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# Self-similar solutions of the radially symmetric relativistic Euler equations

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The study of radially symmetric motion is important for the theory of explosion waves. We construct rigorously self-similar entropy solutions to Riemann initial-boundary value problems for the radially symmetric relativistic Euler equations. We use the assumption of self-similarity to reduce the relativistic Euler equations to a system of nonlinear ordinary differential equations, from which we obtain detailed structures of solutions besides their existence. For the ultra-relativistic Euler equations, we also obtain the uniqueness of the self-similar entropy solution to the Riemann initial-boundary value problems.

**Key words:** Relativistic Euler equations, radial symmetry, Riemann problem, self-similar solution

**2010 Mathematics Subject Classification:** Primary: 35L65. Secondary: 35L60; 35L67.

## 1 Introduction

The motion of a relativistic fluid in the Minkowski space-time is governed by the relativistic Euler equations

$$\begin{cases} \left(\dfrac{n}{\sqrt{1-q^2}}\right)_t + \left(\dfrac{nu_1}{\sqrt{1-q^2}}\right)_{x_1} + \left(\dfrac{nu_2}{\sqrt{1-q^2}}\right)_{x_2} + \left(\dfrac{nu_3}{\sqrt{1-q^2}}\right)_{x_3} = 0, \\ \left(\dfrac{iu_1}{1-q^2}\right)_t + \left(\dfrac{iu_1^2}{1-q^2}+p\right)_{x_1} + \left(\dfrac{iu_1u_2}{1-q^2}\right)_{x_2} + \left(\dfrac{iu_1u_3}{1-q^2}\right)_{x_3} = 0, \\ \left(\dfrac{iu_2}{1-q^2}\right)_t + \left(\dfrac{iu_1u_2}{1-q^2}\right)_{x_1} + \left(\dfrac{iu_2^2}{1-q^2}+p\right)_{x_2} + \left(\dfrac{iu_2u_3}{1-q^2}\right)_{x_3} = 0, \\ \left(\dfrac{iu_3}{1-q^2}\right)_t + \left(\dfrac{iu_1u_3}{1-q^2}\right)_{x_1} + \left(\dfrac{iu_2u_3}{1-q^2}\right)_{x_2} + \left(\dfrac{iu_3^2}{1-q^2}+p\right)_{x_3} = 0, \\ \left(\dfrac{iq^2}{1-q^2}+\rho\right)_t + \left(\dfrac{iu_1}{1-q^2}\right)_{x_1} + \left(\dfrac{iu_2}{1-q^2}\right)_{x_2} + \left(\dfrac{iu_3}{1-q^2}\right)_{x_3} = 0, \end{cases} \tag{1.1}$$

where $(u_1, u_2, u_3)$ denotes the velocity of the fluid in three-dimensional space and $q = \sqrt{u_1^2 + u_2^2 + u_3^2}$ is required to be less than the speed of light which is normalised to be one.

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In system (1.1), the thermodynamic quantities $n, p, e, T, S$, $\rho = n + e$, and $i = \rho + p$ denote the average rest-mass density, the pressure, the internal energy per unit volume, the absolute temperature, the entropy per particle, the total mass-energy per unit volume, and the enthalpy per unit volume, respectively. The readers can see [13, 14, 23] for the details.

There are many results about Cauchy problems for the one-dimensional (1D) relativistic Euler equations. Smoller and Temple [21] considered the system of conservation laws in energy and momentum in special relativity

$$\begin{cases} \left(\frac{iu^2}{1-u^2} + \rho\right)_t + \left(\frac{iu}{1-u^2}\right)_x = 0, \\ \left(\frac{iu}{1-u^2}\right)_t + \left(\frac{iu^2}{1-u^2} + p\right)_x = 0 \end{cases} \tag{1.2}$$

with the equation of state $p = \sigma^2 \rho$, where $\sigma$ is a positive constant smaller than one. They solved the Riemann problem and the Cauchy problem for the system. Martí and Müller [17] also studied the Riemann problem for (1.2) with the equation of state $p = \sigma^2 \rho$. Chen [5], Hsu et al. [10], and Li et al. [1, 15] extended Smoller and Temple's results to more general equations of state $p = p(\rho)$. Chen and Yang [3] studied the Riemann problem of (1.2) for a Chaplygin gas equation of state.

