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Vector Solutions for Azimuth

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Vector based methods applied to navigational problems offer some advantages in the calculation of distance and azimuth. In this paper, the methods of vector analysis are used to develop a variety of expressions for azimuth that are suited to both the syntax of computer algorithms and commercial mathematical software. The solutions presented here do not require recourse to the rules of spherical trigonometry - a distinct advantage when dealing with the spheroid. In the process of preparing this paper, an error of principle that occurred in a previous work was revealed and is repaired here.

KEY WORDS

1. Azimuth. 2. Great Circle. 3. Great Ellipse. 4. Vectors.

1. GENERAL. On the spheroidal earth generated by a reference ellipse rotated about its minor axis, two known positions lying on the great ellipse (GE) at positions P_1 and P_2 define two vectors of position $\mathbf{X}_1$ and $\mathbf{X}_2$ that are in turn specified by their geocentric latitude and longitude co-ordinates ϕ_1, θ_1 and ϕ_2, θ_2 respectively. The general vector of position is given as:

$$\mathbf{X} = r[\hat{\mathbf{i}}\cos\phi\cos\theta + \hat{\mathbf{j}}\cos\phi\sin\theta + \hat{\mathbf{k}}\sin\phi] \tag{1}$$

$$\text{where} \qquad r = a\sqrt{\frac{1-\varepsilon^2}{1-\varepsilon^2\cos^2\phi}} \tag{2}$$

Here, a is the equatorial radius, ε is the ellipticity of the generating ellipse and $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ are unit vectors. The geodetic latitude ψ and the geocentric latitude are related by $tan\phi = (1-\varepsilon^2)tan\psi$; the longitude is θ and all are in radians. Three unit vectors associated with the position vector are the meridian tangent vector $\hat{\mathbf{u}}_\phi$, the parallel tangent vector $\hat{\mathbf{u}}_\theta$ and the unit normal $\hat{\mathbf{n}}$. These are defined as:

$$\hat{\mathbf{u}}_\phi = -\hat{\mathbf{i}}\sin\psi\cos\theta - \hat{\mathbf{j}}\sin\psi\sin\theta + \hat{\mathbf{k}}\cos\psi$$

$$\hat{\mathbf{u}}_\theta = -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta$$

$$\hat{\mathbf{n}} = \hat{\mathbf{i}}\cos\psi\cos\theta + \hat{\mathbf{j}}\cos\psi\sin\theta + \hat{\mathbf{k}}\sin\psi$$

In other expressions to follow, there is a unit polar vector $\mathbf{X}_p = \hat{\mathbf{k}}$. Also, latitude and longitude co-ordinates are designated as N, S, E or W and are assigned algebraic significance with N = E = 1 and S = W = −1.

2. EQUATION OF THE GREAT CIRCLE OR GREAT ELLIPSE. From two surface positions it is a simple matter to determine the equation of the GE upon which they lie. The GE is a solution[2] to $\mathbf{X}\cdot(\mathbf{X}_1\times\mathbf{X}_2)=0$ and has the following form in geocentric co-ordinates:

$$\tan\phi=-(\lambda\cos\theta+\mu\sin\theta) \tag{3}$$

and in geodetic co-ordinates on the spheroid

$$\tan\psi=-\frac{(\lambda\cos\theta+\mu\sin\theta)}{1-\varepsilon^2} \tag{4}$$

Equation 3 also describes the great circle (GC) on the sphere.

The coefficients λ and μ are: $\lambda=m/p$ and $\mu=n/p$, where the latter terms arise from the vector $\hat{\mathbf{i}}m+\hat{\mathbf{j}}n+\hat{\mathbf{k}}p=\mathbf{X}_1\times\mathbf{X}_2$, which is perpendicular to the plane of the GE and has a unit vector is given by:

$$\hat{\mathbf{X}}_{12}=\frac{\mathbf{X}_1\times\mathbf{X}_2}{|\mathbf{X}_1\times\mathbf{X}_2|}=\frac{\hat{\mathbf{i}}\lambda+\hat{\mathbf{j}}\mu+\hat{\mathbf{k}}}{\sqrt{\lambda^2+\mu^2+1}} \tag{5}$$

