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Forward Clusters for Degenerate Random Environments

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We consider connectivity properties and asymptotic slopes for certain random directed graphs on $\mathbb{Z}^2$ in which the set of points $\mathcal{C}_o$ that the origin connects to is always infinite. We obtain conditions under which the complement of $\mathcal{C}_o$ has no infinite connected component. Applying these results to one of the most interesting such models leads to an improved lower bound for the critical occupation probability for oriented site percolation on the triangular lattice in two dimensions.

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1. Introduction and main results

The main objects of study in this paper are the *two-dimensional orthant model* (one of the most interesting examples within a class of models called *degenerate random environments*), and its dual model, a version of *oriented site percolation*. Part of the motivation for studying degenerate random environments is an interest in the behaviour of random walks in random environments that are non-elliptic. Indeed, many of the results of this paper and of [6] have immediate implications for the behaviour (in particular, directional transience) of random walks in certain non-elliptic environments (see *e.g.* [7, 8]).

For fixed $d \geqslant 2$, let $\mathcal{E} = \{\pm e_i : i = 1,\dots,d\}$ be the set of unit vectors in $\mathbb{Z}^d$, and let $\mathcal{P}$ denote the power set of $\mathcal{E}$. Let μ be a probability measure on $\mathcal{P}$. A *degenerate random environment* (DRE) is a random directed graph, that is, an element $\mathcal{G} = \{\mathcal{G}_x\}_{x\in\mathbb{Z}^d}$ of $\mathcal{P}^{\mathbb{Z}^d}$. We equip $\mathcal{P}^{\mathbb{Z}^d}$ with the product σ-algebra and the product measure $\mathbb{P} = \mu^{\otimes\mathbb{Z}^d}$, so that $\{\mathcal{G}_x\}_{x\in\mathbb{Z}^d}$ are i.i.d. under $\mathbb{P}$. We denote the expectation of a random variable Z with respect to $\mathbb{P}$ by $\mathbb{E}[Z]$.

We say that the DRE is 2-*valued* when μ charges exactly two points, that is, there exist distinct $E_1, E_2 \in \mathcal{P}$ and $p \in (0,1)$ such that $\mu(\{E_1\}) = p$ and $\mu(\{E_2\}) = 1 - p$. As in the percolation setting, there is a natural coupling of graphs for all values of p as follows. Let $\{U_x\}_{x\in\mathbb{Z}^d}$ be i.i.d. standard uniform random variables under $\mathbb{P}$. Setting

$$\mathcal{G}_x = \begin{cases} E_1 & \text{if } U_x < p, \\ E_2 & \text{otherwise,} \end{cases} \tag{1.1}$$

it is easy to see that for any fixed p, $\mathcal{G}$ has the correct law, and that the set of E_1 sites is increasing in p, $\mathbb{P}$-almost surely.

For the most part, in this paper, we will consider two-dimensional models. Interesting examples of two-dimensional, 2-valued models include the following.

Model 1.1. (↑→ ←↓) *Let* $E_1 = \{\uparrow, \rightarrow\}$ *and* $E_2 = \{\downarrow, \leftarrow\}$ (*and set* $\mu(\{E_1\}) = p$, $\mu(\{E_2\}) = 1 - p$).

We call the generalization to d dimensions the *orthant model* (so this is the *two-dimensional orthant model*).

Model 1.2. (←↓→ ↑) *Let* $E_1 = \{\leftarrow, \downarrow, \rightarrow\}$ *and* $E_2 = \{\uparrow\}$.

Model 1.3. (↕↔) *Let* $E_1 = \{\uparrow, \downarrow\}$ *and* $E_2 = \{\rightarrow, \leftarrow\}$.

We will focus on Model 1.1 in this paper, but we believe extensions of our results to Model 1.2 are possible. A central limit theorem for random walk in the degenerate random environment of Model 1.3 is proved in [2]. See [6] for more on these and other examples of two-dimensional 2-valued models, and their properties. For any set A, let $|A|$ denote its cardinality.

Definition. Given an environment $\mathcal{G}$:

- We say that x is *connected* to y, and write $x \to y$ if: there exists an $n \geqslant 0$ and a sequence $x = x_0, x_1, \ldots, x_n = y$ such that $x_{i+1} - x_i \in \mathcal{G}_{x_i}$ for $i = 0, \ldots, n-1$. We say that x and y are *mutually connected*, or that they *communicate*, and write $x \leftrightarrow y$ if $x \to y$ and $y \to x$.
- Define $\mathcal{C}_x = \{y \in \mathbb{Z}^d : x \to y\}$ (the *forward* cluster), $\mathcal{B}_x = \{y \in \mathbb{Z}^d : y \to x\}$ (the *backward* cluster), and $\mathcal{M}_x = \{y \in \mathbb{Z}^d : x \leftrightarrow y\} = \mathcal{B}_x \cap \mathcal{C}_x$ (the *bi-connected* cluster). Let o denote the origin in $\mathbb{Z}^d$ and set

$$\theta_+ = \mathbb{P}(|\mathcal{C}_o| = \infty), \quad \theta_- = \mathbb{P}(|\mathcal{B}_o| = \infty), \quad \text{and} \quad \theta = \mathbb{P}(|\mathcal{M}_o| = \infty).$$

- A nearest-neighbour path in $\mathbb{Z}^d$ is *open in* $\mathcal{G}$ if that path consists of directed edges in $\mathcal{G}$.

It is interesting to consider percolation-type questions where $\theta_+ < 1$, and indeed in the true percolation settings where $\mu(\{\emptyset\}) = 1 - \mu(\{E\})$ for some configuration E, there is a simple relation between the directed percolation probabilities of the form $\theta_+ = \mu(\{E\})\theta_-$

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(see [6, Lemma 2.5]). Our current interest lies in those cases where μ is such that

$$\boxed{\theta_+ = 1}. \tag{1.2}$$

This is precisely the condition which ensures that a random walk on this random graph visits infinitely many sites [7, Lemma 2.2]. It is equivalent to the condition that there exists a set of orthogonal directions $V \subset \mathcal{E}$ such that $\mu(\mathcal{G}_o \cap V \neq \varnothing) = 1$, that is, almost surely at every site, some element of V occurs [6, Lemma 2.2]. Models 1.1–1.3 above satisfy this condition (*e.g.* by taking $V = \{\uparrow, \leftarrow\}$, so one of those directions is always available to the walker). Note that $\theta_- = 1$ if and only if there exists $e \in \mathcal{E}$ such that $\mu(e \in \mathcal{G}_o) = 1$, and that none of these three models satisfy this condition. In fact $\theta_- \in (0,1)$ for each of these models when $p \in (0,1)$ [6, Corollary 3.5].

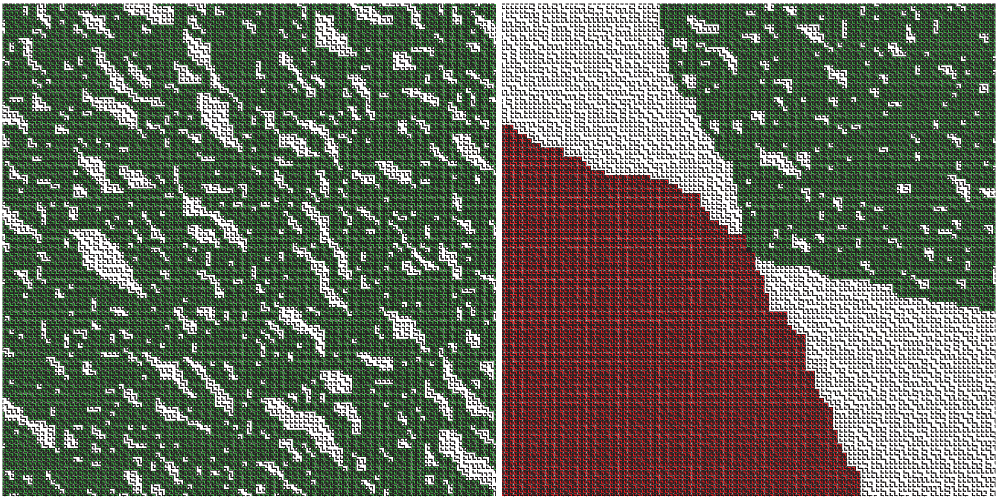
Model 1.1 exhibits a phase transition (in fact two phase transitions by symmetry) when the parameter p changes. While $\theta_+ = 1$ and $\theta_- > 0$ for all p, the geometry of an infinite $\mathcal{B}_x$ changes from having a non-trivial boundary to being all of $\mathbb{Z}^2$ as p decreases from 1 to 1/2 [6, Theorem 3.12]. Moreover, the critical point p_c at which this transition takes place is also the critical point $p_c^{\triangle}$ for oriented site percolation on the triangular lattice (OTSP). This connection arises because of a duality between the two models: when $\mathcal{B}_o$ is infinite but not all of $\mathbb{Z}^2$, its boundary is a path in the oriented triangular lattice (see the proof of [6, Theorem 3.12]).

Some information about the geometry of an infinite $\mathcal{M}_x$ can then be inferred, based on the geometry of $\mathcal{B}_x$ [6, Theorem 4.9], which strongly suggests that the same critical point p_c determines the transition from $\theta = 0$ to $\theta > 0$. The first goal of the present paper is to establish this. To do so we will obtain an asymptotic slope for the outer boundaries of $\mathcal{C}_o$, using oriented percolation arguments. See Theorem 1.4 below. Rigorous bounds for p_c already exist in the literature. Using the random environment structure (specifically, the webs of coalescing random walks this environment provides) we obtained an improved lower bound in [6, Theorem 4.12]. A second goal of the present paper is to further improve on this bound; see Theorem 1.5 below. Similar results hold for Model 1.2, which is dual to a partially oriented site percolation model on the triangular lattice. However, for Model 1.3, if $|\mathcal{B}_x| = \infty$ then $\mathcal{B}_x = \mathbb{Z}^2$ almost surely, regardless of p (see [6, Theorems 3.13 and 4.11]).

Simulations indicate that $\mathcal{C}_o$ and infinite $\mathcal{B}_o$ clusters have similar geometry, except that $\mathcal{C}_o$ typically has 'holes' whereas $\mathcal{B}_o$ does not. In order to give a clearer description of this weak kind of duality, we study the geometry of $\bar{\mathcal{C}}_x \supset \mathcal{C}_x$, defined by

$$\bar{\mathcal{C}}_x = \{z \in \mathbb{Z}^d : \text{every infinite nearest-neighbour self-avoiding path starting at } z \text{ passes through } \mathcal{C}_x\}. \tag{1.3}$$

The results of this paper can be separated into two groups. Section 2 concerns some relatively general results about connected clusters, beginning with a discussion of positive correlation (or lack thereof) between connection events, and the nature of holes in $\mathcal{C}_o$ for two-dimensional models. The remainder of Section 2 considers the possible geometries of $\bar{\mathcal{C}}_x$ clusters for a class of two-dimensional models, resulting in Propositions 2.3–2.4, whose statements (and proofs) are natural adaptations of the corresponding results for $\mathcal{B}_x$ clusters in [6, Proposition 3.8, Corollary 3.10].



