THE RESERVE UNCERTAINTIES IN THE CHAIN LADDER MODEL OF MACK REVISITED

BY

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ABSTRACT

We revisit the “full picture” of the claims development uncertainty in Mack’s (1993) distribution-free stochastic chain ladder model. We derive the uncertainty estimators in a new and easily understandable way, which is much simpler than the derivation found so far in the literature, and compare them with the well known estimators of Mack and of Merz–Wüthrich.

Our uncertainty estimators of the one-year run-off risks are new and different to the Merz–Wüthrich formulas. But if we approximate our estimators by a first order Taylor expansion, we obtain equivalent but simpler formulas. As regards the ultimate run-off risk, we obtain the same formulas as Mack for single accident years and an equivalent but better interpretable formula for the total over all accident years.

KEYWORDS

claims reserving, distribution free chain-ladder model, Bayesian chain-ladder model, conditional mean square error of prediction, ultimate run-off uncertainty, one-year run-off uncertainties, Mack’s formula, Wüthrich–Merz formulas, cost of capital loading, market value margin.

1. INTRODUCTION

In the last decades many papers on stochastic claims reserving have been and still are published (see for instance Merz and Wüthrich (2008a) and the continuing flow of publications in the recent actuarial literature and in neighbouring fields like Harnau and Nielsen (2017)). Moreover machine learning algorithms seem to become popular in actuarial science (see the many presentations at the ICA 2018 in Berlin).

Despite this development, chain ladder (CL) and Bornhuetter Ferguson (BF) methods are still the most used and the most popular methods in the

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insurance practice. One of the reason is that in practice one has to be *sufficiently accurate*, but it is not necessary to always use the "theoretically optimal" model. A methodology whose mechanism is understood and which is sufficiently accurate is mostly preferred to a highly sophisticated methodology whose mechanism is difficult to understand and to see through. In our opinion, which is based on many years of practical experience, the BF and the CL methods are in many situations sufficiently accurate and, therefore, still useful methods. In the following we concentrate on the CL method.

Assume we are currently at the end of calendar year I. Under *run-off risk* we understand the *risk of an adverse claims development.* Thereby we distinguish between *the ultimate run-off risk* Z^{ult}, which is the deviation of the ultimate claim from its forecast at the end of calendar year I, and the *one-year run-off risk* $Z^{(I+k)}$ in accounting year $I+k$, which is the deviation of the forecast of the ultimate claim at the end of the accounting year $I+k$ from its forecast at the beginning of the accounting year $I+k$. The latter is reflected in the P&L account of the accounting year $I+k$ by the so called *claims development result (CDR)*.

As common in the actuarial literature, we will take *the conditional mean square error of prediction (msep)* as a measure for the reserve uncertainty. For best estimate reserves it is by definition the conditional expected value of the square of the run-off risk.

In 1993 Mack (1993) presented a stochastic CL-model and derived a formula to estimate the msep of the *ultimate run-off risk*. This formula is well known and widely used in the insurance practice. With the emergence of the new solvency regulation (Swiss Solvency Test and Solvency II) there arose the need to assess another kind of reserve risk. The risk considered is the change of the risk bearing capital within the next accounting year (one-year time horizon). Hence the *relevant reserve risk for solvency purposes* is the *one-year run-off risk in the next accounting year*. A formula for estimating the corresponding msep was first published by Merz and Wüthrich in 2008.

As market consistent valuation is a basic element of the new solvency regulation and of IFRS 17, the best estimate reserves have to be complemented by a *market value margin* corresponding to the discounted costs of the risk capital needed for the run-off until the final settlement of all claims. For this purpose, one also needs estimators of the msep of the *one-year run-off risk in later accounting years* until the end of the claims development. Merz and Wüthrich (2014) considered the estimation of the msep of the ultimate run-off risk as well the estimation of the msep of the one-year run-off risks of all future accounting years ("full picture") within a specific Bayesian CL-model, the so called Gamma–Gamma BCL model, which is similar to a model considered in Gisler (2006).

However, the Bayesian CL-model is different to the distribution free CL-model of Mack. For instance, in the model considered in Merz and Wüthrich (2014) the msep does only exist if the observed triangle fulfils specific conditions (see Merz and Wüthrich (2014), assumptions in Theorem 3.8). Thus we do not know, whether the results derived in Merz and Wüthrich (2014) also apply to the classical Mack model.

Formulas for the full picture of the CL reserve uncertainties derived strictly within the classical CL-model of Mack (Mack (1993)) were first published in 2016 in three different papers (Diers *et al.* (2016), Gisler (2016), Röhr (2016)). Gisler (2016) is an earlier version of the present paper. In Diers *et al.* (2016) the authors estimate the msep by the use of bootstrap-techniques similar to the bootstrap approach in Buchwalder *et al.* (2006), which gave rise to several controversial discussions (see Buchwalder *et al.* (2006), Gisler (2006), Mack *et al.* (2006)). In Röhr (2016) the author rewrites the msep of the one-year and the ultimate run-off risks as a function of the individual future CL development factors, takes its first order Taylor approximation at their predicted values and derives estimators of this "linearized msep". The formulas derived in Röhr (2016) and in Diers *et al.* (2016) are equivalent to the Mack and the Merz–Wüthrich formulas.

In the present paper we derive the msep estimators in a new way. Different from Röhr we consider the msep of the run-off risks and not a linearized approximation and different from Diers *et al.* we do not use any bootstrap technique. For the one-year run-off risk, the formulas obtained are new and different to the Merz–Wüthrich formulas and to the ones in Röhr (2016) and Diers *et al.* (2016). But if we approximate our formulas by a first order Taylor expansion, we find equivalent but simpler formulas than the ones in Merz and Wüthrich (2008b) and in Merz and Wüthrich (2014). For the ultimate run-off risk, and comparing with Mack (1993), we obtain the same formulas for single accident years and an equivalent but easier interpretable formula for the total over all accident years.

In our opinion, the derivation of the results in the above mentioned recent papers (Merz and Wüthrich (2014), Diers *et al.* (2016), Röhr (2016)) as well as in Merz and Wüthrich (2008b) are rather difficult to follow, whereas the present paper is very simple and easy to understand. We only need straightforward mathematics and can see behind the formulas, as they can be interpreted in an intuitively comprehensible way. This simplicity is perhaps the biggest merit of the present paper.

Organisation of the paper. In Section 2 we introduce some notation and the data structure. In Section 3 we review the CL-reserving method and the stochastic CL-model of Mack. At the end of this section a natural estimation principle and the so called telescope formula are presented. In Section 4 we consider the one-year run-off uncertainties for all future accounting years until the end of the claims development and compare the results with the Merz–Wüthrich formulas . In Section 5 we derive the formulas for the ultimate run-off uncertainty and compare them with the Mack-formula. In Section 6 we briefly consider the relationship between the one-year and the ultimate run-off risks. Finally, a numerical example is presented in Section 7.

2. NOTATION AND DATA STRUCTURE

The starting point of claims reserving are cumulative claim figures (usually claim payments or incurred losses) $C_{i,j} > 0$ from accident periods $i = i_0, \ldots, I$ at the end of development periods $j = j_0, \ldots, J$ arranged in a table with i on the vertical axes and j on the horizontal axes. We assume that all claims are settled at the end of development year J and that therefore $C_{i,J}$ denotes the ultimate claim of accident year i. We further assume that the number of accident years is greater or equal than the number of development years, that is $J - j_0 \geq I - i_0$. The index j_0 is introduced because in the actuarial literature the first development year is sometimes denoted by zero and sometimes by 1. Hence j_0 is either zero or one.

At the end of accident year I the data

$$\mathcal{D}_I = \left\{ C_{i,j} : i + j \leq j_0 + I \right\}$$

are known. We will call $\mathcal{D}_I$ a claims development triangle also in the case where the shape is a trapezoid. Our aim is to forecast

$$\mathcal{D}_I^c = \left\{ C_{i,j} : i + j > j_0 + I, j \leq J \right\}.$$

Some notations:

- diagonal functions

  Definition 2.1. *diagonal functions*

$$j_i := \max\{j \text{ such that } C_{i,j} \in \mathcal{D}_I\}, \tag{2.1}$$
$$i_j := \max\{i \text{ such that } C_{i,j} \in \mathcal{D}_I\}. \tag{2.2}$$

  They were already introduced in Röhr (2016) and are convenient to simplify notation. Note that C_{i,j_i} is the diagonal element in row i and that $C_{i_j,j}$ is the diagonal element in column j. Note also that the accident years $i = i_0, \ldots, i_J$ are already fully developed.

  Regardless of the above definition, i_0 and j_0 also denote the first accident and the first development year, what might formally conflict with Definition 2.1. This conflicting notation is consciously taken into account, as the interpretation throughout the paper is always clear: j_i and i_j always refer to Definition 2.1, whereas j_0 and i_0 always denote the first development year and the first accident year respectively.
- We denote the set of observations known at the end of development year j by

$$\mathcal{B}_j := \{C_{i,k} : C_{i,k} \in \mathcal{D}_I,\ k \leq j\}. \tag{2.3}$$

- coefficient of variation

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We denote the coefficient of variation of a random variable (r.v.) X by

$$\mathrm{CoV}\,(X) := \frac{\sqrt{\mathrm{Var}\,(X)}}{E\,[X]},$$

provided the moments exist and $E\,[X] > 0$.

- In this paper empty sums and empty products are defined by

$$\sum\nolimits_{k=l}^{u} x_k := 0 \ \text{ if } u < l,$$

$$\prod\nolimits_{k=l}^{u} x_k := 1 \ \text{ if } u < l.$$

- The following weights will be used later in the paper.

Definition 2.2. *The weights $w_{i,j}$ are defined by*

$$w_{i,j} := \begin{cases} C_{i,j}, & \text{if } \ C_{i,j} \in \mathcal{D}_I, \\ \widehat{C}_{i,j}^{CL}, & \text{otherwise,} \end{cases} \tag{2.4}$$

where $\widehat{C}_{i,j}^{CL}$ is the CL-forecast of $C_{i,j}$ given by Equation (3.3).

- The index "tot" denotes summation over all accident years. For example

$$w_{tot,j} := \sum\nolimits_{i=i_0}^{I} w_{i,j}.$$

3. THE CHAIN LADDER METHOD

3.1. The chain ladder method

The CL method is a pragmatic method, which has been used for decades for estimating reserves. The basic assumption behind CL is that the columns in the development triangle are proportional to each other up to random fluctuations, i.e. there exist constants f_j, $j = j_0, \ldots, J-1$, such that

$$C_{i,j+1} \approx f_j C_{i,j}. \tag{3.1}$$

The constants f_j are called claims-development factors, CL factors or age to age factors. Given the information $\mathcal{D}_I$ it is natural to estimate these unknown constants by

$$\widehat{f}_j^{CL} = \frac{\sum_{i=i_0}^{i_j-1} C_{i,j+1}}{\sum_{i=i_0}^{i_j-1} C_{i,j}}. \tag{3.2}$$

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Due to Equation (3.1) the $C_{i,j} \in \mathcal{D}_I^c$ are forecasted by

$$\widehat{C}_{i,j}^{CL} = C_{i,j_i} \prod_{k=j_i}^{j-1} \widehat{f}_k^{CL}. \tag{3.3}$$

The CL reserve $\widehat{R}_i^{CL}$ of accident year i is an estimate of the *outstanding liabilities,* i.e.

$$\widehat{R}_i^{CL} = \widehat{C}_{i,J}^{CL} - C_{i,j_i}. \tag{3.4}$$

3.2. The stochastic CL-model of Mack

The following distribution-free stochastic model underlying the CL method was presented in Mack (1993).

Model Assumptions 3.1 (Mack-model)

M1 *$C_{i,j}$ belonging to different accident years are independent.*
M2 *There exist positive parameters $f_{j_0}, \ldots, f_{J-1}$ and $\sigma_{j_0}^2, \ldots, \sigma_{J-1}^2$ such that for $i = i_0, \ldots, I$, and $j = j_0, \ldots, J-1$*

$$E[C_{i,j+1} | C_{i,j_0}, \ldots, C_{i,j}] = f_j C_{i,j}, \tag{3.5}$$

$$\text{Var}(C_{i,j+1} | C_{i,j_0}, \ldots, C_{i,j}) = \sigma_j^2 C_{i,j}. \tag{3.6}$$

An interpretation of σ_j^2 can be found in the remarks after Properties 3.2.

As an example let us consider a claims development triangle of private liability insurance. For confidentiality reasons the original data were multiplied by a constant factor. Table 1 shows the observed cumulative claim-payments $C_{i,j}$ in the upper left part and the corresponding CL forecasts in the lower right part together with the resulting CL-reserves.

