i,n-i+1} | H_n] = X_{i,n-i+1}$, and hence $\hat{X}_{i,n-i+1} = X_{i,n-i+1}$ forms the starting point in this recursive formula.

3. Bootstrap Methodology

The Schnieper method presents an interesting exercise for bootstrapping in that there are two separate triangles that have to be resampled independently. This is different from most other applications of bootstrapping for claims reserving, when a single triangle is considered. In this section, we describe how the resampling procedure can be adapted to this novel situation, and in the Appendix we set out the algorithm in detail.

In order to apply the bootstrapping methodology, we require data which can be assumed to be independent and identically distributed (iid). Since the data themselves are not iid, we resample from the residuals rather than the raw data. Also, since the Schnieper method is based on recursive models, we use residuals of the ratios, $\frac{N_{ij}}{E_i}$ and $\frac{D_{ij}}{X_{i,j-1}}$, rather than the observed data, N_{ij} and D_{ij}. This has been discussed in detail in England and Verrall (2006). In order to calculate residuals (suitably normalized), we require the mean and variance of each of the ratios. Following Liu and Verrall (2009), the mean and variance assumptions for the Schnieper model are:

$$E\left[\frac{N_{ij}}{E_i}\middle| H_{i+j-2}\right] = \lambda_j \quad \text{and} \quad E\left[\frac{D_{ij}}{X_{i,j-1}}\middle| H_{i+j-2}\right] = \delta_j,$$

and

$$Var\left[\frac{N_{ij}}{E_i}\middle| H_{i+j-2}\right] = \frac{\sigma_j^2}{E_i} \quad \text{and} \quad Var\left[\frac{D_{ij}}{X_{i,j-1}}\middle| H_{i+j-2}\right] = \frac{\tau_j^2}{X_{i,j-1}}.$$

The idea of bootstrapping is to generate new triangles of data (“bootstrap samples”) which are representative of the underlying distributions of the estimates. When this has been done a reasonable number of times and the required results saved, the sampling properties may be estimated by simply looking at the properties of the bootstrap samples. So, for example, to obtain a bootstrap estimate of the estimation error of the overall reserve, we generate a reasonable number (in most cases we use 10,000) of new sets of data from the original data and estimate the reserve for each of these.

The aim is to estimate the MSEP for a future observation, $X_{im}(n-i+1 < m \le n)$:

$$\begin{aligned} MSEP\left[\hat{X}_{im} \mid H_n\right] &= E\left[(X_{im}-\hat{X}_{im})^2 \mid H_n\right] \\ &= E\left[((X_{im}-E[X_{im} \mid H_n]) - (\hat{X}_{im}-E[X_{im} \mid H_n]))^2 \mid H_n\right] \\ &= E\left[X_{im}-E[X_{im} \mid H_n])^2 \mid H_n\right] + (\hat{X}_{im}-E[X_{im} \mid H_n])^2 \\ &= \textit{process variance} + \textit{estimation error}. \end{aligned}$$

The process variance is straightforward due to the assumptions of independence between accident years and uncorrelatedness between development years. In the context of bootstrapping, it requires the inclusion of a final simulation from the process distribution, details of which are given below. For the estimation error, there are two approximations that can be used, as discussed in detail in Liu and Verrall (2009). The first approach, which was used by England and Verrall (2002), approximates the estimation error by the estimation variance, $Var[\hat{X}_{im} | F_{n-i+1}]$. The other approach is to derive an approximation to the estimation error directly. Corresponding to these two approaches, there are two procedures that can be used in the bootstrapping process. If the estimation variance approximation approach which is adopted by England and Verrall (2002) and Buchwalder et al (2006) is followed, the bootstrap estimate of the approximation is obtained by calculating the sample variance of the bootstrap reserves. However, if the approach of Mack (1993) is followed, the bootstrap estimate of the estimation variance is obtained by calculating the average squared difference between the bootstrap reserve estimate and the original reserve estimate. The rationale for the first approach is clear: we simply estimate the estimation variance by the variance of the bootstrap samples. The rationale for the second approach is that we require a bootstrap estimate of $(\hat{X}_{i,m} - E[X_{i,m} | H_n])^2$, and this can be obtained by looking at the average squared difference between the bootstrap value, $X_{i,m}^B$ and $\hat{X}_{i,m}$.