(a) (b)

Figure 1. Simulations of $\mathcal{C}_o$ when $p = 0.5$ (a) and of $\mathcal{C}_o$ and $\mathcal{B}_o$ when $p = 0.7$ (b) for Model 1.1.

The second group of results concerns Model 1.1 (and its dual OTSP model). Proposition 3.1 gives the existence of phase transitions for the geometry of $\bar{\mathcal{C}}_x$ and its statement (and proof) is again a natural modification of [6, Theorem 3.12]. The remainder of the results in Sections 3–6 are entirely new, and exploit much more deeply the duality between $\mathcal{B}_o$, $\bar{\mathcal{C}}_o$ and OTSP for this model. An analysis of OTSP (see Section 4 and also [4], for example) is used to prove the following result (see also Figure 1) in Sections 3 and 5.

Theorem 1.4. *For Model 1.1* ($\uparrow\rightarrow$ $\leftarrow\downarrow$)*, the following hold.*

(I) $1 - p_c \leqslant p \leqslant p_c$ *if and only if* $\mathcal{M}_o$ *is infinite with positive probability, that is,* $\theta > 0$.

(II) $\bar{\mathcal{C}}_o(p_1) \supsetneq \bar{\mathcal{C}}_o(p_2)$ $\mathbb{P}$*-a.s. whenever* $p_c \leqslant p_1 < p_2 \leqslant 1$*, under the coupling* (1.1).

(III) *when* $p > p_c$*, the northwest-pointing boundary of* $\bar{\mathcal{C}}_o$ *has an asymptotic slope* $\rho_p < -1$*, with* ρ_p *strictly increasing to* -1 *as* $p \downarrow p_c$*. The southeast-pointing boundary of* $\bar{\mathcal{C}}_o$ *has asymptotic slope* $1/\rho_p$*.*

This result is natural given the results in [6], and was suggested by coupled simulations (such as that used to produce Figure 1, for example) for all p. However, its proof requires a much more detailed analysis of the boundary of connected clusters than is needed in [6].

Note that results of [6] already show $\Rightarrow$ for (I). But $\Leftarrow$ is only verified there, assuming a stronger condition on $\mathcal{M}_o$ (that $\mathcal{M}_o$ be 'gigantic'). In contrast to (II), the cluster $\mathcal{C}_o(p)$ is *not* monotone in p under the natural coupling (1.1). For example, letting $\phi_x(p) = \mathbb{P}_p(x \in \mathcal{C}_o)$, we see that $\phi_{(-1,1)}(p) = 0$ if $p = 0$ or $p = 1$, and $\phi_{(-1,1)}(p) > 0$ otherwise (in fact $\phi_{(-1,1)}(1/2) > 1/2$). Even when restricting attention to $p \geqslant 1/2$, finite holes may open or close in $\mathcal{C}_o$ as p increases, so monotonicity of $\mathcal{C}_o(p)$ fails. Finally note that a corollary of (III) is that random walks in random environments whose supports are $\uparrow\rightarrow$ with

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probability p and $\leftarrow\!\urcorner\!\downarrow$ with probability $1-p$ are transient in direction $(1,1)$ when $p > p_c$ [7, Corollary 2.9]. In fact one can prove that the walk is ballistic in direction $(1,1)$ for $p > p_c$ [8].

As noted above, the proof of Theorem 1.4 relies on first proving results about OTSP (Section 4), and exploiting the duality to obtain results for Model 1.1. On the other hand, by finding arbitrarily large connected circuits in Model 1.1 when $0.427 \leqslant p \leqslant 0.573$, we obtain the following result (proved in Section 6) about OTSP.

Theorem 1.5. *The critical occupation probability for oriented site percolation on the triangular lattice* $(d = 2)$ *is at least* 0.5730.

This improves on the best rigorous bounds that we have found in the literature, namely: $0.5466 \leqslant p_c^{\leftarrow\uparrow\searrow} \leqslant 0.7491$ (see [6, 1]). Note that the estimated value is $p_c \approx 0.5956$ [3, 11]. We believe that an adaptation of these arguments to Model 1.2 yields a bound on the critical occupation probability $p_c^{\leftrightarrow\updownarrow\searrow}$ for a partially oriented site percolation model.

2. General models: the forward cluster $\mathcal{C}_o$

In this section we investigate properties of the random sets $\mathcal{C}_x$. Typically these clusters are rather different from connected clusters in percolation models (where $E_2 = \varnothing$). For example, in the 2-valued setting the sets $\mathcal{C}_y$ are not increasing under the coupling (1.1) unless $E_1 \supset E_2$. In particular, for Models 1.1–1.3 the cluster $\mathcal{C}_o$ is not monotone in p. A natural question is to ask whether or not the connection events $\{x \in \mathcal{C}_o\}_{x\in\mathbb{Z}^d}$ are positively correlated, that is, whether

$$\mathbb{P}(y \in \mathcal{C}_o, x \in \mathcal{C}_o) \geqslant \mathbb{P}(y \in \mathcal{C}_o)\mathbb{P}(x \in \mathcal{C}_o).$$

While such a property is true in the percolation setting, this fails in general for degenerate random environments. The easiest example is the case $E_1 = \{e_1\}$ and $E_2 = \{e_2\}$, where

$$\mathbb{P}(e_1 \in \mathcal{C}_o, e_2 \in \mathcal{C}_o) = 0 \neq p(1-p) = \mathbb{P}(e_1 \in \mathcal{C}_o)\mathbb{P}(e_2 \in \mathcal{C}_o).$$

We believe that it fails for Models 1.1–1.3 (and many others) as well.

On the other hand, for a fixed y, the events $\{y \in \mathcal{C}_x\}_{x\in\mathbb{Z}^d}$ are positively correlated. To see why this ought to be the case, note that by translation invariance and relabelling of vertices, this statement is equivalent to $\mathbb{P}(x \in \mathcal{B}_o, y \in \mathcal{B}_o) \geqslant \mathbb{P}(x \in \mathcal{B}_o)\mathbb{P}(y \in \mathcal{B}_o)$, so roughly speaking, knowing that something connects to o makes it more likely that other things connect to o.

Lemma 2.1. *For* $x, y \in \mathbb{Z}^d$,

$$\mathbb{P}(y \in \mathcal{C}_o, y \in \mathcal{C}_x) \geqslant \mathbb{P}(y \in \mathcal{C}_o)\mathbb{P}(y \in \mathcal{C}_x).$$

Proof. It is sufficient to prove that $\mathbb{P}(y \in \mathcal{C}_o, y \notin \mathcal{C}_x) \leqslant \mathbb{P}(y \in \mathcal{C}_o)\mathbb{P}(y \notin \mathcal{C}_x)$. Let $B(n)$ be the set of lattice sites in a ball of radius n, centred at o. Let $E_n = \{o \to y \text{ in } B(n)\}$ be the event that there is a path from o to y lying entirely in $B(n)$. Let $F_n = \{x \to y \text{ in } B(n)\}^c$ be

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the event that there is no path connecting x to y that lies entirely in $B(n)$. Then E_n and F_n are increasing and decreasing in n respectively, with

$$E \equiv \{o \to y\} = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad F \equiv \{x \nrightarrow y\} = \bigcap_{n=1}^{\infty} F_n.$$

Therefore,

$$\mathbb{P}(E \cap F) = \lim_{M\to\infty} \mathbb{P}(E_M \cap F) \leqslant \liminf_{M\to\infty} \mathbb{P}(E_M \cap F_M).$$

Let $\mathcal{C}_{x,M}$ be the set of sites that can be reached from x using only sites in $B(M)$. Observe that if $z \in \mathcal{C}_{x,M}$ and $z \to y$ in $B(M)$ then $x \to y$ in $B(M)$. Thus for any $C \subset B(M)$ with $y \notin C$, on the event $\{\mathcal{C}_{x,M} = C\}$, we have that $\{o \to y \text{ in } B(M)\}$ occurs if and only if $\{o \to y \text{ in } B(M) \setminus C\}$ occurs. This latter event depends only on the random variables $\{\mathcal{G}_z : z \in B(M) \setminus C\}$, while $\{\mathcal{C}_{x,M} = C\}$ depends only on the random variables $\{\mathcal{G}_z : z \in C\}$. Thus we have

$$\begin{aligned}
\mathbb{P}(E_M \cap F_M) &= \sum_{C \subset B(M) \text{ s.t. } y \notin C} \mathbb{P}(0 \to y \text{ in } B(M), \mathcal{C}_{x,M} = C) \\
&= \sum_{C \subset B(M) \text{ s.t. } y \notin C} \mathbb{P}(0 \to y \text{ in } B(M) \setminus C, \mathcal{C}_{x,M} = C) \\
&= \sum_{C \subset B(M) \text{ s.t. } y \notin C} \mathbb{P}(0 \to y \text{ in } B(M) \setminus C)\mathbb{P}(\mathcal{C}_{x,M} = C) \\
&\leqslant \sum_{C \subset B(M) \text{ s.t. } y \notin C} \mathbb{P}(0 \to y)\mathbb{P}(\mathcal{C}_{x,M} = C) = \mathbb{P}(0 \to y)\mathbb{P}(x \nrightarrow y \text{ in } B(M)).
\end{aligned}$$

In other words, $\mathbb{P}(E_M \cap F_M) \leqslant \mathbb{P}(0 \to y)\mathbb{P}(F_M)$, and taking the limit as $M \to \infty$ establishes the result. □

Let $C \subset \mathbb{Z}^2$. We say that C has a *finite hole* G if $G \subset \mathbb{Z}^2 \setminus C$, G is finite, G is connected in $\mathbb{Z}^2$, and every $z \in \mathbb{Z}^2 \setminus G$ that is a neighbour of G must belong to C. The following elementary lemma implies that $\bar{\mathcal{C}}_o$ is obtained from $\mathcal{C}_o$ by filling in all finite holes, and that the backward cluster of a finite hole G is simply G.

