

THE STOKES STRUCTURE OF A GOOD MEROMORPHIC FLAT BUNDLE

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Dedicated to Professor Pierre Schapira

Abstract We give a survey on the Stokes structure of a good meromorphic flat bundle. We also show that a meromorphic flat bundle has the good formal structure if and only if it has a good lattice.

Keywords: meromorphic flat bundle; Stokes structure; Riemann–Hilbert–Birkhoff correspondence

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1. Introduction

In the study on wild harmonic bundles [21], we were led to various interesting progress in the theory of \mathcal{D} -modules and meromorphic flat bundles. In this paper, we shall give a survey on the classification of good meromorphic flat bundles in terms of Stokes filtrations, which can be regarded as a higher-dimensional generalization of Riemann–Hilbert–Birkhoff correspondence (see [5, 14, 15, 25]).

Such a classification has been well established and classical for meromorphic flat bundles on curves. It is done in two steps.

- (i) We take a pullback via a ramified covering and the formal completion along the pole, and then we obtain a nice decomposition, called the Hukuhara–Levelt–Turrittin decomposition.
- (ii) Although the decomposition is just formal in general, it can be lifted to a flat decomposition on each small sector, and the ambiguity of lifting leads us to the notion of Stokes structure.

In the higher-dimensional case, Majima [12] initiated a systematic study on asymptotic analysis for meromorphic flat bundles, and Sabbah [23] revisited it. In some sense, they studied the step (ii). Sabbah pointed out the significance of the step (i). In general, a meromorphic flat bundle on a higher-dimensional variety has a bad singularity called a

turning point, around which we cannot obtain a naive generalization of Hukuhara–Levelt–Turrittin decomposition only after taking the formal completion and the pullback via a ramified covering. So he proposed a conjecture which says that there exists a resolution of turning points for meromorphic flat bundles. In [19] and [21], we have established it in the algebraic case. (See also a survey paper [20].) Later, with a completely different methods, Kedlaya established it in a more general situation [9, 10].

For good meromorphic flat bundles, i.e. meromorphic flat bundles without turning points, we can naturally generalize the step (ii), as studied in [12] and [23] (see also [21]). In [21], we put our stress on a slightly different point from that in [12] and [23]. Although a flat decomposition on a small sector is not canonical, it canonically determines the filtration, called Stokes filtration. So we obtain a system of filtrations on multi-sectors, which is Stokes structure. It gives us a method for classification of good meromorphic flat bundles, i.e. the Riemann–Hilbert–Birkhoff correspondence. (See [23] and the more ambitious [24] for different formulations.) We hope that it would be a part of the foundation in the further study on meromorphic flat bundles and holonomic \mathcal{D} -modules.

In this paper, we shall give a review on Riemann–Hilbert–Birkhoff correspondence for good meromorphic flat bundles. Because our monograph [21] is long, contains several topics, and studies Stokes structure with additional data in a slightly generalized situation, the author expects that it might be useful to collect the related part from [21] in a simplified presentation. We will briefly mention applications of the Riemann–Hilbert–Birkhoff correspondence. One is the deformation of a good meromorphic flat bundle caused by variation of irregular values, which may be regarded as a higher-dimensional generalization in some results in [7]. Besides, it is important in our study on wild harmonic bundles, the author thinks that it is of independent interest. The other is an application to conjugate of holonomic \mathcal{D} -modules, which is essentially due to Sabbah [23].

We will also give a small complementary result on good formal structure and good lattice. In [23], Sabbah introduced the notion of good formal structure. Let X be a complex manifold, and let D be a simple normal crossing hypersurface of X . Let $D = \bigcup_{i \in \Lambda} D_i$ be the irreducible decomposition. For each $I \subset \Lambda$, we put $D_I^\circ := \bigcap_{i \in I} D_i \setminus \bigcup_{j \notin I} D_j$. We have the decomposition $D = \coprod D_I^\circ$. Let (\mathcal{E}, ∇) be a meromorphic flat bundle on (X, D) . Then, we say that (\mathcal{E}, ∇) has the unramifiedly good formal structure, if the formal completion of (\mathcal{E}, ∇) along D_I° has a Hukuhara–Levelt–Turrittin decomposition for each $I \subset \Lambda$, and we say that (\mathcal{E}, ∇) has the good formal structure, if it is locally the descent of a meromorphic flat bundle with the unramifiedly good formal structure. We have natural variants of this condition. For example, we have a weaker variant of good formal structure: the formal completion of (\mathcal{E}, ∇) at each point has a Hukuhara–Levelt–Turrittin decomposition after a ramified extension (see §2.1.4). A stronger variant is the existence of a good lattice. (See §2.3 of [21] for a good lattice. We will review a more specific notion of good Deligne–Malgrange lattice in §2.2.2 below.) In this paper, we shall show that these conditions are equivalent. It may be useful to clarify the theory of meromorphic flat bundles.

The author also gave a survey on good meromorphic flat bundles and Stokes structure in a different way in §5 of [22], based on an earlier version of [21]. In particular, an

inductive use of partial Stokes structure is explained. It might look complicated, it seems more convenient in some situation.

Contents of the paper

In §2, after reviewing the notion of good Deligne–Malgrange lattices in both the formal and complex analytic cases (§§2.1–2.2), we show that a meromorphic flat bundle has a good Deligne–Malgrange lattice if and only if its formal completion at each point has a good Deligne–Malgrange lattice (Proposition 2.18). In §3, we review the notion of Stokes structure. In §4, we discuss Riemann–Hilbert–Birkhoff correspondence. In §4.1, we review that a Stokes structure is associated to an unramifiedly good meromorphic flat bundle. In §4.2, we state the Riemann–Hilbert–Birkhoff correspondence for unramifiedly good meromorphic flat bundles. In §4.3, we explain the deformation of a good meromorphic flat bundles caused by variation of irregular values. In §4.4, we consider good meromorphic flat bundles on the conjugate complex manifolds, and its application to the theory of \mathcal{D} -modules.

2. Good meromorphic flat bundle

2.1. Formal case

2.1.1. Good set of irregular values

We use the partial order $\leq_{\mathbb{Z}^n}$ of \mathbb{Z}^n given by the comparison of each component, i.e. $\mathbf{a} \leq_{\mathbb{Z}^n} \mathbf{b} \iff a_i \leq b_i \ (\forall i)$. Let $\mathbf{0}$ denote the zero in \mathbb{Z}^n .

Let R_0 denote the ring of the formal power series $\mathbb{C}[[z_1, \dots, z_n]]$, and let R be the localization of R_0 with respect to $z_j \ (j = 1, \dots, \ell)$ for some $1 \leq \ell \leq n$, i.e. $R := R_0[z_1^{-1}, \dots, z_\ell^{-1}]$. For $\mathbf{m} = (m_i) \in \mathbb{Z}^n$, we put $\mathbf{z}^{\mathbf{m}} := \prod_{i=1}^n z_i^{m_i}$. For $f = \sum_{\mathbf{m} \in \mathbb{Z}^n} f_{\mathbf{m}} \mathbf{z}^{\mathbf{m}} \in R_0$, we put $\mathcal{S}(f) := \{\mathbf{m} \mid f_{\mathbf{m}} \neq 0\} \cup \{\mathbf{0}\}$. Let $\text{ord}(f)$ denote the minimum of $\mathcal{S}(f)$, if it exists.

For any $\mathbf{a} \in R/R_0$, we take any lift $\tilde{\mathbf{a}}$ to R , and we set $\text{ord}(\mathbf{a}) := \text{ord}(\tilde{\mathbf{a}})$, if the right-hand side exists. It is independent of the choice of a lift $\tilde{\mathbf{a}}$. (We will often use the same symbol \mathbf{a} to denote a lift to R in the subsequent argument.) Recall that a finite subset $\mathcal{I} \subset R/R_0$ is called a good set of irregular values, if the following conditions are satisfied:

- $\text{ord}(\mathbf{a})$ exists for each element $\mathbf{a} \in \mathcal{I}$;
- $\text{ord}(\mathbf{a} - \mathbf{b})$ exists for any two distinct $\mathbf{a}, \mathbf{b} \in \mathcal{I}$;
- the set $\{\text{ord}(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathcal{I}\}$ is totally ordered with respect to the partial order $\leq_{\mathbb{Z}^n}$ on \mathbb{Z}^n .

Remark 2.1. This kind of condition appeared in [23], but the third condition above is slightly stronger than that in [23]. We use it to simplify our inductive arguments in our study on Stokes structure as in [21].

Let \mathcal{I} be a good set of irregular values. Note that the set $\{\text{ord}(\mathbf{a}) \mid \mathbf{a} \in \mathcal{I}\}$ is totally ordered, because $\text{ord}(\mathbf{a}) \not\leq \text{ord}(\mathbf{b})$ and $\text{ord}(\mathbf{a}) \not\geq \text{ord}(\mathbf{b})$ imply that $\text{ord}(\mathbf{a} - \mathbf{b})$ does not exist.

We set $\mathbf{m}(0) := \min\{\text{ord}(\mathbf{a}) \mid \mathbf{a} \in \mathcal{I}\}$. We have the set $\mathcal{T}(\mathcal{I}) := \{\text{ord}(\mathbf{a} - \mathbf{b}) \mid \mathbf{a}, \mathbf{b} \in \mathcal{I}\}$ contained in $\mathbb{Z}_{\leq 0}^n$. Note $\mathbf{m}(0) \leq_{\mathbb{Z}^n} \mathbf{m}$ for any $\mathbf{m} \in \mathcal{T}(\mathcal{I})$, because $\mathbf{a}_{\mathbf{m}} \neq 0$ for some $\mathbf{a} \in \mathcal{I}$. Since $\mathcal{T}(\mathcal{I})$ is assumed to be totally ordered with respect to the partial order $\leq_{\mathbb{Z}^n}$, we can take a sequence $\mathcal{M} := (\mathbf{m}(0), \mathbf{m}(1), \mathbf{m}(2), \dots, \mathbf{m}(L), \mathbf{m}(L + 1))$ in $\mathbb{Z}_{\leq 0}^n$ with the following properties:

- $\mathcal{T}(\mathcal{I}) \subset \mathcal{M}$ and $\mathbf{m}(L + 1) = \mathbf{0}_n$;
- for each $p \leq L$, there exists $1 \leq \mathfrak{h}(p) \leq n$ such that $\mathbf{m}(p + 1) = \mathbf{m}(p) + \boldsymbol{\delta}_{\mathfrak{h}(p)}$, where the j th entry of $\boldsymbol{\delta}_j$ is 1, and the other entries are 0.

Note that such a sequence is not uniquely determined for \mathcal{I} . Put $i := \mathfrak{h}(0)$. For each $\mathbf{a} \in \text{Irr}(\nabla)$, we have the expansion $\mathbf{a} = \sum \mathbf{a}_j z_i^j$. We denote $\mathbf{a}_{m_i} z_i^{m_i}$ by $\bar{\eta}_{\mathbf{m}(0)}(\mathbf{a})$, although it depends on the choice of $\mathbf{m}(1)$. We obtain the map $\bar{\eta}_{\mathbf{m}(0)}: \mathcal{I} \rightarrow R$. For $\mathbf{a}, \mathbf{b} \in \mathcal{I}$, we have $\bar{\eta}_{\mathbf{m}(0)}(\mathbf{a}) = \bar{\eta}_{\mathbf{m}(0)}(\mathbf{b})$ if and only if $\mathbf{a}_{\mathbf{m}(0)} = \mathbf{b}_{\mathbf{m}(0)}$.

It is often convenient to use a coordinate system such that $\text{ord}(\mathbf{a} - \mathbf{b})$ and $\text{ord}(\mathbf{a})$ are contained in the set $\prod_{i=0}^{\ell} \mathbb{Z}_{<0}^i \times \mathbf{0}_{\ell-i}$ for any $\mathbf{a}, \mathbf{b} \in \mathcal{I}$. Such a coordinate system is called admissible for \mathcal{I} .

Let \mathcal{I} be a good set of irregular values with an auxiliary sequence $\mathbf{m}(0), \dots, \mathbf{m}(L)$. For each $\mathbf{a} \in \mathcal{I}$, we have the expansion $\mathbf{a} - \bar{\eta}_{\mathbf{m}(0)}(\mathbf{a}) = \sum_{\mathbf{m} \geq \mathbf{m}(1)} \mathbf{a}_{\mathbf{m}} \mathbf{z}^{\mathbf{m}}$. We obtain a finite set $\mathcal{S} := \{\mathbf{a}_{\mathbf{m}(1)} \mid \mathbf{a} \in \mathcal{I}\} \subset \mathbb{C}$. The following lemma is clear.

Lemma 2.2. *We take $c \in \mathbb{C} \setminus \mathcal{S}$. For each $\mathfrak{c} \in R$, the set $\mathcal{I}_{\mathfrak{c},c} := \{\mathbf{a} - \bar{\eta}_{\mathbf{m}(0)}(\mathbf{a}) - c\mathbf{z}^{\mathbf{m}(1)} \mid \bar{\eta}_{\mathbf{m}(0)}(\mathbf{a}) = \mathfrak{c}\}$ is a good set of irregular values.*

2.1.2. *Unramifiedly good meromorphic flat bundle*

Let \mathcal{M} be a finitely generated R -module. Recall that a connection of \mathcal{M} is a linear map $\nabla: \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_{R/\mathbb{C}}^1$ such that $\nabla(f \cdot s) = \text{d}f \cdot s + f \nabla s$ for $f \in R$ and $s \in \mathcal{M}$. It is called flat, if the curvature $\nabla \circ \nabla: \mathcal{M} \rightarrow \mathcal{M} \otimes \Omega_{R/\mathbb{C}}^2$ is 0. The inner product of ∇s and a derivative v of R over \mathbb{C} is denoted by $\nabla(v)s$.

A finitely generated R_0 -submodule $\mathcal{L} \subset \mathcal{M}$ is called a lattice, if $\mathcal{L} \otimes_{R_0} R = \mathcal{M}$.

Definition 2.3. A lattice \mathcal{L} of \mathcal{M} is called logarithmic, if $\nabla(z_i \partial_i) \mathcal{L} \subset \mathcal{L}$ for $i = 1, \dots, \ell$, and $\nabla(\partial_i) \mathcal{L} \subset \mathcal{L}$ for $i = \ell + 1, \dots, n$.

A lattice \mathcal{L} of \mathcal{M} is called \mathfrak{a} -logarithmic for $\mathfrak{a} \in R/R_0$ if (i) \mathcal{L} is R_0 -free and (ii) $\nabla - \text{d}\tilde{\mathfrak{a}}$ is logarithmic for a lift $\tilde{\mathfrak{a}}$ of \mathfrak{a} to R . If \mathcal{M} has an \mathfrak{a} -logarithmic lattice, it is called \mathfrak{a} -regular.

Definition 2.4. A lattice \mathcal{L} of \mathcal{M} is called unramifiedly good, if there exist a good set of irregular values $\text{Irr}(\nabla) \subset R/R_0$ and a decomposition

$$(\mathcal{L}, \nabla) = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla)} (\mathcal{L}_{\mathfrak{a}}, \nabla_{\mathfrak{a}}) \tag{2.1}$$

such that $\nabla_{\mathfrak{a}}$ are \mathfrak{a} -logarithmic. If \mathcal{M} has an unramifiedly good lattice, we say that \mathcal{M} is an unramifiedly good meromorphic flat bundle on R .

The decomposition (2.1) induces

$$\mathcal{M} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla)} \mathcal{L}_{\mathfrak{a}} \otimes_{R_0} R. \tag{2.2}$$

The decompositions (2.1) and (2.2) are called irregular decomposition of \mathcal{L} and \mathcal{M} , respectively.

Lemma 2.5. *Let \mathcal{L} and \mathcal{L}' be unramifiedly good lattices of \mathcal{M} with irregular decompositions $\mathcal{L} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla)} \mathcal{L}_{\mathfrak{a}}$ and $\mathcal{L}' = \bigoplus_{\mathfrak{a} \in \text{Irr}'(\nabla)} \mathcal{L}'_{\mathfrak{a}}$. Then, we have $\mathcal{L}_{\mathfrak{a}} \otimes_{R_0} R = \mathcal{L}'_{\mathfrak{a}} \otimes_{R_0} R$ for any $\mathfrak{a} \in \text{Irr}(\nabla) \cup \text{Irr}'(\nabla)$. In particular, the decomposition (2.1) is uniquely determined for \mathcal{L} , and the decomposition (2.2) is uniquely determined for \mathcal{M} .*

Proof. The claims are well known in the one-variable case. The several-variables case can be easily reduced to the one-variable case. □

Let \mathcal{L} be an unramifiedly good lattice of (\mathcal{M}, ∇) with the irregular decomposition (2.1). Let $\mathbf{m}(0)$ be the minimum of $\{\text{ord}(\mathfrak{a}) \mid \mathfrak{a} \in \text{Irr}(\nabla)\}$. We put $T := \{\mathfrak{a}_{\mathbf{m}(0)} \mid \mathfrak{a} \in \mathcal{I}\}$. Let $\pi: \text{Irr}(\nabla) \rightarrow T$ be the naturally defined map. For $\alpha \in T$, we put $\mathcal{L}_{\alpha} := \bigoplus_{\pi(\mathfrak{a})=\alpha} \mathcal{L}_{\mathfrak{a}}$. We have the ∇ -flat decomposition $\mathcal{L} = \bigoplus_{\alpha \in T} \mathcal{L}_{\alpha}$. We will implicitly use the following standard characterization of this decomposition.

Lemma 2.6. *Assume that the i th component of $\mathbf{m}(0)$ is negative. Let $\mathcal{L} = \bigoplus_{\alpha} \mathcal{L}'_{\alpha}$ be a decomposition such that*

- (i) $z^{-\mathbf{m}(0)} \nabla(z_i \partial_i) \mathcal{L}'_{\alpha} \subset \mathcal{L}'_{\alpha}$ for each $\alpha \in T$ and
- (ii) $\mathcal{L}'_{\alpha} \otimes_R \mathbb{C} = \mathcal{L}_{\alpha} \otimes_R \mathbb{C}$.

Then, we have $\mathcal{L}'_{\alpha} = \mathcal{L}_{\alpha}$.

Proof. By considering the eigendecomposition of the endomorphism of $\mathcal{L}/z_i \mathcal{L}$ induced by $z^{\mathbf{m}(0)} \nabla(z_i \partial_i)$, we obtain that $\mathcal{L}'_{\alpha}/z_i \mathcal{L}'_{\alpha} = \mathcal{L}_{\alpha}/z_i \mathcal{L}_{\alpha}$ in $\mathcal{L}/z_i \mathcal{L}$. By using Corollary 2.14 below, we obtain $\mathcal{L}_{\alpha} = \mathcal{L}'_{\alpha}$. □

2.1.3. Residue

If we are given an unramifiedly good lattice \mathcal{L} , we obtain an endomorphism $\text{Res}_i(\nabla)$ of $\mathcal{L}/z_i \mathcal{L}$ in a standard way. Namely, for any $f \in \mathcal{L}_{\mathfrak{a}}/z_i \mathcal{L}_{\mathfrak{a}}$, we take a lift $\tilde{f} \in \mathcal{L}$, and let $\text{Res}_i(\nabla_{\mathfrak{a}})f$ be induced by $\nabla_{\mathfrak{a}}^{\text{reg}}(z_i \partial_i) \tilde{f}$, where $\nabla_{\mathfrak{a}}^{\text{reg}} := \nabla_{\mathfrak{a}} - d\tilde{\mathfrak{a}}$ for a lift $\tilde{\mathfrak{a}}$ of \mathfrak{a} . We set $\text{Res}_i(\nabla) := \bigoplus \text{Res}_i(\nabla_{\mathfrak{a}}) \in \text{End}(\mathcal{L}/z_i \mathcal{L})$. It is well defined for (\mathcal{L}, ∇) in the sense that it is independent of the choice of lifts \tilde{f} , $\tilde{\mathfrak{a}}$ and the choice of the coordinate function z_i . It is well known and easy to see that the eigenvalues of $\text{Res}_i(\nabla)$ are contained in \mathbb{C} .

Definition 2.7. Let \mathcal{L} be an unramifiedly good lattice of \mathcal{M} . If the eigenvalues α of $\text{Res}_i(\nabla)$ satisfy $0 \leq \text{Re}(\alpha) < 1$, it is called an unramifiedly good Deligne–Malgrange lattice of \mathcal{M} .

It is standard and easy to show that such a lattice is unique, if it exists.

2.1.4. *Good meromorphic flat bundle*

For a positive integer, let $R^{(e)} := \mathbb{C}[[\zeta_1, \dots, \zeta_\ell, z_{\ell+1}, \dots, z_n]]$ for e th roots ζ_i of z_i . If a finitely generated R -module \mathcal{M} is equipped with a flat connection, $\mathcal{M}^{(e)} := R^{(e)} \otimes_R \mathcal{M}$ is equipped with an induced flat connection. It is naturally equipped with an action of the Galois group G of $R^{(e)}/R$. If $\mathcal{M}^{(e)}$ has an unramifiedly good Deligne–Malgrange lattice $\mathcal{L}^{(e)}$, it is also equipped with a natural G -action, because of the uniqueness of unramifiedly good Deligne–Malgrange lattice. The G -invariant part of $\mathcal{L}^{(e)}$ is called the descent of $\mathcal{L}^{(e)}$.

Definition 2.8. (\mathcal{M}, ∇) is called a good meromorphic flat bundle, if $(\mathcal{M}^{(e)}, \nabla^{(e)})$ is an unramifiedly good meromorphic flat bundle for some $e > 0$. In that case, the descent of the unramifiedly good Deligne–Malgrange lattice of $\mathcal{M}^{(e)}$ is called a good Deligne–Malgrange lattice of \mathcal{M} .

We remark that a meromorphic flat bundle does not have a good Deligne–Malgrange lattice, in general. If it exists, it is unique, which follows from the uniqueness of unramifiedly good Deligne–Malgrange lattice and the following standard and easy lemma.