How accurate are these reserves? The future outcomes of $C_{i,j}$ in the lower right part will deviate from these forecasts. However, when looking at this table we cannot gain a feeling about the expected range of deviation and about the reserve uncertainties.

Table 2 shows in the upper left part the triangle of the observed *individual CL ratios* defined by

$$F_{i,j} := \frac{C_{i,j+1}}{C_{i,j}}. \tag{3.7}$$

In each column j the not yet observed $\{F_{i,j} : i = i_j, \ldots, I\}$ in the lower right part are forecasted by $\widehat{f}_j^{CL}$. Again the future realisations $F_{i,j}$ will deviate from these forecasts. But contrary to Table 1 we can get a feeling about the expected range of deviation when looking at Table 2. When applying CL, therefore, one should always consider the individual CL-factors.

TABLE 1

TRIANGLE OF CUMULATIVE PAYMENTS $C_{i,j}$ AND CL-FORECASTS

	0	1	2	3	4	5	6	7	8	9	10	**Reserves**
2004	1'454	2'240	2'354	2'409	2'448	2'485	2'499	2'601	2'661	2'670	2'670	
2005	1'346	2'094	2'210	2'291	2'353	2'380	2'390	2'397	2'398	2'424	2'425	
2006	1'306	2'110	2'269	2'355	2'386	2'412	2'445	2'456	2'456	2'482	2'519	
2007	1'309	2'035	2'178	2'210	2'263	2'277	2'290	2'292	2'314	2'353	2'353	
2008	1'413	2'276	2'399	2'471	2'512	2'533	2'562	2'600	2'659	2'659	2'669	10
2009	1'448	2'203	2'384	2'505	2'517	2'523	2'531	2'538	2'542	2'562	2'572	30
2010	1'458	2'363	2'574	2'621	2'668	2'678	2'706	2'711	2'738	2'760	2'770	59
2011	1'564	2'284	2'404	2'473	2'561	2'584	2'589	2'615	2'640	2'661	2'672	83
2012	1'648	2'542	2'683	2'737	2'760	2'782	2'802	2'829	2'857	2'880	2'891	109
2013	1'675	2'570	2'786	2'892	2'948	2'972	2'993	3'023	3'053	3'077	3'089	141
2014	1'693	2'529	2'670	2'740	2'790	2'813	2'833	2'860	2'889	2'912	2'923	183
2015	1'667	2'529	2'670	2'749	2'798	2'822	2'841	2'870	2'898	2'921	2'932	262
2016	1'805	2'736	2'914	3'000	3'054	3'079	3'101	3'132	3'162	3'188	3'200	464
2017	1'973	3'042	3'240	3'336	3'396	3'424	3'448	3'482	3'517	3'545	3'558	1'585
Total												**2'926**
f_j^{CL}	1.542	1.065	1.029	1.018	1.008	1.007	1.010	1.010	1.008	1.004		

TABLE 2

TRIANGLE OF OBSERVED CL-RATIOS $F_{i,j}$ AND CL-FORECASTS f_j^{CL}

	0	1	2	3	4	5	6	7	8	9
2004	1.541	1.051	1.023	1.016	1.015	1.006	1.041	1.023	1.003	1.000
2005	1.556	1.055	1.037	1.027	1.011	1.004	1.003	1.000	1.011	1.000
2006	1.616	1.075	1.038	1.013	1.011	1.014	1.004	1.000	1.011	1.015
2007	1.555	1.070	1.015	1.024	1.006	1.006	1.001	1.010	1.017	1.000
2008	1.611	1.054	1.030	1.017	1.008	1.011	1.015	1.023	1.000	1.004
2009	1.521	1.082	1.051	1.005	1.002	1.003	1.003	1.002	1.008	1.004
2010	1.621	1.089	1.018	1.018	1.004	1.010	1.002	1.010	1.008	1.004
2011	1.460	1.053	1.029	1.036	1.009	1.002	1.010	1.010	1.008	1.004
2012	1.542	1.055	1.020	1.008	1.008	1.007	1.010	1.010	1.008	1.004
2013	1.534	1.084	1.038	1.019	1.008	1.007	1.010	1.010	1.008	1.004
2014	1.494	1.056	1.026	1.018	1.008	1.007	1.010	1.010	1.008	1.004
2015	1.517	1.056	1.029	1.018	1.008	1.007	1.010	1.010	1.008	1.004
2016	1.516	1.065	1.029	1.018	1.008	1.007	1.010	1.010	1.008	1.004
f_j^{CL}	**1.542**	**1.065**	**1.029**	**1.018**	**1.008**	**1.007**	**1.010**	**1.010**	**1.008**	**1.004**

It is well known (see for instance Mack (1993)) that Model Assumptions 3.1 imply the following properties.

Properties 3.2.

i) The estimator (3.2) can be written as a weighted mean

$$\widehat{f}_j^{CL} = \sum_{i=i_0}^{i_j-1} \frac{w_{i,j}}{w_{\bullet,j}} F_{i,j}, \tag{3.8}$$

where $w_{\bullet,j} = \sum_{i=i_0}^{i_j-1} w_{i,j}$ and where the $w_{i,j}$ are defined in Equation (2.4).

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ii)

$$E[F_{i,j}|C_{i,j_0},\dots,C_{i,j}] = E[F_{i,j}|C_{i,j}] = f_j, \tag{3.9}$$

$$\mathrm{Var}(F_{i,j}|C_{i,j_0},\dots,C_{i,j}) = \mathrm{Var}(F_{i,j}|C_{i,j}) = \frac{\sigma_j^2}{C_{i,j}}, \tag{3.10}$$

$$\mathrm{CoV}\left(F_{i,j}\middle| C_{i,j_0},\dots,C_{i,j}\right) = \mathrm{CoV}\left(F_{i,j}\middle| C_{i,j}\right) = \sqrt{\frac{\sigma_j^2}{f_j^2}\frac{1}{C_{i,j}}}. \tag{3.11}$$

iii) Conditional on $\mathcal{B}_j$ it holds that

$$E\left[\widehat{f}_j^{CL}\middle| \mathcal{B}_j\right] = f_j, \tag{3.12}$$

$$\mathrm{Var}\left(\widehat{f}_j^{CL}\middle| \mathcal{B}_j\right) = \frac{\sigma_j^2}{\sum_{i=i_0}^{i_j-1} w_{i,j}}, \tag{3.13}$$

$$\mathrm{CoV}\left(\widehat{f}_j^{CL}\middle| \mathcal{B}_j\right) = \sqrt{\frac{\sigma_j^2}{f_j^2}\frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}}}. \tag{3.14}$$

iv) σ_j^2 can be estimated by the unbiased estimator

$$\widehat{\sigma}_j^2 = \frac{1}{i_j - i_0 - 1}\sum_{i=i_0}^{i_j-1} w_{i,j}\left(F_{i,j} - \widehat{f}_j^{CL}\right)^2, \ \text{if } i_j - i_0 \geqq 2. \tag{3.15}$$

If $i_{J-1} - i_0 < 2$ we can estimate $\widehat{\sigma}_{J-1}^2$ as suggested in Mack (1993) by

$$\widehat{\sigma}_{J-1}^2 = \min\left(\widehat{\sigma}_{J-2}^4/\widehat{\sigma}_{J-3}^2, \min\left(\widehat{\sigma}_{J-2}^2, \widehat{\sigma}_{J-3}^2\right)\right).$$

v)

$$E\left[F_{i,k}\middle| \mathcal{B}_j\right] = f_k \quad \text{for } j \le k \le J-1. \tag{3.16}$$

vi)

$$\left\{F_{i,k}\middle| \mathcal{B}_j : k = j,\dots,J-1\right\} \ \text{are uncorrelated.} \tag{3.17}$$

Remarks

- Note that $C_{i,j}$ in the denominator of the right hand side of Equation (3.10) is a known constant and plays the role of a weight. Thus (3.10) is analogous to the variance condition in a weighted regression or in credibility. Analogously as in weighted regression and in credibility σ_j^2 can be interpreted as the conditional variance of $F_{i,j}$ normalized for weight one. The quantities $C_{i,j}$ are in most cases monetary amounts, and as such have a dimension, as for instance CHF. The $F_{i,j}$ have no dimension, and hence the left hand side of Equation (3.10) has no dimension either. Therefore the σ_j^2 on the right hand side must have the same dimension as $C_{i,j}$.

- As already mentioned in Röhr (2016), the coefficients of variation (3.11) and (3.14) can be estimated by

$$\widehat{\mathrm{CoV}}\left(F_{i,j}\middle|\, C_{i,j}\right) = \sqrt{\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\frac{1}{w_{i,j}}}, \tag{3.18}$$

$$\widehat{\mathrm{CoV}}\left(\widehat{f}_j^{CL}\middle|\, \mathcal{B}_j\right) = \sqrt{\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\frac{1}{\sum\limits_{i=i_0}^{i_j-1} w_{i,j},}}. \tag{3.19}$$

- They are intuitively comprehensible "uncertainty measures" for the relative deviation of $F_{i,j}$ and of $\widehat{f}_j^{CL}$ from the "true" CL-factors f_j.

3.3. Estimation principle

The msep contain by definition the unknown CL factors f_j. To find an estimator of the msep, we have to replace them by suitable estimators. The following estimation principle is commonly used in the relevant actuarial literature as for instance in Mack (1993), Röhr (2016), Diers *et al.* (2016), Buchwalder *et al.* (2006) and in many other papers.

Estimation Principle 3.3

a) quadratic difference terms

$$\textit{estimator of } \left(\widehat{f}_j^{CL} - f_j\right)^2 = \widehat{\mathrm{Var}}\left(\widehat{f}_j^{CL}\middle|\, \mathcal{B}_j\right) = \frac{\widehat{\sigma}_j^2}{\sum_{i=i_0}^{i_j-1} w_{i,j}}. \tag{3.20}$$

b) other functions of f_j such as $\prod\limits_{j=j_i}^{J-1} f_j^2$ are estimated by replacing the unknown parameters f_j by $\widehat{f}_j^{CL}$.

Remarks

- Even though the estimation principle 3.3 is commonly used, we should be aware that it is not simply the plug-in technique found in statistics and that it is not well defined because of (3.20). For instance

$$f_j^2 = \left(f_j + \widehat{f}_j^{CL} - f_j\right)^2 = f_j^2 + 2f_j\left(\widehat{f}_j^{CL} - f_j\right) + \left(\widehat{f}_j^{CL} - f_j\right)^2.$$

If we apply the estimation principle 3.3 to the left hand side, we estimate f_j^2 by $\left(\widehat{f}_j^{CL}\right)^2$, and if we apply it to the right hand side, we estimate f_j^2 by $\left(\widehat{f}_j^{CL}\right)^2 + \widehat{\sigma}_j^2 / \sum_{i=i_0}^{i_j-1} w_{i,j}$. Thus we have to use the estimation principle in an appropriate way. In the above example we estimate f_j^2 by $\left(\widehat{f}_j^{CL}\right)^2$, because

$$E\left[\left(\widehat{f}_j^{CL}\right)^2 \middle| \mathcal{B}_j\right] = f_j^2 + \sigma_j^2 / \sum\nolimits_{i=i_0}^{i_j-1} w_{i,j}$$

is upward biased and because this upward bias would even be increased if the estimator $\left(\widehat{f}_j^{CL}\right)^2 + \widehat{\sigma}_j^2 / \sum\nolimits_{i=i_0}^{i_j-1} w_{i,j}$ was used.

3.4. The telescope formula

The following formula will be used in the later sections.

Lemma 3.4 (telescope formula). *For any real numbers x_j and y_j, $j = 1, 2, \ldots, J$, it holds that*

$$\prod\nolimits_{j=1}^{J} x_j - \prod\nolimits_{j=1}^{J} y_i = \sum_{j=1}^{J} \left(\prod\nolimits_{k=1}^{j-1} x_k\right) (x_j - y_j) \left(\prod\nolimits_{m=j+1}^{J} y_m\right). \tag{3.21}$$

Proof. This result is well known. We show it for a product with $J = 3$. The extension to any number J is self evident.

$$\begin{aligned} x_1x_2x_3 - y_1y_2y_3 = & \; x_1x_2x_3 - x_1x_2y_3 + x_1x_2y_3 - x_1y_2y_3 + x_1y_2y_3 - y_1y_2y_3 \\ = & \; (x_1 - y_1)\, y_2y_3 + x_1\,(x_2 - y_2)\, y_3 + x_1x_2\,(x_3 - y_3)\,. \end{aligned}$$

□

4. THE ONE-YEAR RUN-OFF PREDICTION UNCERTAINTY

In the new solvency regulation (SST, Solvency II) a time horizon of one year is considered. Therefore the reserve risk relevant for solvency purposes is the one-year run-off risk in the next accounting year. Moreover, the best estimate reserves have to be augmented by a market value margin corresponding to the discounted costs of capital needed for the run-off until final settlement. For these two purposes we need estimates of the msep of the one year run-off risks of all future accounting years $I + k$, $k = 1, \ldots, J - j_0$.