Lemma 2.2. *Suppose that $x \in \bar{\mathcal{C}}_o \setminus \mathcal{C}_o$. Then x belongs to a finite hole G of $\mathcal{C}_o$ and $\mathcal{B}_x \subset G$.*

Proof. Let $x \in \bar{\mathcal{C}}_o \setminus \mathcal{C}_o$, and let $G \subset \mathbb{Z}^2 \setminus \mathcal{C}_o$ be the $\mathbb{Z}^2$-connected cluster of x in $\mathbb{Z}^2 \setminus \mathcal{C}_o$. Clearly every neighbour $z \in \mathbb{Z}^2 \setminus G$ of G is in $\mathcal{C}_0$. Let $y \in G$. Then there is a finite self-avoiding path in G from $x \in \bar{\mathcal{C}}_o$ to y, so $y \in \bar{\mathcal{C}}_o$. Thus $G \subset \bar{\mathcal{C}}_o \setminus \mathcal{C}_o$. Every infinite connected subset of $\mathbb{Z}^2$ contains an infinite nearest-neighbour self-avoiding path (it is easy to construct this iteratively: if x_0 is connected to infinity then there exists some neighbour x_1 of x_0 that connects to infinity off $\{x_0\}$, *etc.*) Therefore G is finite. Thus G is a finite hole containing x. Finally, since $x \in \mathcal{B}_x$ and $\mathcal{B}_x \cap \mathcal{C}_o = \emptyset$, it follows that $\mathcal{B}_x \subset G$. □

The above relationships between $\mathcal{B}$ and $\mathcal{C}$ clusters are rather weak. We can however prove results about the $\mathcal{C}$ clusters which are analogous to results about $\mathcal{B}$ clusters in [6],

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using modifications of the arguments from [6]. To state these results, we need the notion of blocking functions, and we introduce the notation $x^{[i]}$ with $i \in \{1, 2\}$ to denote the ith coordinate of $x \in \mathbb{Z}^2$, that is, $x = (x^{[1]}, x^{[2]})$.

Definition. Given $f : \mathbb{Z} \to \mathbb{Z}$, define $f_{\leqslant} \subset \mathbb{Z}^2$ and $f_{>} \subset \mathbb{Z}^2$ by

$$f_{\leqslant} = \{y \in \mathbb{Z}^2 : y^{[2]} \leqslant f(y^{[1]})\} \quad \text{and} \quad f_{>} = \{y \in \mathbb{Z}^2 : y^{[2]} > f(y^{[1]})\}.$$

We say that y is *below* f if $y^{[2]} \leqslant f(y^{[1]})$, and *strictly below* f if $y^{[2]} < f(y^{[1]})$. We define $f_{\geqslant}$ and $f_{<}$ similarly, and speak likewise of y being *above* f or *strictly above* f.

We say that $f : \mathbb{Z} \to \mathbb{Z}$ is a *forward lower blocking function* (flbf) for $\mathcal{G}$ if there is no open path in $\mathcal{G}$ from $f_{\geqslant}$ to $f_{<}$, that is, if $f_{\geqslant} \nrightarrow f_{<}$. Similarly, f is a *forward upper blocking function* (fubf) for $\mathcal{G}$ if $f_{\leqslant} \nrightarrow f_{>}$. Note that these notions are slightly different from the (backward) lower blocking function $f_{<} \nrightarrow f_{\geqslant}$ and (backward) upper blocking function $f_{>} \nrightarrow f_{\leqslant}$ in [6]. Since the model is defined via edges directed out from a vertex, for forward clusters we look at the boundary inside the cluster (whose edges must not connect to outside the cluster), whereas for backward clusters we look at the boundary outside the cluster (whose edges must not connect to the cluster). Thus, f is a flbf if and only if $f - 1$ is a bubf, and f is a fubf if and only if $f + 1$ is a blbf.

We write $\mathcal{A}_E = \{\mathcal{G}_o \cap E \neq \varnothing\}$ for $E \subset \mathcal{E}$ and use shorthand such as $\mathcal{A}_{\leftarrow\uparrow} = \mathcal{A}_{\{\leftarrow,\uparrow\}}$. The following Propositions are $\mathcal{C}$ versions of the $\mathcal{B}$ results [6, Proposition 3.8 and Corollary 3.10].

Proposition 2.3. *Fix $d = 2$. Assume that $\mu(\mathcal{A}_{\leftarrow\uparrow}) = 1$, $\mu(\mathcal{A}_{\downarrow\rightarrow}) = 1$, $\mu(\mathcal{A}_{\leftarrow}) > 0$, $\mu(\mathcal{A}_{\rightarrow}) > 0$, $\mu(\mathcal{A}_{\uparrow}) > 0$, and $\mu(\mathcal{A}_{\downarrow}) > 0$.*

(a) *The following $\mathbb{P}$-a.s. exhaust the possibilities for $\bar{\mathcal{C}}_x$:*

(i) $\bar{\mathcal{C}}_x = \mathbb{Z}^2$,

(ii) *there exists a decreasing flbf $f : \mathbb{Z} \to \mathbb{Z}$ such that $\bar{\mathcal{C}}_x = f_{\geqslant}$,*

(iii) *there exists a decreasing fubf $f : \mathbb{Z} \to \mathbb{Z}$ such that $\bar{\mathcal{C}}_x = f_{\leqslant}$.*

(b) *Only one of* (i), (ii), (iii) *can have probability different from* 0.

Proposition 2.4. *Fix $d = 2$. Assume that $\mu(\mathcal{A}_{\uparrow\rightarrow}) = 1$, $\mu(\mathcal{A}_{\leftarrow\uparrow}) = 1$, $\mu(\mathcal{A}_{\leftarrow}) > 0$, $\mu(\mathcal{A}_{\rightarrow}) > 0$, and $\mu(\mathcal{A}_{\uparrow}) > 0$.*

(a) *The following $\mathbb{P}$-a.s. exhaust the possibilities for $\bar{\mathcal{C}}_x$:*

(i) $\bar{\mathcal{C}}_x = \mathbb{Z}^2$,

(ii) *there exists a flbf $f : \mathbb{Z} \to \mathbb{Z}$ such that $\bar{\mathcal{C}}_x = f_{\geqslant}$,*

(b) *only one of* (i) *or* (ii) *can have probability different from* 0.

Proposition 2.3 applies to Model 1.1, while Proposition 2.4 applies to Model 1.2. When $\bar{\mathcal{C}}_x = f_{\geqslant}$ (so f is a flbf) we say that $\bar{\mathcal{C}}_x$ is *blocked below*. Similarly, when $\bar{\mathcal{C}}_x = f_{\leqslant}$ (so f is a fubf) we say that $\bar{\mathcal{C}}_x$ is *blocked above*.

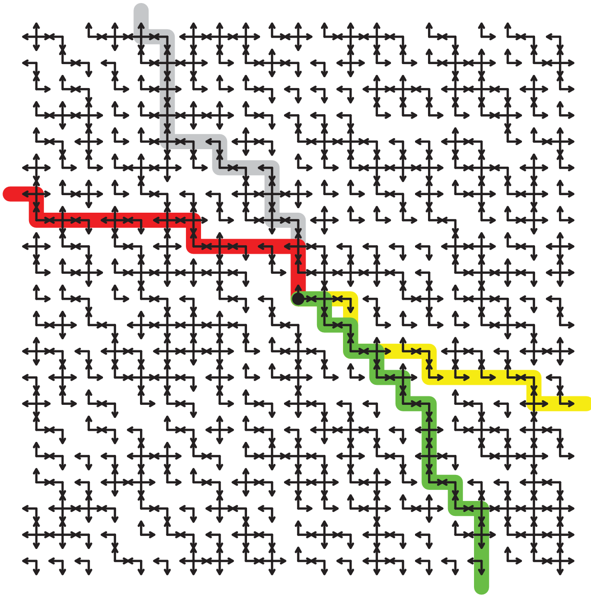

Figure 2. Part of each of the *Nw* (grey), *nW* (red), *Se* (green) and *sE* (yellow) paths from the origin for a 3-valued DRE (↑↳ ↰↓ ↔↕).

An important notion that arises in the proofs of these results (and elsewhere throughout this paper) is the asymptotic slope of a path.

Definition. A nearest-neighbour path $x_0, x_1, \dots$ with $x_i = (x_i^{[1]}, x_i^{[2]}) \in \mathbb{Z}^2$ is said to have *asymptotic slope* σ if

$$\lim_{n\to\infty} \frac{x_n^{[2]}}{x_n^{[1]}} = \sigma.$$

Proof of Proposition 2.3. Define the *Nw* path ($\subset \mathcal{C}_x$) from x to be that path starting from x obtained by following $\uparrow$ whenever possible, and otherwise following $\leftarrow$ (see Figure 2). On a set of $\mathbb{P}$-measure 1, this path exists (since $\mu(\mathcal{A}_{\leftarrow\uparrow}) = 1$), and has asymptotic slope

$$\sigma_{Nw} = -\frac{\mu(\mathcal{A}_{\uparrow})}{1-\mu(\mathcal{A}_{\uparrow})}.$$

Similarly, the nW path from x, defined to be that path starting from x obtained by following $\leftarrow$ whenever possible and otherwise following $\uparrow$, exists and has asymptotic slope

$$\sigma_{nW} = -\frac{1-\mu(\mathcal{A}_{\leftarrow})}{\mu(\mathcal{A}_{\leftarrow})}.$$

We can also define the *Se* and *sE* paths from x and find their asymptotic slopes.

We claim that *Se* paths from any two points y' and z' intersect almost surely (as long as $\mu(\mathcal{A}_{\downarrow}) \neq 1$). Similarly, the *Nw* (resp. *sE* and *nW*) paths from any y' and z' meet. To see that the *Se* paths meet, assume first that for some k, y' and z' both belong to the line

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$\{(i,j) : i = j+k\}$. Follow the (unique) *Se* path from y' (resp. z'), and let Y_n (resp. Z_n) be the location after n steps. Then Y_n is a space-time random walk path started from y' (with spatial axis $\{(i,j) : i = j+k\}$ and temporal axis in the direction of $\{(i,j) : i = -j\}$) that steps down (south with respect to the lattice $\mathbb{Z}^d$) with probability $\mu(\mathcal{A}_\downarrow)$ and up (east/right with respect to the lattice $\mathbb{Z}^d$ with probability $1 - \mu(\mathcal{A}_\uparrow)$. Similarly Z_n is a random walk with the same law that is independent of Y_n until the paths intersect. Thus the difference $Y_n - Z_n$ is a symmetric random walk (up to the first time it hits 0), with step sizes 0, 2, or -2, started at the point $(y'^{[1]} - z'^{[1]}) + (y'^{[2]} - z'^{[2]}) \in 2\mathbb{Z}$. This process reaches 0 (*i.e.*, the Y_n and Z_n paths eventually coincide in space-time) after some finite time almost surely. If y' and z' do not belong to the same line $\{(i,j) : i = j+k\}$, just follow the path from one point till it reaches the diagonal line the other starts on, and then apply the same argument.