Lemma 2.9. *Assume that (\mathcal{M}, ∇) is an unramifiedly good meromorphic flat bundle. Then, $(\mathcal{M}^{(e)}, \nabla^{(e)})$ is also an unramifiedly good meromorphic flat bundle for any e , and the unramifiedly good Deligne–Malgrange lattice of \mathcal{M} is the descent of that of $\mathcal{M}^{(e)}$.*

We recall that ramification of a good meromorphic flat bundle can be controlled by that of its irregular values.

Lemma 2.10. *Assume that $(\mathcal{M}^{(e)}, \nabla^{(e)})$ is unramifiedly good for $e > 0$. If $\text{Irr}(\nabla^{(e)}) \subset R/R_0$, (\mathcal{M}, ∇) is also unramifiedly good.*

Proof. Let G be the Galois group of $R^{(e)}/R$. Let $\mathcal{L}^{(e)} \subset \mathcal{M}^{(e)}$ be a G -equivariant unramifiedly good lattice of $\mathcal{M}^{(e)}$. Because the G -action on $\text{Irr}(\nabla^{(e)})$ is trivial, the irregular decomposition $\mathcal{L}^{(e)} = \bigoplus \mathcal{L}_\alpha^{(e)}$ is preserved by G . Hence, we have the decomposition of the G -invariant part $(\mathcal{L}^{(e)})^G = \bigoplus (\mathcal{L}_\alpha^{(e)})^G$, which gives the irregular decomposition of $(\mathcal{L}^{(e)})^G$, i.e. $(\mathcal{L}^{(e)})^G$ is an unramifiedly good lattice of \mathcal{M} . □

We recall a bound of ramification index, which is also standard and well known.

Lemma 2.11. *If (\mathcal{M}, ∇) is good, $(\mathcal{M}^{(e_0)}, \nabla^{(e_0)})$ is unramifiedly good, where $e_0 := (\text{rank } \mathcal{M})!$.*

Proof. It is well known in the one-variable case. Let us consider the several-variables case. Take e such that $(\mathcal{M}^{(e)}, \nabla^{(e)})$ is unramifiedly good. We may assume that e is divisible by e_0 . According to Lemma 2.10, we have only to show that $\text{Irr}(\nabla^{(e)}) \subset R^{(e_0)}/R_0^{(e_0)}$. Take $\alpha \in \text{Irr}(\nabla^{(e)})$ and $1 \leq i \leq \ell$. We have the expansion $\alpha = \sum \alpha_p \zeta_i^p$, where ζ_i is an e th root of z_i . By using the result in the one-variable case, we can observe that $\alpha_p = 0$ unless p is divisible by e/e_0 . Hence, we obtain that $\alpha \in R^{(e_0)}/R_0^{(e_0)}$. □

2.1.5. Preliminary from the one-variable case (appendix)

Let k be an integral domain over \mathbb{C} . We consider $\mathcal{R}_0 := k[[t]]$ and $\mathcal{R} := k((t))$, which are naturally equipped with a derivation ∂_t . An \mathcal{R} -module \mathcal{M} is called differential module, if it is equipped with the action of ∂_t such that $\partial_t(fs) = \partial_t(f)s + f\partial_t s$ for $f \in \mathcal{R}$ and $s \in \mathcal{M}$. We recall some basic facts on differential \mathcal{R} -modules from [11] for reference in our argument.

Extension of decomposition. Let \mathcal{M} be a finitely generated differential \mathcal{R} -free module with an \mathcal{R}_0 -free lattice \mathcal{L} such that $t^{M+1}\partial_t\mathcal{L} \subset \mathcal{L}$ for some $M > 0$. Note that we have an induced endomorphism G of $\mathcal{L} \otimes_{\mathcal{R}_0} k$. Assume that there exists decomposition $(\mathcal{L} \otimes_{\mathcal{R}_0} k, G) = (V_1, G_1) \oplus (V_2, G_2)$. For $i \neq j$, we have the endomorphism $\tilde{G}_{i,j}$ of $\text{Hom}(V_i, V_j)$ given by $\tilde{G}_{i,j}(f) = f \circ G_i - G_j \circ f$.

Lemma 2.12. *If $\tilde{G}_{i,j}$ are invertible for $(i, j) = (1, 2), (2, 1)$, then we have a decomposition $\mathcal{L} = \mathcal{L}_1 \oplus \mathcal{L}_2$ such that*

- (i) $t^{M+1}\partial_t\mathcal{L}_i \subset \mathcal{L}_i$ and
- (ii) $\mathcal{L}_i \otimes k = V_i$.

Proof. We give only a sketch of a proof, by following [11]. Let \mathbf{v} be a frame of \mathcal{L} with a decomposition $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ such that $\mathbf{v}_{i|t=0}$ give frames of V_i . Let A be the \mathcal{R}_0 -valued matrices determined by $t^{M+1}\partial_t\mathbf{v} = \mathbf{v}A$. Then, A has the following decomposition corresponding to $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$:

$$A = \begin{pmatrix} \Omega_1 & 0 \\ 0 & \Omega_2 \end{pmatrix} + \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix}.$$

Here, Ω_i are k -valued matrices determined by $G_i\mathbf{v}_i = \mathbf{v}_i\Omega_i$, and $A_{i,j}$ are $t\mathcal{R}_0$ -valued matrices. We consider a change of the base of the following form:

$$\mathbf{v}' = \mathbf{v}G, \quad G = I + \begin{pmatrix} 0 & X \\ Y & 0 \end{pmatrix}.$$

Here, the entries of X and Y are contained in $t\mathcal{R}_0$. We would like to take G such that

$$t^{M+1}\partial_t\mathbf{v}' = \mathbf{v}'B, \quad B = \begin{pmatrix} \Omega_1 + Q_1 & 0 \\ 0 & \Omega_2 + Q_2 \end{pmatrix}. \tag{2.3}$$

The relation of A , G and B are given by $AG + t^{M+1}\partial_tG = GB$. We obtain the equations $A_{11} + A_{12}Y + Q_1 = 0$ and $\Omega_2Y + A_{21} + A_{22}Y + t^{M+1}\partial_tY = Y\Omega_1 + YQ_1$. By eliminating Q_1 , we obtain the equation

$$\Omega_2Y - Y\Omega_1 + A_{21} + A_{22}Y + t^{M+1}\partial_tY + Y(A_{11} + A_{12}Y) = 0. \tag{2.4}$$

By the assumption, we have the invertibility of the endomorphism on the space of k -valued (r_2, r_1) -matrices, given by $Z \mapsto \Omega_2Z - Z\Omega_1$, where $r_i := \text{rank } \mathcal{L}_i$ ($i = 1, 2$). By using a t -expansion, we can find a solution of (2.4) in the space of $t\mathcal{R}_0$ -valued matrices. Similarly, we can find desired X and Q_2 . □

Uniqueness.

Lemma 2.13. *Let \mathcal{M} be an \mathcal{R} -free differential module. Assume that there exists an \mathcal{R}_0 -free lattice $\mathcal{L} \subset \mathcal{M}$ and $\mathfrak{a} \in \mathcal{R} \setminus \mathcal{R}_0$ such that $t\partial_t - t\partial_t\mathfrak{a}$ preserves \mathcal{L} . Then, any flat section of \mathcal{M} is 0.*

Proof. Take $f \in \mathcal{M}$ such that $\partial_t f = 0$. Assume $f \neq 0$, and we will deduce a contradiction. We can take $N \in \mathbb{Z}$ such that $t^N f \in \mathcal{L}$ and the induced element of $\mathcal{L}/t\mathcal{L}$ is non-zero. By the assumption, we have

$$\mathcal{L} \ni (t\partial_t - t\partial_t\mathfrak{a})(t^N f) = (N - t\partial_t\mathfrak{a})t^N f.$$

But, it is easy to see that $(N - t\partial_t\mathfrak{a})t^N f \notin \mathcal{L}$, and thus we have arrived at a contradiction. □

Let \mathcal{M}_i ($i = 1, 2$) be differential \mathcal{R} -free modules with \mathcal{R}_0 -free lattices \mathcal{L}_i such that $t\partial_t - t\partial_t\mathfrak{a}_i$ preserve \mathcal{L}_i .

Corollary 2.14. *Assume $\mathfrak{a}_1 - \mathfrak{a}_2 \neq 0$ in $\mathcal{R}/\mathcal{R}_0$. Then, any flat morphism $\mathcal{M}_1 \rightarrow \mathcal{M}_2$ is 0.*

Let \mathcal{M} be a differential \mathcal{R} -module. Let E be an \mathcal{R}_0 -lattice of \mathcal{M} such that $t^{m+1}\partial_t E \subset E$ for some $m > 0$. We have the induced endomorphism G of $E|_{t=0}$.

Lemma 2.15. *Let $s \in \mathcal{M}$. If G is invertible, we have $\partial_t s = 0$ if and only if $s = 0$.*

Let E_i ($i = 1, 2$) be lattices of \mathcal{M} such that $t^{m_i+1}\partial_t E_i \subset E_i$ for some $m_i > 0$. Let G_i be the endomorphism of $E_i|_{t=0}$ induced by $t^{m_i+1}\partial_t$.

Lemma 2.16. *Assume that G_i are semisimple and non-zero. Let T_i be the set of eigenvalues of G_i . Then, we have $m_1 = m_2$ and $T_1 = T_2$.*

Proof. By extending k , we may assume that the eigenvalues of G_i are contained in k . We have ∂_t -decomposition $E_i = \bigoplus_{\mathfrak{b} \in T_i} E_{i,\mathfrak{b}}$ such that $E_{i,\mathfrak{b}|t=0}$ is the eigenspace of G_i corresponding to \mathfrak{b} . We have the induced map $\varphi_{\mathfrak{c},\mathfrak{b}} : E_{1,\mathfrak{b}} \otimes \mathcal{R} \rightarrow E_{2,\mathfrak{c}} \otimes \mathcal{R}$. If $m_1 \neq m_2$ or if $m_1 = m_2$ but $\mathfrak{b} \neq \mathfrak{c}$, we have $\varphi_{\mathfrak{c},\mathfrak{b}} = 0$ by Lemma 2.15. Then, the claim of Lemma 2.16 follows. □

2.2. Complex analytic case

Let X be a complex manifold, and let D be a simple normal crossing hypersurface. Let $\mathcal{O}_X(*D)$ denote the sheaf of meromorphic functions whose poles are contained in D . For a point P of X , let $\mathcal{O}_{\hat{P}}$ be the completion of \mathcal{O}_X at P . Let $\mathcal{O}_{\hat{P}}(*D) := \mathcal{O}_{\hat{P}} \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)$. For an \mathcal{O}_X -module \mathcal{M} , let $\mathcal{M}|_{\hat{P}} := \mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{\hat{P}}$. We denote by $M(X, D)$ (respectively $H(X)$) the space of meromorphic functions on X whose poles are contained in D (respectively holomorphic functions on X).

2.2.1. *Unramifiedly good meromorphic flat bundle*

Let (\mathcal{E}, ∇) be a meromorphic flat bundle on (X, D) , \mathcal{E} is a locally free $\mathcal{O}_X(*D)$ -coherent sheaf with a flat connection $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$. Let E be a lattice of \mathcal{E} , i.e. \mathcal{O}_X -coherent subsheaf of \mathcal{E} such that $E \otimes \mathcal{O}_X(*D) = \mathcal{E}$.

Definition 2.17. (\mathcal{E}, ∇) is called an unramifiedly good meromorphic flat bundle, if and only if $(\mathcal{E}, \nabla)|_{\hat{P}}$ is an unramifiedly good meromorphic flat bundle for each $P \in D$.

We will show the following proposition in § 2.3.

Proposition 2.18. (\mathcal{E}, ∇) is an unramifiedly good meromorphic flat bundle, if and only if it has a lattice $E \subset \mathcal{E}$ such that $E|_{\hat{P}}$ is an unramifiedly good Deligne–Malgrange lattice of $(\mathcal{E}, \nabla)|_{\hat{P}}$ for each $P \in D$.

A lattice E as in the proposition is called an unramifiedly good Deligne–Malgrange lattice of (\mathcal{E}, ∇) . It is unique, if it exists. We will also prove the following proposition in § 2.3.

Proposition 2.19. Let (\mathcal{E}, ∇) be a good meromorphic flat bundle on (X, D) . Then, $\text{Irr}(\nabla, P)$ ($P \in D$) are contained in $\mathcal{O}_X(*D)_P/\mathcal{O}_{X,P}$, i.e. convergent. Moreover, the system of good set of irregular values satisfies the following condition.

- Take a sufficiently small neighbourhood X_P of P such that $\text{Irr}(\nabla, P) \subset M(X_P, D_P)$. Then, for $P' \in D_P$, $\text{Irr}(\nabla, P')$ is the image of $\text{Irr}(\nabla, P) \rightarrow \mathcal{O}_X(*D)_{P'}/\mathcal{O}_{X,P'}$.

In other words, $(\text{Irr}(\nabla, P) \mid P \in D)$ is a good system of irregular values on (X, D) in the following sense.

Definition 2.20. A good system of irregular values on (X, D) is a tuple of finite subsets $\mathcal{I}_P \subset \mathcal{O}_X(*D)_P/\mathcal{O}_{X,P}$ ($P \in D$) satisfying the property in Proposition 2.19. Namely,

- Take a sufficiently small neighbourhood X_P of P such that $\mathcal{I}_P \subset M(X_P, D_P)$. Then, for $P' \in D_P$, $\mathcal{I}_{P'}$ is the image of $\mathcal{I}_P \rightarrow \mathcal{O}_X(*D)_{P'}/\mathcal{O}_{X,P'}$.

Remark 2.21. We can obtain formal decompositions along the intersection of divisors. See Proposition 2.28 below. We can also obtain formal decompositions in various levels along the union of divisors. See §§ 2.4.3–2.4.4 of [21].

Remark 2.22. After we proved Proposition 2.18 and Proposition 2.19 for this paper, they are also included in the latest version of [21] because they are useful to simplify and clarify the theory.

2.2.2. *Good meromorphic flat bundle*

Definition 2.23. (\mathcal{E}, ∇) is called a good meromorphic flat bundle, if $(\mathcal{E}, \nabla)|_{\hat{P}}$ is a good meromorphic flat bundle for each $P \in D$.

For a point $P \in X$, let X_P denote a small neighbourhood of P in X , and we put $D_P := X_P \cap D$.

Corollary 2.24. *The following conditions are equivalent.*

- (A) (\mathcal{E}, ∇) is a good meromorphic flat bundle on (X, D) .
- (B) There uniquely exists a lattice $E \subset \mathcal{E}$ such that $E|_{\hat{P}}$ is a good Deligne–Malgrange lattice of $(\mathcal{E}, \nabla)|_{\hat{P}}$ for each $P \in D$.
- (C) There uniquely exists a lattice $E \subset \mathcal{E}$ with the following property.
 - For each $P \in D$, if we take a small neighbourhood X_P , there exist a ramified covering $\varphi_P : (X'_P, D'_P) \rightarrow (X_P, D_P)$ such that (i) $\varphi_P^*(\mathcal{E}, \nabla)$ is unramifiedly good and (ii) $E|_{X_P}$ is the descent of the unramifiedly good Deligne–Malgrange lattice of $\varphi_P^*(\mathcal{E}, \nabla)$.

Proof. The implications (C) \implies (B) \implies (A) are clear. Let us consider the implication (A) \implies (C). Let $P \in D$. If a small neighbourhood X_P is sufficiently small, we have a ramified covering $\varphi : (X'_P, D'_P) \rightarrow (X_P, D_P)$ such that $\varphi^*(\mathcal{E}, \nabla)$ has an unramifiedly good Deligne–Malgrange lattice E'_P , according to Lemma 2.11 and Proposition 2.18. We obtain the lattice E_P as the descent of E'_P . If we have ramified coverings $\varphi^{(i)} : (X_P^{(i)}, D_P^{(i)}) \rightarrow (X_P, D_P)$ ($i = 1, 2$) such that $\varphi^{(i)*}(\mathcal{E}, \nabla)$ have unramifiedly good Deligne–Malgrange lattices $E_P^{(i)}$, we can find ramified coverings $\psi^{(i)} : (X_P^{(3)}, D_P^{(3)}) \rightarrow (X_P^{(i)}, D_P^{(i)})$ ($i = 1, 2$) such that $\varphi^{(1)} \circ \psi^{(1)} = \varphi^{(2)} \circ \psi^{(2)} =: \psi$. Then, by Lemma 2.9, there exists an unramifiedly good Deligne–Malgrange lattice $E_P^{(3)}$ of $\psi^*(\mathcal{E}, \nabla)$, and $E_P^{(i)}$ ($i = 1, 2$) are the descent of $E_P^{(3)}$ with respect to $\psi^{(i)}$. Hence, we obtain that the descent of $E_P^{(i)}$ with respect to $\varphi^{(i)}$ are the same. Hence, by varying P and gluing E_P , we obtain the desired global lattice E . □

A lattice with the property in the theorem is called a good Deligne–Malgrange lattice.

2.3. Proof of Proposition 2.18 and Proposition 2.19

2.3.1. Deligne–Malgrange lattice

The ‘if’ part of Proposition 2.18 is clear. Our starting point for the proof of the ‘only if’ part is a result due to Malgrange, which we review here.

Let X be a complex manifold with a normal crossing hypersurface D . The singular part of D is denoted by $D^{[2]}$. For a torsion-free sheaf F on X , let $N(F)$ denote the closed subset of X determined by the condition that $Q \in N(F)$ if and only if the stalk of F at Q is not locally free.

Let E be a lattice of a meromorphic flat bundle (\mathcal{E}, ∇) on (X, D) . Let Q be a smooth point of D . Let X_Q be a small neighbourhood of Q in X . We put $D_Q := X_Q \cap D$. If there exist a good set of irregular values $\text{Irr}(\nabla, Q) \subset M(X_Q, D_Q)/H(X_Q)$ and a decomposition

$$(E, \nabla)|_{\hat{D}_Q} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, Q)} (\hat{E}_{\mathfrak{a}}, \hat{\nabla}_{\mathfrak{a}}), \tag{2.5}$$

such that $\hat{\nabla}_{\mathfrak{a}}$ are \mathfrak{a} -logarithmic with respect to $\hat{E}_{\mathfrak{a}}$, then E is called an unramifiedly ‘good’ lattice of (\mathcal{E}, ∇) around Q . In that case, we have the residue $\text{Res}(\nabla) \in \text{End}(E|_{D_Q})$. If the

eigenvalues α of $\text{Res}(\nabla)$ satisfy $0 \leq \text{Re } \alpha < 1$, E is called an unramifiedly ‘good’ Deligne–Malgrange lattice of (\mathcal{E}, ∇) around Q . An unramifiedly ‘good’ Deligne–Malgrange lattice is unique, if it exists. If there exists a ramified covering $\varphi_Q: (X'_Q, D'_Q) \rightarrow (X_Q, D_Q)$ such that $\varphi_Q^*(\mathcal{E}, \nabla)$ has an unramifiedly ‘good’ Deligne–Malgrange lattice, its descent with respect to φ_Q is called a ‘good’ Deligne–Malgrange lattice of (\mathcal{E}, ∇) around Q . A ‘good’ Deligne–Malgrange lattice is unique, if it exists.

Remark 2.25. The condition ‘unramifiedly “good” Deligne–Malgrange’ implies ‘unramifiedly good Deligne–Malgrange’, clearly. It is easy and standard to show that they are actually equivalent. (See Lemma 2.32, for example. Note that Proposition 2.27 and Proposition 2.28 can be shown much more easily if the divisor is smooth.)

We recall the work due to Malgrange on lattices of meromorphic flat bundles. (See also the work due to Mebkhout [17, 18] for a construction of lattices of regular singular meromorphic flat bundles whose poles are not necessarily normal crossing.)

Proposition 2.26 (Malgrange [16]). *Let (\mathcal{E}, ∇) be a meromorphic flat bundle. There exists a unique \mathcal{O}_X -reflexive lattice $E \subset \mathcal{E}$ which is generically ‘good’ Deligne–Malgrange lattice, i.e. there exists a closed analytic subset $Z \subset D$ with $\text{codim}_X(Z) \geq 2$ and $Z \supset D^{[2]} \cup N(E)$, such that $E|_{X \setminus Z}$ is ‘good’ Deligne–Malgrange lattice around any $Q \in D \setminus Z$. In particular, $E|_{X \setminus Z}$ is a good Deligne–Malgrange lattice of $\mathcal{E}|_{X \setminus Z}$.*

See also Proposition 2.7.6 of [21], where we give a small complement that Z can be taken as a closed analytic subset.

Such a lattice is called the Deligne–Malgrange lattice of (\mathcal{E}, ∇) . Our goal is to show that the Deligne–Malgrange lattice is an unramifiedly good Deligne–Malgrange lattice if (\mathcal{E}, ∇) is unramifiedly good. (It also implies that the Deligne–Malgrange lattice is good Deligne–Malgrange, if (\mathcal{E}, ∇) is good, by Corollary 2.24.)

2.3.2. Openness of the good Deligne–Malgrange property

Let X be a complex manifold with a normal crossing hypersurface D . Take a point $p \in D$. We denote by \hat{D} the formal complex analytic space obtained as the completion of X along D (see [2] and [3]). Let (\mathcal{E}, ∇) be a meromorphic flat bundle on (\hat{D}, D) . Let E be a lattice of \mathcal{E} . Make the following assumptions.

- $E|_{\hat{P}}$ is an unramifiedly good Deligne–Malgrange lattice of $(\mathcal{E}, \nabla)|_{\hat{P}}$.

Proposition 2.27. *The elements of $\text{Irr}(\nabla, P)$ are convergent, i.e. there exists a small neighbourhood X_P of P in X such that $\text{Irr}(\nabla, P) \subset M(X_P, D_P)/H(X_P)$. Moreover, the following holds for any $P' \in D_P$.*

- $E|_{\hat{P}'}$ is an unramifiedly good Deligne–Malgrange lattice of $(\mathcal{E}, \nabla)|_{\hat{P}'}$, and $\text{Irr}(\nabla, P')$ is the image of the natural map $\text{Irr}(\nabla, P) \rightarrow \mathcal{O}_X(*D)_{P'}/\mathcal{O}_{X, P'}$.

Refinement. To show Proposition 2.27, we have only to consider the case $X = \Delta^n$, $D = \bigcup_{i=1}^{\ell} \{z_i = 0\}$, and $P = (0, \dots, 0)$. In this case, we shall give a more refined statement.