The highest latitude N or S reached by the GC or GE is at a turning point or vertex ϕ_v, which occurs when $\frac{d\phi}{d\theta}=0$ i.e. when

$$\frac{\lambda\sin\theta-\mu\cos\theta}{\sec^2\phi}=0 \tag{6}$$

whose solution is the vertex longitude $\theta_v=\arctan\left(\frac{\mu}{\lambda}\right)$. Consequently,

$$\mu=\lambda\tan\theta_v\ :\ \lambda=\mu\,\text{cotan}\theta_v \tag{7}$$

Substitution of these two terms back into Equation 3 leads to alternatives for Equations 3 and 4 so then the geocentric latitude on the GC or GE becomes:

$$\tan\phi=\sqrt{\lambda^2+\mu^2}\cos(\theta_v-\theta) \tag{8}$$

and when $\theta=\theta_v$

$$\tan\phi_v=\sqrt{\lambda^2+\mu^2} \tag{9}$$

thus for the GC or GE in geocentric co-ordinates

$$\tan\phi=\tan\phi_v\cos(\theta_v-\theta) \tag{10}$$

and for the GE in geodetic co-ordinates

$$\tan\psi=\frac{\tan\phi_v\cos(\theta_v-\theta)}{1-\varepsilon^2} \tag{11}$$

3. AZIMUTH SOLUTION FOR THE GREAT CIRCLE. For any track including a GE or a GC, the azimuth angle γ as defined here, is the angle measured from north to π or 180° in either direction towards the track where it crosses a meridian going towards P_2. In the case of the GC, it is also the angle between the unit vector normal to the meridian plane and the unit vector normal to

the plane of the GC. Azimuth on the GC at a departure point P_1 can therefore be conveniently expressed an terms of position vectors as:

$$\gamma_1 = arcos\left(\frac{(\mathbf{X}_1 \times \mathbf{X}_\mathrm{p})}{|\mathbf{X}_1 \times \mathbf{X}_\mathrm{p}|} \cdot \frac{(\mathbf{X}_1 \times \mathbf{X}_2)}{|\mathbf{X}_1 \times \mathbf{X}_2|}\right) \tag{12}$$

In a more general case, using the normal to the meridian of any point other than P_1, then azimuth can be established by changing Equation 12 as follows:

$$\gamma = arcos\left(\frac{(\mathbf{X} \times \mathbf{X}_\mathrm{p})}{|\mathbf{X} \times \mathbf{X}_\mathrm{p}|} \cdot \frac{(\mathbf{X}_1 \times \mathbf{X}_2)}{|\mathbf{X}_1 \times \mathbf{X}_2|}\right) \tag{13}$$

This equation avoids a discontinuity that can occur as $\mathbf{X}$ meets $\mathbf{X}_2$ and is therefore more versatile. It is easily shown that when expanded, this equation reduces to the conventional expression for azimuth[1].

4. AZIMUTH SOLUTIONS FOR THE GREAT ELLIPSE.

Equation 12 or 13 will give a good approximate solution for azimuth along the GE on the spheroid but they are not however a correct formulation of the geometry of the problem as it applies to the spheroid. The minute numerical error that would result exists because, unlike on the sphere, where the vectors normal to the meridian and the GC both lie in the tangent plane, only the unit vector perpendicular to the meridian lies in the tangent plane, while the unit vector normal to the GE, though very close, does not. This error of principle occurred in the Author's earlier paper published in the *Journal* [2] and opportunity is taken here to repair that error.

Figure 1 shows the tangent plane at the point of tangency P_x located at the end of vector $\mathbf{X}$. The tangent plane is a small rectilinear area at the surface perpendicular to the unit normal $\hat{\mathbf{n}}$. Also shown are position vectors $\mathbf{X}_1$ and $\mathbf{X}_2$ that define the unit vector $\hat{\mathbf{X}}_{12}$ that lies perpendicular to the plane of the GE but not perpendicular to the unit normal vector $\hat{\mathbf{n}}$ and which is also elevated by a small angle δ above the tangent plane. Also,$\hat{\mathbf{X}}_{12}$ lies in the same vertical plane containing $\hat{\mathbf{X}}_\mathbf{u}$ and the unit normal $\hat{\mathbf{n}}$. The proper definition for azimuth in this case is:

$$\gamma = \arccos\left[\frac{\mathbf{X} \times \mathbf{X}_p}{|\mathbf{X} \times \mathbf{X}_p|} \cdot \hat{\mathbf{X}}_\mathbf{u}\right] \tag{14}$$

Here, the unit vector $\hat{\mathbf{X}}_\mathbf{u}$ is the image of $\hat{\mathbf{X}}_{12}$ in the tangent plane and $\mathbf{X}_p$ is the previously defined polar vector.