Suppose that $w \notin \mathcal{C}_o$, $\bar{y} \in \bar{\mathcal{C}}_o$, $\bar{z} \in \bar{\mathcal{C}}_o$, $w^{[1]} = \bar{y}^{[1]} = \bar{z}^{[1]}$, but $\bar{y}^{[2]} < w^{[2]} < \bar{z}^{[2]}$. Because $\bar{y}$ is either in $\mathcal{C}_o$ or it is enclosed by $\mathcal{C}_o$, we can find $y \in \mathcal{C}_o$ such that $y^{[1]} = \bar{y}^{[1]}$ but $y^{[2]} \leqslant \bar{y}^{[2]}$. Likewise there is a $z \in \mathcal{C}_o$ with $z^{[1]} = \bar{z}^{[1]}$ and $z^{[2]} \geqslant \bar{z}^{[2]}$. The *Se* paths from y and z above meet, as do *Nw* paths from y and z. These four paths enclose w, so $w \in \bar{\mathcal{C}}_o$. Letting

$$L_i = \inf\{j : (i,j) \in \bar{\mathcal{C}}_o\} \quad \text{and} \quad U_i = \sup\{j : (i,j) \in \bar{\mathcal{C}}_o\},$$

it follows that $\bar{\mathcal{C}}_o = \{(i,j) : L_i \leqslant j \leqslant U_i\}$.

If $\mu(\mathcal{A}_\uparrow) = 1$ or $\mu(\mathcal{A}_\downarrow) = 1$ then trivially U or L is infinite. Otherwise we have that $-\infty < \sigma_{Nw} \leqslant \sigma_{nW} \leqslant 0$ and $-\infty < \sigma_{Se} \leqslant \sigma_{sE} \leqslant 0$. Moreover, since every direction is possible, we have that $\sigma_{Nw} < 0$ and $\sigma_{Se} < 0$.

Case (i) corresponds to $L \equiv -\infty$ and $U \equiv \infty$. We can rule out the possibility that L (or U) jumps from finite to infinite values, or *vice versa*, since this would imply the existence of an infinite vertical boundary in $\mathcal{C}_o$, which cannot occur since almost surely every infinite vertical line in $\mathbb{Z}^2$ contains infinitely many sites containing $\leftarrow$ edges and infinitely many sites containing $\rightarrow$ edges. To see that if L takes finite values, it must be decreasing, consider $\mathcal{G}_{(i,L_i)}$. By definition, $\downarrow \notin \mathcal{G}_{(i,L_i)}$. Since $\mu(\mathcal{A}_{\downarrow\!\rightarrow}) = 1$, it follows that $\rightarrow \in \mathcal{G}_{(i,L_i)}$. This implies that $L_{i+1} \leqslant L_i$. Similarly, U_i is decreasing if finite.

If $-\infty < L$ then $f = L$ is a flbf, and similarly if $U < \infty$ then $f = U$ is a fubf. Thus it remains to prove that one of L or U must be infinite, and it suffices to do this for L_0 and U_0.

Let $k \geqslant 0$. If $\sigma_{Nw} < \sigma_{sE}$, we may start from the origin and follow the *sE* path until reaching a vertex x_k on this path from which the *Nw* path includes a vertex $(0, k+j)$ for some $j \geqslant 0$. This shows that U_0 is almost surely infinite. If $\sigma_{nW} > \sigma_{Se}$ then follow the *Se* path from the origin until reaching a vertex y_k on this path from which the *nW* path includes a vertex $(0, -k-j)$ for some $j \geqslant 0$. This shows that L is almost surely infinite.

It remains to verify the claim when $-\infty < \sigma_{Nw} = \sigma_{nW} = \sigma_{Se} = \sigma_{sE} < 0$, which implies that $\mu(\mathcal{A}_\uparrow \cap \mathcal{A}_\leftarrow) = 0$ and $\mu(\mathcal{A}_\downarrow \cap \mathcal{A}_\rightarrow) = 0$. In this case we define a new path, called the nW_s path. From a vertex x, it follows whichever of $\leftarrow$ or $\uparrow$ is possible (now only one will be), except when at a vertex y such that $\downarrow \in \mathcal{G}_y$ and $\leftarrow \in \mathcal{G}_{y-e_2}$ (*i.e.*, such that $\mathcal{G}_y \in \mathcal{A}_\downarrow$ and $\mathcal{G}_{y-e_2} \in \mathcal{A}_\leftarrow$). At such a vertex y, it follows the $\downarrow$ step, followed by the $\leftarrow$ step (see Figure 3 for an example of such a path). This path has a slope $\sigma_{nW_s} > \sigma_{Nw} = \sigma_{Se}$. We may now proceed as before by following the *Se* path from o until reaching a vertex y_k

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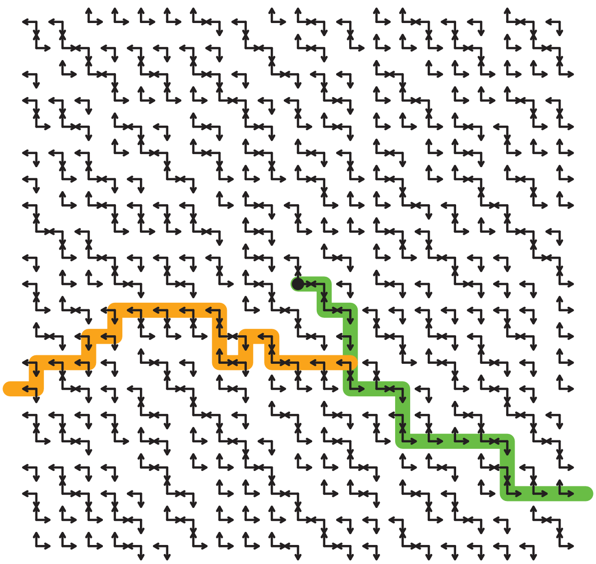

Figure 3. The *Se* path (green) from the origin together with the nW_s path (orange) from a point along the *Se* path, in the two-dimensional orthant model.

from which the nW_s path passes through $(0, -k-j)$ for some $j \geqslant 0$. This shows that L is almost surely infinite, and completes the proof of part (a).

Part (b) now follows from standard arguments; see [6, Proposition 3.8] for details. □

The proof of Proposition 2.4 is a modification of that of [6, Corollary 3.10], in exactly the same way as the proof of Proposition 2.3 adapts that of [6, Proposition 3.8]. We therefore omit it.

3. Critical probabilities and coupling for Model 1.1

For any site $x \in \mathbb{Z}^2$, let $\mathbf{C}_x$ denote the set of sites $y \in \mathbb{Z}^2$ for which there is some $N \geqslant 0$ and a sequence $\{x = y_0, y_1, y_2, \dots, y_N = y\}$ such that $\mathcal{G}_{y_i} = \uparrow\!\rightarrow$ for $0 \leqslant i \leqslant N$ and $y_{i+1} - y_i \in \{\leftarrow, \uparrow, \nwarrow\} = \{-e_1, e_2, e_2 - e_1\}$ for $0 \leqslant i \leqslant N-1$. Note that $\mathbf{C}_x$ is analogous to $\mathcal{C}_x$ but is a *forward cluster for oriented site percolation on the triangular lattice* rather than a forward cluster for the degenerate random environment. Similarly let $\mathbf{B}_x$ denote the set of sites $y \in \mathbb{Z}^2$ for which there is some $N \geqslant 0$ and a sequence $\{y = y_0, y_1, y_2, \dots, y_N = x\}$ such that $\mathcal{G}_{y_i} = \uparrow\!\rightarrow$ and $y_{i+1} - y_i \in \{\leftarrow, \uparrow, \nwarrow\}$ for each i.

Let

$$\mathbf{A} = \{x \in \mathbb{Z}^2 : |\mathbf{C}_x| = |\mathbf{B}_x| = \infty\},$$

that is, $\mathbf{A}$ is the set of sites that are in a bi-infinite cluster (in the sense of oriented site percolation on the triangular lattice (OTSP)) of $\uparrow\!\rightarrow$ sites.

For Model 1.1, the conclusions of Proposition 2.3 can be extended to the following.

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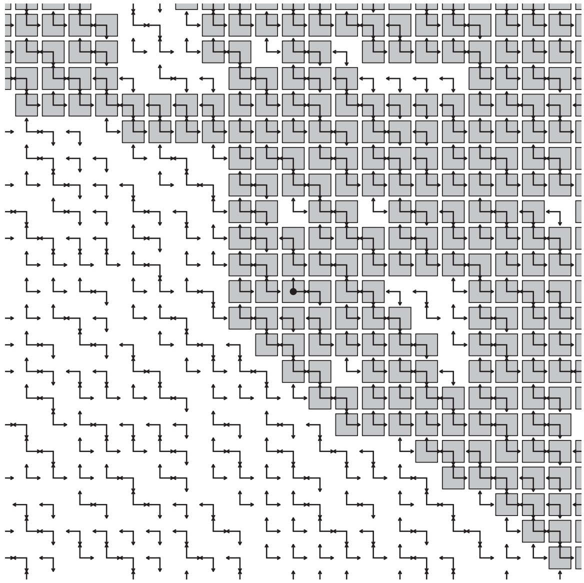

Figure 4. A realization of $\mathcal{C}_o$ in a box, for the orthant model with $p = 0.6$.

Proposition 3.1. *For Model 1.1 (↑→, ←↓):*

(a) $0 \leqslant p < 1 - p_c \;\Rightarrow\; \bar{\mathcal{C}}_o$ *is a.s. blocked above,*
(b) $1 - p_c \leqslant p \leqslant p_c \;\Rightarrow\; \bar{\mathcal{C}}_o = \mathbb{Z}^2$ *a.s.,*
(c) $p_c < p \leqslant 1 \;\Rightarrow\; \bar{\mathcal{C}}_o$ *is a.s. blocked below.*

Proof. We proceed as in the proof of [6, Theorem 3.12], and will reiterate part of the latter in order to explain the role of the triangular lattice.

Let $f(n)$ be decreasing. For f to be a flbf for $\mathcal{G}$, the vertices in $f_{\geqslant}$ that have a (square-lattice) nearest-neighbour in $(f_{\geqslant})^c$ can be enumerated naturally as $\{y_t\}_{t\in\mathbb{Z}}$ to form a sequence of vertices moving upwards and to the left (see Figure 4).

The possible transitions in this sequence of vertices are as follows.

- Upwards, *e.g.* from (n,k) to $(n,k+1)$. This happens if $f(n) \leqslant k < f(n-1) - 1$.
- Leftwards, *e.g.* from (n,k) to $(n-1,k)$. This happens if $f(n) = f(n-1) = k$.
- Diagonally to the northwest, *e.g.* from (n,k) to $(n-1,k+1)$. This happens if $f(n) \leqslant k = f(n-1) - 1$.

These are three of the six possible transitions in a triangular lattice, whose families of lines are horizontal, vertical, and diagonal with slope -1 (the set of lattice points is still $\mathbb{Z}^2$). For $f(n)$ to be a flbf, it is necessary and sufficient that each vertex y_t in this sequence has local environment ↑→. Calling ↑→ vertices 'open' and ←↓ vertices 'closed', this sequence defines a bi-infinite oriented (triangular lattice) nearest-neighbour path such that $\mathcal{G}_{y_t} =$ ↑→. It follows that each $y_t \in \mathbf{A}$ and in particular $(n, f(n)) \in \mathbf{A}$.