For any subset $I \subset \{1, \dots, \ell\}$, we set

$$D_I := \bigcap_{i \in I} \{z_i = 0\} \quad \text{and} \quad D(I) := \bigcup_{i \in I} \{z_i = 0\}.$$

We denote by \hat{D}_I the formal complex analytic space obtained as the completion of X along D_I (see [2] and [3]). We also put $I^c := \{1, \dots, \ell\} \setminus I$. Once we know $\text{Irr}(\nabla, P) \subset M(X_P, D_P)/H(X_P)$, let $\text{Irr}(\nabla, I)$ denote the image of $\text{Irr}(\nabla, P) \rightarrow M(X_P, D_P)/M(X_P, D_P(I^c))$.

Proposition 2.28. *The elements of $\text{Irr}(\nabla, P)$ are convergent. Moreover, if X_P is a sufficiently small neighbourhood of P in X , for any subset $I \subset \ell$, we have a decomposition*

$$(E, \nabla)|_{\hat{D}_{I,P}} = \bigoplus_{\mathfrak{b} \in \text{Irr}(\nabla, I)} ({}^I\hat{E}_{\mathfrak{b}}, {}^I\hat{\nabla}_{\mathfrak{b}}) \tag{2.6}$$

such that

$$({}^I\hat{\nabla}_{\mathfrak{b}} - \text{db})({}^I\hat{E}_{\mathfrak{b}}) \subset {}^I\hat{E}_{\mathfrak{b}} \otimes (\Omega_X^1(\log D(I)) + \Omega_X^1(*D(I^c)))|_{X_P},$$

where we take a lift of \mathfrak{b} to $M(X_P, D_P)$.

Proof of Proposition 2.27. Let us show Proposition 2.27 by assuming Proposition 2.28. It is easy to observe that the decomposition (2.6) induces the irregular decomposition of $E|_{\hat{P}'}$, for any $P' \in D_I \setminus D(I^c)$. We obtain the residue $\text{Res}_i(\nabla)$ of $E|_{D_i}$ from (2.6) as in § 2.1.3, and the eigenvalues are constant on D_i . Hence, if the eigenvalues α of $\text{Res}_i(\nabla)|_{P'}$ satisfy $0 \leq \text{Re}(\alpha) < 1$, we can conclude that the eigenvalues of $\text{Res}_i(\nabla)|_{\hat{P}'}$ ($i \in I$) also satisfy the condition. Thus, we obtain Proposition 2.27. \square

Proof of Proposition 2.28. In the following, instead of considering a neighbourhood X_P , we will replace X by a small neighbourhood of P without mention, if it is necessary.

Step 1. We fix $I \subset \ell$ for a moment. Let E be a free $\mathcal{O}_{\hat{D}_I}$ -module with a meromorphic flat connection $\nabla: E \rightarrow E \otimes \Omega_{\hat{D}_I}^1(*D)$. Assume that we are given the following.

- $\mathbf{m} \in \mathbb{Z}_{\leq 0}^{\ell}$ and $i \in I$ such that $m_i < 0$. We set $\mathbf{m}' := \mathbf{m} + \delta_i$.
- $\mathcal{I} \subset \mathcal{O}_{\hat{P}}(*D)$ such that, for any $\mathfrak{a} \in \mathcal{I}$, (i) $z_i^{-m_i} \mathfrak{a}$ is independent of the variable z_i and (ii) $z^{-\mathbf{m}} \mathfrak{a} \in \mathcal{O}_{\hat{P}}$.
- A decomposition $E|_{\hat{P}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} {}^P E_{\mathfrak{a}}$ such that $z^{-\mathbf{m}'}(\nabla - \text{da})({}^P E_{\mathfrak{a}}) \subset {}^P E_{\mathfrak{a}} \otimes \Omega_{\hat{P}}^1(\log D)$.

We set $T := \{(z^{-\mathbf{m}} \mathfrak{a})(P) \mid \mathfrak{a} \in \mathcal{I}\} \subset \mathbb{C}$. We have a naturally defined map $\pi: \mathcal{I} \rightarrow T$. We set ${}^P E_{\mathfrak{b}} := \bigoplus_{\pi(\mathfrak{a})=\mathfrak{b}} {}^P E_{\mathfrak{a}}$. Let $H(D_I)$ denote the space of holomorphic functions on D_I . Let R denote the localization of $H(D_I)[[z_i \mid i \in I]]$ with respect to $\prod_{i=1}^{\ell} z_i$.

Lemma 2.29. *\mathcal{I} is contained in R , and we have a flat decomposition $E = \bigoplus_{\mathfrak{b} \in T} E_{\mathfrak{b}}$ such that $E_{\mathfrak{b}}|_{\hat{P}} = {}^P E_{\mathfrak{b}}$.*

Proof. First, we remark that $z^{-m}\nabla(z_i\partial_i)^PE_a \subset {}^PE_a$, and thus $z^{-m}\nabla(z_i\partial_i)E \subset E$. Let F be the endomorphism of $E|_{\hat{D}_I \cap D_i}$ induced by $z^{-m}\nabla(z_i\partial_i)$. The eigendecomposition of $F|_P$ is given by

$$E|_P = \bigoplus_{\mathfrak{b} \in T} {}^PE_{\mathfrak{b}|_P}.$$

We obtain the unique decomposition $E|_{\hat{D}_I \cap D_i} = \bigoplus_{\mathfrak{b} \in T} G_{\mathfrak{b}}$ such that (i) $F(G_{\mathfrak{b}}) \subset G_{\mathfrak{b}}$ and (ii) $G_{\mathfrak{b}|_P} = {}^PE_{\mathfrak{b}|_P}$. By comparing F and its completion at P , we obtain that $\mathcal{I} \subset R$. By using an argument in the proof of Lemma 2.12, we obtain the decomposition $E = \bigoplus_{\mathfrak{b} \in T} E_{\mathfrak{b}}$ such that (i) $E_{\mathfrak{b}|_{\hat{D}_I \cap D_i}} = G_{\mathfrak{b}}$ and (ii) it is preserved by $z^{-m}\nabla(z_i\partial_i)$. We obtain $E_{\mathfrak{b}|_{\hat{P}}} = {}^PE_{\mathfrak{b}}$ by Lemma 2.6. In particular, the decomposition is ∇ -flat. \square

Step 2. For $1 \leq p \leq \ell$, we put $\underline{p} := \{1, \dots, p\}$. We denote by $\hat{D}(\underline{p})$ the formal complex analytic space obtained as the completion of X along $D(\underline{p})$. Let E be a free $\mathcal{O}_{\hat{D}(\underline{p})}$ -module with a meromorphic flat connection $\nabla: E \rightarrow E \otimes \Omega^1_{\hat{D}(\underline{p})}(*D)$. Assume that we are given a good set of irregular values $\text{Irr}(\nabla) \subset \mathcal{O}_{\hat{P}}(*D(\underline{p}))/\mathcal{O}_{\hat{P}}$ and a decomposition

$$(E, \nabla)|_{\hat{P}} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla)} ({}^PE_{\mathfrak{a}}, {}^P\nabla_{\mathfrak{a}})$$

such that ${}^P\nabla_{\mathfrak{a}}$ are \mathfrak{a} -logarithmic. For $I \subset \underline{p}$, let $\text{Irr}(\nabla, I)$ denote the image of $\text{Irr}(\nabla)$ via the natural map $p_I: \mathcal{O}_{\hat{P}}(*D(\underline{p}))/\mathcal{O}_{\hat{P}} \rightarrow \mathcal{O}_{\hat{P}}(*D(\underline{p}))/\mathcal{O}_{\hat{P}}(*D(I_1))$, where $I_1 := \underline{p} \setminus I$. For each I and $\mathfrak{b} \in \text{Irr}(\nabla, I)$, we set

$${}^PE_{\mathfrak{b}} := \bigoplus_{\substack{\mathfrak{a} \in \text{Irr}(\nabla) \\ p_I(\mathfrak{a}) = \mathfrak{b}}} {}^PE_{\mathfrak{a}}.$$

Lemma 2.30. *If we shrink X appropriately, $\text{Irr}(\nabla)$ is contained in the image of $M(X, D(\underline{p}))/H(X) \rightarrow \mathcal{O}_{\hat{P}}(*D(\underline{p}))/\mathcal{O}_{\hat{P}}$. For each $I \subset \underline{p}$, we have a flat decomposition $E|_{\hat{D}_I} = \bigoplus_{\mathfrak{b} \in \text{Irr}(\nabla, I)} {}^IE_{\mathfrak{b}}$ such that ${}^IE_{\mathfrak{b}|_{\hat{P}}} = {}^PE_{\mathfrak{b}}$.*

Proof. We use an induction on the rank of E . We take an auxiliary sequence $\mathfrak{m}(0), \dots, \mathfrak{m}(L)$ for $\text{Irr}(\nabla)$. (We use $\mathfrak{m}(0)$ and $\mathfrak{m}(1)$ for $\bar{\eta}_{\mathfrak{m}(0)}$.) We put $T := \{(z^{-m(0)}\mathfrak{a})(P) \mid \mathfrak{a} \in \text{Irr}(\nabla)\}$. We have the naturally defined map $q: \text{Irr}(\nabla) \rightarrow T$. For each $\alpha \in T$, we put ${}^PE_{\alpha} = \bigoplus_{q(\mathfrak{a})=\alpha} {}^PE_{\mathfrak{a}}$. Then, $E|_{\hat{P}} = \bigoplus {}^PE_{\alpha}$ is a flat decomposition.

Applying Lemma 2.29 with $I = \{\mathfrak{h}(0)\}$, we obtain that $\bar{\eta}_{\mathfrak{m}(0)}(\mathfrak{a})$ are meromorphic functions for any $\mathfrak{a} \in \text{Irr}(\nabla)$. Hence, by considering the tensor product with a meromorphic flat line bundle, we have only to consider the case in which T contains at least two elements. (We remark that $\{\mathfrak{b} - \mathfrak{a} \mid \mathfrak{b} \in \text{Irr}(\nabla)\}$ is a good set of irregular values for any fixed $\mathfrak{a} \in \text{Irr}(\nabla)$.) For simplicity, we assume that the coordinate system is admissible for $\text{Irr}(\nabla)$, and let k be determined by $\mathfrak{m}(0) \in \mathbb{Z}_{<0}^k \times \mathbf{0}_{\ell-k}$.

Let $I \subset \underline{p}$. If $I \cap \underline{k} = \emptyset$, then the trivial decomposition is the desired one. Let us consider the case $I \cap \underline{k} \neq \emptyset$. By taking $i \in I \cap \underline{k}$, and applying Lemma 2.29, we obtain a flat decomposition $E|_{\hat{D}_I} = \bigoplus_{\alpha \in T} {}^IE_{\alpha}$ such that ${}^IE_{\alpha} = {}^PE_{\alpha}$. For $I \subset J \subset \underline{k}$ as above,

we obtain ${}^I E_{\alpha|_{\hat{D}_J}} = {}^J E_{\alpha}$ from ${}^I E_{\alpha|_{\hat{P}}} = {}^J E_{\alpha|_{\hat{P}}}$. Due to Lemma 2.31 below, we obtain the flat decomposition

$$(E, \nabla)_{|\hat{D}(\underline{k})} = \bigoplus_{\alpha \in T} (E_{\alpha}, \nabla_{\alpha}), \quad \text{such that } E_{\alpha|_{\hat{D}_I}} = {}^I E_{\alpha}.$$

We may apply the hypothesis of the induction to $(E_{\alpha}, \nabla_{\alpha})$ on $D(\underline{k})$, and we obtain Lemma 2.30. □

We have used the following general lemma.

Lemma 2.31. *Let \hat{V} be a free $\mathcal{O}_{\hat{D}}$ -module on X . Assume that we are given a decomposition $\hat{V}_{|\hat{D}_I} = \bigoplus {}^I \hat{V}_{\alpha}$ for each $I \subset \ell$, such that ${}^I \hat{V}_{\alpha|_{\hat{D}_J}} = {}^J \hat{V}_{\alpha}$ for any $I \subset J$. Then, we have a unique decomposition $\hat{V} = \bigoplus \hat{V}_{\alpha}$ on \hat{D} , which induces the decompositions on \hat{D}_I .*

Proof. Let ${}^I \pi_{\alpha}$ be the projection of $\hat{V}_{|\hat{D}_I}$ onto ${}^I V_{\alpha}$. Then, we have ${}^I \pi_{\alpha|_{\hat{D}_J}} = {}^J \pi_{\alpha}$. Let \mathbf{v} be a frame of V . Let ${}^I \Pi_{\alpha} \in M_r(\mathcal{O}_{\hat{D}_I})$ be determined by ${}^I \pi_{\alpha}(\mathbf{v}) = \mathbf{v} \cdot {}^I \Pi_{\alpha}$, where $r = \text{rank}(V)$. Because ${}^I \Pi_{\alpha|_{\hat{D}_J}} = {}^J \Pi_{\alpha}$, we have $\Pi_{\alpha} \in M_r(\mathcal{O}_{\hat{D}})$ such that $\Pi_{\alpha|_{\hat{D}_I}} = {}^I \Pi_{\alpha}$. (Use the exact sequence in the proof of Proposition 4.1 in [6], for example.) Let π_{α} be the endomorphism of \hat{V} given by $\pi_{\alpha}(\mathbf{v}_{|\hat{D}}) = \mathbf{v}_{|\hat{D}} \cdot \Pi_{\alpha}$, and let \hat{V}_{α} be the image of π_{α} . Then, $\hat{V} = \bigoplus \hat{V}_{\alpha}$ gives the desired decomposition. □

Step 3. We can complete the proof of Proposition 2.28 by applying Lemma 2.30 to $(E, \nabla)_{|\hat{D}}$. □

2.3.3. *The smooth divisor case*

Let us prove Proposition 2.18 and Proposition 2.19 in the case that D is smooth. Because of the uniqueness of a Deligne–Malgrange lattice, we have only to consider the local case. Hence, we set $X := \Delta^n$ and $D := \{z_1 = 0\}$.

Step 1. Let \hat{D} denote the formal complex analytic space obtained as the completion of X along D . We consider a meromorphic flat bundle (\mathcal{E}, ∇) on (\hat{D}, D) satisfying the following.

(C) $(\mathcal{E}, \nabla)_{|\hat{P}}$ has an unramifiedly good Deligne–Malgrange lattice ${}^P E$ for each $P \in D$.

Lemma 2.32. *Let E be an $\mathcal{O}_{\hat{D}}$ -locally free lattice of \mathcal{E} such that $E_{|\hat{P}} = {}^P E$ for any $P \in D$. Then, the following hold.*

- There exists $\mathcal{I} \in z_1^{-1} H(D)[z_1^{-1}]$ such that $\mathcal{I}_{|\hat{P}} = \text{Irr}(\nabla, P)$ for any $P \in D$.
- We have a flat decomposition $E = \bigoplus_{\alpha \in \mathcal{I}} E_{\alpha}$ whose restriction to \hat{P} is the same as the irregular decomposition of ${}^P E$ for any $P \in D$.

Proof. Let $P \in D$. Let X_P be a small neighbourhood of P as in Proposition 2.27. Namely, we have $\text{Irr}(\nabla, P) \subset M(X_P, D_P)/H(X_P)$, and $\text{Irr}(\nabla, P')$ is the image of

$\text{Irr}(\nabla, P)$ by the map $M(X_P, D_P)/H(X_P) \rightarrow \mathcal{O}_{\hat{P}'}(*D_P)/\mathcal{O}_{\hat{P}'}$ for each $P' \in D_P$. Then, the first claim is clear. As in Proposition 2.28, we have a formal decomposition

$$E|_{\hat{D}_P} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, P)} E_{\mathfrak{a}, \hat{D}_P}$$

whose restriction to \hat{P}' is the same as the irregular decomposition of $(E, \nabla)|_{\hat{P}'}$, where $P' \in D_P$. For $P_i \in D$ ($i = 1, 2$), we can take X_{P_i}, D_{P_i} and a decomposition

$$E|_{\hat{D}_{P_i}} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, P_i)} E_{\mathfrak{a}, \hat{D}_{P_i}}.$$

The decomposition is the same on $D_{P_1} \widehat{\cap} D_{P_2}$. Hence, we can glue them, and we obtain the desired decomposition $E = \bigoplus_{\mathfrak{a} \in \mathcal{I}} E_{\mathfrak{a}}$. □

Step 2. Let (\mathcal{E}, ∇) be a meromorphic flat bundle on (\hat{D}, D) satisfying the condition (C) above. We put $Z := \{z_1 = z_2 = 0\}$.

Lemma 2.33. *Assume there exists an $\mathcal{O}_{\hat{D}}$ -free lattice E of \mathcal{E} such that $E|_{\hat{P}} = {}^P E$ for each $P \in D \setminus Z$. Then, the following hold.*

- *There exists $\mathcal{I} \subset z_1^{-1}H(D)[z_1^{-1}]$ such that $\mathcal{I}|_{\hat{Q}} = \text{Irr}(\nabla, Q)$ for any $Q \in D$.*
- *We have a flat decomposition $E = \bigoplus_{\mathfrak{a} \in \mathcal{I}} E_{\mathfrak{a}}$ whose restriction to \hat{Q} is the same as the irregular decomposition of ${}^Q E$ for any $Q \in D$.*

Proof. According to Lemma 2.32, we have only to show that $E|_{\hat{Q}} = {}^Q E$ for any $Q \in Z$.

Fix a point $P \in D \setminus Z$. Let γ be a loop in $D \setminus Z$ starting and ending at P . By Lemma 2.32, for each $P' \in \gamma$, we have a neighbourhood $X_{P'}$ such that $\text{Irr}(\nabla, P') \subset M(X_{P'}, D_{P'})/H(X_{P'})$ and $\text{Irr}(\nabla, P'')$ is the image of $\text{Irr}(\nabla, P')$ for any $P'' \in D_{P'}$. Hence, we obtain a map $\text{Irr}(\nabla, P) \rightarrow \text{Irr}(\nabla, P)$ induced by the analytic continuation along γ . It depends only on the homotopy class of γ . Hence, we obtain a naturally induced action of the fundamental group $\pi_1(D \setminus Z, P)$ on $\text{Irr}(\nabla, P)$. In other words, the family $\{\text{Irr}(\nabla, P) \mid P \in D \setminus Z\}$ gives a covering space of $D \setminus Z$. Note that if the action of $\pi_1(D \setminus Z, P_0)$ on $\text{Irr}(\nabla, P_0)$ is trivial for a point $P_0 \in D \setminus Z$, the action of $\pi_1(D \setminus Z, P)$ on $\text{Irr}(\nabla, P)$ is trivial for any $P \in D \setminus Z$.

Lemma 2.34. *Let $P_0 \in D \setminus Z$. Assume that the action of $\pi_1(D \setminus Z, P_0)$ on $\text{Irr}(\nabla, P_0)$ is trivial. Then, we have $E|_{\hat{Q}} = {}^Q E$ for any $Q \in Z$. In particular, by Lemma 2.32, the conclusion of Lemma 2.33 holds under the assumption.*

Proof. Because the action of $\pi_1(D \setminus Z, P)$ on $\text{Irr}(\nabla, P)$ is trivial, we have $\mathcal{I} \subset z^{-1}H(D \setminus Z)[z^{-1}]$ such that $\mathcal{I}|_{\hat{P}'} = \text{Irr}(\nabla, P')$ for any $P' \in D \setminus Z$. We set $m := \min\{\text{ord}_{z_1}(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$. We use a descending induction on m . If $m = 0$, we can deduce that ∇ is logarithmic with respect to E , and hence the claim is obvious. Let us consider the step $m + 1 \implies m$. We put

$$T := \{(z_1^{-m} z_1 \partial_1 \mathfrak{a})|_D \mid \mathfrak{a} \in \mathcal{I}\} \subset H(D \setminus Z).$$

For any $P \in D \setminus Z$, we have $z_1^{-m} \nabla(z_1 \partial_1)(^P E) \subset ^P E$. Hence, we have $z_1^{-m} \nabla(z_1 \partial_1) E_{|D \setminus Z} \subset E_{|D \setminus Z}$. We obtain $z_1^{-m} \nabla(z_1 \partial_1) E \subset E$. Let G be the endomorphism of $E_{|D}$ induced by $z_1^{-m} \nabla(z_1 \partial_1)$. Because the elements of T are the eigenvalues of $G_{|D \setminus Z}$, they are algebraic over $H(D)$. Hence, we obtain $T \subset H(D)$.

Let $Q \in Z$. In the following, we will shrink X around Q without mention. Let N be the $H(D)((z_1))$ -module corresponding to \mathcal{E} , i.e. the space of the global sections of \mathcal{E} . We may assume that it is a free $H(D)((z_1))$ -module. Let L be the $H(D)[[z_1]]$ -lattice of N corresponding to E . We put $N' := N \otimes M(D, Z)((z_1))$ and $L' := L \otimes M(D, Z)[[z_1]]$. We have the eigendecomposition of $L'/z_1 L'$ with respect to G . By an argument as in Lemma 2.12, it is extended to a decomposition $L' = \bigoplus_{\mathfrak{b} \in T} L'_\mathfrak{b}$ such that $(z_1^{-m+1} \partial_1 - \mathfrak{b})L'_\mathfrak{b} \subset L'_\mathfrak{b}$.