In order to determine the vector $\hat{\mathbf{X}}_\mathbf{u}$, we note first that the vector $\hat{\mathbf{X}}_{12}$ can be considered as a linear combination of $\hat{\mathbf{X}}_\mathbf{u}$, which is to be determined, and $\hat{\mathbf{n}}$ i.e.:

$$\hat{\mathbf{X}}_{12} = \cos\delta\ \hat{\mathbf{X}}_u + \sin\delta\ \hat{\mathbf{n}}$$

Then, on re-arranging

$$\hat{\mathbf{X}}_u = \hat{\mathbf{X}}_{12}\sec\delta - \hat{\mathbf{n}}\tan\delta \tag{15}$$

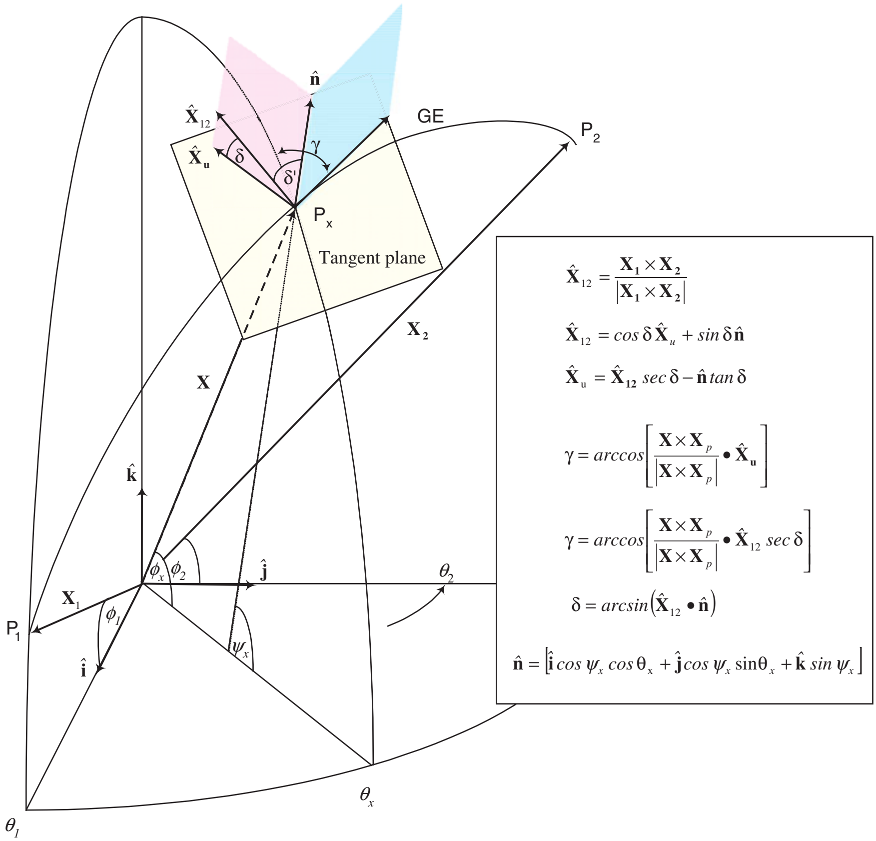

Figure 1.

Substituting Equation 15 into Equation 14, the expression for azimuth becomes:

$$\gamma = \arccos\left[\frac{\mathbf{X}\times\mathbf{X}_p}{|\mathbf{X}\times\mathbf{X}_p|}\cdot(\hat{\mathbf{X}}_{12}\sec\delta - \hat{\mathbf{n}}\tan\delta)\right] \tag{16}$$

But since $\frac{\mathbf{X}\times\mathbf{X}_p}{|\mathbf{X}\times\mathbf{X}_p|}$ and $\hat{\mathbf{n}}$ are orthogonal, their scalar product vanishes leaving:

$$\gamma = \arccos\left[\frac{\mathbf{X}\times\mathbf{X}_p}{|\mathbf{X}\times\mathbf{X}_p|}\cdot\hat{\mathbf{X}}_{12}\sec\delta\right] \tag{17}$$

$$i.e. \qquad \gamma = arcos\left(\frac{(\mathbf{X}\times\mathbf{X}_{\mathrm{p}})}{|\mathbf{X}\times\mathbf{X}_{\mathrm{p}}|}\cdot\frac{(\mathbf{X}_1\times\mathbf{X}_2)}{|\mathbf{X}_1\times\mathbf{X}_2|}\sec\delta\right) \tag{18}$$

To repair the error [2, Equation 29] where the nomenclature there employed **V** as the symbol for vector quantity, it is only necessary to include the term *sec*δ or to restructure the equation to the form of Equation 18 above.