Now continue exactly as in the proof of [6, Theorem 3.12] to obtain (c), substituting Proposition 2.3 for [6, Proposition 3.8]. A similar argument (or just symmetry) gives part (a), and part (b) follows immediately from the above argument and Proposition 2.3. □

For $p > p_c$, let $f(n)$ be the forward lower blocking function such that $\bar{\mathcal{C}}_x = f_{\geqslant}$, and define $\partial^* \mathcal{C}_o \subset \mathcal{C}_o$ by

$$\partial^* \mathcal{C}_o = \{(n,m) \in \mathbb{Z}^2 : m = f(n) \text{ or } f(n) \leqslant m < f(n-1)\}.$$

The above sequence y_t traces out $\partial^* \mathcal{C}_o$ sequentially as a (triangular lattice) path. Clearly o lies in or above the set $\partial^* \mathcal{C}_o$, since $o \in \mathcal{C}_o$ implies that $f(0) \leqslant 0$. We will need the following later, in which we recall that, for example, $\mathbf{C}_{(0,K_0)}$ denotes the forward cluster for oriented site percolation on the triangular lattice from the point $(0,K_0) \in \mathbb{Z}^2$.

Corollary 3.2. *For Model 1.1 (⮥ ⮧), when $p > p_c$ the flbf $f(n)$ satisfies*

$$f(n) = \begin{cases} K_0 \equiv \sup\{k \leqslant 0 : (0,k) \in \mathbf{A}\} & \text{if } n = 0, \\ \sup\{k \in \mathbb{Z} : (n,k) \in \mathbf{B}_{(0,K_0)} \cap \mathbf{A}\} & \text{if } n \in \mathbb{N}, \\ \sup\{k \in \mathbb{Z} : (n,k) \in \mathbf{C}_{(0,K_0)} \cap \mathbf{A}\} & \text{if } n \in -\mathbb{N}. \end{cases} \tag{3.1}$$

Proof. Define $f^*(n)$ to be the right-hand side of (3.1). Note that K_0 is finite as in the proof of [6, Theorem 3.12] and Proposition 3.1). Moreover, if $x \in \mathbf{A}$ then $x + e, x - e' \in \mathbf{A}$ for some $e, e' \in \{e_1, -e_2, e_1 - e_2\}$. It follows that $f^* : \mathbb{Z} \to \mathbb{Z}$ is well defined. Our objective is to show that $f = f^*$.

Note that f^* is decreasing, since if $n \geqslant 1$ and $f^*(n) = k$, for example, then there exists $j \in [k, K_0]$ such that $(n-1, j) \in \mathbf{C}_{(n,k)} \cap \mathbf{B}_{(0,K_0)}$, and then $(n-1, j) \in \mathbf{B}_{(0,K_0)} \cap \mathbf{A}$. Clearly then the origin cannot connect to anything below f^*, so $f^* \leqslant f$. We have seen in the proof of Proposition 3.1 that $(n, f(n)) \in \mathbf{A}$ for every $n \in \mathbb{Z}$. Since $f(0) \leqslant 0$ and $(0, f(0)) \in \mathbf{A}$, we get $f(0) \leqslant f^*(0)$. Therefore $f(0) = f^*(0)$. As in the proof of Proposition 3.1, $(n, f(n)) \in \mathbf{B}_{(0,f(0))}$ or $(n, f(n)) \in \mathbf{C}_{(0,f(0))}$ (depending on the sign of $n \in \mathbb{Z}$) via the path y_t. Since $(n, f(n)) \in \mathbf{A}$, we get that $f(n) \leqslant f^*(n)$, and the result follows. □

We will need a corresponding result for an infinite $\mathcal{B}_o$ cluster. Indeed, by [6, Theorem 3.12] there exists a decreasing function $V : \mathbb{Z} \to \mathbb{Z}$ with $\mathcal{B}_o = V_<$ (in [6], $V - 1$ is a bubf, so V is a flbf). The proof of the following proceeds just as in that of Corollary 3.2, so will be omitted.

Corollary 3.3. *Consider Model 1.1 (⮥ ⮧) with $p > p_c$. If $\mathcal{B}_o$ is infinite, then $\mathcal{B}_o = g_<$ for a decreasing flbf $g : \mathbb{Z} \to \mathbb{Z}$ satisfying*

$$g(n) = \begin{cases} K_0' \equiv \inf\{k \geqslant 0 : (0,k) \in \mathbf{A}\} & \text{if } n = 0, \\ \inf\{k \in \mathbb{Z} : (n,k) \in \mathbf{B}_{(0,K_0')} \cap \mathbf{A}\} & \text{if } n \in \mathbb{N}, \\ \inf\{k \in \mathbb{Z} : (n,k) \in \mathbf{C}_{(0,K_0')} \cap \mathbf{A}\} & \text{if } n \in -\mathbb{N}. \end{cases}$$

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As p increases from 0 to 1, simulation suggests that the cluster $\mathcal{B}_o$ can change from infinite to finite and back again arbitrarily many times, while 'holes' in $\mathcal{C}_o$ can expand and contract. Nevertheless, both clusters have a kind of monotonicity. Theorem 1.4(II) gives the appropriate statement for forward clusters.

Proof of Theorem 1.4(II). Couple the environments for all p as in (1.1), so that as p decreases we switch $\uparrow\rightarrow$ to $\leftarrow\downarrow$. Due to Proposition 3.1, the conclusion of the theorem is trivial if $p_1 = p_c$. So assume $p_c < p_1 < p_2 \leqslant 1$, and let f be the flbf with $\bar{\mathcal{C}}_o(p_1) = f_{\geqslant}$. This implies that every site $x \in \partial^*\mathcal{C}_o(p_1)$ satisfies $x \in \mathcal{C}_o(p_1)$ and $\mathcal{G}_x(p_1) = \uparrow\rightarrow$. Therefore $\mathcal{G}_x(p_2) = \uparrow\rightarrow$ for all such x, which implies that regardless of $\{\mathcal{G}_y(p_2) : y \notin \partial^*\mathcal{C}_o(p_1)\}$, there can be no open path in $\mathcal{G}(p_2)$ from o to any site below $\partial^*\mathcal{C}_o(p_1)$. In other words, $f_< \subset (\mathcal{C}_o(p_2))^c$, and by definition of $\bar{\mathcal{C}}_o$ also $f_< \subset (\bar{\mathcal{C}}_o(p_2))^c$. This implies that $(\bar{\mathcal{C}}_o(p_1))^c \subseteq (\bar{\mathcal{C}}_o(p_2))^c$, which establishes the desired conclusion except for showing that the inequality is strict.

It remains to prove strictness, that is, $\bar{\mathcal{C}}_o(p_2) \neq \bar{\mathcal{C}}_o(p_1)$. Let

$$G_x = \{U_x \leqslant p_2\} = \{\mathcal{G}_x(p_2) = \uparrow\rightarrow\}.$$

Let $\mathcal{H}_2 = \sigma(\{G_x, x \in \mathbb{Z}^2\})$. Then $\mathcal{C}_o(p_2)$ and $\partial^*\mathcal{C}_o(p_2)$ are $\mathcal{H}_2$-measurable random sets, and

$$\partial^*\mathcal{C}_o(p_2) \subset H_2 = \{x \in \mathbb{Z}^2 : \mathcal{G}_x(p_2) = \uparrow\rightarrow\}.$$

Conditional on $\mathcal{H}_2$, for any $\mathcal{H}_2$-measurable subset H' of H_2 we have that $\{U_x : x \in H'\}$ are i.i.d. $U[0, p_2]$ random variables. In particular, $\{U_x : x \in \partial^*\mathcal{C}_o(p_2)\}$ are i.i.d. $U[0, p_2]$ random variables under this conditioning. Thus,

$$\mathbb{P}\big(\{x \in \partial^*\mathcal{C}_o(p_2) : U_x > p_1\} = \varnothing | \mathcal{H}_2\big) = 0$$

almost surely, so

$$\mathbb{P}\big(\{x \in \partial^*\mathcal{C}_o(p_2), U_x > p_1\} = \varnothing\big) = 0.$$

This says that (almost surely) there exists $u \in \partial^*\mathcal{C}_o(p_2)$ with $U_u \in (p_1, p_2]$. Therefore $\mathcal{G}_u(p_1) = \leftarrow\downarrow$, so $u \notin \partial^*\mathcal{C}_o(p_1)$, hence $\partial^*\mathcal{C}_o(p_1) \neq \partial^*\mathcal{C}_o(p_2)$. □

The following is a version of Theorem 1.4(II) for the clusters $\mathcal{B}_o(p)$. The proof is similar, and will be omitted.

Corollary 3.4. *Consider Model 1.1* ($\uparrow\rightarrow$ $\leftarrow\downarrow$), *with the coupling* (1.1). *Let* $p_c \leqslant p_1 < p_2 \leqslant 1$. *Then* $\mathcal{B}_o(p_1) \supsetneq \mathcal{B}_o(p_2)$ $\mathbb{P}$*-a.s. on* $\{|\mathcal{B}_o(p_1)| = \infty\}$.

4. Oriented site percolation on the triangular lattice

In this section we state without proof a number of results about the OTSP model ($\swarrow\uparrow\rightarrow$, $\cdot$) that follow using the methods of [4] for two-dimensional oriented percolation models. Details may be found in [9]. In this model we have local environment $\mathbf{G}_x = \swarrow\uparrow\rightarrow$ with probability p, and $\mathbf{G}_x = \emptyset$ with probability $1 - p$, both on the triangular lattice described in Section 3. Recall that forward clusters in this model are denoted $\mathbf{C}_x$, and backward

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clusters $\mathbf{B}_x$. The natural coupling (1.1) gives a probability space on which the sets $\mathbf{C}_o(p)$ are increasing in p almost surely, so

$$\Theta_+(p) = \mathbb{P}(|\mathbf{C}_o(p)| = \infty) \quad \text{is increasing in } p,$$

giving the critical value $p_c^{\leftarrow\searrow\uparrow} = \inf\{p : \Theta_+(p) > 0\} \in (0, 1)$.

The estimated value is $p_c^{\leftarrow\searrow\uparrow} \approx 0.5956$ (see De'Bell and Essam [3] or Jensen and Guttmann [11]). The best rigorous bounds that we have found in the literature are $0.5466 \leqslant p_c^{\leftarrow\searrow\uparrow} \leqslant 0.7491$, where the latter comes from the fact that $p_c^{\leftarrow\searrow\uparrow} \leqslant p_c^{\uparrow\rightarrow} \leqslant 0.7491$ (the latter, referring to oriented percolation on the square lattice, is in Balister, Bollobás and Stacey [1]). Similarly, if p_c^{TSP} denotes the critical threshold for (un-oriented) triangular site percolation, then $p_c^{\leftarrow\searrow\uparrow} \geqslant p_c^{\mathrm{TSP}} = 1/2$ (see Hughes [10]). The lower bound of 0.5466 comes from [6] based on estimates of the critical value for the model $(\uparrow\rightarrow, \leftarrow\downarrow)$. In Section 6 we improve this lower bound to $0.5731 \leqslant p_c^{\leftarrow\searrow\uparrow}$, by finding arbitrarily large connected circuits in the dual model $(\uparrow\rightarrow, \leftarrow\downarrow)$ when $0.4269 \leqslant p \leqslant 0.5731$.