To include the process variance, we add an extra simulation after each bootstrap, using the appropriate process distribution. This is the most straightforward way to include the process variance, and more details can be found in England and Verrall (2006).

Let $f_{ij} = \dfrac{N_{ij}}{E_i}$ and $g_{ij} = \dfrac{D_{ij}}{X_{i,j-1}}$.

Then the scaled Pearson residuals for the two triangles are given by:

$$r_{PS}(f_{ij}, \hat{\lambda}_j, E_i, \hat{\sigma}_j) = \frac{\sqrt{E_i}\left(f_{ij} - \hat{\lambda}_j\right)}{\hat{\sigma}_j} \text{ and } r_{PS}(g_{ij}, \hat{\delta}_j, X_{i,j-1}, \hat{\tau}_j) = \frac{\sqrt{X_{i,j-1}}\left(g_{ij} - \hat{\delta}_j\right)}{\hat{\tau}_j}.$$

It is well known that a bias correction is required in the context of bootstrap estimation. In order to include this, these residuals are adjusted by multiplying by $\sqrt{\dfrac{n-j}{n-j+1}}$. This gives the adjusted residuals:

$$r_{ij} = \sqrt{\frac{n-j}{n-j+1}}\, r_{PS}(f_{ij}, \hat{\lambda}_j, E_i, \hat{\sigma}_j) \text{ and } s_{ij} = \sqrt{\frac{n-j}{n-j+1}}\, r_{PS}(g_{ij}, \hat{\delta}_j, X_{i,j-1}, \hat{\tau}_j).$$

These adjusted residuals are sampled, with replacement, to generate bootstrap samples of residuals, r_{ij}^B and s_{ij}^B, for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n-i+1$. The triangles of pseudo data are then calculated by inverting the residual definition:

$$f_{ij}^B = r_{ij}^B \frac{\hat{\sigma}_j}{\sqrt{E_i}} + \hat{\lambda}_j \text{ and } g_{ij}^B = s_{ij}^B \frac{\hat{\tau}_j}{\sqrt{X_{i,j-1}}} + \hat{\delta}_j.$$

The appealing aspect of bootstrapping is that the calculations now only involve the simple spreadsheet operations used in the original method to calculate the loss reserves. In other words, they can be based on the original Schnieper paper, rather than involving any more complex statistical analysis similar to that in Liu and Verrall (2009). Thus, for each bootstrap sample, the bootstrap estimates of the parameters in the mean, λ_j^B and δ_j^B, are calculated using the usual weighted averages of the individual development factors. These are given in the following equations:

$$\lambda_j^B = \frac{\sum_{i=1}^{n-j+1} f_{ij}^B E_i}{\sum_{i=1}^{n-j+1} E_i} \quad \text{and} \quad \delta_j^B = \frac{\sum_{i=1}^{n-j+1} g_{ij}^B X_{i,j-1}}{\sum_{i=1}^{n-j+1} X_{i,j-1}}.$$

Note that the observed data, $X_{i,j-1}$, and the exposure E_i act as the weights here: it is not correct to use bootstrapped data for the weights.

The bootstrap estimates of the reserves for each row and the overall total can be obtained by applying the bootstrap values of the parameters, λ_j^B and δ_j^B, to the original formula of Schnieper for the outstanding incurred claims:

$\hat{X}_{i,j} = (1-\delta_j^B)\,\hat{X}_{i,j-1} + E_i \lambda_j^B$, for $j \in \{n-i+2,\ n-i+3, \ldots, n\}$ (with the initial point $\hat{X}_{i,n-i+1} = X_{i,n-i+1}$).