We put $m(Q) := \min\{\text{ord}_{z_1}(\mathfrak{a}) \mid \mathfrak{a} \in \text{Irr}(\nabla, Q)\}$ and

$$T(Q) := \{(z_1^{-m(Q)+1} \partial_1 \mathfrak{a})_{|D} \mid \mathfrak{a} \in \text{Irr}(\nabla, Q)\}.$$

Lemma 2.35. *We have $m(Q) = m$, and $T(Q) = T$ in the completion of $\mathcal{O}_{D,Q}$.*

Proof. We may assume $Q = (0, \dots, 0)$. We put $\mathcal{N} := N \otimes \mathcal{O}_{\hat{Q}}$. It is equipped with an unramifiedly good Deligne–Malgrange lattice ${}^Q \mathcal{L}$ with the irregular decomposition

$${}^Q \mathcal{L} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, Q)} {}^Q \mathcal{L}_\mathfrak{a}. \tag{2.7}$$

Let $\kappa: \text{Irr}(\nabla, Q) \rightarrow T(Q)$ be the naturally defined map. For $\mathfrak{b} \in T(Q)$, we put

$${}^Q \mathcal{L}_\mathfrak{b} = \bigoplus_{\mathfrak{a} \in \kappa^{-1}(\mathfrak{b})} {}^Q \mathcal{L}_\mathfrak{a}.$$

Then, we obtain the decomposition ${}^Q \mathcal{L} = \bigoplus_{\mathfrak{b} \in T(Q)} {}^Q \mathcal{L}_\mathfrak{b}$ such that $(z_1^{-m(Q)+1} \partial_1 - \mathfrak{b}) {}^Q \mathcal{L}_\mathfrak{b} \subset {}^Q \mathcal{L}_\mathfrak{b}$ for any $\mathfrak{b} \in T(Q)$. By considering the extension to the field $\mathbb{C}((z_n)) \cdots ((z_2))((z_1))$, and by using Lemma 2.16, we obtain Lemma 2.35. \square

Let us return to the proof of Lemma 2.34. By Lemma 2.35, we obtain that $\mathfrak{b}_1 - \mathfrak{b}_2$ are nowhere vanishing on D for distinct $\mathfrak{b}_i \in T$. Hence, we have the eigendecomposition of $E_{|D}$ with respect to G on D . By Lemma 2.12, it is extended to a decomposition $E = \bigoplus_{\mathfrak{b}} E_\mathfrak{b}$ such that $(z_1^{-m} \nabla(z_1 \partial_1) - \mathfrak{b}) E_\mathfrak{b} \subset E_\mathfrak{b}$. We have $E_{\mathfrak{b}|\hat{Q}} = {}^Q \mathcal{L}_\mathfrak{b}$, and hence the decomposition is ∇ -flat. Put $\mathcal{E}_\mathfrak{b} = E_\mathfrak{b}(*D)$. We can apply the hypothesis of the induction to $\mathcal{E}_\mathfrak{b} \otimes L(-z_1^{-m} \mathfrak{b}/m)$, and the proof of Lemma 2.34 is finished. \square

Then, Lemma 2.33 follows from the next lemma.

Lemma 2.36. *The action of $\pi_1(D \setminus Z, P)$ on $\text{Irr}(\nabla, P)$ is trivial. In particular, by Lemma 2.34, the claim of Lemma 2.33 holds.*

Proof. Because $\text{Irr}(\nabla, P)$ is finite, we can find a ramified covering $\varphi: X' \rightarrow X$ given by $\varphi(z_1, \zeta_2, z_3, \dots, z_n) = (z_1, \zeta_2^e, z_3, \dots, z_n)$ such that we can apply Lemma 2.34 to $\varphi^*(\mathcal{E}, \nabla)$ and $\varphi^* E$. Then, we have $\varphi^* \text{Irr}(\nabla, P) \subset z_1^{-1} H(D')[z_1^{-1}]$ and $\varphi^* \text{Irr}(\nabla, P)_{|\hat{Q}} = \varphi^* \text{Irr}(\nabla, Q)$. Hence, we can conclude that the action of $\pi_1(D \setminus Z, P)$ is trivial. \square

Step 3. Let us observe that we can ignore the subsets whose codimension in X is larger than 3. Let $X := \Delta^n$ and $D := \{z_1 = 0\}$. A subset $\mathcal{I} \subset M(X, D)/H(X)$ is called good, if its image $\mathcal{I}|_{\hat{P}} \subset \mathcal{O}_{\hat{P}}(*D)/\mathcal{O}_{\hat{P}}$ is good for each $P \in D$. The following lemma is easy to observe.

Lemma 2.37. *Let $Z \subset D$ be a closed analytic subset with $\text{codim}_D(Z) \geq 2$. Let $\mathcal{I} \subset M(X \setminus Z, D \setminus Z)/H(X \setminus Z)$ be a finite subset such that its image $\mathcal{I}|_{\hat{P}} \subset \mathcal{O}_{\hat{P}}(*D)/\mathcal{O}_{\hat{P}}$ is good for each $P \in D \setminus Z$. Then, we have $\mathcal{I} \subset M(X, D)/H(X)$, and it is good.*

Proof. Let $\mathfrak{a} \in \mathcal{I}$. By Hartogs property, we obtain that $\mathfrak{a} \in M(X, D)/H(X)$. By the assumption, $\mathfrak{a}_{\text{ord}(\mathfrak{a})}$ is nowhere vanishing on $D \setminus Z$. Because $\text{codim}_D(Z) \geq 2$, we obtain that $\mathfrak{a}_{\text{ord}(\mathfrak{a})}$ is nowhere vanishing. We can check the other claims similarly. \square

Let (\mathcal{E}, ∇) be a meromorphic flat connection on (X, D) , i.e., \mathcal{E} is a (not necessarily locally free) coherent $\mathcal{O}_X(*D)$ -module with a meromorphic flat connection $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$. Let E be the Deligne–Malgrange lattice of (\mathcal{E}, ∇) . Make the following assumption.

- There exists a closed analytic subset $Z \subset D$ with $\text{codim}_D(Z) \geq 2$ such that $E|_{X \setminus Z}$ is an unramifiedly good Deligne–Malgrange lattice of $(\mathcal{E}, \nabla)|_{X \setminus Z}$.

Lemma 2.38. *If the above condition is satisfied, E is an unramifiedly good Deligne–Malgrange lattice of (\mathcal{E}, ∇) .*

Proof. Since $\text{codim}_Z(D) \geq 2$, we have a good set of irregular values $\mathcal{I} \subset M(X \setminus Z, D \setminus Z)/H(X \setminus Z)$ and the decomposition

$$(E, \nabla)|_{\widehat{D \setminus Z}} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} (F_{\mathfrak{a}, D \setminus Z}, \nabla_{\mathfrak{a}})$$

such that each $(F_{\mathfrak{a}, D \setminus Z}, \nabla_{\mathfrak{a}})$ is \mathfrak{a} -regular. By Lemma 2.37, we have $\mathcal{I} \subset M(X, D)/H(X)$. Let $\pi_{\mathfrak{a}}$ denote the projection onto $F_{\mathfrak{a}, D \setminus Z}$, which gives a section of $\text{End}(E)|_{\widehat{D \setminus Z}}$.

Let us observe that $\pi_{\mathfrak{a}}$ is extended to a section of $\text{End}(E)|_{\hat{D}}$. It is easy to show the following claim by using Hartogs theorem.

- Any section of $\mathcal{O}_{\widehat{D \setminus Z}}$ is extended to a section of $\mathcal{O}_{\hat{D}}$.

Since E is reflexive, we can (locally) take an injection $i: E \rightarrow \mathcal{O}_X^{\oplus N}$ for some large N such that the cokernel $\text{Cok}(i)$ is torsion-free. We can also take a surjection $\varphi: \mathcal{O}_X^{\oplus M} \rightarrow E$. The morphisms i, φ and $\pi_{\mathfrak{a}}$ induce a morphism

$$F_{\mathfrak{a}}: \mathcal{O}_{X|\widehat{D \setminus Z}}^{\oplus M} \rightarrow \mathcal{O}_{X|\widehat{D \setminus Z}}^{\oplus N}.$$

It is extended to a morphism $\tilde{F}_{\mathfrak{a}}: \mathcal{O}_{\hat{D}}^{\oplus M} \rightarrow \mathcal{O}_{\hat{D}}^{\oplus N}$. Since $\text{Cok}(i)$ is torsion free, $\tilde{F}_{\mathfrak{a}}$ factors through $E|_{\hat{D}}$. Let $\mathcal{K} := \text{Ker}(\varphi)$. The restriction of $\tilde{F}_{\mathfrak{a}}$ to \mathcal{K} on $D \setminus Z$ is 0. Then, we obtain $\tilde{F}_{\mathfrak{a}}|_{\mathcal{K}} = 0$ because $\mathcal{O}_{\hat{D}}^{\oplus N}$ is torsion-free. Thus, we obtain the induced maps $\pi_{\mathfrak{a}}: E|_{\hat{D}} \rightarrow E|_{\hat{D}}$ for $\mathfrak{a} \in \mathcal{I}$, which satisfy $\pi_{\mathfrak{a}} \circ \pi_{\mathfrak{a}} = \pi_{\mathfrak{a}}, \pi_{\mathfrak{a}} \circ \pi_{\mathfrak{b}} = 0$ ($\mathfrak{a} \neq \mathfrak{b}$), and $\sum \pi_{\mathfrak{a}} = \text{id}$. They give a decomposition $E = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \hat{E}_{\mathfrak{a}}$. Let us show that $\hat{E}_{\mathfrak{a}}$ are \mathfrak{a} -logarithmic. We have only to consider the case $\mathfrak{a} = 0$.

Take a point $P \in D \setminus Z$. We have the vector space $V := \hat{E}_{0|P}$. We have the endomorphism f of V induced by the residue. Let $E'_0 := V \otimes \mathcal{O}_X$ and $\nabla'_0 = d + f \cdot dz_1/z_1$. We have the natural flat isomorphism $(E'_0, \nabla'_0)|_{\pi^{-1}(P)} \simeq (\hat{E}_0, \nabla_0)|_{\pi^{-1}(P)}$. Since the codimension of Z in D is larger than 2, we obtain a flat isomorphism $\Phi_{0,D \setminus Z}: (E'_0, \nabla'_0)|_{\widehat{D \setminus Z}} \simeq (\hat{E}_0, \nabla_0)|_{\widehat{D \setminus Z}}$. Since E_0 and E'_0 are reflexive, by the above argument, we can show that $\Phi_{0,D \setminus Z}$ and its inverse are extended to a morphism on \hat{D} . Thus, we are done. \square

End of the proof in the smooth divisor case. Let (\mathcal{E}, ∇) be a meromorphic flat sheaf on (X, D) . Assume that $(\mathcal{E}, \nabla)|_{\hat{P}}$ has an unramifiedly good Deligne–Malgrange lattice for each $P \in D$.

Lemma 2.39. *The Deligne–Malgrange lattice E of (\mathcal{E}, ∇) is unramifiedly good Deligne–Malgrange. Namely, the claim of Proposition 2.18 holds if D is smooth. The claim of Proposition 2.19 also holds.*

Proof. There exists a closed analytic subset $Z \subset D$ with $\text{codim}_D(Z) \geq 2$ such that $E|_{X \setminus Z}$ is locally free. There exists a closed analytic subset $Z_1 \subset D$ with $\text{codim}_D(Z_1) \geq 1$ such that $E|_{X \setminus Z_1}$ is unramifiedly good Deligne–Malgrange. By Lemma 2.33, we obtain that E is unramifiedly good Deligne–Malgrange, around any smooth point of Z_1 . Hence, we obtain that there exists a closed analytic subset $Z'' \subset D$ with $\text{codim}_D(Z'') \geq 2$ such that $E|_{X \setminus Z''}$ is unramifiedly good Deligne–Malgrange. Then, by Lemma 2.38, we obtain that E is unramifiedly good Deligne–Malgrange, i.e. the claim of Proposition 2.18 holds. It is also clear that the claim of Proposition 2.19 holds. \square

2.3.4. *The normal crossing case*

Since the claim is local, we set $X := \Delta^n$ and $D := \bigcup_{i=1}^{\ell} \{z_i = 0\}$. We put $\partial D_1 := D_1 \cap \bigcup_{2 \leq j \leq \ell} D_j$. We put $D_1^\circ := D_1 \setminus \partial D_1$.

Step 1. We regard $M(D_1, \partial D_1)((z_1))$ as a differential ring equipped with the differential $\partial_1 := \partial/\partial z_1$. Let \mathcal{N} be a differential $M(D_1, \partial D_1)((z_1))$ -module with a $M(D_1, \partial D_1)[[z_1]]$ -free lattice \mathcal{L} . We put $\mathcal{L}' := \mathcal{L} \otimes H(D_1^\circ)[[z_1]]$. Assume that we have $\mathcal{I} \subset z_1^{-1}H(D_1^\circ)[z_1^{-1}]$ and a decomposition $\mathcal{L}' = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \mathcal{L}'_{\mathfrak{a}}$ such that (i) $(z_1 \partial_1 - z_1 \partial_1 \mathfrak{a})\mathcal{L}'_{\mathfrak{a}} \subset \mathcal{L}'_{\mathfrak{a}}$ and (ii) the eigenvalues α of the induced morphism of $\mathcal{L}'_{\mathfrak{a}}/z_1 \mathcal{L}'_{\mathfrak{a}}$ satisfy $0 \leq \text{Re}(\alpha) < 1$.

Lemma 2.40. *\mathcal{I} is contained in $z_1^{-1}M(D_1, \partial D_1)[z_1^{-1}]$, and we have a decomposition $\mathcal{L} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \mathcal{L}_{\mathfrak{a}}$ such that (i) $(z_1 \partial_1 - z_1 \partial_1 \mathfrak{a})\mathcal{L}_{\mathfrak{a}} \subset \mathcal{L}_{\mathfrak{a}}$ and (ii) the eigenvalues α of the induced morphism of $\mathcal{L}_{\mathfrak{a}}|_{z_1=0}$ satisfy $0 \leq \text{Re}(\alpha) < 1$. Moreover, we have $\mathcal{L}_{\mathfrak{a}} \otimes H(D_1^\circ)[[z_1]] = \mathcal{L}'_{\mathfrak{a}}$.*

Proof. We use a descending induction on $m(\mathcal{L}) := \min\{\text{ord}_{z_1}(\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{I}\}$. If $m(\mathcal{L}) = 0$, there is nothing to prove. Let us consider the case $m(\mathcal{L}) = m < 0$. We put $T(\mathcal{L}) := \{(mz_1^{-m} \mathfrak{a})|_{z_1=0} \mid \mathfrak{a} \in \mathcal{I}\}$. Let us consider the endomorphism G of $\mathcal{L}/z_1 \mathcal{L}$ induced by $z_1^{-m} \nabla(z_1 \partial_1)$. Because the elements of $T(\mathcal{L})$ are the eigenvalues of G , they are algebraic over $M(D_1, \partial D_1)$. Then, we can deduce $T(\mathcal{L}) \subset M(D_1, \partial D_1)$ from $T(\mathcal{L}) \subset H(D_1^\circ)$. If $|T(\mathcal{L})| = 1$, by considering the tensor product with a meromorphic flat bundle of rank one, we can reduce the issue to the case $m(\mathcal{L}) = m + 1$. Let us consider the case

$|T(\mathcal{L})| \geq 2$. It is standard that the eigendecomposition of $\mathcal{L}/z_1\mathcal{L}$ is uniquely extended to a ∇ -flat decomposition $\mathcal{L} = \bigoplus_{\mathfrak{b} \in T(\mathcal{L})} \mathcal{L}_{\mathfrak{b}}$ (see Lemma 2.12). It is easy to observe that $m(\mathcal{L}_{\mathfrak{b}}) \geq m$, and $|T(\mathcal{L}_{\mathfrak{b}})| \leq 1$ if $m(\mathcal{L}_{\mathfrak{b}}) = m$. Thus, we are done. \square

Step 2. Let (\mathcal{E}, ∇) be a meromorphic flat bundle on (X, D) . Let (\mathcal{E}, ∇) be a meromorphic flat bundle on (X, D) . Make the following assumptions.

- For each $P \in D$, $(\mathcal{E}, \nabla)|_{\hat{P}}$ has an unramifiedly good Deligne–Malgrange lattice.
- The Deligne–Malgrange lattice E of (\mathcal{E}, ∇) is \mathcal{O}_X -locally free.

Let us show that E is unramifiedly good Deligne–Malgrange under the assumption. We put $D^{[2]} := \bigcup_{i \neq j} (D_i \cap D_j)$. We can take a ramified covering $\varphi: (X, D) \rightarrow (X, D)$ with the following property.

- For each $P \in D_i \setminus D^{[2]}$, the action of $\pi_1(D_i \setminus D^{[2]}, P)$ on $\text{Irr}(\varphi^*\nabla, P)$ is trivial.

By the argument in the proof of Lemma 2.36, we may and will assume that the above property holds for (\mathcal{E}, ∇) from the beginning. We have already known that $E|_{X \setminus D^{[2]}}$ is unramifiedly good Deligne–Malgrange (Lemma 2.39). In particular, we have $\mathcal{I} \subset z_1^{-1}H(D_1^\circ)[z_1^{-1}]$ and a decomposition $E|_{\hat{D}_1^\circ} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \hat{E}_{\mathfrak{a}}$ such that $(\nabla(z_1\partial_1) - z_1\partial_1\mathfrak{a})\hat{E}_{\mathfrak{a}} \subset \hat{E}_{\mathfrak{a}}$.

Let \mathcal{M} be the differential $M(D_1, \partial D_1)((z_1))$ -module corresponding to \mathcal{E} , and let \mathcal{L} be the $M(D_1, \partial D_1)[[z_1]]$ -lattice induced by E . Applying Lemma 2.40, we obtain $\mathcal{I} \subset z_1^{-1}M(D_1, \partial D_1)[z_1^{-1}]$ and a decomposition $\mathcal{L} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \mathcal{L}_{\mathfrak{a}}$ such that $(z_1\partial_1 - z_1\partial_1\mathfrak{a})\mathcal{L}_{\mathfrak{a}} \subset \mathcal{L}_{\mathfrak{a}}$. Let $\mathfrak{K} := \mathbb{C}((z_n)) \cdots ((z_2))$. By the natural extension $M(D_1, \partial D_1) \subset \mathfrak{K}$, $\mathcal{L} \otimes \mathfrak{K}[[z_1]]$ is the Deligne–Malgrange lattice of the differential module $\mathfrak{M} := (\mathcal{N} \otimes \mathfrak{K}((z_1)), \partial_1)$.

Let ${}^O E$ be the unramifiedly good Deligne–Malgrange lattice of $\mathcal{E}|_{\hat{O}}$ with the irregular decomposition ${}^O E = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, O)} {}^O E_{\mathfrak{a}}$. Let $\text{Irr}(\nabla, 1)$ be the image of $\text{Irr}(\nabla, O)$ via the map $\mathcal{O}_{\hat{O}}(*D)/\mathcal{O}_{\hat{O}} \rightarrow \mathcal{O}_{\hat{O}}(*D)/\mathcal{O}_{\hat{O}}(*D(\neq 1))$, where $D(\neq 1) := \bigcup_{2 \leq j \leq \ell} D_j$. It is easy to see that $\mathfrak{K}[[z_1]] \otimes {}^O E$ is the good Deligne–Malgrange lattice of $(\mathfrak{M}_1, \partial_1)$, and the set of the irregular values is given by $\text{Irr}(\nabla, 1)$. Hence, we obtain $\text{Irr}(\nabla, 1) = \mathcal{I}_1$ in $z_1^{-1}\mathfrak{K}[[z_1^{-1}]]$ and ${}^O E \otimes \mathfrak{K}[[z_1]] = \mathcal{L}_1 \otimes \mathfrak{K}[[z_1]]$. We can deduce a similar relation for each $i = 2, \dots, \ell$.

Lemma 2.41. We have $\text{Irr}(\nabla, O) \subset M(X, D)/H(X)$.

Proof. We set $\mathcal{S} := \{\mathfrak{m} \in \mathbb{Z}^\ell \mid \mathfrak{m} \not\leq_{\mathbb{Z}^\ell} \mathbf{0}\}$. (See § 2.1.1 for $\leq_{\mathbb{Z}^\ell}$.) For $i = 1, \dots, \ell$, we put

$$\begin{aligned} \mathcal{S}_i &:= \{\mathfrak{m} = (m_j) \in \mathcal{S} \mid m_i < 0\}, \\ \mathcal{S}_{\leq i} &:= \{\mathfrak{m} = (m_j) \in \mathcal{S} \mid m_j \geq 0 \ (j < i), \ m_i < 0\}. \end{aligned}$$

We have $\mathcal{S} = \coprod \mathcal{S}_{\leq i}$. For any $\mathfrak{a} \in \text{Irr}(\nabla, O)$, we have the expansion $\mathfrak{a} = \sum_{\mathfrak{m} \in \mathcal{S}} \mathfrak{a}_{\mathfrak{m}} z^{\mathfrak{m}}$. Because its image to $\mathcal{O}_{\hat{O}}(*D)/\mathcal{O}_{\hat{O}}(D(\neq 1))$ is convergent, we obtain the convergence of $\sum_{\mathfrak{m} \in \mathcal{S}_1} \mathfrak{a}_{\mathfrak{m}} z^{\mathfrak{m}}$. Similarly, we obtain the convergence of $\sum_{\mathfrak{m} \in \mathcal{S}_i} \mathfrak{a}_{\mathfrak{m}} z^{\mathfrak{m}}$ for $i = 2, \dots, \ell$. Then, we obtain the convergence of $\sum_{\mathfrak{m} \in \mathcal{S}_{\leq i}} \mathfrak{a}_{\mathfrak{m}} z^{\mathfrak{m}}$ for $i = 1, \dots, \ell$. Then, we obtain the convergence of \mathfrak{a} . \square

We take a frame v of ${}^{\mathcal{O}}E$. Let f be a section of E . We have the expression $f = \sum f_p v_p$. We obtain $f_p \in \mathfrak{K}[[z_1]]$, and hence f_p is z_1 -regular, i.e. f_p does not contain the negative power of z_1 . Similarly, we obtain that f_p are z_j -regular for $j = 2, \dots, \ell$. Thus, we obtain $E|_{\hat{\mathcal{O}}} \subset {}^{\mathcal{O}}E$. Similarly, we obtain ${}^{\mathcal{O}}E \subset E|_{\hat{\mathcal{O}}}$, and hence $E|_{\hat{\mathcal{O}}} = {}^{\mathcal{O}}E$. Thus, we obtain that E is unramifiedly good Deligne–Malgrange lattice.