In order to describe the shape of an infinite $\mathbf{C}_x$ cluster, define $\mathbf{w}_n = \sup\{x : (-n, x) \in \mathbf{C}_o\}$ and $\mathbf{v}_n = \inf\{x : (-n, x) \in \mathbf{C}_o\}$. The following proposition is proved using subadditivity of quantities related to $\mathbf{w}_n$. Minor modifications arise from the proofs in [4], because the latter treats oriented bond percolation on the square lattice, while we need oriented site percolation on the triangular lattice.

Proposition 4.1. *For the percolation model* $(\leftarrow\searrow\uparrow, \cdot)$ *with* $1 > p > p_c^{\leftarrow\searrow\uparrow}$*, there exists* $\rho = \rho_p < -1$ *such that almost surely on the event* $\{|\mathbf{C}_o| = \infty\}$*, the upper and lower boundaries of* $\mathbf{C}_o$ *have asymptotic slopes* ρ *and* $1/\rho$ *respectively. In other words,* $\mathbf{w}_n/(-n) \to \rho$ *and* $\mathbf{v}_n/(-n) \to 1/\rho$ *almost surely as* $n \to \infty$. □

Since $\mathbf{w}_n$ is bounded above by a sum of $n + 1$ independent Geometric$(1 - p)$ random variables, we get the inequality $-p/(1 - p) \leqslant \rho_p$. The following two additional lemmas can be proved as in [4].

Lemma 4.2. ρ_p *is continuous and strictly decreasing in* $p > p_c^{\leftarrow\searrow\uparrow}$*, with* $\rho_p \uparrow -1$ *as* $p \downarrow p_c^{\leftarrow\searrow\uparrow}$.

Let $\tau = \sup\{y - x : (x, y) \in \mathbf{C}_o\}$, which measures the furthest diagonal line reached by the forward cluster of the origin. More generally, if $z = (x_0, y_0)$, let

$$\tau_z = \sup\{(y - y_0) - (x - x_0) : (x, y) \in \mathbf{C}_z\}.$$

Note that $|\mathbf{C}_o| = \infty \Leftrightarrow \tau = \infty$.

Lemma 4.3. *If* $p > p_c^{\leftarrow\searrow\uparrow}$*, then there exist constants* $C, \gamma > 0$ *such that*

$$\mathbb{P}(n \leqslant \tau < \infty) \leqslant Ce^{-\gamma n}.$$

On the event $\{|\mathbf{C}_o| = \infty\}$, for $n \in \mathbb{Z}_+$ let

$$\mathbf{a}_n = \sup\{x_n^{[2]} : x_n \in \mathbf{C}_o, |\mathbf{C}_{x_n}| = \infty, x_n^{[1]} = -n\}.$$

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for $n \in \mathbb{Z}_+$. The following result says that $\mathbf{a}_n$ has the same asymptotic slope as the upper boundary of $\mathbf{C}_o$.

Corollary 4.4. *For the percolation model* $(\leftarrow\uparrow\nwarrow, \cdot)$ *with* $p > p_c^{\leftarrow\uparrow\nwarrow}$, $\lim \mathbf{a}_n/(-n) = \rho_p$ *almost surely on the event* $\{|\mathbf{C}_o| = \infty\}$.

Proof. Let $p > p_c^{\leftarrow\uparrow\nwarrow}$. Since $\mathbf{a}_n \leqslant \mathbf{w}_n$ for every n, and $\mathbf{w}_n/(-n) \to \rho_p > -\infty$, it suffices to prove that for each $\epsilon > 0$,

$$\mathbb{P}(\mathbf{w}_n - \mathbf{a}_n > \epsilon n \text{ infinitely often}, |\mathbf{C}_o| = \infty) = 0.$$

Let $R > -\rho_p$. Then $\mathbf{w}_n \leqslant Rn$ for all sufficiently large n, so we may find N such that $\mathbf{w}_n \leqslant Rn$ for every $n \geqslant N$. It will therefore suffice to show that

$$\mathbb{P}(|\mathbf{C}_o| = \infty, \text{ and } \mathbf{w}_n - \mathbf{a}_n > \epsilon n \text{ for infinitely many } n \geqslant N(1+R)) = 0. \tag{4.1}$$

Call $y - x$ the *generation* of a point $(x, y) \in \mathcal{L}$. Along any open path in this model, the generation increases at each step, by 1 or by 2. Fix $\epsilon > 0$, and $n > 4/\epsilon$. Suppose that $|\mathbf{C}_o| = \infty$, and $N(1+R) \leqslant n$, and $\mathbf{w}_n - \mathbf{a}_n > \epsilon n$. Let T be the triangle with vertices $a = (-n, 0)$, $b = (-n, Rn)$, and $c = (-N, RN)$, and let $T_0 \supset T$ be the triangle with vertices a, b, and o. Any open path Γ from o to $(-n, \mathbf{w}_n)$ enters T along the side ac. Since $n \geqslant N(1+R)$, ac has slope $\leqslant 1$, so entry to T occurs no later than generation n. Therefore from generation n through $n + \mathbf{w}_n$ the path Γ lies entirely within T. Consider the lattice point $z = (x, y)$ on this path whose generation first exceeds $n + \mathbf{w}_n - \epsilon n$. Then $\mathbf{w}_n - \epsilon n > \mathbf{a}_n \geqslant 0$, so $n + \mathbf{w}_n - \epsilon n > n$, and hence $z \in T$. The generation of $(-n, \mathbf{w}_n)$ is at least $\epsilon n - 2 > \epsilon n/2$ larger than that of z, so $\tau_z \geqslant \epsilon n/2$. On the other hand, $x \geqslant -n$, so $y = (y - x) + x \geqslant n + \mathbf{w}_n - \epsilon n + x \geqslant \mathbf{w}_n - \epsilon n > \mathbf{a}_n$. Therefore we cannot have $|\mathbf{C}_z| = \infty$. In other words, $\tau_z < \infty$. We have shown that for $n > 4/\epsilon$,

$$\{|\mathbf{C}_o| = \infty, N(1+R) \leqslant n, \text{ and } \mathbf{w}_n - \mathbf{a}_n > \epsilon n\} \subset \bigcup_{z \in T_0} \left\{\frac{\epsilon n}{2} \leqslant \tau_z < \infty\right\}.$$

There are at most Rn^2 lattice points in T_0, so by Lemma 4.3, the probability of this event is at most $CRn^2 e^{-\gamma n\epsilon/2}$, which sums. Therefore (4.1) follows by Borel–Cantelli. □

5. Asymptotic slopes for Model 1.1

To complete this circle of results, it simply remains to show that the remaining parts of Theorem 1.4 follow from the results of the previous section.

Proof of Theorem 1.4(III). By Proposition 3.1, when $p > p_c$, $\bar{\mathcal{C}}_o$ is bounded below by a flbf $f(n)$. As in Corollary 3.2, we construct $y \equiv (0, K_0) \in \mathbf{A}$. So $\{|\mathbf{C}_y| = \infty = |\mathbf{B}_y|\}$ and for $n \in \mathbb{N}$, we may define $\mathbf{a}_n(y) = \sup\{z_n^{[2]} : z_n \in \mathbf{C}_y, |\mathbf{C}_{z_n}| = \infty, z_n^{[1]} = -n\}$. Corollary 3.2 now implies that $f(-n) = \mathbf{a}_n(y)$, so by Corollary 4.4 and translation invariance, we get that $f(-n)/(-n) \to \rho_p$, where ρ_p is as in Proposition 4.1. By that result and Lemma 4.2, $\rho_p \uparrow -1$ as $p \downarrow p_c$. This establishes the desired statements for the northwest boundary. The results for the southeast boundary follow by symmetry. □

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Proof of Theorem 1.4(I). By [6, Theorem 4.9(b)], it simply remains to show that $\mathcal{M}_o$ is a.s. finite when $p > p_c$. Let $p > p_c$. Then as above, $\bar{\mathcal{C}}_o$ is bounded below by f, which satisfies $f(-n)/(-n) \to \rho < -1$, and by symmetry in the model $(\uparrow\!\rightarrow, \leftarrow\!\downarrow)$ also $f(n)/n \to 1/\rho > -1$ for $n \in \mathbb{N}$. Similarly, by Corollary 3.3, $\mathcal{B}_o$ is bounded above by g, and $g(-n)/(-n) \to 1/\rho$ by symmetry in the dual oriented site percolation model. Then also $g(n)/n \to \rho$ by symmetry in the model $(\uparrow\!\rightarrow, \leftarrow\!\downarrow)$. Since $\rho < -1$ it follows that $g(n) < f(n)$ for all but finitely many n, so $\mathcal{M}_o$ is finite. See Figure 1. □

6. Lower bounds on p_c

We say that a path $\vec{x} = x_0, x_1, x_2, \ldots$ is *in* $\mathcal{C}_x$ if $x_0 = x$ and $x_{i+1} - x_i \in \mathcal{G}_{x_i}$ for each $i \in \mathbb{Z}_+$. For $\sigma \in \mathbb{R}$ and $x \in \mathbb{Z}^2$ we write $\mathcal{S}_x^-(\sigma)$ (resp. $\mathcal{S}_x^+(\sigma)$) for the set of self-avoiding walks $\vec{x}$ in $\mathcal{C}_x$ such that $x_n^{[2]}/x_n^{[1]} \to \sigma$ and $x_n^{[1]} \to -\infty$ (resp. $x_n^{[1]} \to +\infty$), and set $\mathcal{S}^-(\sigma) = \mathcal{S}_o^-(\sigma)$ (resp. $\mathcal{S}^+(\sigma) = \mathcal{S}_o^+(\sigma)$). The following lemma gives a strategy for generating better one-sided bounds for the critical point for the orthant model, and hence also for oriented site percolation on the triangular lattice.

Lemma 6.1. *Consider the model* $(\uparrow\!\rightarrow\ \leftarrow\!\downarrow)$, *with* $p \geqslant 0.5$. *Suppose that for some* $\sigma > -1$, *there exists a path* $\vec{x} \in \mathcal{S}^-(\sigma)$ *almost surely. Then* $p \leqslant p_c$.

Proof. Suppose that $p > p_c$. Then $\bar{\mathcal{C}}_o$ is bounded below by f, where $f(-n)/(-n) \to \rho_p < -1$ as in Theorem 1.4(III).