At the end of accounting year $I + k$ there will be available the data $\mathcal{D}_{I+k}$. The CL-factors and the prediction of the ultimate claim will then be made on the basis of $\mathcal{D}_{I+k}$.

As in the previous section we denote by $\widehat{f}_j^{CL}$ the estimated CL factors and by $w_{i,j} = \widehat{C}_{i,j}^{CL}$ the CL forecasts of $C_{i,j}$ at time I. For future accounting years $I + k$, $k = 1, \ldots, J - j_0$, we will use the following notation:

$\mathcal{D}_{I+k}$:= the claims development triangle available at the end of accounting year $I + k$,

$\widehat{f}_j^{CL(I+k)}$, $\widehat{C}_{i,J}^{CL(I+k)}$:= estimated CL factors and CL forecasts at time $I + k$ based on $\mathcal{D}_{I+k}$.

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From the perspective at time I, $\widehat{f}_j^{CL(I+k)}$ and $\widehat{C}_{i,J}^{CL(I+k)}$ are r.v., whereas $\widehat{f}_j^{CL}$ and $\widehat{C}_{i,j}^{CL}$ are known constants.

The one-year run-off risk in accounting year $I+k+1$, which will be reflected by the *claims development result* in the profit and loss account of accounting year $I+k+1$, is defined by

$$Z_{\text{tot}}^{(I+k+1)} := \widehat{C}_{\text{tot},J}^{CL(I+k+1)} - \widehat{C}_{\text{tot},J}^{CL(I+k)} \text{ for the total over all accident years,} \quad (4.1)$$

$$Z_{i}^{(I+k+1)} := \widehat{C}_{i,J}^{CL(I+k+1)} - \widehat{C}_{i,J}^{CL(I+k)} \text{ for accident years } i = i_J + k + 1, \ldots, I. \quad (4.2)$$

4.1. The one-year run-off prediction uncertainty in the next accounting year

In this subsection we consider the msep of the one-year run-off risk in the next accounting year $I+1$.

Result 4.1. *The msep of the one year run-off risk in the next accounting year $I+1$ can be estimated by*

i) total over all accident years

$$\widehat{\text{msep}}_{\text{tot},I+1} = w_{\text{tot},J}^2 \left\{ \prod_{j=j_0}^{J-1} \left(1 + b_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right) - 1 \right\}, \quad (4.3)$$

where

$$b_j = \frac{w_{i_j,j}}{\left(\sum_{i=i_0}^{i_j-1} w_{i,j}\right)\left(\sum_{i=i_0}^{i_j} w_{i,j}\right)}, \quad (4.4)$$

$w_{i,j}$ as defined in Equation (2.4).

ii) single accident year i

$$\widehat{\text{msep}}_{i,I+1} = w_{i,J}^2 \frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2} \left\{ \frac{1}{w_{i,j_i}} + \frac{1}{\sum_{k=i_0}^{i-1} w_{k,j_i}} \right\}$$
$$+ w_{i,J}^2 \left(1 + \frac{1}{w_{i,j_i}} \frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2} \right) \left(\prod_{j=j_i+1}^{J-1} \left(1 + b_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right) - 1 \right). \quad (4.5)$$

Remarks

- (4.3) and (4.5) are new and different to the Merz–Wüthrich formulas and to the estimators in the existing literature (e.g. the estimators in Merz and Wüthrich (2008b), Results 3.1 and 3.2.).
- Formula (4.3) for the total over all accident years is surprisingly simple and even simpler than formula (4.5) for a single accident year.

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- The first summand in Equation (4.5) reflects the risk that the observation on the next diagonal will deviate from the forecast at time I, whereas the second summand in Equation (4.5) reflects the risk of updating the forecasts of later development years due to an update of the estimated CL-factors from time I to time $I+1$.
- Intuitively comprehensible interpretation.
  From (4.12) and (4.18) we see that

$$b_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} = \frac{\widehat{\text{msep}}\left(\Delta \widehat{f}_j^{CL)}\right)}{\left(\widehat{f}_j^{CL}\right)^2}, \tag{4.6}$$

where

$$\Delta \widehat{f}_j^{CL} := \widehat{f}_j^{CL(I+1)} - \widehat{f}_j^{CL}, \tag{4.7}$$

$$\widehat{\text{msep}}\left(\Delta \widehat{f}_j^{CL)}\right) := \widehat{E}\left(\left(\Delta \widehat{f}_j^{CL)}\right)^2 \middle| \mathcal{D}_I\right). \tag{4.8}$$

Note that Equation (4.7) is the change of the CL-estimate of f_j from time I to time $I+1$, which can be interpreted as the one year development result of the chain-ladder factor estimate. Analogously we define

$$\Delta w_{\text{tot},J} := w_{\text{tot},J}^{(I+1)} - w_{\text{tot},J}, \tag{4.9}$$

$$\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}\right) := \widehat{E}\left(\left(\Delta w_{\text{tot},J}\right)^2 \middle| \mathcal{D}_I\right). \tag{4.10}$$

Equation (4.3) can now be written as

$$\frac{\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}\right)}{w_{\text{tot},J}^2} = \prod_{j=j_0}^{J-1}\left(1 + \frac{\widehat{\text{msep}}\left(\Delta \widehat{f}_j^{CL)}\right)}{\left(\widehat{f}_j^{CL)}\right)^2}\right) - 1, \tag{4.11}$$

which is a comprehensible interpretation, as it links the square of the relative one-year development result uncertainty with the squared relative one year development uncertainties of the chain ladder factor estimates.

Derivation of Result 4.1

a) **total over all accident years** (formula (4.3)).
   We complement the observed triangle by filling up the not yet observed lower right part $\mathcal{D}_I^c$ with the CL-forecasts $w_{i,j} = \widehat{C}_{i,J}^{CL}$ and take the total over each column, that is we define

$$\widehat{C}_{\text{tot},j}^{CL} := w_{\text{tot},j} = \sum\nolimits_{i=i_0}^{I} w_{i,j}.$$

   Analogously we define

$$\widehat{C}_{\text{tot},j}^{CL(I+1)} := w_{\text{tot},j}^{(I+1)} = \sum\nolimits_{i=i_0}^{I} w_{i,j}^{(I+1)},$$

where

$$w_{i,j}^{(I+1)} = \begin{cases} C_{i,j} & \text{if } C_{i,j} \in \mathcal{D}_{I+1}, \\ \widehat{C}_{i,j}^{CL(I+1)} & \text{otherwise.} \end{cases}$$

By definition of $\widehat{f}_j^{CL}$ and $\widehat{f}_j^{CL(I+1)}$ it holds that

$$\begin{aligned} w_{\text{tot},j} &= w_{\text{tot},j_0} \prod\nolimits_{j=j_0}^{j-1} \widehat{f}_j^{CL}, \\ w_{\text{tot},j}^{(I+1)} &= w_{\text{tot},j_0} \prod\nolimits_{j=j_0}^{j-1} \widehat{f}_j^{CL(I+1)}. \end{aligned}$$

Next we note that

$$\begin{aligned} \widehat{f}_j^{CL(I+1)} - \widehat{f}_j^{CL} &= \frac{\sum_{i=i_0}^{i_j} w_{i,j} F_{i,j}}{\sum_{i=i_0}^{i_j} w_{i,j}} - \frac{\sum_{i=i_0}^{i_j-1} w_{i,j} F_{i,j}}{\sum_{i=i_0}^{i_j-1} w_{i,j}} \\ &= a_j \left(F_{i_j,j} - \widehat{f}_j^{CL} \right), \end{aligned} \tag{4.12}$$

where

$$a_j = \frac{w_{i_j,j}}{\sum_{i=i_0}^{i_j} w_{i,j}}. \tag{4.13}$$

Since the observations of different accident years are independent, it follows that

$$\left\{ F_{i,j_i}, \widehat{f}_{j_i+1}^{CL(I+1)}, \ldots, \widehat{f}_{J-1}^{CL(I+1)} \right\} \quad \text{are independent given } \mathcal{D}_I, \tag{4.14}$$

$$\left\{ \widehat{f}_{j_0}^{CL(I+1)}, \widehat{f}_{j_0+1}^{CL(I+1)}, \ldots, \widehat{f}_{J-1}^{CL(I+1)} \right\} \quad \text{are independent given } \mathcal{D}_I. \tag{4.15}$$

With (4.15) we obtain by applying the estimation principle 3.3

$$\begin{aligned} \widehat{E}\left[\left(Z_{\text{tot}}^{(I+1)} \right)^2 \middle| \mathcal{D}_I \right] &= w_{\text{tot},j_0}^2 \widehat{E}\left[\left(\prod\nolimits_{j=j_0}^{j-1} \widehat{f}_j^{CL(I+1)} - \prod\nolimits_{j=j_0}^{j-1} \widehat{f}_j^{CL} \right)^2 \middle| \mathcal{D}_I \right] \\ &= w_{\text{tot},j_0}^2 \left\{ \widehat{E}\left[\left(\prod\nolimits_{j=j_0}^{j-1} \widehat{f}_j^{CL(I+1)} \right)^2 \middle| \mathcal{D}_I \right] - \left(\prod\nolimits_{j=j_0}^{j-1} \widehat{f}_j^{CL} \right)^2 \right\}. \end{aligned} \tag{4.16}$$

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From the model assumptions 3.1 and from (4.12) and (4.14) follows

$$E\left[\left.\widehat{f}_j^{CL(I+1)}\right|\mathcal{D}_I\right]=\widehat{f}_j^{CL}+a_j\left(f_j-\widehat{f}_j^{CL}\right),$$

$$\mathrm{Var}\left(\left.\widehat{f}_j^{CL(I+1)}\right|\mathcal{D}_I\right)=a_j^2\frac{\sigma_j^2}{w_{i_j,j}},$$

$$E\left[\left.\left(\widehat{f}_j^{CL(I+1)}\right)^2\right|\mathcal{D}_I\right]=\left(\widehat{f}_j^{CL}+a_j\left(f_j-\widehat{f}_j^{CL}\right)\right)^2+a_j^2\frac{\sigma_j^2}{w_{i_j,j}},$$

and, by applying the estimation principle 3.3,

$$\widehat{E}\left[\left.\widehat{f}_j^{CL(I+1)}\right|\mathcal{D}_I\right]=\widehat{f}_j^{CL}, \tag{4.17}$$

$$\begin{aligned}\widehat{E}\left[\left.\left(\widehat{f}_j^{CL(I+1)}\right)^2\right|\mathcal{D}_I\right]&=\left(\widehat{f}_j^{CL}\right)^2+a_j^2\widehat{\sigma}_j^2\left(\frac{1}{\sum_{i=i_0}^{i_j-1}w_{i,j}}+\frac{1}{w_{i_j,j}}\right)\\&=\left(\widehat{f}_j^{CL}\right)^2+b_j\widehat{\sigma}_j^2,\end{aligned} \tag{4.18}$$

where

$$b_j=\frac{w_{i_j,j}}{\left(\sum_{i=i_0}^{i_j-1}w_{i,j}\right)\left(\sum_{i=i_0}^{i_j}w_{i,j}\right)}. \tag{4.19}$$

From Equations (4.16) and (4.18) we obtain Equation (4.3).

b) **Single accounting year** i.

$$Z_i^{(I+1)}=\widehat{C}_{i,J}^{CL(I+1)}-\widehat{C}_{i,J}^{CL}=w_{i,j_i}\left\{F_{i,j_i}\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL(I+1)}-\widehat{f}_{j_i}^{CL}\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL}\right\}. \tag{4.20}$$

With the telescope formula (3.21) we can write Equation (4.20) as

$$Z_i^{(I+1)}=\underbrace{w_{i,j_i}\left(F_{i,j_i}-\widehat{f}_{j_i}^{CL}\right)\left(\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL}\right)}_{A_i}+\underbrace{w_{i,j_i}F_{i,j_i}\left(\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL(I+1)}-\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL)}\right)}_{B_i}.$$