Bootstrapping only addresses the estimation error for the model. If the aim of the exercise is to obtain a bootstrap estimate of the estimation error, then this is all that is needed. However, for claims reserving purposes, we also require the prediction error and the full predictive distribution of the reserves. To obtain these, it is necessary to include the process variance, using the process distributions. The most straightforward option here, since we are only specifying the first two moments, is to use normal distributions for both N_{ij} and D_{ij}. (Note that it would be possible to use other models, such as the over-dispersed Poisson distribution.) Thus, the final step in the process to obtain simulations of the loss reserves suitable for calculating prediction errors and the predictive distribution is to simulate from these process distributions, using the bootstrap sample values for the means. In other words, for each triangle, we obtain simulated values of the incrementals, using the appropriate process distributions:

$$\frac{N_{ij}}{E_i}\bigg| H_{i+j-2} \sim Normal\left(\lambda_j^B, \frac{\sigma_j^2}{E_i}\right) \text{ and } \frac{D_{ij}}{X_{i,j-1}}\bigg| H_{i+j-2} \sim Normal\left(\delta_j^B, \frac{\tau_j^2}{X_{i,j-1}}\right).$$

These simulated values can then be used to obtain estimates of the outstanding liabilities. The variances of these distributions can be used to quantify the process variance, either analytically (as in Liu and Verrall, 2009) or using simulation

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as in this paper. An approximation to the full predictive distribution of the outstanding liabilities can be obtained from these simulated values in conjunction with the bootstrapping procedure.

Note that we have chosen here to use normal distributions for the process distributions. It would also be possible make other choices for these distributions, including using the distributions of the residuals to construct a process distribution, or to use a distribution-free approach. For more details of this, see Peters et al (2009), which demonstrates how to use bootstrap procedures for the distribution-free chain ladder model, without making parametric assumptions. When considering the tails of the distributions, for example when looking at measures such as VaR, the specific assumptions made about the process distribution can have an impact. We would recommend that these issues are considered in any implementation, but in this paper we show how to construct the bootstrapping procedure, illustrate the results when using normal process distributions, and compare them with the analytical results in Liu and Verrall (2009).

The algorithm bootstrapping Schnieper's model is set out in the Appendix. In section 4, we provide illustrations of the bootstrapping method, and compare with the analytical results.

4. Illustration

In this section, we illustrate the results by applying the bootstrapping methodology to the data from Schnieper (1991). The results are compared with the analytical methods, as well as the bootstrap estimation of the prediction error using Mack's approximation.

The data used by Schnieper consisted of an IBNR triangle, X_{ij}, and exposure, E_i, which are shown in Table 1. Tables 2 and 3 show the more detailed data, consisting of the new claims, N_{ij}, and the changes in the existing claims, $-D_{ij}$. These data were taken from a practical motor third party liability excess-of-loss pricing problem.

TABLE 1.

Cumulative IBNR (X_{ij}) and Exposure (E_i) for both new and existing claims.

Dev year / Accident year	1	2	3	4	5	6	7	Exposure
1	7.5	28.9	52.6	84.5	80.1	76.9	79.5	10,224
2	1.6	14.8	32.1	39.6	55.0	60.0		12,752
3	13.8	42.4	36.3	53.3	96.5			14,875
4	2.9	14.0	32.5	46.9				17,365
5	2.9	9.8	52.7					19,410
6	1.9	29.4						17,617
7	19.1							18,129

TABLE 2.

INCREMENTAL INCURRED CLAIMS FROM NEW CLAIMS (N_{ij}).

Accident year \ Dev year	1	2	3	4	5	6	7
1	7.5	18.3	28.5	23.4	18.6	0.7	5.1
2	1.6	12.6	18.2	16.1	14	10.6	
3	13.8	22.7	4	12.4	12.1		
4	2.9	9.7	16.4	11.6			
5	2.9	6.9	37.1				
6	1.9	27.5					
7	19.1						

TABLE 3.

INCREMENTAL INCURRED CLAIMS FROM EXISTING CLAIMS (D_{ij}).

Accident year \ Dev year	2	3	4	5	6	7
1	–3.1	4.8	–8.5	23	3.9	2.5
2	–0.6	0.9	8.6	–1.4	5.6	
3	–5.9	10.1	–4.6	–31.1		
4	–1.4	–2.1	–2.8			
5	0	–5.8				
6	0					

TABLE 4.