It follows that for any $\sigma > \rho_p$, the set

$$\{x \in \mathcal{C}_o : x_i^{[1]} < 0, x_i^{[2]}/x_i^{[1]} > \sigma\}$$

is finite. Since $\rho_p < -1$, this implies that there can be no infinite path $\vec{x} \in \mathcal{S}^-(\sigma)$ for $\sigma > -1$. □

For the next result we turn our attention to the model $(\uparrow\!\rightarrow, \leftarrow)$, which by coupling can be considered as a subgraph of the model $(\uparrow\!\rightarrow\ \leftarrow\!\downarrow)$ (for each fixed p). In particular, then, any paths present in model $(\uparrow\!\rightarrow, \leftarrow)$ also exist in the model $(\uparrow\!\rightarrow\ \leftarrow\!\downarrow)$.

For $\sigma \in \mathbb{R}$ and $x \in \mathbb{Z}^2$ we write $\mathcal{S}_x(\sigma)$ for the set of self-avoiding paths in $\mathcal{C}_x$ such that $x_n^{[2]} \to \infty$ and $x_n^{[2]}/x_n^{[1]} \to \sigma$ as $n \to \infty$. Let $\mathcal{S}(\sigma) = \mathcal{S}_o(\sigma)$. Note that

$$\mathcal{S}(\sigma) = \begin{cases} \mathcal{S}^-(\sigma) & \text{if } \sigma < 0, \\ \mathcal{S}^+(\sigma) & \text{if } \sigma > 0. \end{cases}$$

Let

$$h_1(p) = \frac{p^2}{1-p} - \frac{1-p}{p} = \frac{p^3 - (1-p)^2}{p(1-p)} = \frac{p^3 - p^2 + 2p - 1}{p(1-p)}.$$

Lemma 6.2. *For the model* $(\uparrow\!\rightarrow, \leftarrow)$, *there exist (self-avoiding) paths* $L_o \in \mathcal{S}(-p/(1-p))$ *and* $R_o \in \mathcal{S}(h_1(p)^{-1})$ *in* $\mathcal{C}_o$. *Moreover, environment-connected components of* $\mathcal{C}_o^c$ *between these two paths are finite.*

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Proof. The asymptotic slope of the nW path (follow $\leftarrow$ when possible and otherwise follow $\uparrow$) from the origin is $-p/(1-p)$. Call this path L_o to establish the first claim. For the second claim, consider the path R_o from the origin that evolves as follows. Whenever the environment at the current location is $\leftarrow$, the path follows this west arrow. Whenever the environment at the current location x is ↑↳, the path follows the east arrow to $x+(1,0)$ if the environment at $x+(1,0)$ is also ↑↳, otherwise the path follows the north arrow. By definition this path never backtracks, so it is self-avoiding. After each northern step taken by the path, the environment thereafter encountered has never been viewed before, hence each northern step constitutes a renewal. Thus the path moves upwards through the set of horizontal bands $\mathbb{Z}\times\{k\}$, $k=0,1,2,\dots$. Let (X_k,k) be the point where our path first enters the kth band. Then we can represent X_{k+1} as follows:

$$X_{k+1} = \sup\{j : \exists \text{ path from } (X_k,k) \text{ to } (j,k+1) \text{ consistent with the environment and lying within the } k\text{th band, except for the final step}\}.$$

Then the $\Delta_k = X_{k+1} - X_k$ are i.i.d. with

$$\mathbb{P}(\Delta_0 = m) = \begin{cases} p^{m+1}(1-p) & \text{if } m \geqslant 0, \\ (1-p)^m p & \text{if } m < 0. \end{cases}$$

It follows that

$$\mathbb{E}[\Delta_0] = \sum_{m=0}^{\infty} mp^{m+1}(1-p) - \sum_{m=1}^{\infty} m(1-p)^m p = \frac{p^2}{1-p} - \frac{1-p}{p} = h_1(p),$$

and that the asymptotic slope of the path R_o is $1/h_1(p)$ as claimed.

By construction, a vertex $(x,n)\in R_o$ never lies strictly to the left of any vertex $(j,n)\in L_o$ (the paths R_o and L_o may meet, but not cross). An elementary comparison of the asymptotic slopes shows that $R_o=\{r_0,r_1,\dots\}$ eventually lies strictly to the right of $L_o=\{l_0,l_1,\dots\}$, in the sense that there exists some m such that for all $n\geqslant m$,

$$\inf\{j : (j,n)\in R_o\} > \sup\{j : (j,n)\in L_o\}.$$

Trivially also by construction $r_n^{[2]}\to\infty$ and similarly for $l_n^{[2]}$. To prove the last claim observe that the nW path from each r_n eventually hits the path L_o (since nW paths from any two vertices eventually meet), and that these nW paths are all in $\mathcal{C}_o$. □

Note that it follows immediately from this result that we can find similar paths (up to symmetry) in $\mathcal{C}_o$ for the models containing this as a submodel. This implies results for other models. For example, it gives the following improvement on the lower bound on p_c in [6, Theorem 4.12] (which we improve further in Theorem 1.5).

Corollary 6.3. *In Model 1.1* (↑↳ ←↓), $\bar{\mathcal{C}}_o=\mathbb{Z}^2$ *for* $p\in[0.5,0.5699)$. *Therefore* $p_c^{↑↳} \geqslant 0.5699$.

Proof. By Lemma 6.2, the model (↑↳ $\leftarrow$) contains a path $R_o\in\mathcal{S}(h_1(p)^{-1})$. By interchanging p and $1-p$, the model ($\leftarrow$ ↑↳) contains a path $R_o'\in\mathcal{S}(h_1(1-p)^{-1})$. Rotating ($\leftarrow$ ↑↳) by $\pi/2$ anticlockwise shows that the model ($\downarrow$ ↲↑) contains a path $R_o''\in\mathcal{S}^-(-h_1(1-p))$.

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Reflecting the $(\downarrow \,{\hookleftarrow}\!\uparrow)$ model about the horizontal axis shows that the model $(\uparrow \,{\hookleftarrow}\!\downarrow)$ contains a path $R_o''' \in \mathcal{S}^-(h_1(1-p))$. Since Model 1.1 contains $(\uparrow \,{\hookleftarrow}\!\downarrow)$, $\mathcal{C}_o$ contains a self-avoiding path $P_1 = R_o''' \in \mathcal{S}^-(h_1(1-p))$. So if $p \geqslant 0.5$ is such that $h_1(1-p) > -1$ then $p \leqslant p_c^{\nearrow}$ by Lemma 6.1 and $\bar{\mathcal{C}}_o = \mathbb{Z}^2$ by Proposition 3.1.

The condition $h_1(1-p) > -1$ holds for $p \in [1/2, p_1)$, where $p_1^3 - p_1^2 + 2p_1 - 1 = 0$, that is, for $p \in [1/2, 0.5699)$ (the condition is actually equivalent to $h_1(p) < 0$ since $h_1(p) + h_1(1-p) = -1$). □

The true value we are aiming for is $p_c^{\nearrow}$, which is estimated as 0.5956. The path R_o above is defined (see Lemma 6.2) in terms of expected horizontal displacements that can be achieved for paths consistent with the environment, staying within a horizontal band of width 1. In fact, there is a sequence of estimates that in principle should converge to the true value. Repeat the above argument, but using bands of width K. The resulting bounds p_K should converge to $p_c^{\nearrow}$. We will content ourselves with computing p_2. The remainder of the paper involves a relatively routine but technical calculation of p_2, which is given purely because it improves the bound on $p_c^{\nearrow}$ to $p_c^{\nearrow} \geqslant 0.5730$ (*i.e.*, the bound appearing in Theorem 1.5).

Proof of Theorem 1.5. In analogy with the proof of Lemma 6.2, we let $X_0 = 0$ and $\Delta_k = X_{k+1} - X_k$, where

$$X_{k+1} = \sup\{j : \exists \text{ path from } (X_k, 2k) \text{ to } (j, 2k+2) \text{ consistent with the environment and lying within the band } \mathbb{Z} \times \{2k, 2k+1\}, \text{ except for the final step}\}.$$

We set $h_2(p) = \mathbb{E}[\Delta_0]/2$ so that the path using horizontal bands has slope $1/h_2(p)$. The strategy of Corollary 6.3 works with h_2 in place of h_1, provided $h_2(1-p) > -1$. Therefore we are left to compute $h_2(p)$. This is done below, ultimately leading to $p_2 = 1 - q$ where

$$q^{11} - 6\,q^{10} + 18\,q^9 - 38\,q^8 + 64\,q^7 - 90\,q^6 + 104\,q^5 - 94\,q^4 + 66\,q^3 - 34\,q^2 + 12\,q - 2 = 0. \tag{6.1}$$

Using Newton's method (with a starting point of $q = 0.4$, in the computer algebra system *Maxima*) gives $p_2 \approx 0.5730$, which completes the proof. In the remainder of this section, we show how to obtain (6.1).

We write $\mathcal{G}_0 = \binom{\mathcal{G}_{(0,1)}}{\mathcal{G}_{(0,0)}}$ for the pair of local environments at $(0,0)$ and $(0,1)$. First consider the case $\mathcal{G}_0 = \binom{\Lsh\!\to}{\Lsh\!\to}$. Our first goal will be to compute

$$\mu_{\substack{\uparrow\to\\ \uparrow\to}} = \mathbb{E}\left[\Delta_0 \mid \mathcal{G}_0 = \begin{pmatrix} \uparrow\to \\ \uparrow\to \end{pmatrix}\right].$$

Paths are not actually unique, but we take the convention that in this situation the path moves $\uparrow$ from $(0,0)$ to $(0,1)$, and then considers its next move. If $\mathcal{G}_1 = \binom{\uparrow\to}{\cdot}$ (the $\cdot$ simply means an arbitrary environment) the path moves $\rightarrow$ to $(1,1)$. If $\mathcal{G}_1 = \binom{\leftarrow\downarrow}{\leftarrow\downarrow}$ then we have hit an impassable obstacle, and the path has no choice but to exit using $\uparrow$, in which case $X_1 = 0$. The remaining possibility is $\mathcal{G}_1 = \binom{\leftarrow\downarrow}{\uparrow\to}$. There could in fact be a sequence of

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such pairs, followed either by a $\begin{pmatrix}\cdot\\ \text{⬐}\end{pmatrix}$ or by $\begin{pmatrix}\text{↳}\\ \text{↳}\end{pmatrix}$. For example,

$$\mathcal{G}_0\mathcal{G}_1\ldots\mathcal{G}_5=\begin{matrix}\text{↳}\text{⬐} & \text{⬐} & \text{⬐} & \text{⬐} & \cdot\\ \text{↳} & \text{↳} & \text{↳} & \text{↳} & \text{↳}\text{⬐}\end{matrix}$$

or

$$\mathcal{G}_0\mathcal{G}_1\ldots\mathcal{G}_5=\begin{matrix}\text{↳}\text{⬐} & \text{⬐} & \text{⬐} & \text{⬐} & \text{↳}\\ \text{↳} & \text{↳} & \text{↳} & \text{↳} & \text{↳} & \text{↳}\end{matrix}$$

The first case also represents an impassable obstacle, so our path chooses to exit using $\uparrow$, making $X_1=0$, In the second case, we move $\rightarrow$, $\downarrow$, take a sequence of $\rightarrow$'s, and then move $\uparrow$ to reach the top row of the band again.