Hence

$$E\left[\left.\left(Z_i^{(I+1)}\right)^2\right|\mathcal{D}_I\right]=E\left[\left.A_i^2\right|\mathcal{D}_I\right]+2E\left[\left.A_iB_i\right|\mathcal{D}_I\right]+E\left[\left.B_i^2\right|\mathcal{D}_I\right]. \tag{4.21}$$

For the first summand in Equation (4.21) we get

$$\begin{aligned}E\left[\left.A_i^2\right|\mathcal{D}_I\right]&=w_{i,j_i}^2\left\{E\left[\left.\left(F_{i,j_i}-f_{j_i}\right)^2\right|\mathcal{D}_I\right]+\left(f_{j_i}-\widehat{f}_{j_i}^{CL}\right)^2\right\}\left(\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL}\right)^2\\&=w_{i,j_i}\sigma_{j_i}^2\left(\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL}\right)^2+w_{i,j_i}^2\left(f_{j_i}-\widehat{f}_{j_i}^{CL}\right)^2\left(\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL}\right)^2,\end{aligned}$$

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which by use of the estimation principle 3.3 is estimated by

$$\widehat{E}\left[A_i^2\middle|\mathcal{D}_I\right]=w_{i,J}^2\frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2}\left\{\frac{1}{w_{i,j_i}}+\frac{1}{\sum_{k=i_0}^{i-1}w_{k,j_i}}\right\}. \tag{4.22}$$

From the independence property (4.14) and by applying the estimation principle 3.3 we also obtain

$$\widehat{E}\left[A_iB_i\middle|\mathcal{D}_I\right]=0 \tag{4.23}$$

for the second summand in Equation (4.21), since

$$\widehat{E}\left[\left(\prod_{j=j_i+1}^{J-1}\widehat{f}_j^{CL(I+1)}-\prod_{j=j_i+1}^{J-1}f_j^{CL)}\right)\middle|\mathcal{D}_I\right]=0.$$

From the independence property (4.14), (4.17) and by applying the estimation principle 3.3 follows that the third summand in Equation (4.21) can be estimated by

$$\widehat{E}\left[B_i^2\middle|\mathcal{D}_I\right]=w_{i,j_i}^2\left(\left(\widehat{f}_{j_i}^{CL}\right)^2+\frac{\widehat{\sigma}_{j_i}^2}{w_{i,j_i}}\right)\left(\prod_{j=j_i+1}^{J-1}\left(\left(\widehat{f}_j^{CL}\right)^2+b_j\widehat{\sigma}_j^2\right)-\prod_{j=j_i+1}^{J-1}\left(\widehat{f}_j^{CL}\right)^2\right)$$

$$=w_{i,j_i}^2\prod_{j=j_i}^{J-1}\left(\widehat{f}_j^{CL}\right)^2\left(1+\frac{1}{w_{i,j_i}}\frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2}\right)\left(\prod_{j=j_i+1}^{J-1}\left(1+b_j\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\right)-1\right). \tag{4.24}$$

By plugging Equations (4.22)–(4.24) into Equation (4.21) we get Equation (4.5). □

To compare with the Merz–Wüthrich formulas we approximate the estimators in Result 4.1 by a first order Taylor expansion.

Result 4.2 (Taylor Approximation). *The msep of the one year run-off risk in the next accounting year I+1 can be estimated by*

i) total over all accident years

$$\widehat{\text{msep}}_{\text{tot},I+1}^{\text{TA}}=w_{\text{tot},J}^2\left\{\sum_{j=j_0}^{J-1}b_j\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\right\}, \tag{4.25}$$

where b_j is defined in Equation (4.4).

ii) single accident year

$$\widehat{\text{msep}}_{i,I+1}^{\text{TA}}=w_{i,J}^2\frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2}\left\{\frac{1}{w_{i,j_i}}+\frac{1}{\sum_{k=i_0}^{i-1}w_{k,j_i}}\right\}+w_{i,J}^2\left\{\sum_{j=j_i+1}^{J-1}b_j\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\right\}. \tag{4.26}$$

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iii) Equations (4.25) and (4.26) are equivalent to the Merz–Wüthrich formulas (formulas (1.2) and (2.3) in Merz and Wüthrich (2014)*).*

Remarks

- **Merz–Wüthrich formulas** (formulas (1.2) and (2.3) in Merz and Wüthrich (2014)) written in our notation:
  i) total over all accident years

$$\widehat{\text{msep}}^{\text{MW}}_{\text{tot},I+1} = \sum_{i=i_J+1}^{I} w_{i,J}^2 \left\{ \frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2} \left(\frac{1}{w_{i,j_i}} + \frac{1}{\sum_{k=i_0}^{i_j-1} w_{k,j_i}} \right) + \sum_{j=j_i+1}^{J-1} b_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right\}$$
$$+ 2 \sum_{i=i_J+1}^{I} \sum_{m=i+1}^{I} w_{i,J} w_{m,J} \left\{ \frac{\widehat{\sigma}_{j_i}^2}{\left(\widehat{f}_{j_i}^{CL}\right)^2} \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i}} + \sum_{j=j_i+1}^{J-1} b_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right\}, \quad (4.27)$$

  ii) single accident year

$$\widehat{\text{msep}}^{\text{MW}}_{i,I+1} = \widehat{\text{msep}}^{\text{TA}}_{i,I+1} \text{ given by Equation (4.26).} \quad (4.28)$$

  We see that Equation (4.25) for the total over all accident years is a more concise and easier representation of Equation (4.27), which has in addition an intuitively accessible interpretation (see next bullet point).

  Side remark: The formulas in Merz and Wüthrich (2014) are different but equivalent to the formulas first published in Merz and Wüthrich (2008b) and the ones in Bühlmann *et al.* (2009).
- Intuitively comprehensible interpretation.
  Analogously as in Result 4.1 we can look behind the formulas and we can write (4.25) as

$$\frac{\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}\right)}{w_{\text{tot},J}^2} = \sum_{j=j_0}^{J-1} \frac{\widehat{\text{msep}}\left(\Delta \widehat{f}_j^{CL)}\right)}{\left(\widehat{f}_j^{CL)}\right)^2}, \quad (4.29)$$

  where $\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}\right)$ and $\widehat{\text{msep}}\left(\Delta \widehat{f}_j^{CL)}\right)$ are defined in Equations (4.10) and (4.8). Hence, the square of the "relative" one year development result uncertainty (with the ultimate loss as scaling basis) equals the sum of the squared "relative" one year development uncertainties of the chain-ladder factor estimates (with the chain ladder factor estimates at time I as scaling basis).
- The numerical results obtained by the Taylor approximations are very close to the ones obtained by the estimators of Result 4.1 if $b_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} << 1$, what is the case in most practical situations.
- The proof of iii) (equivalence of (4.25) with (4.27)) is given in Appendix A.

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- Other equivalent expressions of the Merz Wüthrich formulas were published in Diers *et al.* (2016) (special case of Corollary 3.9) and in Röhr (2016) (special case of main result 5.3).

4.2. The one-year run-off prediction uncertainty in future accounting years

The msep of the one year run-off risk in future accounting years are defined by

$$\text{msep}_{\text{tot},I+k+1} := E\left[\left(Z_{\text{tot}}^{(I+1+k)}\right)^2 \middle| \mathcal{D}_I\right], \tag{4.30}$$

$$\text{msep}_{i,I+k+1} := E\left[\left(Z_{i}^{(I+1+k)}\right)^2 \middle| \mathcal{D}_I\right], \tag{4.31}$$

where $Z_{\text{tot},I+k}$ and $Z_{i,I+k}$ are defined in Equations (4.1) and (4.2).

Result 4.3. *The msep of the one-year run-off risk in future accounting years can be estimated by*

i) total over all accident years, accounting years $I+k+1$, $k=0,\dots,J-j_0-1$

$$\widehat{\text{msep}}_{\text{tot},I+k+1} = w_{\text{tot},J}^2 \left\{ \prod_{j=j_i+k}^{J-1} \left(1+\widehat{b}_j^{(I+k)} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\right) - 1 \right\}, \tag{4.32}$$

where

$$\widehat{b}_j^{(I+k)} = \frac{w_{i_j+k,j}}{\left(\sum_{i=i_0}^{i_j+k-1} w_{i,j}\right)\left(\sum_{l=i_0}^{i_j+k} w_{l,j}\right)}, \tag{4.33}$$

weights $w_{i,j}$ as defined in Equation (2.4) .

ii) single accident year i, accounting years $I+k+1$, $k=0,\dots,J-j_i-1$

$$\widehat{\text{msep}}_{i,I+k+1} = w_{i,J}^2 \frac{\widehat{\sigma}_{j_i+k}^2}{\left(\widehat{f}_{j_i+k}^{CL}\right)^2} \left\{ \frac{1}{w_{i,j_i+k}} + \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}} \right\} \tag{4.34}$$
$$+ w_{i,J}^2 \left(1+\frac{1}{w_{i,j_i+k}} \frac{\widehat{\sigma}_{j_i+k}^2}{\left(\widehat{f}_{j_i+k}^{CL}\right)^2}\right) \left(\prod_{j=j_i+k+1}^{J-1} \left(1+\widehat{b}_j^{(I+k)} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\right) - 1\right).$$

Remarks

- Result 4.3 has the same structure as Result 4.1. The estimators in Result 4.3 are obtained from the estimators in Result 4.1 by simply replacing b_j by $\widehat{b}_j^{(I+k)}$ and the not yet known weights in $\widehat{b}_j^{(I+k)}$ by the forecasts at time I.

- For $k=0$ Result 4.3 coincides with Result 4.1.
- Intuitively comprehensible interpretation.
  Analogously as in Result 4.1 we can see from the derivation of Result 4.3 that

$$\widehat{b}_j^{(I+k)} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} = \frac{\widehat{\text{msep}}\left(\Delta\widehat{f}_j^{CL(I+k)}\right)}{\left(\widehat{f}_j^{CL}\right)^2}, \tag{4.35}$$

where

$$\Delta\widehat{f}_j^{CL(I+k)} := \widehat{f}_j^{CL(I+k+1)} - \widehat{f}_j^{CL(I+k)} \tag{4.36}$$

is the change of the CL-estimate of f_j from time $I+k$ to time $I+k+1$, and where

$$\widehat{\text{msep}}\left(\Delta\widehat{f}_j^{CL(I+k)}\right) := \widehat{E}\left(\left(\Delta\widehat{f}_j^{CL}\right)^2 \middle| \mathcal{D}_I\right). \tag{4.37}$$

Analogously we define

$$\Delta w_{\text{tot},J}^{(I+k)} := w_{\text{tot},J}^{(I+k+1)} - w_{\text{tot},J}^{(I+k)}, \tag{4.38}$$

$$\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}^{(I+k)}\right) := \widehat{E}\left(\left(\Delta w_{\text{tot},J}^{(I+k)}\right)^2 \middle| \mathcal{D}_I\right) \tag{4.39}$$

Result (4.32) can now be written as

$$\frac{\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}^{(I+k)}\right)}{w_{\text{tot},J}^2} = \prod_{j=j_0}^{J-1}\left(1 + \frac{\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}^{(I+k)}\right)}{\left(\widehat{f}_j^{CL}\right)^2}\right) - 1, \tag{4.40}$$

which is a comprehensible interpretation, as it links the square of the relative one-year development result uncertainty with the squared relative one year development uncertainties of the chain ladder factor estimates.