A COMPARISON OF BOOTSTRAP AND ANALYTICAL RESULTS.

	Reserves		**Prediction Errors**		**Prediction Errors %**	
	Analytical	**Bootstrap**	**Analytical**	**Bootstrap**	**Analytical**	**Bootstrap**
$i = 2$	4.4	4.3	9.5	9.4	215%	217%
$i = 3$	4.8	4.8	14.3	14.4	298%	299%
$i = 4$	32.9	33.2	29.8	31.4	91%	95%
$i = 5$	60.3	61.1	41.2	43.0	68%	70%
$i = 6$	77.2	77.6	43.5	45.6	56%	59%
$i = 7$	104.3	104.8	49.2	51.5	47%	49%
Overall Total	283.9	285.8	121.9	122.9	43%	43%

Table 4 shows a comparison of the results using the analytical methods derived in Liu and Verrall (2009) and the bootstrap results. The bootstrap results were obtained using the estimation variance approach which is adopted by England and Verrall (2002) and Buchwalder et al (2006), so that the bootstrap estimate of the approximation was obtained by calculating the sample variance of the bootstrap reserves.

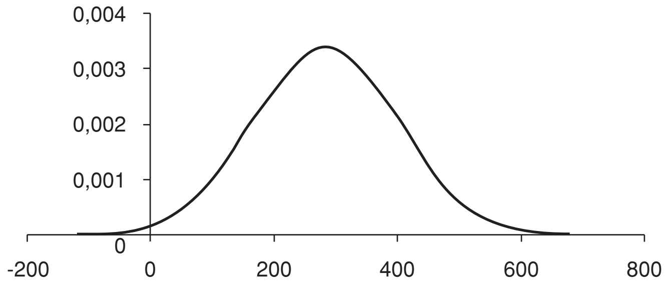

FIGURE 1. Bootstrap Predictive Distribution of the Schnieper Overall Reserve.

It can be seen that there is a good agreement between the analytical results and those obtained using bootstrapping (allowing for the fact that bootstrapping is a simulation-based method).

A major advantage of using bootstrapping over the analytical approach is that it is also possible to obtain a simulation of the predictive distribution. This is illustrated in Figure 1, which shows the predictive distribution of the overall reserve for the Schnieper method, smoothed using a Kernel smoother with bandwidth 50.

As mentioned in Section 3, there are two approximation approaches described in Liu and Verrall (2009): Tables 5 and 6 compare the differences when following these two approaches. The column labeled E&V (2002) corresponds to the approach adopted by England and Verrall (2002) and Buchwalder et al (2006). The second column shows the results using the approach of Mack (1993). In the first approach, the estimation error is approximated using the sample variance of the bootstrap reserves, and in the second approach, the bootstrap estimate of the estimation error is obtained by calculating the average squared difference between the bootstrap reserve estimate and the original reserve estimate. In both cases, this is done before the sampling from the process distribution is carried out in order to obtain an estimate of the MSEP.

TABLE 5.

A COMPARISON OF BOOTSTRAP ESTIMATION ERRORS.

	E & V (2002)	**Mack (1993)**
$i = 2$	6.929	6.938
$i = 3$	10.040	10.061
$i = 4$	16.183	16.384
$i = 5$	23.689	23.883
$i = 6$	23.629	23.897
$i = 7$	27.677	27.976
Overall Total	98.017	99.020

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TABLE 6.

A COMPARISON OF BOOTSTRAP PREDICTION ERRORS.

	E & V (2002)	**Mack (1993)**
$i = 2$	9.361	9.266
$i = 3$	14.399	14.330
$i = 4$	31.414	31.735
$i = 5$	43.017	43.333
$i = 6$	45.553	45.598
$i = 7$	51.490	51.817
Overall Total	122.893	124.116

5. CONCLUSION

This paper has shown how bootstrapping can be applied in the context of the Schnieper method of claims reserving. This is a novel application, because it involves bootstrapping two separate triangles. The illustration shows that it is possible to reproduce the MSEP of the analytical methods that were derived in Liu and Verrall (2009). There are a number of practical advantages of the bootstrapping approach. Firstly, it is straightforward to implement in a spreadsheet, giving it great practical appeal. Secondly, and perhaps more importantly, it is also possible to obtain the full predictive distribution. In the context of capital modeling and solvency, it is important to have the full predictive distribution in order to examine quantiles as well as the first two moments of the distribution. In this context, we would emphasise again that the effect of the process distribution should also be considered.