Step 3. Let us consider the case in which we do not assume that E is \mathcal{O}_X -locally free. We have a closed analytic subset $Z \subset D$ with $\text{codim}_D(Z) \geq 2$ such that $E|_{X \setminus Z}$ is \mathcal{O}_X -locally free. Then, it is an unramifiedly good Deligne–Malgrange lattice of $(\mathcal{E}, \nabla)|_{X \setminus Z}$, according to Step 2. We put $D_1^* := D_1 \setminus Z$ and $\partial D_1^* := \partial D_1 \setminus Z$. We have $\mathcal{I} \subset z_1^{-1}M(D_1^*, \partial D_1^*)[[z_1^{-1}]]$ and the irregular decomposition $E|_{\hat{D}_1^*} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \hat{E}_{\mathfrak{a}, D_1^*}$. By using the Hartogs property and the argument in the proof of Lemma 2.38, we obtain $\mathcal{I} \subset z_1^{-1}M(D_1, \partial D_1)[z_1^{-1}]$ and a decomposition $E|_{\hat{D}_1} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \hat{E}_{\mathfrak{a}}$ such that (i) $(\nabla(z_1 \partial_1) - z_1 \partial_1 \mathfrak{a})\hat{E}_{\mathfrak{a}} \subset \hat{E}_{\mathfrak{a}}$ and (ii) the eigenvalues α of the induced endomorphism of $\hat{E}_{\mathfrak{a}|D_1}$ satisfy $0 \leq \text{Re}(\alpha) < 1$. We have $E \otimes \mathfrak{K}[[z_1]] = {}^{\mathcal{O}}E \otimes \mathfrak{K}[[z_1]]$. Let v be a frame of ${}^{\mathcal{O}}E$. Let f be a section of E . We have the expression $f = \sum f_p v_p$. Then, we obtain that f_p is z_1 -regular. Similarly, we obtain that f_p are z_j -regular ($j = 1, \dots, \ell$) and hence $E|_{\hat{\mathcal{O}}} \subset {}^{\mathcal{O}}E$.

To show ${}^{\mathcal{O}}E \subset E|_{\hat{\mathcal{O}}}$, we consider the dual. Put $\mathcal{E}^{\vee} := \text{Hom}_{\mathcal{O}_X(*D)}(\mathcal{E}, \mathcal{O}_X(*D))$, which is equipped with a naturally induced flat connection ∇ . Put $E^{\vee} := \text{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_X)$, which is a lattice of \mathcal{E}^{\vee} . It is generically unramifiedly good lattice, and the eigenvalues α of the residue satisfy $-1 < \text{Re}(\alpha) \leq 0$. Put ${}^{\mathcal{O}}E^{\vee} := \text{Hom}_{\mathcal{O}_{\hat{\mathcal{O}}}}({}^{\mathcal{O}}E, \mathcal{O}_{\hat{\mathcal{O}}})$ which is an unramifiedly good lattice of $(\mathcal{E}^{\vee}, \nabla)|_{\hat{\mathcal{O}}}$. The eigenvalues α of the residues satisfy $-1 < \text{Re}(\alpha) \leq 0$. Then, we obtain $E|_{\hat{\mathcal{O}}}^{\vee} \subset {}^{\mathcal{O}}E^{\vee}$ by the above argument. We have $E|_{\hat{\mathcal{O}}}^{\vee} \simeq \text{Hom}_{\mathcal{O}_X}(E, \mathcal{O}_{\hat{\mathcal{O}}}) \simeq \text{Hom}_{\mathcal{O}_{\hat{\mathcal{O}}}}(E|_{\hat{\mathcal{O}}}, \mathcal{O}_{\hat{\mathcal{O}}})$. Hence, we can conclude that ${}^{\mathcal{O}}E = E|_{\hat{\mathcal{O}}}$. Thus, we obtain that E is an unramifiedly good Deligne–Malgrange lattice of (\mathcal{E}, ∇) at O , and the proof of Proposition 2.18 is finished. Then, Proposition 2.19 follows from Proposition 2.27. \square

3. Stokes structure

3.1. Preliminary

3.1.1. Filtration indexed by a finite ordered set

Let (I, \leq) be a finite ordered set. Let V be a vector space. In this section, a filtration F of V indexed by (I, \leq) means a family of subspaces $F_a \subset V$ ($a \in I$) with the following properties.

- $F_a \subset F_b$ if $a \leq b$.
- There exists a splitting $V = \bigoplus V_a$ such that $F_a = \bigoplus_{b \leq a} V_b$.

We put $F_{<a} := \sum_{b < a} F_b$ and $\text{Gr}_a^F(V) = F_a / F_{<a} \simeq V_a$. For a given subset $S \subset I$, we set $F_S := \sum_{a \in S} F_a$.

Remark 3.1. Note that we assume the existence of splitting, which is unusual. We consider the above type of filtration just for Stokes filtration.

Let $\varphi: (I, \leq) \rightarrow (I', \leq')$ be a morphism of ordered sets, and let F be a filtration of V indexed by (I, \leq) . Then, we have the induced filtration F^φ indexed by (I', \leq') constructed inductively as follows:

$$F_b^\varphi = F_{<b}^\varphi + \sum_{a \in \varphi^{-1}(b)} F_a.$$

We set $V_b^\varphi := \bigoplus_{a \in \varphi^{-1}(b)} V_a$. Then, $V = \bigoplus_{b \in I'} V_b^\varphi$ gives a splitting of F^φ .

Definition 3.2. Let F and F' be filtrations of V indexed by (I, \leq) and (I', \leq') , respectively. Let $\varphi: (I, \leq) \rightarrow (I', \leq')$ be a morphism of ordered sets. We say that F and F' are compatible over φ , if F' is the same as F^φ above. If $I = I'$ (but possibly $(I, \leq) \neq (I', \leq')$) and $\varphi = \text{id}$, we just say F and F' are compatible.

In the case $I = I'$, we have the natural isomorphism $\text{Gr}_a^F(V) \simeq \text{Gr}_a^{F'}(V)$.

Lemma 3.3. Let F be a filtration of V indexed by (I, \leq) . Let \leq_i ($i \in \Lambda$) be orders on I such that (i) the identity $\varphi_i: (I, \leq) \rightarrow (I, \leq_i)$ are order preserving and (ii) $a \leq b$ if and only if $a \leq_i b$ for any $i \in \Lambda$. Then, F can be reconstructed from F^{φ_i} ($i \in \Lambda$) in the sense $F_a = \bigcap_{i \in \Lambda} F_a^{\varphi_i}$.

Proof. We take a splitting $V = \bigoplus_{a \in I} V_a$ of the filtration F . Recall $F_a^{\varphi_i} = \bigoplus_{b \leq_i a} V_b$. Then, the claim of the lemma is clear. □

Let (I, \leq) be an ordered set, and let V be a finite-dimensional vector space equipped with a filtration F indexed by (I, \leq) . Let us give an induced filtration F^\vee on the dual vector space V^\vee . We set $I^\vee := I$ and let \leq^\vee be the order of I^\vee defined by $a \leq^\vee b \iff a \geq b$. For distinction, we use the symbol $-a$ if we regard $a \in I$ as an element of I^\vee . And, ' $-a \leq^\vee -b$ ' is denoted by $-a \leq -b$.

We take a splitting $V = \bigoplus_{a \in I} V_a$ of the filtration F . In general, for a vector subspace $U \subset V$, let $U^\perp \subset V^\vee$ be $\{f \in V^\vee \mid f(v) = 0 \ \forall v \in U\}$. For each $a \in I$, let $S(a)$ denote the set of $b \in I$ such that $b \not\geq a$. We have the subspaces of V^\vee given as follows:

$$V_{-a}^\vee := \left(\bigoplus_{b \neq a} V_b \right)^\perp, \quad F_{-a}^\vee(V^\vee) := \left(\bigoplus_{b \in S(a)} V_b \right)^\perp.$$

The subspaces $\{F_{-a}^\vee(V^\vee) \mid -a \in I^\vee\}$ are well defined, and give a filtration of V^\vee indexed by (I^\vee, \leq) . The decomposition $V^\vee = \bigoplus_{-a \in I^\vee} V_{-a}^\vee$ gives a splitting of the filtration F^\vee .

3.1.2. Induced orders on good set of irregular values

Let X be a complex manifold with a simple normal crossing hypersurface D . Let $\pi: \tilde{X}(D) \rightarrow X$ be the real blowup. (In this paper, the real blowup along D means the fibre products of the real blowup along the irreducible components, taken over X .)

Let $\mathcal{I}_P \subset \mathcal{O}_{X,P}(*D)/\mathcal{O}_{X,P}$ be a good set of irregular values, where $P \in D$. For each $Q \in \pi^{-1}(P)$, we shall introduce an order \leq_Q on the set \mathcal{I}_P . We can take a coordinate neighbourhood (X_P, z_1, \dots, z_n) around P such that $D_P := X_P \cap D$ is expressed as

$\bigcup_{i=1}^{\ell} \{z_i = 0\}$, and that $\mathcal{I}_P \subset M(X_P, D_P)/H(X_P)$. We take a lift $\tilde{\mathbf{a}} \in M(X_P, D_P)$ for each $\mathbf{a} \in \mathcal{I}_P$. For each distinct $\mathbf{a}, \mathbf{b} \in \mathcal{I}_P$, we put

$$F_{\mathbf{a}, \mathbf{b}} := -\operatorname{Re}(\tilde{\mathbf{a}} - \tilde{\mathbf{b}}) |z^{-\operatorname{ord}(\mathbf{a}-\mathbf{b})}|. \tag{3.1}$$

It naturally induces a C^∞ -function on $\tilde{X}_P(D_P)$.

Definition 3.4. Let $Q \in \pi^{-1}(P)$. We say $\mathbf{a} <_Q \mathbf{b}$ for distinct $\mathbf{a}, \mathbf{b} \in \mathcal{I}_P$, if $F_{\mathbf{a}, \mathbf{b}}(Q) < 0$. We say $\mathbf{a} \leq_Q \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathcal{I}_P$, if $\mathbf{a} <_Q \mathbf{b}$ or $\mathbf{a} = \mathbf{b}$. The relation \leq_Q is a partial order on \mathcal{I}_P .

It is easy to check that the condition is independent of the choice of a coordinate system (z_1, \dots, z_n) and lifts $\tilde{\mathbf{a}}$. The following lemma is clear.

Lemma 3.5. For any $Q \in \pi^{-1}(D)$, there exists a neighbourhood \mathcal{N} of Q in $\pi^{-1}(D)$ such that, for any $Q' \in \mathcal{N}$, the map $(\mathcal{I}_{\pi(Q)}, \leq_Q) \rightarrow (\mathcal{I}_{\pi(Q')}, \leq_{Q'})$ is order preserving.

3.2. Stokes data

Let X be a complex manifold, and let D be a simple normal crossing hypersurface of X . Let $\pi: \tilde{X}(D) \rightarrow X$ be the real blowup. Let $\mathcal{I} = (\mathcal{I}_P \mid P \in D)$ be a good system of irregular values on (X, D) . Let \mathcal{U} be a locally connected subset of $\tilde{X}(D)$, and let \mathfrak{V} be a local system on \mathcal{U} .

Definition 3.6. A Stokes datum of \mathfrak{V} over \mathcal{I} is a tuple of filtrations $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}^Q \mid Q \in \mathcal{U} \cap \pi^{-1}(D))$ of germs \mathfrak{V}_Q indexed by $(\mathcal{I}_{\pi(Q)}, \leq_Q)$ satisfying the following compatibility condition.

- Let $Q \in \mathcal{U} \cap \pi^{-1}(D)$. Take a small neighbourhood \mathcal{N} as in Lemma 3.5 such that $\mathcal{N} \cap \mathcal{U}$ is connected. For any $Q' \in \mathcal{U} \cap \mathcal{N}$, we have the induced filtration $\tilde{\mathcal{F}}^{Q'}$ of $\mathfrak{V}_{Q'}$. Then, $(\mathfrak{V}_{Q'}, \tilde{\mathcal{F}}^{Q'}) \rightarrow (\mathfrak{V}_Q, \tilde{\mathcal{F}}^Q)$ is compatible over $(\mathcal{I}_{\pi(Q)}, \leq_Q) \rightarrow (\mathcal{I}_{\pi(Q')}, \leq_{Q'})$.

Let Q and Q' be as in Definition 3.6. If $Q' \in \pi^{-1}(P)$, we have $\mathcal{I}_{\pi(Q)} = \mathcal{I}_{\pi(Q')}$, and we have an induced isomorphism

$$\operatorname{Gr}^{\tilde{\mathcal{F}}^Q}(\mathfrak{V}|_{\mathcal{U}_Q})|_{\mathcal{U}_{Q'}} \simeq \operatorname{Gr}^{\tilde{\mathcal{F}}^{Q'}}(\mathfrak{V}|_{\mathcal{U}_{Q'}}).$$

Hence, we have the associated graded sheaf on a neighbourhood of $\pi^{-1}(P) \cap \mathcal{U}$ in \mathcal{U} , denoted by $\operatorname{Gr}^{\tilde{\mathcal{F}}}(\mathfrak{V}_{\pi^{-1}(P) \cap \mathcal{U}})$.

Let \mathfrak{V}_i ($i = 1, 2$) be local systems on \mathcal{U} with Stokes data $\tilde{\mathcal{F}}_i$ over \mathcal{I} . A morphism $F: (\mathfrak{V}_1, \tilde{\mathcal{F}}_1) \rightarrow (\mathfrak{V}_2, \tilde{\mathcal{F}}_2)$ is defined to be a morphism of sheaves such that the induced morphisms $\mathfrak{V}_{1Q} \rightarrow \mathfrak{V}_{2Q}$ preserve filtrations for any $Q \in \mathcal{U} \cap \pi^{-1}(D)$.

Remark 3.7. We are given only the filtrations indexed by $(\mathcal{I}_{\pi(Q)}, \leq_Q)$ for $Q \in \pi^{-1}(D) \cap \mathcal{U}$ in the definition of Stokes data. We shall observe that we can obtain more refined filtrations in Proposition 3.16.

Remark 3.8. ‘Stokes data’ in this paper is called ‘full pre-Stokes data’ in [21]. It was useful to consider filtrations, called ‘partial Stokes filtrations’ in the level $\mathfrak{m}(0)$, which are indexed by the image of \mathcal{I}_P via $\tilde{\eta}_{\mathfrak{m}(0)}$. We also have partial Stokes filtrations in various levels. It explains the meaning of the adjective ‘full’. In [21], we are interested in not only meromorphic flat bundles but also their lattices. To describe an unramifiedly good lattices, we need an additional data with a system of Stokes filtrations. It is called full Stokes data in [21]. It explains the meaning of the prefix ‘pre’.

Because we are concerned with good meromorphic flat bundles in this paper, we use the terminology ‘Stokes data’ in the sense of Definition 3.6.

3.3. Extension and uniqueness of Stokes structure

3.3.1. Category of Stokes data

Let X be a complex manifold with a simple normal crossing hypersurface D . Let G be a finite group acting on (X, D) . Let \mathcal{I} be a good system of irregular values on (X, D) which is G -equivariant in the sense $g^*\mathcal{I}_{g(P)} = \mathcal{I}_P$ for any $g \in G$ and $P \in D$.

Let \mathfrak{V} be a local system on $\tilde{X}(D)$ with a G -action, i.e. for each $g \in G$, we are given an isomorphism $g^*\mathfrak{V} \simeq \mathfrak{V}$ compatible with the group law. Let $\tilde{\mathcal{F}}$ be a Stokes data of \mathfrak{V} . For each $g \in G$, we have the induced Stokes data $g^*\tilde{\mathcal{F}}$ of \mathfrak{V} . The Stokes data is called G -equivariant if $g^*\tilde{\mathcal{F}} = \tilde{\mathcal{F}}$. The category of G -equivariant local system with Stokes data on $\tilde{X}(D)$ is denoted by $\text{SD}(X, D, \mathcal{I})^G$. If $G = \{1\}$, it is denoted by $\text{SD}(X, D, \mathcal{I})$.

3.3.2. Statement

We consider the following situation. Let $p: X \rightarrow B$ be a smooth fibration of complex manifolds with a normal crossing hypersurface D . For simplicity, we make the following assumptions.

- B is simply connected.
- We put $X^b := X \times_B b$ and $D^b := D \times_B b$ for any $b \in B$. Then, (X, D) is topologically a product of (X^b, D^b) and B .

For example, we would like to consider the case $(X, D) = (X^b, D^b) \times B$ as complex manifolds.

Let \mathcal{I} be a good system of irregular values on (X, D) . Its restriction to X^b is denoted by \mathcal{I}^b . For an object $(\mathfrak{V}, \tilde{\mathcal{F}})$ in $\text{SD}(X, D, \mathcal{I})$, we have a naturally induced object $(\mathfrak{V}^b, \tilde{\mathcal{F}}^b)$ in $\text{SD}(X^b, D^b, \mathcal{I}^b)$, obtained as the restriction. Although the following theorem is a special case of Corollary 4.4.4 in [21], we shall give a proof in §3.3.6, to explain some more detailed property of Stokes data.

Theorem 3.9. *For any $b \in B$, the restriction $\Upsilon: \text{SD}(X, D, \mathcal{I}) \rightarrow \text{SD}(X^b, D^b, \mathcal{I}^b)$ is equivalent.*

Theorem 3.9 says that a Stokes data of \mathfrak{V}^b over \mathcal{I}^b is uniquely extended to a data of \mathfrak{V} over \mathcal{I} in a functorial way.

Remark 3.10. Theorem 3.9 (and Theorem 4.13 below) may be regarded as a higher-dimensional generalization of Theorems 1 and 2 for the one-dimensional case in [7]. (See also [26] for the local one-dimensional case.) These theorems imply that a variation of irregular values causes a deformation of a Stokes data, or equivalently a good meromorphic flat bundle. In the one-dimensional case, or locally in the higher-dimensional case, the coefficients of the irregular values make a universal family of such deformations. It would be interesting to have a universal family when X is a projective variety.*

Assume that a finite group G acts on (X, D) over B , and \mathcal{I} is G -equivariant. By using the uniqueness, we obtain the following.

Corollary 3.11. *The restriction $\text{SD}(X, D, \mathcal{I})^G \rightarrow \text{SD}(X^b, D^b, \mathcal{I}^b)^G$ is an equivalence for any $b \in B$.*

3.3.3. *Preliminary*

We mention easy property of Stokes data. Let X be a complex manifold with a normal crossing hypersurface D . Let $\pi: \tilde{X}(D) \rightarrow X$ be the real blowup. Let U be a locally connected subset of $\pi^{-1}(D)$. Let \mathfrak{V} be a local system on U .

Lemma 3.12. *Let $\tilde{\mathcal{F}}_i$ ($i = 1, 2$) be Stokes data of \mathfrak{V} .*

- *If there exists a dense subset $U' \subset U$ such that $\tilde{\mathcal{F}}_1^Q = \tilde{\mathcal{F}}_2^Q$ for $Q \in U'$. Then, we have $\tilde{\mathcal{F}}_1 = \tilde{\mathcal{F}}_2$.*
- *Let Z be any subset of U . If $\tilde{\mathcal{F}}_1^Q = \tilde{\mathcal{F}}_2^Q$ for any $Q \in Z$, there exists a neighbourhood Z' of Z such that $\tilde{\mathcal{F}}_1^Q = \tilde{\mathcal{F}}_2^Q$ for any $Q \in Z'$.*

Proof. The first claim follows from Lemma 3.3. The second claim follows from the compatibility of the system of filtrations. □

Let \mathfrak{V}_i ($i = 1, 2$) be local systems on U with a morphism $F: \mathfrak{V}_1 \rightarrow \mathfrak{V}_2$. It is easy to deduce the following corollary.

Corollary 3.13. *Let $\tilde{\mathcal{F}}_i$ ($i = 1, 2$) be Stokes data of \mathfrak{V}_i .*

- *If there exists a dense subset $U' \subset U$ such that F preserves $\tilde{\mathcal{F}}^Q$ for $Q \in U'$. Then, F preserves $\tilde{\mathcal{F}}$.*
- *Let Z be any subset of U . If F preserves $\tilde{\mathcal{F}}^Q$ for any $Q \in Z$, there exists a neighbourhood Z' of Z such that F preserves $\tilde{\mathcal{F}}^Q$ for any $Q \in Z'$.*

3.3.4. *Filtration on a small convex set*

We put $X := \Delta^n$, $D_i = \{z_i = 0\}$, $D := \bigcup_{i=1}^\ell D_i$ and $D_\ell := \bigcap_{i=1}^\ell D_i$. Let $\pi: \tilde{X}(D) \rightarrow X$ be the real blowup. We have the natural identification $\pi^{-1}(D_\ell) = (S^1)^\ell \times D_\ell$ by the coordinate (z_1, \dots, z_n) . We use the polar coordinate $(\theta_1, \dots, \theta_\ell)$ for $(S^1)^\ell$, induced by (z_1, \dots, z_ℓ) .

* This remark is thanks to the referee.

Let $\mathcal{I} \subset M(X, D)/H(X)$ be a good set of irregular values. For $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, let $F_{\mathfrak{a}, \mathfrak{b}}$ be given by (3.1). For a subset $A \subset \pi^{-1}(D_\ell)$, the order \leq_A on \mathcal{I} is given as in Definition 3.4. Namely, we say $\mathfrak{a} <_A \mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ if $F_{\mathfrak{a}, \mathfrak{b}} < 0$ on A , and we say $\mathfrak{a} \leq_A \mathfrak{b}$ if we have $\mathfrak{a} <_A \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{b}$.

Condition 3.14. Let $P \in D_\ell$. Let \mathcal{C} be a closed convex subset of $(S^1)^\ell$ satisfying the following.

- There exist $(\theta_1^{(0)}, \dots, \theta_\ell^{(0)})$ such that \mathcal{C} is contained in $\{(\theta_1, \dots, \theta_\ell) \mid |\theta_i - \theta_i^{(0)}| < \pi/2\}$. In particular, we can identify \mathcal{C} with a closed region in \mathbb{R}^n .
- We naturally regard $\mathcal{C}(P) := \mathcal{C} \times \{P\}$ as a subset of $\pi^{-1}(D_\ell)$. Then, for each distinct pair $(\mathfrak{a}, \mathfrak{b})$ of \mathcal{I} , if $\mathcal{C}(P) \cap F_{\mathfrak{a}, \mathfrak{b}}^{-1}(0) \neq \emptyset$, it divides $\mathcal{C}(P)$ into two closed regions.