Derivation of Result 4.3

Equations (4.30) and (4.31) can be written as

$$\text{msep}_{\text{tot},I+k+1} = E\left[E\left[\left(Z_{\text{tot}}^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right] \middle| \mathcal{D}_I\right], \tag{4.41}$$

$$\text{msep}_{i,I+k+1} = E\left[E\left[\left(Z_i^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right] \middle| \mathcal{D}_I\right]. \tag{4.42}$$

By applying the estimation principle 3.3 on the inner expected values of Equations (4.41) and of Equation (4.42) we obtain (see Result 4.1, but with σ_j^2 instead of $\widehat{\sigma}_j^2$)

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$$\widetilde{E}\left[\left(Z_{\text{tot}}^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right] := \left(w_{\text{tot},J}^{(I+k)}\right)^2 \left\{\prod_{j=j_0+k}^{J-1}\left(1+b_j^{(I+k)}\frac{\sigma_j^2}{\left(\widehat{f}_j^{CL(I+k)}\right)^2}\right)-1\right\}, \tag{4.43}$$

$$\widetilde{E}\left[\left(Z_i^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right] := \left(w_{i,J}^{(I+k)}\right)^2 \frac{\sigma_{j_i+k}^2}{\left(\widehat{f}_{j_i+k}^{CL(I+k)}\right)^2}\left\{\frac{1}{w_{i,j_i+k}^{(I+k)}}+\frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}^{(I+k)}}\right\} \tag{4.44}$$

$$+\left(w_{i,J}^{(I+k)}\right)^2\left(1+\frac{1}{w_{i,j_i+k}^{(I+k)}}\frac{\sigma_j^2}{\left(\widehat{f}_{j_i+k}^{CL(I+k)}\right)^2}\right)\left(\prod_{j=j_i+1+k}^{J-1}\left(1+b_j^{(I+k)}\frac{\sigma_j^2}{\left(\widehat{f}_j^{CL(I+k)}\right)^2}\right)-1\right),$$

where

$$b_j^{(I+k)} = \frac{w_{i_j+k,j}^{(I+k)}}{\left(\sum_{i=i_0}^{i_j+k-1} w_{i,j}^{(I+k)}\right)\left(\sum_{i=i_0}^{i_j+k} w_{i,j}^{(I+k)}\right)},$$

$$\widehat{f}_j^{CL(I+k)} = \sum_{i=i_0}^{i_j+k-1}\frac{w_{i,j}^{(I+k)}}{w_{\bullet,j}^{(I+k)}}F_{i,j}, \quad \text{where } w_{\bullet,j}^{(I+k)} = \sum_{i=i_0}^{i_j+k-1} w_{i,j}^{(I+k)},$$

$$w_{i,j}^{(I+k)} = \begin{cases} C_{i,j} & \text{if } C_{i,j}\in\mathcal{D}_{I+k} \\ \widehat{C}_{i,j}^{CL(I+k)} & \text{otherwise} \end{cases}.$$

$\widetilde{E}\left[\left(Z_{\text{tot}}^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right]$ and $\widetilde{E}\left[\left(Z_i^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right]$ are r.v., since some of the weights are not yet known at time I. To simplify notation and to indicate that these are functions of the entries in $\mathcal{D}_{I+k}$ we define

$$g_{\text{tot}}\left(\mathcal{D}_{I+k}\right) := \widetilde{E}\left[\left(Z_{\text{tot}}^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right],$$

$$g_i\left(\mathcal{D}_{I+k}\right) := \widetilde{E}\left[\left(Z_i^{(I+1+k)}\right)^2 \middle| \mathcal{D}_{I+k}\right].$$

To estimate the msep we consider

$$\widetilde{\text{msep}}_{\text{tot},I+k+1} := E\left[g_{\text{tot}}\left(\mathcal{D}_{I+k}\right)\middle|\, \mathcal{D}_I\right], \tag{4.45}$$

$$\widetilde{\text{msep}}_{i,I+k+1} := E\left[g_i\left(\mathcal{D}_{I+k}\right)\middle|\, \mathcal{D}_I\right]. \tag{4.46}$$

The conditional expected values on the right hand side depend on distributional assumptions and can usually not be calculated in a closed form. To find estimators in the distribution-free case, we therefore approximate $g_{\text{tot}}\left(\mathcal{D}_{I+k}\right)$

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and $g_i(\mathcal{D}_{I+k})$ by the following first order Taylor approximation around the forecasts $w_{i,j}$.

$$g_{\text{tot}}^{TA}(\mathcal{D}_{I+k}) := g_{\text{tot}}\left(\widetilde{\mathcal{D}}_{I+k}\right) + \sum_{i=i_J+1}^{I} \sum_{j=j_i+1}^{\min(j_i+k,J)} \left.\frac{\partial g_{\text{tot}}}{\partial C_{i,j}}\right|_{\widetilde{\mathcal{D}}_{I+k}} \left(C_{i,j} - w_{i,j}\right),$$

$$g_i^{TA}(\mathcal{D}_{I+k}) := g_i\left(\widetilde{\mathcal{D}}_{I+k}\right) + \sum_{i=i_J+1}^{I} \sum_{j=j_i+1}^{\min(j_i+k,J)} \left.\frac{\partial g_i}{\partial C_{i,j}}\right|_{\widetilde{\mathcal{D}}_{I+k}} \left(C_{i,j} - w_{i,j}\right),$$

where $\widetilde{\mathcal{D}}_{I+k}$ denotes the triangle obtained by replacing the k newest diagonals in $\mathcal{D}_{I+k}$, which are not yet observed at time I, by the CL-forecasts at time I. Since

$$w_{i,j} = \widehat{E}\left[\left. C_{i,j}\right| \mathcal{D}_I\right],$$

we get

$$E\left[\left. g_{\text{tot}}^{TA}(\mathcal{D}_{I+k})\right| \mathcal{D}_I\right] = g_{\text{tot}}\left(\widetilde{\mathcal{D}}_{I+k}\right), \tag{4.47}$$

$$E\left[\left. g_i^{TA}(\mathcal{D}_{I+k})\right| \mathcal{D}_I\right] = g_i\left(\widetilde{\mathcal{D}}_{I+k}\right). \tag{4.48}$$

Result 4.3 then follows by replacing σ_j^2 in Equations (4.47) and (4.48) by $\widehat{\sigma}_j^2$. □

To compare with the Merz–Wüthrich formulas we again approximate the estimators in Result 4.3 by a first order Taylor expansion.

Result 4.4 (Taylor approximation). *The msep of the one-year run-off risk in future accounting years can be estimated by*

i) total over all accident years, accounting years $I+k+1, \quad k=0,\ldots,J-j_0-1$

$$\widehat{\text{msep}}_{\text{tot},I+k+1}^{\text{TA}} = w_{\text{tot},J}^2 \left\{ \sum_{j=j_0+k}^{J-1} \widehat{b}_j^{(I+k)} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right\}. \tag{4.49}$$

ii) single accident year i, accounting years $I+k+1, \quad k=0,\ldots,J-j_i-1$

$$\widehat{\text{msep}}_{i,I+k+1}^{\text{TA}} = w_{i,J}^2 \frac{\widehat{\sigma}_{j_i+k}^2}{\left(\widehat{f}_{j_i+k}^{CL}\right)^2} \left\{ \frac{1}{w_{i,j_i+k}} + \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}} \right\} + w_{i,J}^2 \left\{ \sum_{j=j_i+k+1}^{J-1} \widehat{b}_j^{(I+k)} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right\}. \tag{4.50}$$

iii) Equations (4.49) and (4.50) are equivalent to the Merz–Wüthrich formulas (formulas (1.4) and (2.4) in Merz and Wüthrich (2014)).

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Remarks

- Equations (4.49) and (4.50) coincide with (4.25) and (4.26) for $k=0$.
- **Merz–Wüthrich formulas** (formulas (1.4) and (2.4) in Merz and Wüthrich (2014)) **written in our notation**

  a) *total over all accident years, accounting years* $I+k+1, \quad k=0,\ldots,J-j_0-1$

$$\widehat{\text{msep}}^{\text{MW}}_{\text{tot},I+k+1} := \widehat{E}\left[\left(Z^{(I+k+1)}_{\text{tot}}\right)^2 \middle| \mathcal{D}_I\right]^{\text{MW}} =$$
$$= \sum_{i=i_J+k+1}^{I}\left\{w^2_{i,J}\frac{\widehat{\sigma}^2_{j_i+k}}{\left(\widehat{f}^{CL}_{j_i+k}\right)^2}\left(\frac{1}{w_{i,j_i+k}}+\frac{1}{\sum_{l=i_0}^{i-k-1} w_{l,j_i+k}}\prod_{m=1}^{k}\left(1-a_{j_i+m}\right)\right)\right\}$$
$$+\sum_{i=i_J+k+1}^{I}\left\{w^2_{i,J}\sum_{j=j_i+k+1}^{J-1}\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}\left(a_{j-k}\frac{1}{\sum_{l=i_0}^{i_j-1} w_{l,j}}\prod_{m=0}^{k-1}\left(1-a_{j-m}\right)\right)\right\}$$
$$+2\sum_{i=i_J+k+1}^{I}\sum_{n=i+1}^{I} w_{i,J}\, w_{n,J}\frac{\widehat{\sigma}^2_{j_i+k}}{\left(\widehat{f}^{CL}_{j_i+k}\right)^2}\frac{1}{\sum_{l=i_0}^{i-k-1} w_{l,j_i+k}}\prod_{m=1}^{k}\left(1-a_{j_i+m}\right) \tag{4.51}$$
$$+2\sum_{i=i_J+k+1}^{I}\sum_{n=i+1}^{I} w_{i,J}\, w_{n,J}\sum_{j=j_i+k+1}^{J-1}\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}\left(a_{j-k}\frac{1}{\sum_{l=i_0}^{i_j-1} w_{l,j}}\prod_{m=0}^{k-1}\left(1-a_{j-m}\right)\right)$$

  where $a_j = \dfrac{w_{i_j,j}}{\sum_{l=i_0}^{i_j} w_{l,j}}$.

  b) *single accident year* i, accounting years $I+k+1, \quad k=0,\ldots,J-j_i-1$

$$\widehat{\text{msep}}^{\text{MW}}_{i,I+k+1} := \widehat{E}\left[\left(Z^{(I+k+1)}_{i}\right)^2 \middle| \mathcal{D}_I\right]^{\text{MW}}$$
$$= w^2_{i,J}\frac{\widehat{\sigma}^2_{j_i+k}}{\left(\widehat{f}^{CL}_{j_i+k}\right)^2}\left(\frac{1}{w_{i,j_i+k}}+\prod_{m=1}^{k}\left(1-a_{j_i+m}\right)\frac{1}{\sum_{l=i_0}^{i-1-k} w_{l,j_i+k}}\right) \tag{4.52}$$
$$+w^2_{i,J}\sum_{j=j_i+k+1}^{J-1}\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}\left(a_{j-k}\prod_{m=0}^{k-1}\left(1-a_{j-m}\right)\frac{1}{\sum_{l=i_0}^{i_j-1} w_{l,j}}\right).$$

  It is astonishing that the Merz–Wüthrich formula (4.51) for the total over all accident years can be written in such a concise way as (4.49). Formula (4.50) too is simpler than Equation (4.52).

- Intuitively comprehensible interpretation.

  Analogously as in Result 4.3 Equation (4.49) can be written as

$$\frac{\widehat{\text{msep}}\left(\Delta w^{(I+k)}_{\text{tot},J}\right)}{w^2_{\text{tot},J}} = \sum_{j=j_0}^{J-1}\frac{\widehat{\text{msep}}\left(\Delta\widehat{f}^{CL(I+k)}_j\right)}{\left(\widehat{f}^{CL}_j\right)^2},$$

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where $\widehat{\text{msep}}\left(\Delta\widehat{f}_j^{CL(I+k)}\right)$ and $\widehat{\text{msep}}\left(\Delta w_{\text{tot},J}^{(I+k)}\right)$ are defined in Equations (4.37) and (4.39). Hence, the square of the relative one-year claims development result uncertainty is equal to the sum of the squared relative one-year development uncertainties of the chain ladder estimates.

- Other estimators of the msep of the one-year run-off risk in future accounting years were first published in Röhr (2016) (special case of main result 5.3 in Röhr (2016)). A formal proof that these estimators are equivalent to the Merz–Wüthrich estimators (4.51) and (4.52) is not given in Röhr (2016), but it is mentioned that numerical examples would indicate this equivalence.

Proof of Result 4.4. Equations (4.49) and (4.50) are obtained by a straightforward first order Taylor approximation of Equations (4.32) and (4.34). A proof of (iii) is given in Appendix B □

5. THE ULTIMATE RUN-OFF PREDICTION UNCERTAINTY

The *ultimate run-off risk* is the deviation of the ultimate claim from its forecast, that is the r.v.

$$Z_i^{ult} := C_{i,J} - \widehat{C}_{i,J}^{CL} \quad \text{for single accident years } i = i_0, \ldots, I,$$
$$Z_{\text{tot}}^{ult} := C_{\text{tot},J} - \widehat{C}_{\text{tot},J}^{CL} \quad \text{for the total over all accident years.}$$

The *msep* of the ultimate run-off risk is defined by

$$\text{msep}_{\text{tot,ult}} := E\left[\left(Z_{\text{tot}}^{ult}\right)^2 \middle| \mathcal{D}_I\right] \text{ for the total over all accident years,}$$
$$\text{msep}_{i,\text{ult}} := E\left[\left(Z_i^{ult}\right)^2 \middle| \mathcal{D}_I\right] \text{ for single accident years } i \in \{i_0, \ldots, I\}.$$

Result 5.1. *The msep of the ultimate run-off risk can be estimated by*

a) total over all accident years

$$\widehat{\text{msep}}_{\text{tot,ult}} = \sum_{j=j_0}^{J-1} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\left(\sum_{i=i_j}^{I} \frac{w_{i,J}^2}{w_{i,j}} + \frac{\left(\sum_{i=i_j}^{I} w_{i,J}\right)^2}{\sum_{i=i_0}^{i_j-1} w_{i,j}}\right), \tag{5.1}$$

where the weights $w_{i,j}$ are defined in Equation (2.4).

b) single accident year i

$$\widehat{\text{msep}}_{i,\text{ult}} = w_{i,J}^2 \left\{\sum_{j=j_i}^{J-1} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\left(\frac{1}{w_{i,j}} + \frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}}\right)\right\}. \tag{5.2}$$

c) Equation (5.2) is the same formula as the one found by Mack and Equation (5.1) is equivalent to the Mack-formula.