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Expanding $\frac{\mathbf{X} \times \mathbf{X}_p}{|\mathbf{X} \times \mathbf{X}_p|}$ gives $\hat{\mathbf{i}} \sin\theta - \hat{\mathbf{j}} \cos\theta$, which is recognized as $-\hat{\mathbf{u}}_\theta$. Then Equation 17 becomes:

$$\gamma = \arccos\left(-\hat{\mathbf{u}}_\theta \cdot \hat{\mathbf{X}}_{12} \sec\delta\right) \tag{19}$$

Also, when Equation 5 is substituted into Equation 17 together with the expansion for $-\hat{\mathbf{u}}_\theta$ above, a scalar equation for azimuth is obtained. e.g.:

$$\gamma = \arccos\left(\frac{(\lambda \sin\theta - \mu \cos\theta) \sec\delta}{\sqrt{\lambda^2 + \mu^2 + 1}}\right) \tag{20}$$

From Figure 1, the angle $\delta' = \arccos\left(\hat{\mathbf{X}}_{12} \cdot \hat{\mathbf{n}}\right)$ and the small angle δ is its complement so

$$\delta = \arcsin\left(\hat{\mathbf{X}}_{12} \cdot \hat{\mathbf{n}}\right) \tag{21}$$

which, on substituting Equation 5, produces an expression for δ namely:

$$\delta = \arcsin\left(\frac{\varepsilon^2 \sin|\psi|}{\sqrt{\lambda^2 + \mu^2 + 1}}\right) \tag{22}$$

Figure 2 is similar to Figure 1 but with the addition of unit vectors tangent to the meridian, the parallel and the GE. Unit vector $\hat{\mathbf{t}}$ tangent to the GE at the point of tangency is the product:

$$\hat{\mathbf{t}} = \frac{\hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}}{\left|\hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}\right|} \tag{23}$$

As Figure 2 shows, the azimuth angle γ is between $\hat{\mathbf{u}}_\phi$ and $\hat{\mathbf{t}}$ therefore

$$\gamma = \arccos\left(\hat{\mathbf{u}}_\phi \cdot \hat{\mathbf{t}}\right) \tag{24}$$

or

$$\gamma = \arccos\left(\hat{\mathbf{u}}_\phi \cdot \left(\frac{\hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}}{\left|\hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}\right|}\right)\right) \tag{25}$$

Since $\hat{\mathbf{X}}_{12}$ and $\hat{\mathbf{u}}_\phi$ are each unit vectors it might be tempting to put $\hat{\mathbf{t}} = \hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}$. This would be incorrect because the two unit vectors are not orthogonal. We note however, that in the denominator, $\left|\hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}\right| = \left|\hat{\mathbf{X}}_{12}\right| |\hat{\mathbf{n}}| \sin\delta' = \sin\delta' = \cos\delta$ since $\left|\hat{\mathbf{X}}_{12}\right|$ and $|\hat{\mathbf{n}}|$ are each unity, thus if one wished, Equation 24 could be written as:

$$\gamma = \arccos\left(\hat{\mathbf{u}}_\phi \cdot (\hat{\mathbf{X}}_{12} \times \hat{\mathbf{n}}) \sec\delta\right) \tag{26}$$

though Equation 24 might be preferred for its compactness.