The following lemma is clear.

Lemma 3.15. Let P and \mathcal{C} be as in Condition 3.14. Then, there exists a small neighbourhood \mathcal{B} of P in D_ℓ such that the following holds.

- For any non-empty subset $\mathcal{B}_0 \subset \mathcal{B}$, the order $\leq_{\mathcal{C} \times \mathcal{B}_0}$ on \mathcal{I} is the same as $\leq_{\mathcal{C}(P)}$.

In particular, for any $P' \in \mathcal{B}$, the orders $\leq_{\mathcal{C}(P)}$ and $\leq_{\mathcal{C}(P')}$ are the same, where $\mathcal{C}(P') := \mathcal{C} \times P'$.

Proposition 3.16. Let P and \mathcal{C} be as in Condition 3.14. Let \mathfrak{V} be a local system on $\mathcal{C}(P)$ with a Stokes data $(\tilde{\mathcal{F}}^Q \mid Q \in \mathcal{C}(P))$. Then, there uniquely exists a global filtration $\tilde{\mathcal{F}}^{\mathcal{C}(P)}$ indexed by $(\mathcal{I}_P, \leq_{\mathcal{C}(P)})$ such that, for any $Q \in \mathcal{C}(P)$, the filtrations $\tilde{\mathcal{F}}^{\mathcal{C}(P)}$ and $\tilde{\mathcal{F}}^Q$ are compatible over $(\mathcal{I}_P, \leq_{\mathcal{C}(P)}) \rightarrow (\mathcal{I}_P, \leq_Q)$. In other words, there exists a decomposition $\mathfrak{V} = \bigoplus_{\mathfrak{a} \in \mathcal{I}} \mathfrak{V}_\mathfrak{a}$, which gives a splitting of $\tilde{\mathcal{F}}^Q$ for any $Q \in \mathcal{C}(P)$.

Proof. In the proof, $\mathcal{C}(P)$ is denoted by \mathcal{C} for simplicity of the description. Let \mathfrak{V} be the space of global sections of \mathfrak{V} . We have natural isomorphisms $\mathfrak{V} \simeq \mathfrak{V}_Q$ for any $Q \in \mathcal{C}$. We regard that we are given filtrations $\tilde{\mathcal{F}}^Q$ ($Q \in \mathcal{C}$) on \mathfrak{V} . We shall show that there uniquely exists a filtration $\tilde{\mathcal{F}}^{\mathcal{C}}$ of \mathfrak{V} such that for any $Q \in \mathcal{C}$, the filtrations $\tilde{\mathcal{F}}^{\mathcal{C}}$ and $\tilde{\mathcal{F}}^Q$ are compatible over $(\mathcal{I}_P, \leq_{\mathcal{C}}) \rightarrow (\mathcal{I}_P, \leq_Q)$.

For $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}_P$, we have $\mathfrak{a} \leq_{\mathcal{C}} \mathfrak{b}$ if and only if $\mathfrak{a} \leq_Q \mathfrak{b}$ for any $Q \in \mathcal{C}$. Hence, the uniqueness of such a filtration follows from Lemma 3.3.

Put $H_{\mathfrak{a}, \mathfrak{b}} := F_{\mathfrak{a}, \mathfrak{b}}^{-1}(0)$ for distinct $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$. A connected component of $\mathcal{C} \setminus \bigcup H_{\mathfrak{a}, \mathfrak{b}}$ is called a chamber. If Q is contained in a chamber, then \leq_Q is totally ordered. If Q and Q' are contained in the same chamber, we have $\leq_Q = \leq_{Q'}$.

Take Q_0 in a chamber, and let \mathfrak{a} be minimal with respect to \leq_{Q_0} . Note that \mathfrak{a} is also minimal with respect to $\leq_{\mathcal{C}}$. Let us observe that $\tilde{\mathcal{F}}_{\mathfrak{a}}^{Q_0}$ is contained in $\tilde{\mathcal{F}}_{\mathfrak{a}}^Q$ for any $Q \in \mathcal{C}$. We take the interval I connecting Q and Q_0 . We take points $R_0 = Q_0, R_1, R_2, \dots, R_{N-1}, R_N = Q$ in I such that the open interval (R_i, R_{i+1}) is contained in a chamber. For $R', R'' \in (R_i, R_{i+1})$, we have $\tilde{\mathcal{F}}_{\mathfrak{a}}^{R'} = \tilde{\mathcal{F}}_{\mathfrak{a}}^{R''}$ by the compatibility condition for Stokes data. For $R \in (R_{i-1}, R_{i+1})$, we have $\tilde{\mathcal{F}}_{\mathfrak{a}}^{R_i} \subset \tilde{\mathcal{F}}_{\mathfrak{a}}^R$. For $\mathfrak{b} \in \mathcal{I}$, $F_{\mathfrak{a}, \mathfrak{b}}$ is monotonously increasing along I from Q_0 to Q around R_i . Hence, $F_{\mathfrak{a}, \mathfrak{b}}(R_i) > 0$ if

and only if $F_{a,b}(R) > 0$ for $R \in (R_{i-1}, R_i)$. It implies $\tilde{\mathcal{F}}_a^R \subset \tilde{\mathcal{F}}_a^{R_i}$ for $R \in (R_{i-1}, R_i)$. Therefore, we obtain $\tilde{\mathcal{F}}_a^{Q_0} \subset \tilde{\mathcal{F}}_a^Q$. We can also deduce that $\tilde{\mathcal{F}}_a^{Q_0} \rightarrow \text{Gr}_a^{\tilde{\mathcal{F}}^Q}$ is an isomorphism for any Q . Hence, in particular, if $\mathfrak{b} \neq \mathfrak{a}$ is minimal with respect to \leq_Q , we have $\tilde{\mathcal{F}}_b^Q \cap \tilde{\mathcal{F}}_a^{Q_0} = 0$.

We put $\mathfrak{V}_0 := \mathfrak{V}/\tilde{\mathcal{F}}_a^{Q_0}$. For any $Q \in \mathcal{C}$ and $\mathfrak{b} \in \mathcal{I}$, let $\tilde{\mathcal{F}}_b^Q(\mathfrak{V}_0)$ be the image of $\tilde{\mathcal{F}}_b^Q(\mathfrak{V})$ to \mathfrak{V}_0 . Let $\mathfrak{V} = \bigoplus \mathfrak{V}_{b,Q}$ be a splitting of $\tilde{\mathcal{F}}^Q$. We remark that we may assume $\mathfrak{V}_{a,Q} = \tilde{\mathcal{F}}_a^{Q_0}$. Then, it is easy to see that the images of $\mathfrak{V}_{b,Q}$ gives a splitting of the filtration $\tilde{\mathcal{F}}^Q(\mathfrak{V}_0)$. We can also easily observe that the system of filtrations $(\tilde{\mathcal{F}}^Q(\mathfrak{V}_0) \mid Q \in \mathcal{C})$ gives a Stokes data of the local system \mathfrak{V}_0 on \mathcal{C} associated to \mathfrak{V}_0 .

Assume that we have filtrations $\tilde{\mathcal{F}}^C$ for \mathfrak{V}_0 and \mathfrak{V} with the desired property. Then, $\tilde{\mathcal{F}}_b^C(\mathfrak{V}_0)$ is obtained as the image of $\tilde{\mathcal{F}}_b^C(\mathfrak{V})$. Actually, let $\mathfrak{V} = \bigoplus_{b \in \mathcal{I}} \mathfrak{V}_b$ be a splitting of $\tilde{\mathcal{F}}^C(\mathfrak{V})$. We may assume $\mathfrak{V}_a = \tilde{\mathcal{F}}_a^{Q_0}$. The decomposition also gives a splitting of $\tilde{\mathcal{F}}^Q(\mathfrak{V})$ for each $Q \in \mathcal{C}$. We have the induced decomposition $\mathfrak{V}_0 = \bigoplus \mathfrak{V}_{0,b}$, which gives a splitting of $\tilde{\mathcal{F}}^Q(\mathfrak{V}_0)$ for each $Q \in \mathcal{C}$. It implies that the decomposition gives a splitting of $\tilde{\mathcal{F}}^C(\mathfrak{V}_0)$ by the uniqueness, and we can conclude that $\tilde{\mathcal{F}}^C(\mathfrak{V}_0)$ is obtained as the image of $\tilde{\mathcal{F}}^C(\mathfrak{V})$.

Let us show the claim of the proposition by using an induction on $|\mathcal{I}|$. The case $|\mathcal{I}| = 1$ is obvious. Let Q_0 be a point in a chamber, and let \mathfrak{a} be the minimal with respect to \leq_{Q_0} . If \mathfrak{a} is the minimum with respect to \leq_C , we can construct the desired filtration of \mathfrak{V} as the pullback via $\mathfrak{V} \rightarrow \mathfrak{V}_0$. Assume that \mathfrak{a} is not the minimum. We can find a point Q_1 in a chamber such that \mathfrak{a} is not minimal with respect to \leq_{Q_1} . Let $\mathfrak{b} \in \mathcal{I}$ be minimal with respect to \leq_{Q_1} . We remark $\tilde{\mathcal{F}}_b^{Q_1} \cap \tilde{\mathcal{F}}_a^{Q_0} = 0$. We put $\mathfrak{V}_1 := \mathfrak{V}/\tilde{\mathcal{F}}_b^{Q_1}$ and $\mathfrak{V}_2 := \mathfrak{V}/(\tilde{\mathcal{F}}_b^{Q_1} \oplus \tilde{\mathcal{F}}_a^{Q_0})$. As remarked above, the associated local systems \mathfrak{V}_i ($i = 0, 1, 2$) are equipped with the induced Stokes structure. By construction, we have

$$\tilde{\mathcal{F}}_c^Q(\mathfrak{V}) = \tilde{\mathcal{F}}_c^Q(\mathfrak{V}_1) \times_{\tilde{\mathcal{F}}_c^Q(\mathfrak{V}_2)} \tilde{\mathcal{F}}_c^Q(\mathfrak{V}_0)$$

for any $Q \in \mathcal{C}$ and $c \in \mathcal{I} \setminus \{\mathfrak{a}, \mathfrak{b}\}$.

By the hypothesis of the induction, \mathfrak{V}_i are equipped with the filtration $\tilde{\mathcal{F}}^C$ with the desired property. Note that $\tilde{\mathcal{F}}^C(\mathfrak{V}_2)$ is obtained as the image of $\tilde{\mathcal{F}}^C(\mathfrak{V}_i)$ ($i = 0, 1$). We put

$$\tilde{\mathcal{F}}_c^C(\mathfrak{V}) := \tilde{\mathcal{F}}_c^C(\mathfrak{V}_0) \times_{\tilde{\mathcal{F}}_c^C(\mathfrak{V}_2)} \tilde{\mathcal{F}}_c^C(\mathfrak{V}_1).$$

Let us check that $\tilde{\mathcal{F}}^C(\mathfrak{V})$ has the desired property. Let $\mathfrak{V}_2 = \bigoplus \mathfrak{V}_{2,c}$ be a splitting of $\tilde{\mathcal{F}}^C$. Let $\mathfrak{V}_c \subset \tilde{\mathcal{F}}_c^C(\mathfrak{V})$ be a lift of $\mathfrak{V}_{2,c}$. We put $\mathfrak{V}_a := \tilde{\mathcal{F}}_a^{Q_0}$ and $\mathfrak{V}_b := \tilde{\mathcal{F}}_b^{Q_1}$. By using that $\tilde{\mathcal{F}}^C(\mathfrak{V}_2)$ is obtained as the image of $\tilde{\mathcal{F}}^C(\mathfrak{V}_i)$ ($i = 0, 1$), we can check that $\mathfrak{V} = \bigoplus \mathfrak{V}_c$ is a splitting of the filtration $\tilde{\mathcal{F}}^C$. Similarly, we can check that it gives a splitting of each $\tilde{\mathcal{F}}^Q(\mathfrak{V})$. Hence, $\tilde{\mathcal{F}}^C$ is compatible with $\tilde{\mathcal{F}}^Q$ for each $Q \in \mathcal{C}$. □

3.3.5. Local extension

We continue to use the notation in §3.3.4. Let \mathfrak{V} be a local system on $\tilde{X}(D)$.

Lemma 3.17. *Let P be a point of D_ℓ . A Stokes data of $\mathfrak{V}|_{\pi^{-1}(P)}$ is uniquely extended to a Stokes data on a neighbourhood of $\pi^{-1}(P)$ in $\tilde{X}(D)$.*

Proof. For any $Q \in \pi^{-1}(P)$, we take a small neighbourhood \mathcal{U}_Q in $\tilde{X}(D)$ such that $\leq_Q = \leq_{\mathcal{U}_Q}$. We can find $Q_1, \dots, Q_N \in \pi^{-1}(P)$ such that $\pi^{-1}(P) \subset \bigcup \mathcal{U}_{Q_i}$. We may

assume \mathcal{U}_{Q_i} are the product $C_i \times B$ where B is a neighbourhood of P in $[0, 1]^\ell \times D_\ell$, and $C_i \subset (S^1)^\ell$. (We use the natural identification $\tilde{X}(D) \simeq ([0, 1] \times S^1)^\ell \times D_\ell$.) As remarked in the second claim of Lemma 3.12, for $Q' \in \mathcal{U}_{Q_i}$, we have the induced filtration $\tilde{\mathcal{F}}^{Q', Q_i}$ of $\mathfrak{V}_{Q'}$ induced by $\tilde{\mathcal{F}}^{Q_i}$ and $(\mathcal{I}_P, \leq_{Q_i}) \rightarrow (\mathcal{I}_{\pi(Q')}, \leq_{Q'})$. For any $R \in P \times (C_i \cap C_j)$, there exists a neighbourhood $\mathcal{U}_R \subset \mathcal{U}_{Q_i} \cap \mathcal{U}_{Q_j}$ such that, for any $Q' \in \mathcal{U}_R$, both $\tilde{\mathcal{F}}^{Q', Q_i}$ and $\tilde{\mathcal{F}}^{Q', Q_j}$ are induced by $\tilde{\mathcal{F}}^R$ and $(\mathcal{I}_P, \leq_R) \rightarrow (\mathcal{I}_{\pi(Q')}, \leq_{Q'})$, and hence they are the same. Therefore, by shrinking B , we obtain a Stokes structure $\mathfrak{V}_{\pi^{-1}(B)}$ whose restriction to $\pi^{-1}(P)$ is the same as the given one. The uniqueness follows from the first claim of Lemma 3.12.* \square

Lemma 3.18. *Let B be an open subset of D_ℓ , and let $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}^Q \mid Q \in \pi^{-1}(B))$ be a Stokes data on $\mathfrak{V}_{|\pi^{-1}(B)}$. Let P be a point in the boundary of B such that, for any small ball B_P around P , the intersection $B \cap B_P$ is connected. Then, there exists a small neighbourhood B_1 of P in D_ℓ such that $\tilde{\mathcal{F}}$ is uniquely extended to a Stokes data of $\mathfrak{V}_{|\pi^{-1}(B \cup B_1)}$.*

Proof. Let $Q \in \pi^{-1}(P)$. We take \mathcal{C} as in Condition 3.14 with the following property.

- $Q \in \mathcal{C}(P)$ and $\leq_Q = \leq_{\mathcal{C}(P)}$.

We take a small neighbourhood \mathcal{B} of P as in Lemma 3.15. We may assume $\mathcal{B} \cap B$ is connected.

Let $P' \in \mathcal{B} \cap B$. We have the unique filtration $\tilde{\mathcal{F}}^{\mathcal{C}(P')}$ of $\mathfrak{V}_{|\mathcal{C}(P')}$ as in Proposition 3.16. It naturally induces a filtration of the restriction of \mathfrak{V} on a neighbourhood of $\mathcal{C}(P')$, denoted by $\tilde{\mathcal{F}}^{\mathcal{C}(P')}$. If $P'' \in B$ is sufficiently close to P' , the restriction of $\tilde{\mathcal{F}}^{\mathcal{C}(P')}$ has the property in Proposition 3.16 for the local system $\mathfrak{V}_{|\mathcal{C}(P'')}$ with the Stokes data. Hence, it is the same as $\tilde{\mathcal{F}}^{\mathcal{C}(P'')}$. Namely, we have the filtration $\tilde{\mathcal{F}}^{(B \cap \mathcal{B}) \times \mathcal{C}}$ of $\mathfrak{V}_{|(B \cap \mathcal{B}) \times \mathcal{C}}$ indexed by (\mathcal{I}, \leq_Q) such that, for any $Q' \in (B \cap \mathcal{B}) \times \mathcal{C}$, the filtrations $\tilde{\mathcal{F}}^{(B \cap \mathcal{B}) \times \mathcal{C}}$ and $\tilde{\mathcal{F}}^{Q'}$ are compatible over $(\mathcal{I}, \leq_Q) \rightarrow (\mathcal{I}, \leq_{Q'})$. Let $\tilde{\mathcal{F}}^Q$ be the filtration of \mathfrak{V}_Q indexed by (\mathcal{I}, \leq_Q) , induced by $\tilde{\mathcal{F}}^{(B \cap \mathcal{B}) \times \mathcal{C}}$. It is independent of the choice of \mathcal{C} . By construction, we obtain that $(\tilde{\mathcal{F}}^Q \mid Q \in \pi^{-1}(P))$ gives a Stokes data of $\mathfrak{V}_{|\pi^{-1}(P)}$. According to Lemma 3.17, if we choose a small neighbourhood B_1 of P in D_ℓ , it is extended to a Stokes data of $\mathfrak{V}_{|\pi^{-1}(B_1)}$. By construction, the restriction of the Stokes data to $\pi^{-1}(B_1 \cap B)$ are the same. Thus, we obtain a desired extension. The uniqueness of the extension can be shown similarly. \square

Let $(\mathfrak{V}_i, \tilde{\mathcal{F}})$ ($i = 1, 2$) be objects in $\text{SD}(X, D, \mathcal{I})$. Let $F: \mathfrak{V}_1 \rightarrow \mathfrak{V}_2$ be a morphism of local systems.

Lemma 3.19. *Let B be an open subset of D_ℓ such that F_Q is compatible with the Stokes filtrations $\tilde{\mathcal{F}}^Q(\mathfrak{V}_{i,Q})$ for any $Q \in \pi^{-1}(B)$. Let P be a point in the boundary of B . Then, there exists a small neighbourhood B_1 of P in D_ℓ such that F_Q is compatible with the Stokes filtrations $\tilde{\mathcal{F}}^Q(\mathfrak{V}_{i,Q})$ for any $Q \in \pi^{-1}(B \cup B_1)$.*

* The author thanks the referee for this simplified proof.

Proof. Let $Q \in \pi^{-1}(P)$. We take \mathcal{C} and \mathcal{B} as in the proof of Lemma 3.18. Then, the Stokes filtration $\tilde{\mathcal{F}}^Q(\mathfrak{A}_i)$ is reconstructed from the filtrations $\tilde{\mathcal{F}}^{Q'}$ from $Q' \in (\mathcal{B} \cap B) \times \mathcal{C}$. Hence, F_Q is compatible with the filtrations $\tilde{\mathcal{F}}^Q(\mathfrak{A}_i)$. Then, the claim of the lemma follows from Corollary 3.13. □

3.3.6. Proof of Theorem 3.9

We use a topological identification $(X, D) = (X^b, D^b) \times B$.

The functor \mathcal{Y} is clearly faithful. Let us show that it is full. Let $(\mathfrak{A}_i, \tilde{\mathcal{F}})$ be objects in $\text{SD}(X, D)$ with a morphism $F^b: (\mathfrak{A}_1^b, \tilde{\mathcal{F}}^b) \rightarrow (\mathfrak{A}_2^b, \tilde{\mathcal{F}}^b)$. We have a unique morphism of local systems $F: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ whose restriction to $\tilde{X}^b(D^b)$ is equal to F^b . Let us show that F gives a morphism in $\text{SD}(X, D)$. Let $P^b \in D^b$. By using Corollary 3.13 and Lemma 3.19, we obtain that F_Q preserves the filtrations $\tilde{\mathcal{F}}^Q(\mathfrak{A}_i)$ for any $Q \in \pi^{-1}(P^b \times B)$. Hence, the functor \mathcal{Y} is full.