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Remarks

- The first summand in Equations (5.1) and (5.2) represent the process variance and the second summand the estimation error.
- Intuitively accessible interpretation.
  Result 5.1 can be written as

$$\widehat{\text{msep}}_{\text{tot,ult}} = \sum_{j=j_0}^{J-1} \left\{ \sum_{i=i_j}^{I} w_{i,J}^2 \, \widehat{\text{CoV}}\left(F_{i,j} \middle| C_{i,j}\right)^2 + \left(\sum_{i=i_j}^{I} w_{i,J} \right)^2 \widehat{\text{CoV}}\left(\widehat{f}_j^{CL} \middle| \mathcal{B}_j\right)^2 \right\}, \tag{5.3}$$

$$\widehat{\text{msep}}_{i,\text{ult}} = \sum_{j=j_i}^{J-1} \left\{ w_{i,J}^2 \, \widehat{\text{CoV}}\left(F_{i,j} \middle| C_{i,j}\right)^2 + w_{i,J}^2 \, \widehat{\text{CoV}}\left(\widehat{f}_j^{CL} \middle| \mathcal{B}_j\right)^2 \right\}, \tag{5.4}$$

  which are very nice and easily interpretable formulas. We can clearly see the impact of the uncertainties originating from the $F_{i,j}$ and from the $\widehat{f}_j^{CL}$ on the msep given by the square of a weight times the square of the coefficients of variation.
- Equation (5.2) is the same as Theorem 3 in Mack (1993), but Equation (5.1) is different albeit equivalent to the one in Mack (Corollary on page 220 in Mack (1993)).

  Mack-formula for the total over all accident years written in our notation.

$$\begin{aligned}\widehat{\text{msep}}_{\text{tot,ult}}^{\text{Mack}} = &\sum_{i=i_J+1}^{I} w_{i,J}^2 \left\{ \sum_{j=j_i}^{J-1} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \left(\frac{1}{w_{i,j}} + \frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}} \right) \right\} \\ &+ 2 \sum_{i=i_J+1}^{I} w_{i,J} \left(\sum_{k=i+1}^{I} w_{k,J} \right) \sum_{j=j_i}^{J-1} \left(\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \frac{1}{\sum_{m=i_0}^{i_j-1} w_{m,j}} \right).\end{aligned} \tag{5.5}$$

  Comparing with Equation (5.1) we see that Equation (5.1) is more concise and has an intuitively accessible interpretation, whereas the second summand (covariance term) in Equation (5.5) is difficult to interpret.
- The process variance for the total over all accident years is just the sum of the process variance of the single accident years (see Equation(5.16)), because the observations in different accident years are independent. But as for the estimation error it holds that

$$\widehat{EE}_{\text{tot}} = \sum_{j=j_0}^{J-1} \left\{ \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \frac{\left(\sum_{i=i_j}^{I} w_{i,J}\right)^2}{\sum_{i=i_0}^{i_j-1} w_{i,j}} \right\} > \sum_{j=j_0}^{J-1} \widehat{EE}_{\text{i}} = \sum_{j=j_0}^{J-1} \left\{ \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \frac{\sum_{i=i_j}^{I} w_{i,J}^2}{\sum_{i=i_0}^{i_j-1} w_{i,j}} \right\},$$

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since the uncertainty due to $\widehat{f}_j^{CL}$ affects the accident years $\{i = i_j, \ldots, I\}$ simultaneously. In the Mack-formula (5.5) the difference $\widehat{EE}_{\text{tot}} - \sum_{j=j_0}^{J-1} \widehat{EE}_{\text{i}}$ is taken into account by the covariance term (second summand of Equation (5.5)).

Derivation of Result 5.1

In the following we derive the estimators of these msep in a slightly different way than in Mack (1993) by making use of the telescope formula (3.21).

It is convenient to introduce

$$\mu_{i,j} := E\left[\left. C_{i,j} \right| \mathcal{D}_I\right] = w_{i,j_i} \prod_{k=j_i}^{j-1} f_k \text{ for } j = j_i + 1, \ldots, J, \tag{5.6}$$

$$\mu_{\text{tot},j} := E\left[\left. C_{\text{tot},j} \right| \mathcal{D}_I\right]. \tag{5.7}$$

The following splitting is well known.

$$\text{msep}_{\text{tot,ult}} = \underbrace{E\left[\left. \left(C_{\text{tot},J} - \mu_{\text{tot},J}\right)^2 \right| \mathcal{D}_I\right]}_{PV_{\text{tot}}} + \underbrace{\left(\mu_{\text{tot},J} - \widehat{C}_{\text{tot},J}^{CL}\right)^2}_{EE_{\text{tot}}}, \tag{5.8}$$

$$\text{msep}_{i,\text{ult}} = \underbrace{E\left[\left. \left(C_{i,J} - \mu_{i,J}\right)^2 \right| \mathcal{D}_I\right]}_{PV_i} + \underbrace{\left(\mu_{i,J} - \widehat{C}_{i,J}^{CL}\right)^2}_{EE_i}, \tag{5.9}$$

where PV_{tot} and PV_i are called *process variance* (deviation of the ultimate claim from its expected value) and EE_i and EE_{tot} *estimation error* (misestimating the expected value).

By applying the telescope formula (3.21) we get

$$\begin{aligned}\left(C_{i,J} - \mu_{i,J}\right)^2 &= w_{i,j_i}^2 \left(\prod_{j=j_i}^{J-1} F_{i,j} - \prod_{j=j_i}^{J-1} f_j\right)^2 \\ &= \left(\sum_{j=j_i}^{J-1} C_{i,j}\left(F_{i,j} - f_j\right) \prod_{k=j+1}^{J-1} f_k\right)^2,\end{aligned} \tag{5.10}$$

$$\begin{aligned}\left(C_{\text{tot},J} - \mu_{\text{tot},J}\right)^2 &= \left(\sum_{i=i_0}^{I} \left(\sum_{j=j_i}^{J-1} C_{i,j}\left(F_{i,j} - f_j\right)\left(\prod_{k=j+1}^{J-1} f_k\right)\right)\right)^2 \\ &= \left(\sum_{j=j_0}^{J-1} \left(\sum_{i=i_j+1}^{I} C_{i,j}\left(F_{i,j} - f_j\right)\left(\prod_{k=j+1}^{J-1} f_k\right)\right)\right)^2.\end{aligned} \tag{5.11}$$

$$\begin{aligned}EE_i &= w_{i,j_i}^2 \left(\prod_{j=j_i}^{J-1} f_j - \prod_{j=j_i}^{J-1} \widehat{f}_j^{CL}\right)^2 \\ &= \left(\sum_{j=j_i}^{J-1} \mu_{i,j}\left(f_j - \widehat{f}_j^{CL}\right)\left(\prod_{k=j+1}^{J-1} \widehat{f}_k^{CL}\right)\right)^2,\end{aligned} \tag{5.12}$$

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$$EE_{\text{tot}} = \left(\sum_{j=j_0}^{J-1} \left(\sum_{i=i_j+1}^{I} \mu_{i,j} \right) \left(f_j - \widehat{f}_j^{CL} \right) \left(\prod_{k=j+1}^{J-1} \widehat{f}_k^{CL} \right) \right)^2 . \tag{5.13}$$

By conditioning on $\mathcal{B}_k$, $k > j$, we see that

$$E \left[C_{i,j} \left(F_{i,j} - f_j \right) C_{i,k} \left(F_{i,k} - f_k \right) \right] = 0. \tag{5.14}$$

From the definition of the process variance, Equations (5.10), (5.11) and the independence of accident years follows:

$$PV_i = \sum_{j=j_i}^{J-1} \mu_{i,j} \left(\prod_{k=j+1}^{J-1} f_k \right)^2 \sigma_j^2 = \sum_{j=j_i}^{J-1} \mu_{i,J}^2 \frac{\sigma_j^2}{f_j^2} \frac{1}{\mu_{i,j}}, \tag{5.15}$$

$$PV_{\text{tot}} = \sum_{i=i_J+1}^{I} PV_i = \sum_{j=j_0}^{J-1} \sum_{i=i_j+1}^{I} \mu_{i,j} \left(\prod_{k=j+1}^{J-1} f_k \right)^2 \sigma_j^2$$

$$= \sum_{j=j_0}^{J-1} \sum_{i=i_j+1}^{I} \mu_{i,J}^2 \frac{\sigma_j^2}{f_j^2} \frac{1}{\mu_{i,j}}. \tag{5.16}$$

By applying the estimation principle 3.3 Result 5.1 is immediately obtained from Equations (5.12), (5.13), (5.15), (5.16).

The following Corollary shows an equivalent way of writing formula (5.1).

Corollary 5.2. *Formula (5.1) for estimating the msep of the ultimate run-off risk for the total over all accident years can also be written as*

$$\widehat{E} \left[\left(Z_{\text{tot}}^{ult} \right)^2 \middle| \mathcal{D}_I \right] = w_{\text{tot},J}^2 \left\{ \sum_{j=j_0}^{J-1} q_j \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL} \right)^2} \frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}} \right\} \tag{5.17}$$

$$= w_{\text{tot},J}^2 \left\{ \sum_{j=j_0}^{J-1} q_j \left(\widehat{\text{CoV}} \left(\widehat{f}_j^{CL} \middle| \mathcal{B}_j \right) \right)^2 \right\}, \tag{5.18}$$

where

$$q_j = \frac{\sum_{i=i_j}^{I} w_{i,J}}{\sum_{i=i_0}^{I} w_{i,J}} = \textit{fraction of } \widehat{C}_{\text{tot},J}^{CL} \textit{ affected by the uncertainty of } \widehat{f}_j^{CL}.$$

Remarks

- Formula (5.17) was already found by Ancus Röhr in Röhr (2016). However, he derived a formula for estimating the msep of $\widetilde{Z}_{\text{tot}}^{ult}$, where $\widetilde{Z}_{\text{tot}}^{ult}$ is a first order Taylor expansion of Z_{tot}^{ult}. Interestingly the resulting estimator is equivalent to the Mack estimator and hence also equivalent to (5.1).

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- Equation (5.17) is an astonishing and surprisingly simple result. Nevertheless we prefer Result 5.1, as we can see there explicitly the impact of the uncertainties originating from the $F_{i,j}$ and from the $\widehat{f}_j^{CL}$ on the msep.
- The proof of the equivalence between (5.17) and (5.1) is given in Appendix C.

6. RELATIONSHIP BETWEEN ONE-YEAR AND ULTIMATE RUN-OFF RISKS

The sum of the one year run-off risks over all future development years is equal to the ultimate run-off risk, i.e.

$$\sum_{k=1}^{J} Z_{\text{tot}}^{(I+k)} = Z_{\text{tot}}^{ult}, \tag{6.19}$$

$$\sum_{k=1}^{J-j_i} Z_i^{(I+k)} = Z_i^{ult}, \tag{6.20}$$

and hence

$$E\left[\left(\sum_{k=1}^{J} Z_{\text{tot}}^{(I+k)}\right)^2 \middle| \mathcal{D}_I\right] = E\left[\left(Z_{\text{tot}}^{ult}\right)^2 \middle| \mathcal{D}_I\right], \tag{6.21}$$

$$E\left[\left(\sum_{k=1}^{J} Z_i^{(I+k)}\right)^2 \middle| \mathcal{D}_I\right] = E\left[\left(Z_i^{ult}\right)^2 \middle| \mathcal{D}_I\right]. \tag{6.22}$$

By definition of best estimate reserves the forecast of the claims development result in any future period is zero. For this reason it is often argued that the process of best estimate forecasts $\left\{\widehat{C}_{i,J}^{BE(I+k)} : k = 0, \ldots, J\right\}$ is a martingale and that therefore the one-year run-off risks, which are the increments of this process, are uncorrelated. Based on this martingale argument it is then required that the estimators of the one-year run-off risk should satisfy the *"splitting" property*, which means that the sum of the estimated msep of the one-year run off risks summed up over all future accounting years until final development should be equal to the estimated msep of the ultimate run-off risk.