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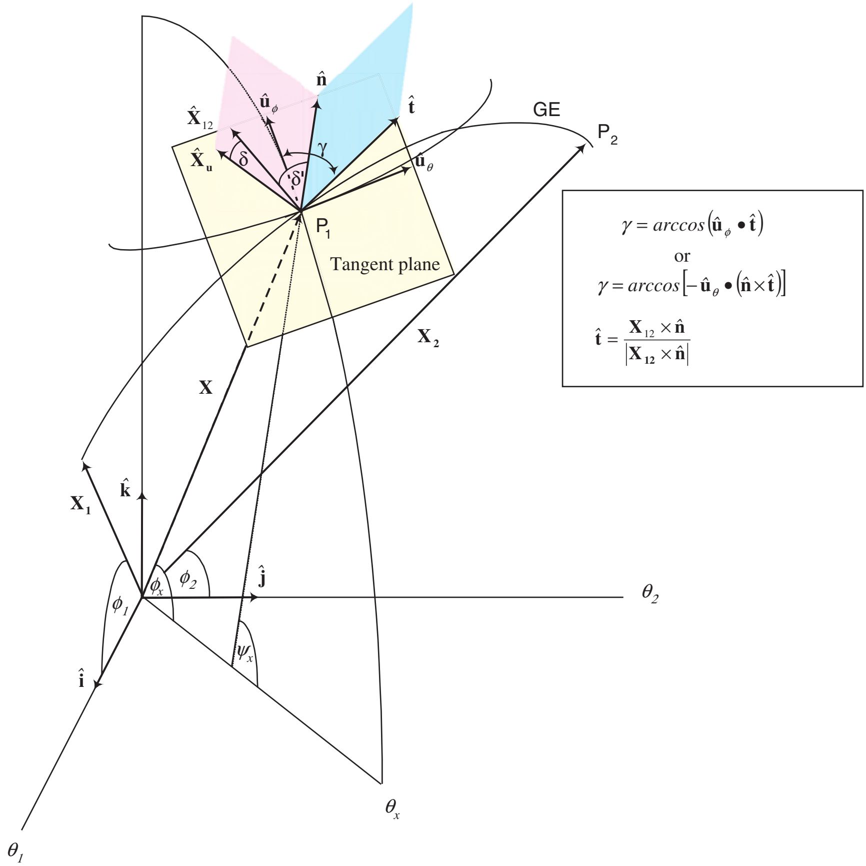

Figure 2.

Another vector statement for azimuth emerges upon re-visiting Equation 14 and considering Figure 2 again. From the figure it is clear that:

$$\hat{\mathbf{X}}_u = \hat{\mathbf{n}} \times \hat{\mathbf{t}} \tag{27}$$

Then, with $-\hat{\mathbf{u}}_\theta$ substituted for $\frac{\mathbf{X} \times \mathbf{X}_p}{|\mathbf{X} \times \mathbf{X}_p|}$, Equation 14 can be written compactly as:

$$\gamma = \arccos[-\hat{\mathbf{u}}_\theta \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{t}})] \tag{28}$$

Vector equations 19, 24 and 28 are compact expressions that do not require special rules for their evaluation.

Another approach is one that makes use of the magnitudes of the orthogonal components of a differential arc on the GE. Figure 3 illustrates the GE curve s and the components of a differential arc on s at P, the terminus of the vector $\mathbf{X}$ whose geocentric co-ordinates are ϕ and θ.

With $\mathbf{X}_\phi = \frac{\partial \mathbf{X}}{\partial \phi}$ and $\mathbf{X}_\theta = \frac{\partial \mathbf{X}}{\partial \theta}$, the following equations provide the needed components for azimuth. The projection of a differential element of arc ds on the meridian is:

$$\mathrm{d}\ s\ \cos\gamma = |\mathbf{X}_\varphi| \mathrm{d}\phi \tag{29}$$

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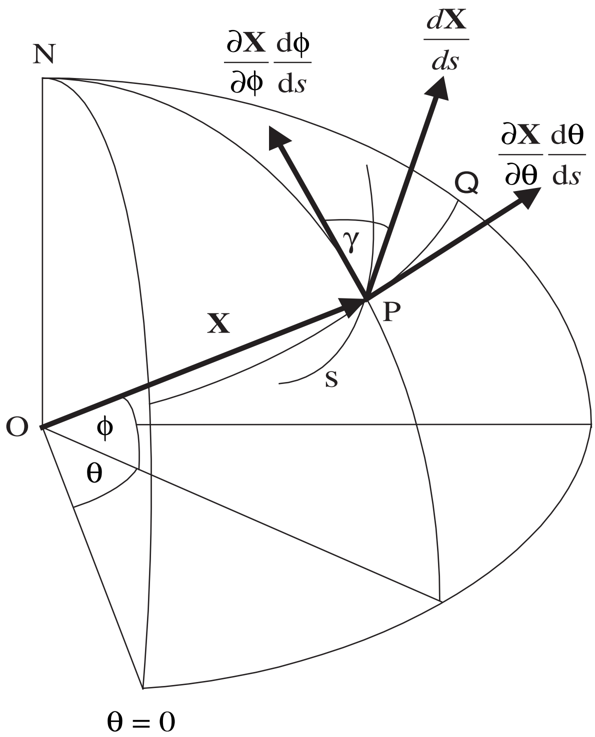