Let us show that \mathcal{Y} is essentially surjective and let $(\mathfrak{A}^b, \tilde{\mathcal{F}}^b)$ be an object in $\text{SD}(X^b, D^b, \mathcal{I}^b)$. We have a local system \mathfrak{A} on $\tilde{X}(D)$ whose restriction to $\tilde{X}^b(D^b)$ is isomorphic to \mathfrak{A}^b . Let $P^b \in D^b$. Let $b_1 \in B$. We take a path γ_{b_1} connecting b and b_1 in B . It naturally gives a path γ_{P^b, b_1} connecting (P^b, b_1) in $P^1 \times B$. By using Lemma 3.17 and Lemma 3.18 along γ_{P^b, b_1} , we obtain a Stokes data of $\mathfrak{A}|_{\pi^{-1}(P^b, b_1)}$. Because B is simply connected, it is independent of the choice of γ_b . Thus, we obtain a system of filtrations $(\tilde{\mathcal{F}}^Q \mid Q \in \pi^{-1}(D))$. Let us check the compatibility condition. Let $P^b \in D^b$ and $b_1 \in B$. We take a path γ_{b_1} connecting b and b_1 , which embeds the interval into B . The image of γ_{P^b, b_1} is also denoted by Γ . We obtain a Stokes data of $\mathfrak{A}|_{\pi^{-1}(\Gamma)}$. If we take a small neighbourhood \mathcal{B} of Γ , it is uniquely extended to a Stokes data $\tilde{\mathcal{F}}'$ of $\mathfrak{A}|_{\pi^{-1}(\mathcal{B})}$. We may assume that \mathcal{B} is of the form $\mathcal{B}_1 \times \mathcal{B}_2$, where \mathcal{B}_1 is a neighbourhood of P^b in X^b , and \mathcal{B}_2 is a neighbourhood of b_1 in B . Let (P_2^b, b_2) be a point in \mathcal{B} . Then, the Stokes filtration $\tilde{\mathcal{F}}^{Q'}$ for $Q \in \pi^{-1}(P_2^b, b_2)$ can be constructed with a path connecting (P_2^b, b_2) and (P_2^b, b) in \mathcal{B} . Hence, it is the same as the filtration obtained from $\tilde{\mathcal{F}}'$, which implies the compatibility condition. □

4. Riemann–Hilbert–Birkhoff correspondence

4.1. Stokes filtration of unramifiedly good meromorphic flat bundle

Let X be a complex manifold with a normal crossing hypersurface D . Let $\pi: \tilde{X}(D) \rightarrow X$ be the real blowup. A holomorphic function on $\tilde{X}(D)$ is a C^∞ -function on $\tilde{X}(D)$ whose restriction to $X \setminus D$ is holomorphic. (See [23] or [21, § 3.1.3] for more details.) Let $\mathcal{O}_{\tilde{X}(D)}$ be the sheaf of holomorphic functions on $\tilde{X}(D)$. We put

$$\mathcal{O}_{\tilde{X}(D)}(*D) := \mathcal{O}_{\tilde{X}(D)} \otimes_{\pi^{-1}\mathcal{O}_X} \pi^{-1}\mathcal{O}_X(*D).$$

Let (\mathcal{E}, ∇) be an unramifiedly good meromorphic flat bundle on (X, D) . We put $\pi^*\mathcal{E} := \pi^{-1}\mathcal{E} \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{O}_{\tilde{X}(D)}$, which is a locally free $\mathcal{O}_{\tilde{X}(D)}(*D)$ -module. For each $Q \in \pi^{-1}(D)$, let $\pi^*\mathcal{E}_Q$ denote the germ at Q , and $\pi^*\mathcal{E}_{|Q}$ denote the formal completion. The irregular

decomposition of $(\mathcal{E}, \nabla)_{|\pi(Q)}$ induces

$$\pi^* \mathcal{E}_{|\hat{Q}} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, \pi(Q))} {}^Q \hat{\mathcal{E}}_{\mathfrak{a}}.$$

We put

$$\hat{\mathcal{F}}_{\mathfrak{a}}^Q(\pi^* \mathcal{E}_{|\hat{Q}}) := \bigoplus_{\mathfrak{b} \leq_Q \mathfrak{a}} {}^Q \hat{\mathcal{E}}_{\mathfrak{a}}.$$

The following is implied in Theorem 3.2.1 of [21].

Theorem 4.1. *For any $Q \in \pi^{-1}(D)$, there exists a unique ∇ -flat filtration $\tilde{\mathcal{F}}^Q$ of $\pi^* \mathcal{E}_Q$ indexed by the ordered set $(\text{Irr}(\nabla, \pi(Q)), \leq_Q)$ with the following property.*

(A) $\text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}^Q}(\pi^* \mathcal{E}_Q)$ are free $\mathcal{O}_{\tilde{X}(D)}$ -modules, and $\tilde{\mathcal{F}}_{\mathfrak{a}}^Q(\pi^* \mathcal{E}_Q)_{|\hat{Q}} = \hat{\mathcal{F}}_{\mathfrak{a}}^Q(\pi^* \mathcal{E}_{|\hat{Q}})$.

We can find a ∇ -flat splitting of $\tilde{\mathcal{F}}^Q$, i.e. a ∇ -flat decomposition $\pi^* \mathcal{E}_Q = \bigoplus \mathcal{E}_{Q, \mathfrak{a}}$ such that $\tilde{\mathcal{F}}_{\mathfrak{a}}^Q(\pi^* \mathcal{E}_Q) = \bigoplus_{\mathfrak{b} \leq_Q \mathfrak{a}} \mathcal{E}_{Q, \mathfrak{b}}$.

The filtration $\tilde{\mathcal{F}}^Q$ is called the Stokes filtration of \mathcal{E} at Q .

The system of filtrations $(\tilde{\mathcal{F}}^Q \mid Q \in \pi^{-1}(D))$ induces a Stokes data as follows. Let \mathfrak{Y} denote the local system on $\tilde{X}(D)$ associated to $(\mathcal{E}, \nabla)_{|X \setminus D}$. For each $Q \in \pi^{-1}(D)$, the stalk \mathfrak{Y}_Q is equipped with the filtration $\tilde{\mathcal{F}}^Q(\mathfrak{Y}_Q)$ induced by $\tilde{\mathcal{F}}^Q$ for $\pi^* \mathcal{E}_Q$. The following theorem is also implied in Theorem 3.2.1 of [21].

Theorem 4.2. *The system of filtrations $(\tilde{\mathcal{F}}^Q \mid Q \in \pi^{-1}(D))$ is a Stokes data of \mathfrak{Y} over the good system of irregular values $\text{Irr}(\nabla) = (\text{Irr}(\nabla, P) \mid P \in D)$.*

Let E be the unramifiedly good Deligne–Malgrange lattice of (\mathcal{E}, ∇) . We put $\pi^* E := \pi^{-1} E \otimes_{\pi^{-1} \mathcal{O}_X} \mathcal{O}_{\tilde{X}(D)}$. For each $Q \in \pi^{-1}(D)$, let $\pi^* E_Q$ denote the germ at Q . The filtration $\tilde{\mathcal{F}}^Q$ in Theorem 4.1 induces a filtration of $\pi^* E_Q$, which is also denoted by $\tilde{\mathcal{F}}^Q$. The following proposition is implied in Proposition 3.2.9 and Proposition 3.2.11 of [21].

Proposition 4.3. *$\text{Gr}^{\tilde{\mathcal{F}}^Q}(\pi^* E_Q)$ is a locally free $\mathcal{O}_{\tilde{X}(D)}$ -module. We can find a ∇ -flat decomposition $\pi^* E_Q = \bigoplus E_{\mathfrak{a}, Q}$ such that $\tilde{\mathcal{F}}_{\mathfrak{a}}^Q(\pi^* E_Q) = \bigoplus_{\mathfrak{b} \leq_Q \mathfrak{a}} E_{\mathfrak{b}, Q}$.*

Remark 4.4. Stokes filtration already appeared in the classical works on the classification of meromorphic flat bundles: see, for example, [14] and [15] (see also [5]).

Remark 4.5. In [21], $\tilde{\mathcal{F}}^Q$ is called full Stokes filtration, because we also consider partial Stokes filtration in various levels.

4.1.1. Some functoriality

We have the following functoriality, which is a special case of Proposition 3.2.3 of [21]. (See also an explanation in § 4.1.3.)

Proposition 4.6. *Let $(\mathcal{E}_i, \nabla_i)$ ($i = 1, 2$) be unramifiedly good meromorphic flat bundles on (X, D) . Let $F: \mathcal{E}_1 \rightarrow \mathcal{E}_2$ be a ∇ -flat morphism. For simplicity, we assume that $\text{Irr}(\nabla_1) \cup \text{Irr}(\nabla_2)$ is also good. Then, for each $Q \in \pi^{-1}(D)$, the induced morphism $\pi^* \mathcal{E}_{1Q} \rightarrow \pi^* \mathcal{E}_{2Q}$ is compatible with the Stokes filtrations.*

We also have the functoriality for dual, which is a special case of Proposition 3.2.5 of [21]. We can easily deduce it by using the uniqueness in Theorem 4.1.

Proposition 4.7. *Let (\mathcal{E}, ∇) be an unramifiedly good meromorphic flat bundle on (X, D) . The Stokes filtration of $(\mathcal{E}^\vee, \nabla^\vee)$ at $Q \in \pi^{-1}(D)$ is given by the procedure in § 3.1.1.*

4.1.2. *The associated graded bundle*

Let $P \in D$. By the compatibility condition of the system of Stokes filtrations, we obtain the following associated graded locally free $\mathcal{O}_{\tilde{X}(D)}(*D)$ -module with a flat connection on a neighbourhood of $\pi^{-1}(P)$:

$$\text{Gr}^{\tilde{\mathcal{F}}}(\pi^* \mathcal{E}_{\pi^{-1}(P)}, \nabla) = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, P)} (\text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\pi^* \mathcal{E}_{\pi^{-1}(P)}), \nabla_{\mathfrak{a}}).$$

By taking the pushforward via π , we obtain an $\mathcal{O}_X(*D)$ -module with a flat connection on a neighbourhood X_P of P :

$$\text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_P, \nabla) = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, P)} (\text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\mathcal{E}_P), \nabla_{\mathfrak{a}}).$$

Similarly, we obtain an $\mathcal{O}_{\tilde{X}(D)}$ -module $\text{Gr}^{\tilde{\mathcal{F}}}(\pi^* E_{|\pi^{-1}(P)})$ and an \mathcal{O}_X -module $\text{Gr}^{\tilde{\mathcal{F}}}(E_P)$ with induced meromorphic connections.

The following proposition is a special case of Proposition 3.2.8 and Proposition 3.2.9 of [21].

Proposition 4.8. *$\text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_P, \nabla)$ is a graded meromorphic flat bundle with an unramifiedly good Deligne–Malgrange lattice $\text{Gr}^{\tilde{\mathcal{F}}}(E_P)$ on (X_P, D_P) satisfying*

$$\text{Gr}^{\tilde{\mathcal{F}}}(\pi^* \mathcal{E}_{\pi^{-1}(P)}, \nabla) \simeq \pi^* \text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_P, \nabla), \quad \text{Gr}^{\tilde{\mathcal{F}}}(\pi^* E_{|\pi^{-1}(P)}) \simeq \pi^* \text{Gr}^{\tilde{\mathcal{F}}}(E_P). \tag{4.1}$$

We have a canonical isomorphism $\text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_P, \nabla)|_{\hat{P}} \simeq (\mathcal{E}, \nabla)|_{\hat{P}}$ and $\text{Gr}^{\tilde{\mathcal{F}}}(E_P)|_{\hat{P}} \simeq E|_{\hat{P}}$ compatible with the irregular decompositions. In particular, $(\text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\mathcal{E}_P), \nabla_{\mathfrak{a}})$ are \mathfrak{a} -regular.

Let $(\mathcal{E}_1, \nabla_1) \rightarrow (\mathcal{E}_2, \nabla_2)$ be a morphism of unramifiedly good meromorphic flat bundles. For simplicity, we assume $\text{Irr}(\nabla_1, P) \cup \text{Irr}(\nabla_2, P)$ is good. Then, we have the induced morphism on a neighbourhood of P :

$$\text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_{1,P}) \rightarrow \text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_{2,P}).$$

For an unramifiedly good meromorphic flat bundle (\mathcal{E}, ∇) , we have the following canonical isomorphism on a neighbourhood of P :

$$\text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_P^\vee) \simeq \text{Gr}^{\tilde{\mathcal{F}}}(\mathcal{E}_P)^\vee.$$

4.1.3. *Splitting and frame*

Let $P \in D$ and $Q \in \pi^{-1}(P)$. Let \mathcal{U} be a sufficiently small neighbourhood of Q in $\tilde{X}(D)$ on which we have a flat decomposition

$$\pi^*(\mathcal{E}, \nabla)|_{\mathcal{U}} = \bigoplus_{\mathfrak{a} \in \text{Irr}(\nabla, P)} (\mathcal{E}_{\mathcal{U}, \mathfrak{a}}, \nabla_{\mathfrak{a}}) \tag{4.2}$$

giving a splitting of the filtration $\tilde{\mathcal{F}}^Q$. The compatibility of the system of the filtrations (Theorem 4.2) means, for each $Q' \in \mathcal{U}$, (4.2) induces a splitting of $\tilde{\mathcal{F}}^{Q'}$ of the germ $\pi^*\mathcal{E}_{Q'}$.

Let (z_1, \dots, z_n) be a local coordinate around P for which D is locally expressed as $\bigcup_{i=1}^{\ell} \{z_i = 0\}$. We can take a frame $\mathbf{u}_{\mathfrak{a}}$ of $\text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\mathcal{E}_P)$ such that

$$\nabla \mathbf{u}_{\mathfrak{a}} = \mathbf{u}_{\mathfrak{a}} \left(d\mathfrak{a} + \sum_{i=1}^{\ell} A_i \frac{dz_i}{z_i} \right), \tag{4.3}$$

where A_i are constant matrices. If we take a flat splitting of $\tilde{\mathcal{F}}^Q$ as in (4.2), it induces a flat isomorphism $\pi^* \text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\mathcal{E})|_{\mathcal{U}} \simeq \pi^*\mathcal{E}|_{\mathcal{U}}$. Hence, we can obtain a frame $\mathbf{v}_{\mathcal{U}} = (\mathbf{v}_{\mathfrak{a}, \mathcal{U}})$ of $\pi^*\mathcal{E}|_{\mathcal{U}}$ such that (4.3) holds for each $\mathbf{v}_{\mathfrak{a}, \mathcal{U}}$.

We can easily deduce Proposition 4.6 by using frames as above. Logically, Proposition 4.6 is more basic than the existence of such frames. But, we argue it for explanation. We take flat splittings $\pi^*\mathcal{E}_i|_{\mathcal{U}} \simeq \pi^* \text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\mathcal{E}_i)|_{\mathcal{U}}$ of the filtration $\tilde{\mathcal{F}}^Q$ on a small neighbourhood \mathcal{U} of Q for $i = 1, 2$. We have the corresponding decomposition $F = \sum F_{\mathfrak{a}, \mathfrak{b}, \mathcal{U}}$, where

$$F_{\mathfrak{a}, \mathfrak{b}, \mathcal{U}}: \pi^* \text{Gr}_{\mathfrak{b}}^{\mathcal{F}}(\mathcal{E}_1)|_{\mathcal{U}} \rightarrow \pi^* \text{Gr}_{\mathfrak{b}}^{\mathcal{F}}(\mathcal{E}_2)|_{\mathcal{U}}.$$

We have only to show that $F_{\mathfrak{a}, \mathfrak{b}} = 0$ unless $\mathfrak{b} \geq_Q \mathfrak{a}$. We take frames $\mathbf{u}_{\mathfrak{a}}^{(i)}$ of $\text{Gr}_{\mathfrak{a}}^{\tilde{\mathcal{F}}}(\mathcal{E}_P)$ as above. We have the expression $F_{\mathfrak{a}, \mathfrak{b}, \mathcal{U}}(\mathbf{u}_{\mathfrak{b}}^{(1)}) = \mathbf{u}_{\mathfrak{a}}^{(2)} B_{\mathfrak{a}, \mathfrak{b}}$, where $B_{\mathfrak{a}, \mathfrak{b}}$ be the matrix valued function on \mathcal{U} . By the flatness of $F_{\mathfrak{a}, \mathfrak{b}, \mathcal{U}}$, we obtain that $B_{\mathfrak{a}, \mathfrak{b}}$ satisfies a differential equation. Because $B_{\mathfrak{a}, \mathfrak{b}}$ is polynomial order in $|z_i^{-1}|$ ($i = 1, \dots, \ell$), we easily obtain $B_{\mathfrak{a}, \mathfrak{b}} = 0$ unless $\mathfrak{a} \leq_Q \mathfrak{b}$.

4.1.4. *Characterization by growth order*

Let (\mathcal{E}, ∇) be an unramifiedly good meromorphic flat bundle on (X, D) . Let $Q \in \pi^{-1}(D)$. Let \mathcal{U} be a small neighbourhood of Q in $\tilde{X}(D)$. Take any frame \mathbf{v} of $\mathcal{E}|_{\mathcal{U}}$. A ∇ -flat section of $\mathcal{E}|_{\mathcal{U} \setminus \pi^{-1}(D)}$ is expressed as $f = \sum f_j v_j$, where f_j are holomorphic functions on $\mathcal{U} \setminus \pi^{-1}(D)$. Let \mathbf{f} denote the tuple (f_j) . Then, the filtration $\tilde{\mathcal{F}}^Q$ can be characterized as follows, which is a special case of Proposition 3.2.6 in [21]. We can easily deduce it by using the frame as in § 4.1.3.

Proposition 4.9. *We have $f \in \tilde{\mathcal{F}}_{\mathfrak{a}}^Q(\pi^*\mathcal{E}_Q)$ if and only if $|e^{\mathfrak{a}} \mathbf{f}|$ is of polynomial order.*

Remark 4.10. Let (z_1, \dots, z_n) be a coordinate system around $\pi(Q)$ such that D is expressed as $\bigcup_{i=1}^{\ell} \{z_i = 0\}$ around $\pi(Q)$. We say that a function F on $\mathcal{U}_Q \setminus \pi^{-1}(D)$ is of polynomial order, if $|F| = O(\prod_{i=1}^{\ell} |z_i|^{-N})$ for some $N > 0$.

4.2. Equivalence

Let X be a complex manifold with a normal crossing hypersurface D . Let $\mathcal{I} = (\mathcal{I}_P \mid P \in D)$ be a good system of irregular values on (X, D) . Let $\text{MF}(X, D, \mathcal{I})$ be the category of unramifiedly good meromorphic flat bundles (\mathcal{E}, ∇) on (X, D) such that $\text{Irr}(\nabla, P) \subset \mathcal{I}_P$ for each $P \in D$. We have a naturally defined functor

$$\text{RHB}: \text{MF}(X, D, \mathcal{I}) \rightarrow \text{SD}(X, D, \mathcal{I}).$$

The following theorem is a special case of Corollary 4.3.2 of [21].

Theorem 4.11. *The functor RHB is an equivalence.*

Proof. We explain only the full faithfulness. It is clearly faithful. Let us show that it is full. Let \mathcal{E}_i ($i = 1, 2$) be unramifiedly good meromorphic flat bundles on (X, D) . Let $F: \mathcal{E}_1|_{X \setminus D} \rightarrow \mathcal{E}_2|_{X \setminus D}$ be a flat morphism preserving Stokes filtration at each $Q \in \pi^{-1}(D)$. We would like to show that F is extended to a morphism $\mathcal{E}_1 \rightarrow \mathcal{E}_2$. We have only to consider the case $X = \Delta^n$ and $D = \bigcup_{i=1}^\ell \{z_i = 0\}$. We take frames \mathbf{v}_i of \mathcal{E}_i . Let A be the matrix determined by $F\mathbf{v}_1 = \mathbf{v}_2A$. We have only to show that A is of polynomial order in $|z_i|^{-1}$ ($i = 1, \dots, \ell$). For each $Q \in \pi^{-1}(D)$, we take flat splittings of $\tilde{\mathcal{F}}^Q(\pi^*\mathcal{E}_{iQ})$ as in (4.2). We take frames $\mathbf{u}_{i\alpha}$ of $\mathcal{E}_{i\mathcal{U}, \alpha}$ as in §4.1.3, which give frames \mathbf{u}_i of $\pi^*\mathcal{E}_i|_{\mathcal{U}}$. Let B be the matrix determined by $F\mathbf{u}_1 = \mathbf{u}_2B$. By a direct computation, we obtain that B is of polynomial order in $|z_i|^{-1}$ ($i = 1, \dots, \ell$) on $\mathcal{U} \setminus \pi^{-1}(D)$. Let G_i be the matrix determined by $\mathbf{v}_i = \mathbf{u}_iG_i$. Then, G_i and G_i^{-1} are of polynomial order in $|z_i|^{-1}$ ($i = 1, \dots, \ell$) on $\mathcal{U} \setminus \pi^{-1}(D)$. Hence, we obtain that A is of polynomial order in $|z_i|^{-1}$ ($i = 1, \dots, \ell$) on $\mathcal{U} \setminus \pi^{-1}(D)$. By varying Q , we obtain the desired estimate for A . As for the essential surjectivity, we refer to [21]. □

Let $\text{MF}(X, D, \mathcal{I})^G$ denote the category of G -equivariant unramifiedly good meromorphic flat bundles over \mathcal{I} . It is easy to deduce the following as a corollary of Theorem 4.11.

Corollary 4.12. *The functor $\text{RHB}: \text{MF}(X, D, \mathcal{I})^G \rightarrow \text{SD}(X, D, \mathcal{I})^G$ is an equivalence.*

Let $\varphi: (X', D') \rightarrow (X, D)$ be a ramified Galois covering with the Galois group G . Let $\mathcal{I}' := \varphi^*\mathcal{I}$. We have naturally defined descent functors $\text{Des}: \text{MF}(X', D', \mathcal{I}')^G \rightarrow \text{MF}(X, D, \mathcal{I})$ and $\text{Des}: \text{SD}(X', D', \mathcal{I}')^G \rightarrow \text{SD}(X, D, \mathcal{I})$. It is easy to obtain an equivalence $\text{Des} \circ \text{RHB} \simeq \text{RHB} \circ \text{Des}$.

4.3. Extension

We consider the situation in §3.3.2. We obtain the following theorem from Theorem 3.9 and Theorem 4.1.

Theorem 4.13. *The restriction $\text{MF}(X, D, \mathcal{I}) \rightarrow \text{MF}(X^b, D^b, \mathcal{I}^b)$ is equivalent.*

Let G be a finite group acting on (X, D) over B . Assume that \mathcal{I} is G -equivariant in the sense of §3.3.1. By using the uniqueness, we easily obtain the following.

Corollary 4.14. *The restriction $\text{MF}(X, D, \mathcal{I})^G \rightarrow \text{MF}(X^b, D^b, \mathcal{I}^b)^G$ is an equivalence.*

In §§4.3.1–4.3.2, we shall explain easy examples of deformation. (See [21, §4.5] for more details.)

4.3.1. Deformation $\mathcal{E}^{(\mathcal{T})}$

Let \mathcal{C} be a simply connected compact region in \mathbb{C}^m with a base point c_0 . We put $(X^\circ, D^\circ) := (X, D) \times \mathcal{C}$. Let \mathcal{T} be a holomorphic function on \mathcal{C} such that $\mathcal{T}(c_0) = 1$. From a good meromorphic flat bundle (\mathcal{E}, ∇) on (X, D) , we shall construct a good meromorphic flat bundle $(\mathcal{E}, \nabla)^{(\mathcal{T})}$ in a functorial way, such that $(\mathcal{E}, \nabla)_{|X \times \{c_0\}}^{(\mathcal{T})} = (\mathcal{E}, \nabla)$. (See [21, §4.5.1] for more details.)