However, best estimate forecasts are usually not a martingale, what is also the case for the CL-forecasts in the Mack-model. The CL forecasts fulfil

$$\widehat{E}\left[\widehat{C}_{i,J}^{CL(I+k+1)} \middle| \widehat{C}_{i,J}^{CL(I+k)}\right] = \widehat{C}_{i,J}^{CL(I+k)},$$

but they do not satisfy the martingale condition

$$E\left[\widehat{C}_{i,J}^{CL(I+k+1)} \middle| \widehat{C}_{i,J}^{CL(I+k)}\right] = \widehat{C}_{i,J}^{CL(I+k)},$$

because the unknown CL factors f_j are replaced in the CL-forecasts by their estimates $\widehat{f}_j^{CL}$ and $\widehat{f}_j^{CL(I+1+k)}$ respectively. Hence there is no mathematical reason that the estimators should fulfil the splitting property.

Theorem 6.1.

i)

$$\sum_{k=1}^{J-j_0} \widehat{\text{msep}}_{\text{tot},I+k} > \widehat{\text{msep}}_{\text{tot,ult}},$$
$$\sum_{k=1}^{J-j_i} \widehat{\text{msep}}_{i,I+k} > \widehat{\text{msep}}_{i,\text{ult}},$$

where $\widehat{\text{msep}}_{\text{tot},I+k}$ *and* $\widehat{\text{msep}}_{i,I+k}$ *are the estimators in Result 4.3 and* $\widehat{\text{msep}}_{\text{tot,ult}}$ *and* $\widehat{\text{msep}}_{i,\text{ult}}$ *the Mack estimators (estimators in Result 5.1).*

ii)

$$\sum_{k=1}^{J-j_0} \widehat{\text{msep}}^{\text{TA}}_{\text{tot},I+k} = \widehat{\text{msep}}_{\text{tot,ult}},$$
$$\sum_{k=1}^{J-j_i} \widehat{\text{msep}}^{\text{TA}}_{i,I+k} = \widehat{\text{msep}}_{i,\text{ult}},$$

where $\widehat{\text{msep}}^{\text{TA}}_{\text{tot},I+k}$ *and* $\widehat{\text{msep}}^{\text{TA}}_{i,I+k}$ *are the estimators in Result 4.4.*

Remarks

- The sum of the one-year msep estimators in Result 4.3 is bigger than the Mack estimators of the ultimate run-off risk.
- The estimators $\widehat{\text{msep}}^{\text{TA}}_{\text{tot},I+k}$ and $\widehat{\text{msep}}^{\text{TA}}_{i,I+k}$ in Result 4.4 fulfil the splitting property. The splitting property entails a splitting of the ultimate run off risk over the future accounting years until final development, which is a convenient property. This might be an argument and a good reason to use the estimators $\widehat{\text{msep}}^{\text{TA}}_{\text{tot},I+k}$ and $\widehat{\text{msep}}^{\text{TA}}_{i,I+k}$ (Taylor approximations) instead of the estimators $\widehat{\text{msep}}_{\text{tot},I+k}$ and $\widehat{\text{msep}}_{i,I+k}$, in particular, since the differences in numerical results between the two estimators are negligible for most situations in practice.

Proof. i) follows from the fact that $\widehat{\text{msep}}_{\text{tot},I+k}$ and $\widehat{\text{msep}}_{i,I+k}$ respectively are bigger than $\widehat{\text{msep}}^{\text{TA}}_{\text{tot},I+k}$ and $\widehat{\text{msep}}^{\text{TA}}_{i,I+k}$ for all k. The splitting property ii) has already been proved by MW (proposition 6.1 in Merz and Wüthrich (2014)) and by Röhr (see Remark 5.6 in Röhr (2016)) . □

7. NUMERICAL EXAMPLE

As a numerical example we consider the data in Table 1 with estimated total reserves of 2'926 at the end of 2017.

Table 3 shows the square root of the estimated msep for the ultimate run-off and for the one year run-off in the next accounting year, where the latter is calculated with Result 4.1 (column A in Table 3) as well as with Result 4.2 (Taylor approximation; column TA in Table 3). We see that in this example the numerical results of the two estimators are practically the same. We can also

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TABLE 3

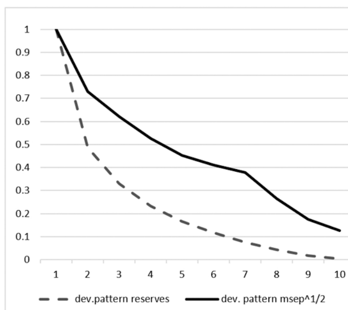
accident year	Reserves	Ultimate run-off msep$^{1/2}$	Ultimate run-off in % reserves	One year run-off next account. year msep$^{1/2}$ A	One year run-off next account. year msep$^{1/2}$ TA	One year run-off in % reserves
2008	10	21.37	210%	21.374	21.374	210.0%
2009	30	27.91	93%	19.006	19.006	63.0%
2010	59	42.91	73%	31.991	31.991	54.2%
2011	83	58.17	70%	40.891	40.891	49.5%
2012	109	62.14	57%	15.749	15.749	14.5%
2013	141	65.65	47%	15.760	15.760	11.2%
2014	183	68.61	38%	27.662	27.661	15.1%
2015	262	75.03	29%	31.821	31.821	12.1%
2016	464	89.07	19%	42.963	42.963	9.3%
2017	1'585	137.85	9%	101.131	101.130	6.4%
total	**2'926**	**303.44**	**10%**	**180.515**	**180.514**	**6.2%**

TABLE 4

total over all accident years

accounting year	Ingoing reserves	Ingoing reserves dev. pattern	One year run-off msep$^{1/2}$ A	One year run-off msep$^{1/2}$ TA	One year run-off dev.pattern
2017	2'916	100%	180.51	180.51	100%
2018	1'423	49%	131.67	131.67	73%
2019	961	33%	111.94	111.94	62%
2020	679	23%	94.94	94.94	53%
2021	483	17%	81.49	81.49	45%
2022	341	12%	74.20	74.20	41%
2023	224	8%	68.14	68.14	38%
2024	125	4%	47.83	47.83	26%
2025	54	2%	31.73	31.73	18%
2026	14	0%	22.98	22.98	13%
		sum of one year msep	**92'079**		
		ultimate msep	**92'079**		

graph development patterns
development pattern reserves and msep^1/2



see that the numerical results in column A are greater than the ones in column TA, but only very minor in the third digit after the decimal point in accident years 2014 and 2017.

Table 4 shows the estimates of the msep of the one-year run-off for the total over all accident years for all future accounting years until final development together with the forecasted ingoing reserves. The numerical results in column A were calculated with Result 4.3 and the ones in column TA with Result 4.4 (Taylor approximation). Again the numerical results obtained by the two estimators are practically the same. We can also check the splitting property. If we sum up the square of the entries in column A, we obtain the same as the square of the estimated msep of the ultimate run-off for the total over all accident years from Table 3.

In the current formula for calculating the risk margin in solvency II it is assumed that the required capital for the remaining one-year run-off risk in

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future accounting years decreases proportionally to the remaining reserves. This was due to the lack of formulas to calculate the prediction uncertainty for future accounting years. But now the formulas have been developed and are here. Comparing the development pattern of the reserves with the development pattern of the square root of the msep (see graph beside Table 4) we see that the latter decreases much slower. This is not a surprise. Complex and complicated claims such as severe bodily injury claims stay open for a long time, whereas "normal" claims can be settled much quicker. Hence the proportion of the reserves stemming from complex claims is bigger in later development years. But the prediction uncertainty of this kind of claims is bigger than for the "normal" claims. This also means that one will need more capital in solvency II with these new formulas, since the risk margin will become bigger.

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APPENDIX A. PROOF OF RESULT 4.2, ITEM (III)

We have to show that

$$\widehat{\text{msep}}^{\text{TA}}_{\text{tot},I+1} = w^2_{\text{tot},J}\left\{\sum_{j=j_0+k}^{J-1} b_j \frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}\right\} \tag{A.23}$$

is equivalent to

$$\begin{aligned}\widehat{\text{msep}}^{\text{MW}}_{\text{tot},I+1} = &\sum_{i=i_J+1}^{I} w^2_{i,J}\left\{\frac{\widehat{\sigma}^2_{j_i}}{\left(\widehat{f}^{CL}_{j_i}\right)^2}\left(\frac{1}{w_{i,j_i}}+\frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j_i}}\right)+w^2_{i,J}\sum_{j=j_i+1}^{J-1}\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}b_j\right\}\\ &+2\sum_{i=i_J+1}^{I}\sum_{m=i+1}^{I} w_{i,J}w_{m,J}\left\{\frac{\widehat{\sigma}^2_{j_i}}{\left(\widehat{f}^{CL}_{j_i}\right)^2}\frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i}}+\sum_{j=j_i+1}^{J-1}\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}b_j\right\}.\end{aligned} \tag{A.24}$$

We rewrite Equation (A.24) by summing first over the development years and afterwards over the corresponding accident years and obtain

$$\begin{aligned}\widehat{\text{msep}}_{\text{tot},I+1} = &\sum_{j=j_0}^{J-1} w^2_{i_j,j}\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}\left(\frac{1}{w_{i_j,j}}+\frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}}\right)\prod_{k=j}^{J-1}\left(\widehat{f}^{CL}_k\right)^2\\ &+\sum_{j=j_0+1}^{J-1}\left\{\sum_{i=i_j+1}^{I} w^2_{i,j}b_j+2\left(\sum_{m=i_j+1}^{I} w_{m,j}\right)\frac{w_{i_j,j}}{\sum_{l=i_0}^{i_j-1} w_{l,j}}\right.\\ &\left.+2\left(\sum_{i=i_j+1}^{I}\sum_{m=i+1}^{I} w_{i,j}w_{m,j}b_j\right)\right\}\prod_{k=j}^{J-1}\left(\widehat{f}^{CL}_k\right)^2\frac{\widehat{\sigma}^2_j}{\left(\widehat{f}^{CL}_j\right)^2}.\end{aligned} \tag{A.25}$$

Furthermore

$$w_{i_j,j}^2\left(\frac{1}{w_{i_j,j}}+\frac{1}{\sum_{i=i_0}^{i_j-1}w_{i,j}}\right)=w_{i_j,j}^2\left(\frac{\sum_{i=i_0}^{i_j}w_{i,j}}{w_{i_j,j}\sum_{i=i_0}^{i_j-1}w_{i,j}}\right)$$

$$=\left(\sum\nolimits_{i=i_0}^{i_j}w_{i,j}\right)^2 b_j. \tag{A.26}$$

$$w_{i_j,j}w_{m,j}\frac{1}{\sum_{l=i_0}^{i_j-1}w_{l,j_i}}=\left(\sum\nolimits_{i=i_0}^{i_j}w_{i,j}\right)\frac{w_{i_j,j}}{\left(\sum_{i=i_0}^{i_j}w_{i,j}\right)\sum_{l=i_0}^{i_j-1}w_{l,j}}w_{m,j} \tag{A.27}$$

$$=\left(\sum\nolimits_{i=i_0}^{i_j}w_{i,j}\right)w_{m,j}b_j. \tag{A.28}$$

Inserting Equations (A.26) and (A.28) into Equation (A.25) yields

$$\widehat{\mathrm{msep}}_{\mathrm{tot},I+1}=\sum_{j=j_0}^{J-1}\left\{\left(\sum_{i=i_0}^{i_j}w_{i,j}\right)^2+\sum_{i=i_j+1}^{I}w_{i,j}^2+2\left(\sum_{i=i_0}^{i_j}w_{i,j}\right)\left(\sum_{m=i_j+1}^{I}w_{m,j}\right)\right.$$

$$\left.+2\sum_{i=i_j+1}^{I}\sum_{m=i+1}^{I}w_{i,j}w_{m,j}\right\}\prod_{k=j}^{J-1}\left(\widehat{f}_k^{CL}\right)^2\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}b_j$$

$$=\sum_{j=j_0}^{J-1}\left\{\left(\sum_{i=i_0}^{I}w_{i,j}\right)^2\prod_{k=j}^{J-1}\left(\widehat{f}_k^{CL}\right)^2\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}b_j\right\}$$

$$=w_{\mathrm{tot},J}^2\left(\sum_{j=j_0}^{J-1}b_j\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\right). \tag{A.29}$$

Thus we have proved that Equation (A.29) is equivalent to the MW-formula (A.24) □

APPENDIX B. PROOF OF RESULT 4.4, ITEM (III)

(i) **Single accident year: equivalence of formula** (4.50) with formula (4.52).
It is convenient to introduce the following notation:

$$s_j^{(I+k)}:=\sum_{i=i_0}^{i_j+k}w_{i,j}^{(I+k)}=\text{sum of observations in column } j \text{ known at time } I+k \ \ (\text{for } k\le I-i_j),$$

$$\widehat{s}_j^{(I+k)}:=\sum_{i=i_0}^{i_j+k}w_{i,j}\ \ =\text{"CL-forecast" of } s_j^{(I+k)} \text{ at time } I,$$

$$\widehat{a}_j^{(I+k)}:=\frac{w_{i_j+k,j}}{\widehat{s}_j^{(I+k)}};$$

$$\widehat{b}_j^{(I+k)}:=\frac{w_{i_j+k,j}}{\widehat{s}_j^{(I+k)}\widehat{s}_j^{(I+k-1)}}$$

Note that

$$1-\widehat{a}_j^{(I+k)} = \frac{\widehat{s}_j^{(I+k-1)}}{\widehat{s}_j^{(I+k)}}, \tag{A.30}$$

$$\widehat{b}_j^{(I+k)} = \widehat{a}_j^{(I+k)} \frac{1}{\widehat{s}_j^{(I+k-1)}}. \tag{A.31}$$

Note also that for $I+k \le I$ all quantities are known at time I and that $\widehat{a}_j^{(I)} = a_j$ and $\widehat{b}_j^{(I)} = b_j$, where a_j and b_j are as defined in Equations (4.13) and (4.19).