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APPENDIX

This Appendix provides the algorithm, step by step, which is needed in order to implement the bootstrap process described in section 2.

1. Calculate the link ratios and the variances of the link ratios for true IBNR and IBNER run-off triangles as $f_{ij} = \frac{N_{ij}}{E_i}$ and $g_{ij} = \frac{D_{ij}}{X_{i,j-1}}$. Note that the variances, σ_j^2 and τ_j^2, remain unchanged throughout: they are not recalculated from the bootstrap samples.

2. Calculate the scaled Pearson residuals:
$$r_{PS}(f_{ij}, \hat{\lambda}_j, E_i, \hat{\sigma}_j) = \frac{\sqrt{E_i}\left(f_{ij} - \hat{\lambda}_j\right)}{\hat{\sigma}_j}$$
and
$$r_{PS}(g_{ij}, \hat{\delta}_j, X_{i,j-1}, \hat{\tau}_j) = \frac{\sqrt{X_{i,j-1}}\left(g_{ij} - \hat{\delta}_j\right)}{\hat{\tau}_j}$$

3. Adjust these two groups of scaled Pearson residuals by multiplying by $\sqrt{\frac{n-j}{n-j+1}}$ to correct the bootstrap bias:
$$r_{ij} = \sqrt{\frac{n-j}{n-j+1}}\, r_{PS}(f_{ij}, \hat{\lambda}_j, E_i, \hat{\sigma}_j) \text{ and } s_{ij} = \sqrt{\frac{n-j}{n-j+1}}\, r_{PS}(g_{ij}, \hat{\delta}_j, X_{i,j-1}, \hat{\tau}_j).$$

   Start the iterative loop to be repeated N times ($N \geq 1000$).

4. Set $B = 1$.

5. Randomly draw, with replacement, from the constructed residual run-off triangles, denoted as $R = \{r_{ij}, i=1,\ldots,n; j=1,\ldots n-i+1\}$ and $S = \{s_{ij}, i=1,\ldots,n; j=1,\ldots n-i+1\}$, respectively. Denote the bootstrap residuals as r_{ij}^B and s_{ij}^B, $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n-i+1$, so that two pseudo samples of the Pearson residuals for true IBNR and IBNER claims are created and denoted as $R^B = \{r_{ij}^B, i=1,\ldots,n; j=1,\ldots n-i+1\}$ and $S^B = \{s_{ij}^B, i=1,\ldots,n; j=1,\ldots n-i+1\}$.

6. Calculate the bootstrap link ratios of the true IBNR and IBNER, f_{ij}^B and g_{ij}^B using equations (3) and (4).

7. Calculate the N_{ij}-weighted and D_{ij}-weighted average bootstrap development factors for the true IBNR and IBNER, λ_j^B and δ_j^B, using equations (1) and (2), respectively.

8. Simulate a future payment for each cell in the lower triangle for both true IBNR and IBNER claims, respectively, from the process distribution with the mean calculated from step 7.

$$\frac{N_{ij}}{E_i}\bigg| H_{i+j-2} \sim Normal\left(\lambda_j^B, \frac{\sigma_j^2}{E_i}\right) \text{ for the true IBNR claims}$$

and $$\frac{D_{ij}}{X_{i,j-1}}\bigg| H_{i+j-2} \sim Normal\left(\delta_j^B, \frac{\tau_j^2}{X_{i,j-1}}\right) \text{ for the future IBNER claims.}$$

9. Use the simulated predicted incremental claims, $-D_{ij} + N_{ij}$, from step 8 to obtain reserve estimates.

10. Store the results, set $B = B + 1$ and return to step 5 (the start of the iterative loop) until $B = N$.

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