Unramified case. Let \mathcal{I} be a good system of irregular values on (X, D) . For each $(P, c) \in D^\circ$, we put

$$\mathcal{I}_{(P,c)}^{(\mathcal{T})} := \{\mathcal{T}\mathbf{a} \mid \mathbf{a} \in \mathcal{I}_P\}.$$

Thus, we obtain a good system of irregular values $\mathcal{I}^{(\mathcal{T})}$ on (X°, D°) . According to Theorem 4.13, the restriction $\text{MF}(X^\circ, D^\circ, \mathcal{I}^\circ) \rightarrow \text{MF}(X \times \{c_0\}, D \times \{c_0\}, \mathcal{I})$ is an equivalence. Hence, for $(\mathcal{E}, \nabla) \in \text{MF}(X, D, \mathcal{I})$, we have $(\mathcal{E}^{(\mathcal{T})}, \nabla^{(\mathcal{T})}) \in \text{MF}(X^\circ, D^\circ, \mathcal{I}^{(\mathcal{T})})$ such that $(\mathcal{E}^{(\mathcal{T})}, \nabla^{(\mathcal{T})})_{|X \times \{c_0\}} = (\mathcal{E}, \nabla)$. It is unique up to canonical isomorphisms.

Let $\varphi: (X', D') \rightarrow (X, D)$ be a ramified Galois covering with the Galois group G . We put $\mathcal{I}' := \varphi^*\mathcal{I}$. Take $(\mathcal{E}', \nabla') \in \text{MF}(X', D', \mathcal{I}')^G$. Let $(\mathcal{E}, \nabla) \in \text{MF}(X, D, \mathcal{I})$ be the descent of (\mathcal{E}', ∇') . According to Corollary 4.14, $(\mathcal{E}', \nabla')^{(\mathcal{T})}$ is also G -equivariant.

Lemma 4.15. $(\mathcal{E}, \nabla)^{(\mathcal{T})}$ is the descent of $(\mathcal{E}', \nabla')^{(\mathcal{T})}$.

Proof. Let $(\mathcal{E}_1, \nabla_1) \in \text{MF}(X, D, \mathcal{I})$ be the descent of $(\mathcal{E}', \nabla')^{(\mathcal{T})}$. By construction, the restrictions of $(\mathcal{E}, \nabla)^{(\mathcal{T})}$ and $(\mathcal{E}_1, \nabla_1)$ to $X \times \{c_0\}$ are naturally isomorphic. By Theorem 4.13, they are isomorphic on X° . □

General case. Let (\mathcal{E}, ∇) be a good meromorphic flat bundle on (X, D) , which is not necessarily unramified. For any $P \in D$, we can take a small neighbourhood X_P and a ramified covering $\varphi_P: (X'_P, D'_P) \rightarrow (X_P, D_P)$ such that $\varphi_P^*(\mathcal{E}, \nabla)$ is unramified. By applying the procedure in the unramified case, we obtain the deformation $(\varphi_P^*(\mathcal{E}, \nabla))^{(\mathcal{T})}$ on (X'_P, D'_P) . By taking the descent, we obtain $(\mathcal{E}, \nabla)^{(\mathcal{T})}_P$ on (X°_P, D°_P) . It is well defined up to canonical isomorphisms as a germ of a good meromorphic flat bundle at $P \times \mathcal{C}$, according to Lemma 4.15. By gluing, we can globalize and obtain a good meromorphic flat bundle $(\mathcal{E}, \nabla)^{(\mathcal{T})}$ on (X°, D°) .

Pullback. We explain the functoriality for pullback. Let X_1 be a complex manifold with a normal crossing hypersurface D_1 . Let $F: X_1 \rightarrow X$ be a morphism such that $F^{-1}(D) \subset D_1$. Let (\mathcal{E}, ∇) be a good meromorphic flat bundle on (X, D) . We obtain a good meromorphic flat bundle $(\mathcal{E}_1, \nabla_1) := F^*(\mathcal{E}, \nabla) \otimes \mathcal{O}_{X_1}(*D_1)$ on (X_1, D_1) . Let F_C be the induced morphism $X^\circ_1 \rightarrow X^\circ$. Then, it is easy to obtain a natural isomorphism $(\mathcal{E}_1, \nabla_1)^{(\mathcal{T})} \simeq F_C^*(\mathcal{E}, \nabla)^{(\mathcal{T})}$. Indeed, we have only to consider the local and unramified case, and we have only to compare their restrictions to $X \times \{c_0\}$ as in the proof of Lemma 4.15.

4.3.2. Deformation $\mathcal{E}^{(T)}$

Take $T \in \mathbb{C} \setminus \{0\}$ such that $|\arg(T)| < \pi/2$. For a given good meromorphic flat bundle (\mathcal{E}, ∇) on (X, D) , we shall construct a good meromorphic flat bundle $(\mathcal{E}^{(T)}, \nabla^{(T)})$ on

(X, D) in a functorial way. We take a compact region $\mathcal{C} \subset \mathbb{C}$ which contains 0 and 1, and take a nowhere vanishing holomorphic function $\mathcal{T}: \mathcal{C} \rightarrow \mathbb{C}$ such that (i) $\mathcal{T}(0) = 1$, (ii) $\mathcal{T}(1) = T$ and (iii) $|\arg(\mathcal{T})| < \pi/2$. Then, we obtain the deformation $(\mathcal{E}^{(\mathcal{T})}, \nabla^{(\mathcal{T})})$ on (X°, D°) . By taking the specialization at $c = 1$, we obtain the desired $(\mathcal{E}^{(T)}, \nabla^{(T)})$. It is easy to show that $(\mathcal{E}^{(T)}, \nabla^{(T)})$ is independent of the choice of $(\mathcal{C}, \mathcal{T})$ up to canonical isomorphisms. (See [21, § 4.5.2] for more details.)

Let \mathcal{I} be a good system of irregular values on (X, D) . For each $P \in D$, we put $\mathcal{I}_P^{(T)} := \{T\mathbf{a} \mid \mathbf{a} \in \mathcal{I}_P\}$, and we obtain a good system of irregular values $\mathcal{I}^{(T)}$ on (X, D) . The above construction gives $\text{MF}(X, D, \mathcal{I}) \rightarrow \text{MF}(X, D, \mathcal{I}^{(T)})$, in the unramified case. If $T_i \in \mathbb{C}$ ($i = 1, 2$) satisfy $|\arg(T_i)| < \pi/2$ and $|\arg(T_1 T_2)| < \pi/2$, then we have a canonical isomorphism $\mathcal{E}^{(T_1 T_2)} \simeq (\mathcal{E}^{(T_1)})^{(T_2)}$.

Pullback. We explain the functoriality for pullback. Let X_1 be a complex manifold with a normal crossing hypersurface D_1 . Let $F: X_1 \rightarrow X$ be a morphism such that $F^{-1}(D) \subset D_1$. Let \mathcal{E} be a good meromorphic flat bundle of (\mathcal{E}, ∇) on (X, D) . We obtain a good meromorphic flat bundle $\mathcal{E}_1 := F^* \mathcal{E} \otimes \mathcal{O}_{X_1}(*D_1)$. Then, we have a natural isomorphism $\mathcal{E}_1^{(T)} \simeq F^* \mathcal{E}^{(T)}$, which follows from the functoriality of the construction in § 4.3.1 via pullback.

4.3.3. *Remark*

This deformation procedure, grown out with the discussion with Sabbah, is one of the key ingredient in our study on wild harmonic bundles [21]. Let $(E, \bar{\partial}_E, \theta, h)$ be a good wild harmonic bundle on (X, D) . We have the associated family of λ -flat bundles $(\mathcal{E}, \mathbb{D})$ on $\mathbb{C}_\lambda \times (X \setminus D)$. It is one of the main task to prolong it to a family of meromorphic λ -flat bundles on $\mathbb{C}_\lambda \times X$.

For each complex number λ , we have the associated λ -flat bundle $(\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ on $X \setminus D$. By considering the holomorphic sections in polynomial orders, we obtain a good meromorphic λ -flat bundle $(\mathcal{P}\mathcal{E}^\lambda, \mathbb{D}^\lambda)$ on (X, D) . However, in the non-tame case, we cannot obtain a nice meromorphic object in family, if we consider holomorphic sections with polynomial growth. Thus, the deformation procedure as above gets into our study on wild harmonic bundles.

4.4. **Conjugate**

4.4.1. *Good meromorphic flat bundle on the conjugate*

Let X be a complex manifold with a normal crossing hypersurface D . Let X^\dagger be the conjugate of X , i.e. $X^\dagger = X$ as a C^∞ -manifold, but the complex structure is the opposite one. Let \mathcal{I} be a good system of irregular values on (X, D) . For each $P \in D$, we put $\bar{\mathcal{I}}_P := \{\bar{\mathbf{a}} \mid \mathbf{a} \in \mathcal{I}_P\}$. Then, we obtain a good system of irregular values $\bar{\mathcal{I}}$ on (X^\dagger, D^\dagger) .

Let $\bar{X}^\dagger(D^\dagger) \rightarrow X^\dagger$ be the real blowup of X^\dagger along D^\dagger . It is naturally identified with $\bar{X}(D)$ as a C^∞ -manifold. For each $Q \in \pi^{-1}(P)$, the orders \leq_Q on \mathcal{I}_P and $\bar{\mathcal{I}}_P$ are the same under the natural bijection. Hence, we have the natural identification $\text{SD}(X, D, \mathcal{I}) \simeq \text{SD}(X^\dagger, D^\dagger, \bar{\mathcal{I}})$. It induces the following equivalences of categories:

$$\text{MF}(X, D, \mathcal{I}) \xrightarrow{\simeq} \text{SD}(X, D, \mathcal{I}) = \text{SD}(X^\dagger, D^\dagger, \bar{\mathcal{I}}) \xleftarrow{\simeq} \text{MF}(X^\dagger, D^\dagger, \bar{\mathcal{I}}).$$

For $\mathcal{E} \in \text{MF}(X, D, \mathcal{I})$, let $\mathcal{E}^c \in \text{MF}(X^\dagger, D^\dagger, \tilde{\mathcal{I}})$ be the corresponding object. It is determined up to canonical isomorphism. We put $\mathcal{C}(\mathcal{E})^{\text{mod } D} := (\mathcal{E}^\vee)^c$. It is naturally isomorphic to $(\mathcal{E}^c)^\vee$, which can be shown by comparison of the associated Stokes structure.

Example 4.16. Let $X = \Delta^n$ and $D = \bigcup_{i=1}^\ell \{z_i = 0\}$. Let $\mathbf{a} \in M(X, D)$. If $\mathcal{E} = \mathcal{O}_X(*D)e$ with the connection $\nabla e = e(\mathbf{da} + \sum \alpha_i dz_i/z_i)$, then $\mathcal{C}(\mathcal{E})^{\text{mod } D} = \mathcal{O}_{X^\dagger}(*D^\dagger)e^\dagger$ with the connection $\nabla e^\dagger = e^\dagger(d(-\bar{\mathbf{a}}) + \sum \alpha_i d\bar{z}_i/\bar{z}_i)$.

Prolongation of the pairing. Recall some sheaves from [23], which we refer to for more detailed property. Let $\iota: X \setminus D \rightarrow X$ and $\tilde{\iota}: X \setminus D \rightarrow \tilde{X}(D)$ be the natural inclusions. Let $\mathcal{A}_{\tilde{X}(D)}^{\text{mod } D}$ be the subsheaf of $\tilde{\iota}_* \mathcal{O}_{X \setminus D}$ which consists of the sections with moderate growth along D . Let $\mathfrak{D}\mathbf{b}_{\tilde{X}(D)}^{\text{mod } D}$ be the image of $\mathfrak{D}\mathbf{b}_{\tilde{X}(D)} \rightarrow \tilde{\iota}_* \mathfrak{D}\mathbf{b}_{X \setminus D}$. Let $\mathfrak{D}\mathbf{b}_X^{\text{mod } D}$ be the image of $\mathfrak{D}\mathbf{b}_X \rightarrow \iota_* \mathfrak{D}\mathbf{b}_{X \setminus D}$.

Let \mathcal{E} be an unramifiedly good meromorphic flat bundle on (X, D) . We put

$$\begin{aligned} \mathcal{E}_{\tilde{X}(D)}^{\text{mod } D} &:= \pi^{-1}(\mathcal{E}) \otimes_{\pi^{-1}(\mathcal{O}_X)} \mathcal{A}_{\tilde{X}(D)}^{\text{mod } D}, \\ \mathcal{C}(\mathcal{E})_{\tilde{X}(D)}^{\text{mod } D} &:= \pi^{-1}(\mathcal{C}(V)^{\text{mod } D}) \otimes_{\pi^{-1}(\mathcal{O}_{X^\dagger})} \mathcal{A}_{\tilde{X}^\dagger(D^\dagger)}^{\text{mod } D^\dagger}. \end{aligned}$$

Note that we have the natural pairing of $V|_{X \setminus D}$ and $\mathcal{C}(V)|_{X \setminus D}^{\text{mod } D}$ to the sheaf of C^∞ -functions on $X \setminus D$.

Proposition 4.17. *It is naturally extended to the pairings*

$$\mathcal{E} \times \mathcal{C}(\mathcal{E})^{\text{mod } D} \rightarrow \mathfrak{D}\mathbf{b}_X^{\text{mod } D}, \quad \mathcal{E}_{\tilde{X}(D)}^{\text{mod } D} \times \mathcal{C}(\mathcal{E})_{\tilde{X}(D)}^{\text{mod } D} \rightarrow \mathfrak{D}\mathbf{b}_{\tilde{X}(D)}^{\text{mod } D}.$$

Proof. Let us consider the second one. Take $P \in D$ and $Q \in \pi^{-1}(P)$. We put $\mathcal{I} := \text{Irr}(\mathcal{E}, P)$. We have $\text{Irr}(\mathcal{C}(\mathcal{E})^{\text{mod } D}, P) = \tilde{\mathcal{I}}^\vee := \{-\bar{\mathbf{a}} \mid \mathbf{a} \in \mathcal{I}\}$. We take a sufficiently small neighbourhood \mathcal{U} of Q in $\tilde{X}(D)$, and a flat splitting $\pi^* \mathcal{E}|_{\mathcal{U}} = \bigoplus_{\mathbf{a} \in \mathcal{I}} \mathcal{E}_{\mathcal{U}, \mathbf{a}}$ of the filtration $\tilde{\mathcal{F}}^Q$. We can take a frame $\mathbf{u}_\mathbf{a}$ of $\mathcal{E}_{\mathcal{U}, \mathbf{a}}^{\text{mod } D} := \mathcal{E}_{\mathcal{U}, \mathbf{a}} \otimes \mathcal{A}_{\tilde{X}(D)}^{\text{mod } D}$ such that $\nabla \mathbf{u}_\mathbf{a} = \mathbf{u}_\mathbf{a} \mathbf{da}$. Similarly, we can take a flat splitting

$$\pi^* \mathcal{C}(\mathcal{E})|_{\mathcal{U}}^{\text{mod } D} = \bigoplus_{\mathbf{b} \in \tilde{\mathcal{I}}^\vee} \mathcal{C}(\mathcal{E})_{\mathcal{U}, \mathbf{b}}^{\text{mod } D}$$

of the filtration $\tilde{\mathcal{F}}^Q$, and a frame $\mathbf{w}_\mathbf{b}$ of

$$\mathcal{C}(\mathcal{E})_{\mathcal{U}, \mathbf{b}}^{\text{mod } D} := \mathcal{C}(\mathcal{E})_{\mathcal{U}, \mathbf{b}}^{\text{mod } D} \otimes \mathcal{A}_{\tilde{X}^\dagger(D^\dagger)}^{\text{mod } D^\dagger}$$

such that $\nabla \mathbf{w}_\mathbf{b} = \mathbf{w}_\mathbf{b} \mathbf{db}$.

Note that the filtration $\tilde{\mathcal{F}}^Q$ for $\mathcal{C}(\mathcal{E})^{\text{mod } D}$ is the same as that for \mathcal{E}^\vee , under the identification of $\mathcal{E}_{X \setminus D}^\vee = \mathcal{C}(\mathcal{E})|_{X \setminus D}^{\text{mod } D}$ as C^∞ -flat bundles. We also recall the functoriality of the Stokes filtration in Proposition 4.7. Hence, the induced pairing between $\mathcal{E}_{\mathcal{U}, \mathbf{a}}^{\text{mod } D}$ and $\mathcal{C}(\mathcal{E})_{\mathcal{U}, \mathbf{b}}^{\text{mod } D}$ is 0 unless $-\text{Re}(\mathbf{a} + \mathbf{b}) \leq_Q 0$. If $-\text{Re}(\mathbf{a} + \mathbf{b}) \leq_Q 0$ is satisfied, by a direct computation, we can check that the pairing between $\mathcal{E}_{\mathcal{U}, \mathbf{a}}^{\text{mod } D}$ and $\mathcal{C}(\mathcal{E})_{\mathcal{U}, \mathbf{b}}^{\text{mod } D}$ is valued in $\mathfrak{D}\mathbf{b}_{\tilde{X}(D)}^{\text{mod } D}$. The first one follows from the second one. \square

4.4.2. Conjugate of holonomic \mathcal{D} -modules

We briefly mention an application to \mathcal{D} -modules. We have a natural correspondence between coherent \mathcal{O}_X -modules with flat connections, and coherent \mathcal{O}_{X^\dagger} -modules with flat connections, through local systems. In [8], Kashiwara studied how to generalize it for holonomic \mathcal{D}_X -modules. Let \mathcal{M} be a regular singular holonomic \mathcal{D}_X -module. Let $\mathfrak{D}\mathfrak{b}_X$ be the sheaf of distributions on X , which is naturally regarded as a bi- $(\mathcal{D}_X, \mathcal{D}_{X^\dagger})$ -module. Hence, $\mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}, \mathfrak{D}\mathfrak{b}_X)$ are naturally \mathcal{D}_{X^\dagger} -modules. In [8], he showed that $\mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}, \mathfrak{D}\mathfrak{b}_X) = 0$ for $i > 0$, and $C_X(\mathcal{M}) := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{D}\mathfrak{b}_X)$ is a regular singular holonomic \mathcal{D}_{X^\dagger} -module. Let \mathbb{D}_{X^\dagger} denote the dual functor in the category of \mathcal{D}_{X^\dagger} -modules. Then, $\mathcal{M} \mapsto \mathbb{D}_{X^\dagger} C_X(\mathcal{M})$ gives an appropriate generalization of the above correspondence.

Sabbah [23] studied its generalization for holonomic \mathcal{D} -modules which are not necessarily regular singular. He essentially established that the problem can be reduced to the existence of resolution of turning points and Riemann–Hilbert–Birkhoff correspondence. We have already known the local existence of resolution of turning points due to Kedlaya [9] (see [21] for the algebraic case). We have also prepared asymptotic analysis for good meromorphic flat bundles over higher-dimensional varieties. Hence, it may be appropriate to mention here that the problems can be solved formally. This is essentially due to Sabbah. We will just indicate where detailed arguments are given. (See [23] and [24] for more details.)

Let X be a complex manifold with a normal crossing hypersurface D . Let \mathcal{E} be an unramifiedly good meromorphic flat bundle on (X, D) . From Proposition 4.17, we obtain the following morphisms:

$$C(\mathcal{E})^{\text{mod } D} \rightarrow \mathcal{H}om_{\mathcal{D}_X}(\mathcal{E}, \mathfrak{D}\mathfrak{b}_X^{\text{mod } D}) \rightarrow R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{E}, \mathfrak{D}\mathfrak{b}_X^{\text{mod } D}), \tag{4.4}$$

$$C(\mathcal{E})_{X(D)}^{\text{mod } D} \rightarrow \mathcal{H}om_{\mathcal{D}_{X(D)}^{\text{mod } D}}(\mathcal{E}_{X(D)}^{\text{mod } D}, \mathfrak{D}\mathfrak{b}_{X(D)}^{\text{mod } D}) \rightarrow R\mathcal{H}om_{\mathcal{D}_{X(D)}^{\text{mod } D}}(\mathcal{E}_{X(D)}^{\text{mod } D}, \mathfrak{D}\mathfrak{b}_{X(D)}^{\text{mod } D}). \tag{4.5}$$

Here, $\mathcal{D}_{X(D)}^{\text{mod } D} := \pi^{-1}\mathcal{D}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{A}_{X(D)}^{\text{mod } D}$.

Proposition 4.18. *The morphisms (4.4) and (4.5) are isomorphisms.*

Proof. We may assume that $X = \Delta^n$ and $D = \bigcup_{i=1}^\ell \{z_i = 0\}$. Let us show (4.5). We have only to consider the case $\mathcal{E} = \mathcal{O}_X(*D)e$ with the connection $\nabla e = e(d\mathfrak{a} + \sum \alpha_i dz_i/z_i)$. Then, the claim for (4.5) can be reduced to the Grothendieck–Dolbeault Lemma [23, (II.1.17)] by the argument in [23, §II.3.3]. (See also the proof of Lemma 7 in [8].) Then, we can formally deduce the claim for (4.4) from that for (4.5). (See the argument in [23, pp. 67–68].) □

Corollary 4.19. *Let \mathcal{M} be a holonomic \mathcal{D}_X -module. We have $\mathcal{E}xt_{\mathcal{D}_X}^i(\mathcal{M}, \mathfrak{D}\mathfrak{b}_X) = 0$ unless $i \neq 0$, and $C_X(\mathcal{M}) := \mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{D}\mathfrak{b}_X)$ is a holonomic \mathcal{D}_{X^\dagger} -module. The functor C_X induces a contravariant equivalence between the derived categories of cohomologically holonomic \mathcal{D} -modules on X and X^\dagger .*

Proof. As remarked on p. 66 of [23], Kashiwara’s argument in [8] permits us to reduce the issue to the case that \mathcal{M} is a meromorphic flat bundle on (X, D) , where D is a normal

crossing hypersurface. Since the claim is local, applying the local existence of resolution of turning points [9] with the argument in [8], we can reduce the issue to the case that \mathcal{M} is a good meromorphic flat bundle on (X, D) . As noted on p. 66 in [23], we have only to show that $R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}\mathfrak{b}_X^{\text{mod } D})$ is a good meromorphic flat bundle on (X, D) . By the argument on p. 67 in [23], it can be reduced to the case that \mathcal{M} is unramifiedly good. Then, the claim follows from Proposition 4.18. \square

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