Lemma B.1. *It holds that*

$$a_{j-k} = \widehat{a}_j^{(I+k)} \tag{A.32}$$

$$\prod_{m=0}^{k-1} \left(1-a_{j-m}\right) \frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}} = \frac{1}{\sum_{i=i_0}^{i_j+k-1} w_{i,j}}. \tag{A.33}$$

Proof of Lemma B.1.

$$w_{i_j+k,j} = w_{i_{j-k},j-k} \prod_{l=j-k}^{j-1} \widehat{f}_l^{CL},$$

$$\widehat{s}_j^{(I+k)} = \left(\sum_{i=i_0}^{i_{j-k}} w_{i,j-k}\right) \prod_{l=j-k}^{j-1} \widehat{f}_l^{CL},$$

$$\widehat{a}_j^{(I+k)} = \frac{w_{i_j+k,j}}{\widehat{s}_j^{(I+k)}} = \frac{w_{i_{j-k},j-k}}{\sum_{i=i_0}^{i_{j-k}} w_{i,j-k}} = a_{j-k}.$$

$$\begin{aligned}
\prod_{m=0}^{k-1} \left(1-a_{j-m}\right) \frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}} &= \prod_{m=k-1}^{0} \left(1-\widehat{a}_j^{(I+m)}\right) \frac{1}{\widehat{s}_j^{(I-1)}} \\
&= \frac{\widehat{s}_j^{(I+k-2)}}{\widehat{s}_j^{(I+k-1)}} \cdot \frac{\widehat{s}_j^{(I+k-3)}}{\widehat{s}_j^{(I+k-2)}} \cdot \ldots \cdot \frac{\widehat{s}_j^{(I)}}{\widehat{s}_j^{(I+1)}} \cdot \frac{\widehat{s}_j^{(I-1)}}{\widehat{s}_j^{(I)}} \cdot \frac{1}{\widehat{s}_j^{(I-1)}} \\
&= \frac{1}{\widehat{s}_j^{(I+k-1)}} \\
&= \frac{1}{\sum_{i=i_0}^{i_j+k-1} w_{i.j}}.
\end{aligned}$$

□

To prove the equivalence of formula (4.50) with formula (4.52) we have to show that

$$\frac{1}{\sum_{l=i_0}^{i-k-1} w_{l,j_i+k}} \prod_{m=1}^{k} \left(1-a_{j_i+m}\right) = \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}} \quad \text{and} \tag{A.34}$$

$$a_{j-k} \prod_{m=0}^{k-1} \left(1-a_{j-m}\right) \frac{1}{\sum_{l=i_0}^{i_j-1} w_{l,j}} = \widehat{b}_j^{(I+k)}. \tag{A.35}$$

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$$\frac{1}{\sum_{l=i_0}^{i-k-1} w_{l,j_i+k}} \prod_{m=1}^{k} \left(1 - a_{j_i+m}\right) = \frac{1}{\sum_{l=i_0}^{i_{j_i+k}-1} w_{l,j_i+k}} \prod_{m=0}^{k-1} \left(1 - a_{j_i+k-m}\right)$$
$$= \frac{1}{\sum_{i=i_0}^{i_{j_i+k}+k-1} w_{i,j_i+k}}$$
$$= \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}},$$

where the second equation follows from Equation (A.33) for $j = j_i + k$.

With Equations (A.32), (A.33) and (A.31) we also obtain

$$a_{j-k} \prod_{m=0}^{k-1} \left(1 - a_{j-m}\right) \frac{1}{\sum_{l=i_0}^{i_j-1} w_{l,j}} = \widehat{a}_j^{(I+k)} \frac{1}{\widehat{s}_j^{(I+k-1)}} = \widehat{b}_j^{(I+k)}.$$

□

ii) **Total over all accident years: equivalence of formula (4.49) with formula (4.51).** Replacing Equation (4.52) by Equation (4.50) and making use of Lemma B.1 we can write Equation (4.51) in the following equivalent way:

$$\begin{aligned}
\widehat{\mathrm{msep}}_{\mathrm{tot},I+k+1} = &\sum_{i=i_J+k+1}^{I} w_{i,J}^2 \frac{\widehat{\sigma}_{j_i+k}^2}{\left(\widehat{f}_{j_i+k}^{CL}\right)^2} \left\{ \left(\frac{1}{w_{i,j_i+k}} + \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}} \right) \right\} \\
&+ \sum_{i=i_J+k+1}^{I} w_{i,J}^2 \left\{ \sum_{j=j_i+k+1}^{J-1} \widehat{b}_j^{(I+k)} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \right\} \\
&+2 \sum_{i=i_J+k+1}^{I} \sum_{n=i+1}^{I} w_{i,J}\, w_{n,J} \frac{\widehat{\sigma}_{j_i+k}^2}{\left(\widehat{f}_{j_i+k}^{CL}\right)^2} \frac{1}{\sum_{l=i_0}^{i-1} w_{l,j_i+k}} \\
&+2 \sum_{i=i_J+k+1}^{I} \sum_{n=i+1}^{I} w_{i,J}\, w_{n,J} \sum_{j=j_i+k+1}^{J-1} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \widehat{b}_j^{(I+k)}.
\end{aligned} \tag{A.36}$$

As in subsection 4.4 we rewrite Equation (A.36) by first summing over the development years and afterwards over the corresponding accident years to obtain

$$\begin{aligned}
\widehat{\mathrm{msep}}_{\mathrm{tot},I+k+1} = &\sum_{j=j_0+k}^{J-1} w_{i_j+k,j}^2 \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \left(\frac{1}{w_{i_j+k,j}} + \frac{1}{\sum_{i=i_0}^{i_j+k-1} w_{i,j}} \right) \prod_{l=j}^{J-1} \left(\widehat{f}_l^{CL}\right)^2 \\
&+ \sum_{j=j_0+k+1}^{J-1} \left\{ \sum_{i=i_j+k+1}^{I} w_{i,j}^2 + 2 \left(\sum_{m=i_j+k+1}^{I} w_{m,j} \right) \frac{w_{i_j+k,j}}{\sum_{l=i_0}^{i_j+k-1} w_{l,j}} \right. \\
&+ \left. \left(\sum_{i=i_j+k+1}^{I} \sum_{m=i+1}^{I} w_{i,j} w_{m,j} \right) \right\} \prod_{l=j}^{J-1} \left(\widehat{f}_l^{CL}\right)^2 \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \widehat{b}_j^{(I+k)}.
\end{aligned} \tag{A.37}$$

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The following calculations are analogous to the ones in subsection A

$$w_{i_j+k,j}^2\left(\frac{1}{w_{i_j+k,j}}+\frac{1}{\sum_{i=i_0}^{i_j+k-1}w_{i,j}}\right)=w_{i_j+k,j}^2\left(\frac{\sum_{i=i_0}^{i_j+k}w_{i,j}}{w_{i_j+k,j}\sum_{i=i_0}^{i_j+k-1}w_{i,j}}\right)$$
$$=\left(\sum_{i=i_0}^{i_j+k}w_{i,j}\right)^2\widehat{b}_j^{(I+k)}. \quad \text{(A.38)}$$

$$w_{i_j+k,j}w_{m,j}\frac{1}{\sum_{l=i_0}^{i_j+k-1}w_{l,j_i}}=\left(\sum_{i=i_0}^{i_j+k}w_{i,j}\right)\frac{w_{i_j+k,j}}{\left(\sum_{i=i_0}^{i_j+k}w_{i,j}\right)\sum_{l=i_0}^{i_j+k-1}w_{l,j}}w_{m,j}$$
$$=\left(\sum_{i=i_0}^{i_j+k}w_{i,j}\right)w_{m,j}\widehat{b}_j^{(I+k)}. \quad \text{(A.39)}$$

By inserting Equations (A.38) and (A.39) into Equation (A.37) we obtain

$$\begin{aligned}\widehat{\text{msep}}_{\text{tot},I+k+1}&=\sum_{j=j_0+k}^{J-1}\left\{\left(\sum_{i=i_0}^{i_j+k}w_{i,j}\right)^2+\sum_{i=i_j+k+1}^{I}w_{i,j}^2+2\left(\sum_{i=i_0}^{i_j+k}w_{i,j}\right)\left(\sum_{m=i_j+k+1}^{I}w_{m,j}\right)\right.\\&\left.\quad+2\sum_{i=i_j+k+1}^{I}\sum_{m=i+1}^{I}w_{i,j}w_{m,j}\right\}\prod_{k=j}^{J-1}\left(\widehat{f}_k^{CL}\right)^2\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\widehat{b}_j^{(I+k)}\\&=\sum_{j=j_0+k}^{J-1}\left\{\left(\sum_{i=i_0}^{I}w_{i,j}\right)^2\prod_{k=j}^{J-1}\left(\widehat{f}_k^{CL}\right)^2\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\widehat{b}_j^{(I+k)}\right\}\\&=w_{\text{tot},J}^2\left\{\sum_{j=j_0+k}^{J-1}\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\widehat{b}_j^{(I+k)}\right\}.\end{aligned}$$

□

APPENDIX C. PROOF OF COROLLARY 5.2

We have to prove that the right hand side of Equation (5.1) can be expressed by the right hand side of Equation (5.17).

$$\begin{aligned}&\sum_{j=j_0}^{J-1}\left\{\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\left(\sum_{i=i_j}^{I}\frac{w_{i,J}^2}{w_{i,j}}+\frac{\left(\sum_{i=i_j}^{I}w_{i,J}\right)^2}{\sum_{i=i_0}^{i_j-1}w_{i,j}}\right)\right\}\\&=\sum_{j=j_0}^{J-1}\left\{\frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2}\left(\prod_{k=j}^{J-1}\widehat{f}_k^{CL}\right)\left(\left(\sum_{i=i_j}^{I}w_{i,J}\right)\left(1+\frac{\sum_{i=i_j}^{I}w_{i,J}}{\sum_{i=i_0}^{i_j-1}w_{i,J}}\right)\right)\right\}\end{aligned}$$

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$$= \sum_{j=j_0}^{J-1} \left\{ \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \left(\prod_{k=j}^{J-1} \widehat{f}_k^{CL} \right) \left(\sum_{i=i_j}^{I} w_{i,J} \right) \left(\frac{w_{\text{tot},J}}{\sum_{i=i_0}^{i_j-1} w_{i,J}} \right) \right\}$$

$$= w_{\text{tot},J}^2 \left(\sum_{j=j_0}^{J-1} \frac{\widehat{\sigma}_j^2}{\left(\widehat{f}_j^{CL}\right)^2} \frac{1}{\sum_{i=i_0}^{i_j-1} w_{i,j}} \frac{\sum_{i=i_j}^{I} w_{i,J}}{w_{\text{tot},J}} \right).$$

□

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