This description makes it clear that there is a renewal structure here, with the construction starting afresh every time we reach a new ↳ environment on the top row of the band. To formalize this, let $(Z_j,1)$ be the site of the jth such new ↳ environment reached by our path (with $Z_0=0$ corresponding to the initial environment). If J denotes the total number of such environments reached, then we take $Z_j=Z_J$, $\forall j\geqslant J$. The dynamics are that

$$Z_{j+1}=\begin{cases}Z_j+1 & \text{if } \mathcal{G}_{Z_j+1}=\begin{pmatrix}\text{↳}\\ \cdot\end{pmatrix}\\[2ex] Z_j+i & \text{if } \mathcal{G}_{Z_j+1}\ldots\mathcal{G}_{Z_j+i}=\begin{matrix}\text{⬐} & \ldots\text{⬐} & \text{↳}\\ \text{↳} & \ldots\ \text{↳} & \text{↳}\end{matrix}\\[2ex] Z_j & \text{if } \mathcal{G}_{Z_j+1}=\begin{pmatrix}\text{⬐}\\ \text{⬐}\end{pmatrix} \text{ or } \mathcal{G}_{Z_j+1}\ldots=\begin{matrix}\text{⬐} & \ldots\text{⬐} & \cdot\\ \text{↳} & \ldots\ \text{↳}\text{⬐}\end{matrix}\end{cases}$$

Let $\eta=p(1-p)$ and

$$\alpha=\frac{p^2}{p^2+1-p}=\frac{p^2}{1-p(1-p)}=\frac{p^2}{1-\eta}$$

denote the probability of encountering a $\begin{pmatrix}\text{↳}\\ \text{↳}\end{pmatrix}$ before a $\begin{pmatrix}\cdot\\ \text{⬐}\end{pmatrix}$. Let $\theta=p+p(1-p)\alpha$ denote the probability that $Z_1>Z_0$. Then

$$\mathbb{E}[Z_1]=p+p(1-p)\sum_{i=1}^{\infty}(i+1)[p(1-p)]^{i-1}p^2=\theta+\frac{\alpha p(1-p)}{1-p(1-p)}$$

and the renewal structure implies that

$$\mu_{\substack{\text{↳}\\ \text{↳}}}=\mathbb{E}\left[\sum_{j=0}^{\infty}[Z_{j+1}-Z_j]1_{j\leqslant J}\right]=\sum_{j=0}^{\infty}\mathbb{E}[Z_1]\theta^j=\frac{\mathbb{E}[Z_1]}{1-\theta}.$$

The second case we consider is that $\mathcal{G}_0=\begin{pmatrix}\text{⬐}\\ \text{↳}\end{pmatrix}$, so our next goal is to compute

$$\mu_{\substack{\text{⬐}\\ \text{↳}}}=\mathbb{E}\left[\Delta_0\mid\mathcal{G}_0=\begin{pmatrix}\text{⬐}\\ \text{↳}\end{pmatrix}\right].$$

If

$$\mathcal{G}_0\ldots\mathcal{G}_j=\begin{matrix}\text{⬐} & \ldots\text{⬐} & \text{↳}\\ \text{↳} & \ldots\ \text{↳} & \text{↳}\end{matrix}$$

then we travel j steps $\rightarrow$ and then $\uparrow$, and find ourselves back in the situation just considered. While if

$$\mathcal{G}_0\mathcal{G}_1\ldots = \begin{smallmatrix} {\leftarrow\!\downarrow} & \cdots & {\leftarrow\!\downarrow} & \\ {\uparrow\!\rightarrow} & \cdots & {\uparrow\!\rightarrow} & {\leftarrow\!\downarrow} \end{smallmatrix}\,.$$

then we are blocked to the right, and instead travel $\uparrow$ and then follow $\leftarrow$'s till reaching a ${\uparrow\!\rightarrow}$ on the top row of the band, at which point we exit from the band via that $\uparrow$. This gives us the expression

$$\begin{aligned}\mu_{\substack{\leftarrow\downarrow\\ \uparrow\rightarrow}} &= \sum_{j=1}^{\infty} j[p(1-p)]^{j-1}p^2 + \alpha\mu_{\substack{\uparrow\rightarrow\\ \uparrow\rightarrow}} - (1-\alpha)\sum_{i=1}^{\infty} i(1-p)^{i-1}p \\ &= \alpha\mu_{\substack{\uparrow\rightarrow\\ \uparrow\rightarrow}} + \frac{\alpha}{1-p(1-p)} - \frac{1-\alpha}{p}.\end{aligned}$$

The third case of interest is

$$\mu_{\substack{\leftarrow\downarrow\\ \leftarrow\downarrow}} = \mathbb{E}\left[\Delta_0 \mid \mathcal{G}_0 = \begin{pmatrix} \leftarrow\downarrow \\ \leftarrow\downarrow \end{pmatrix}\right].$$

We start out by following $\leftarrow$ along the bottom row, till reaching a ${\uparrow\!\rightarrow}$ site, at which point we go $\uparrow$ to the top row. If the site so reached is a ${\leftarrow\!\downarrow}$ then we again proceed $\leftarrow$ till reaching a ${\uparrow\!\rightarrow}$ on the top row, at which point we exit from the band via the $\uparrow$. On the other hand, if the first site reached in the top row is a ${\uparrow\!\rightarrow}$ then we have the opportunity to regain some lost ground. We step $\rightarrow$ along the top row as long as possible, and only go $\uparrow$ just before reaching a ${\leftarrow\!\downarrow}$. For example,

$$\mathcal{G}_{-5}\mathcal{G}_{-4}\mathcal{G}_{-3}\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0 = \begin{smallmatrix} {\uparrow\!\rightarrow} & {\uparrow\!\rightarrow} & {\uparrow\!\rightarrow} & {\uparrow\!\rightarrow} & {\leftarrow\!\downarrow} & {\leftarrow\!\downarrow} \\ {\uparrow\!\rightarrow} & {\leftarrow\!\downarrow} & {\leftarrow\!\downarrow} & {\leftarrow\!\downarrow} & {\leftarrow\!\downarrow} & {\leftarrow\!\downarrow} \end{smallmatrix}$$

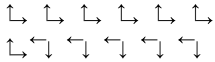
has $X_1 = -2$. This leads to the general expression

$$\begin{aligned}\mu_{\substack{\leftarrow\downarrow\\ \leftarrow\downarrow}} &= \sum_{k=1}^{\infty}(1-p)^{k-1}p\left[-k - \sum_{j=1}^{\infty} j(1-p)^j p + \sum_{j=0}^{k-2} jp^{j+1}(1-p) + (k-1)p^k\right] \\ &= -\frac{1}{p} - \frac{1-p}{p} + \sum_{j=0}^{\infty} jp^{j+2}\sum_{k=j+2}^{\infty}(1-p)^k + \sum_{k=1}^{\infty}(k-1)p^2[p(1-p)]^{k-1} \\ &= -\frac{2-p}{p} + \frac{\alpha(1-p)}{1-p(1-p)}[p+(1-p)^2].\end{aligned}$$

The fourth and final case of interest is

$$\mu_{\substack{\uparrow\rightarrow\\ \leftarrow\downarrow}} = \mathbb{E}\left[\Delta_0 \mid \mathcal{G}_0 = \begin{pmatrix} \uparrow\rightarrow \\ \leftarrow\downarrow \end{pmatrix}\right].$$

Once again, we must go $\leftarrow$ till reaching the first ${\uparrow\!\rightarrow}$ on the bottom row of the band, and then go $\uparrow$. If this leads to a ${\leftarrow\!\downarrow}$ vertex, then all we can do is head $\leftarrow$ on the top row, till reaching a ${\uparrow\!\rightarrow}$ vertex, at which point we may exit via $\uparrow$. However, if we reach the top row at a ${\uparrow\!\rightarrow}$ vertex then we can regain lost ground by heading $\rightarrow$. Either this ends as in the previous case, before making it all the way back to $(0,1)$. Or we do make it back to $(0,1)$ this way, in which case we find ourselves back in the first case examined above. For example,

$$\mathcal{G}_{-5}\mathcal{G}_{-4}\mathcal{G}_{-3}\mathcal{G}_{-2}\mathcal{G}_{-1}\mathcal{G}_0 =$$ 

has $X_1 \geqslant 0$. This leads to the general expression

$$\mu^{\Lsh}_{\Rsh} = \sum_{k=1}^{\infty}(1-p)^{k-1}p\Big[-k - \sum_{j=1}^{\infty} j(1-p)^j p + \sum_{j=0}^{k-2} jp^{j+1}(1-p) + \big(k + \mu^{\uparrow\rightarrow}_{\rightarrow}\big)p^k\Big]$$
$$= -\frac{2-p}{p} + \frac{\alpha}{1-p(1-p)}[1+(1-p)^3] + \alpha\mu^{\uparrow\rightarrow}_{\rightarrow}.$$

Thus, $2h_2(p)$ is equal to

$$\mathbb{E}[\Delta_0] = p^2\mu^{\uparrow\rightarrow}_{\rightarrow} + p(1-p)\mu^{\leftarrow\downarrow}_{\rightarrow} + p(1-p)\mu^{\Lsh}_{\Rsh} + (1-p)^2\mu^{\leftarrow\downarrow}_{\leftarrow}. \tag{6.2}$$

This gives us that

$$h_2(p) = \frac{p^{11} - 6\,p^{10} + 16\,p^9 - 30\,p^8 + 46\,p^7 - 62\,p^6 + 72\,p^5 - 66\,p^4 + 48\,p^3 - 26\,p^2 + 10\,p - 2}{2\,p^9 - 8\,p^8 + 18\,p^7 - 28\,p^6 + 32\,p^5 - 28\,p^4 + 18\,p^3 - 8\,p^2 + 2\,p}.$$

Here the denominator is equal to

$$2\,(p-1)^2\,p\,(p^2+1)\,(p^2-p+1)^2.$$

Solving for $h_2(1-p) = -1$ is equivalent to finding $p = 1-q$, where

$$q^{11} - 6\,q^{10} + 18\,q^9 - 38\,q^8 + 64\,q^7 - 90\,q^6 + 104\,q^5 - 94\,q^4 + 66\,q^3 - 34\,q^2 + 12\,q - 2 = 0,$$

as claimed. □

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