Figure 3.

and its projection on the parallel is

$$\mathrm{d}\ s\ \sin\gamma = |\mathbf{X}_\theta|\mathrm{d}\theta \tag{30}$$

and so

$$\tan\gamma = \frac{|\mathbf{X}_\theta|}{|\mathbf{X}_\phi|}\frac{\mathrm{d}\theta}{\mathrm{d}\phi} \tag{31}$$

Applying these operations to the vector defined in Equation 1, then after some manipulation and simplification, we arrive at:

$$|\mathbf{X}_\phi| = \mathrm{r}\left(1 + \left(\frac{\dot{\mathrm{r}}}{\mathrm{r}}\right)^2\right)^{\frac{1}{2}}$$

$$\text{and} \qquad |\mathbf{X}_\theta| = \mathrm{r}\cos\theta$$

Therefore

$$\gamma = \arctan\left[\frac{\cos\phi}{\sqrt{\left(1+\left(\frac{\dot{\mathrm{r}}}{\mathrm{r}}\right)^2\right)}}\frac{\mathrm{d}\theta}{\mathrm{d}\phi}\right] \tag{32}$$

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Using Equation 2, $\frac{\dot{r}}{r} = \frac{1}{r}\frac{dr}{d\varphi} = \frac{-\varepsilon^2 \tan\phi}{(1-\varepsilon^2) + \tan^2\phi}$ which, when substituted into the denominator of Equation 32, provides a scalar expression for azimuth:

$$\gamma = \arctan\left(\frac{\cos\phi(\alpha + \tan^2\phi)}{\sqrt{(1+\tan^2\phi)(\alpha^2+\tan^2\phi)}}\frac{d\theta}{d\phi}\right) \tag{33}$$

Inserting Equation 6 into Equation 33 then

$$\gamma = \arctan\left(\frac{(\alpha + \tan^2\phi)}{\sqrt{(\alpha^2+\tan^2\phi)}(\lambda\sin\theta - \mu\cos\theta)}\right) \tag{34}$$

Converting this last expression to geodetic latitude, it then becomes:

$$\gamma = \arctan\left(\frac{(1 + \alpha\tan^2\psi)}{\sqrt{(1+\tan^2\psi)}(\lambda\sin\theta - \mu\cos\theta)}\right) \tag{35}$$

When Equations 20 and 35, are combined, other relationships for azimuth on the GE are obtained, e.g.:

$$\sin\gamma = \frac{1+\alpha\tan^2\psi}{\sqrt{1+\tan^2\psi}}\frac{\sec\delta}{\sqrt{\lambda^2+\mu^2+1}} \tag{36}$$

With re-arrangement using Equation 9, Equation 36 can be re-written as:

$$\frac{\cos\psi\sin\gamma}{1-\varepsilon^2\sin^2\psi} = \cos\phi_v\sec\delta \tag{37}$$

Which reduces for the GC ($\varepsilon = 0$, $\Psi = \phi$) to:

$$\cos\phi\sin\gamma = \cos\phi_v \tag{38}$$

These two expressions show that the azimuth γ_0 on crossing the equator ($\phi = 0$) is related to the vertex (N or S) towards which the track is aiming by:

$$\gamma_0 = \frac{\pi}{2} - \phi_v \tag{39}$$

5. CONCLUDING REMARKS.

Methods requiring rules for spherical triangles to obtain solutions have served navigation calculations based on a sphere quite well over the years. Unfortunately, these methods are not helpful when performing navigation calculations on the spheroidal Earth. Results from the analysis shown here demonstrate that vector methods provide a direct path towards acquiring solutions for azimuth. The results obtained provide three compact vector expressions plus other scalar expressions relating latitude and azimuth. The variety of expressions developed here provides the navigator with a selection of methods for calculating azimuth. A by-product of this work has been the repair of an error of principle in a previous paper. It goes without saying of course that the vector solutions presented here work for the sphere just as well when ellipticity is set to zero.

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