Li et al. [2, 16] studied the Cauchy problem for the system of conservation laws in baryon number and momentum in special relativity

$$\begin{cases} \left(\frac{n}{\sqrt{1-u^2}}\right)_t + \left(\frac{nu}{\sqrt{1-u^2}}\right)_x = 0, \\ \left(\frac{iu}{1-u^2}\right)_t + \left(\frac{iu^2}{1-u^2} + p\right)_x = 0, \end{cases} \tag{1.3}$$

with the equation of state $p = p(\rho)$ that satisfies $p'(\rho) > 0$ and $p''(\rho) \geq 0$. Chen and Yang [4] studied the Riemann problem of system (1.3) for the Chaplygin gas.

Chen [6] studied the Riemann problem for the 1D full-relativistic Euler equations

$$\begin{cases} \left(\frac{n}{\sqrt{1-u^2}}\right)_t + \left(\frac{nu}{\sqrt{1-u^2}}\right)_x = 0, \\ \left(\frac{iu}{1-u^2}\right)_t + \left(\frac{iu^2}{1-u^2} + p\right)_x = 0, \\ \left(\frac{iu^2}{1-u^2} + \rho\right)_t + \left(\frac{iu}{1-u^2}\right)_x = 0, \end{cases} \tag{1.4}$$

with the equation of state $p = Sn^\gamma$, where $\gamma$ is the adiabatic constant. More recently, Wissman [24] extended Smoller and Temple's result to the system (1.4) with the equation of state $p = \sigma^2 \rho$.

The global existence of entropy solution to the Cauchy problem for the multi-dimensional relativistic Euler equations is still a complicated open problem. Thus, it has been profitable to consider some special problems, such as multi-dimensional Riemann problems which refer to Cauchy problems with special initial data that are constant along each ray from the origin. However, as far as we know, there are few results about the global existence of entropy solution to the multi-dimensional Riemann problems for the relativistic Euler equations. In this paper, we

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consider Riemann initial-boundary value problems for the multi-dimensional relativistic Euler equations with radial symmetry.

The conservation laws in energy and momentum in special relativity with radial symmetry have the form

$$\begin{cases} \left(\dfrac{iu^2}{1-u^2}+\rho\right)_t+\left(\dfrac{iu}{1-u^2}\right)_r+\dfrac{(N-1)iu}{(1-u^2)r}=0, \\ \left(\dfrac{iu}{1-u^2}\right)_t+\left(\dfrac{iu^2}{1-u^2}+p\right)_r+\dfrac{(N-1)iu^2}{(1-u^2)r}=0, \end{cases} \tag{1.5}$$

where $N=2$ (or 3) represents the dimension of space. We consider (1.5) with the initial and boundary conditions

$$(u,p)(0,r)=(u_0,p_0), \quad (\rho u)(t,0)=0. \tag{1.6}$$

We take the ultra-relativistic fluid, of which the equation of state has the form

$$p=\frac{1}{3}\rho.$$

We look for self-similar solutions that depend only on the self-similar variable $\xi=r/t$. The main result can be stated as the following theorem:

**Theorem 1.1** *For any datum $(u_0,p_0)$ with $p_0>0$ and $u_0\in(-1,1)$, the problem (1.5), (1.6) admits a unique self-similar entropy solution. Moreover,*

- *if $u_0\in(0,1)$, then the solution is continuous (see Figure 1(a));*
- *if $u_0\in(-1,0)$, then the solution contains a single shock followed by a constant state (see Figure 1(c)).*

We also consider the radially symmetric full-relativistic Euler equations

$$\begin{cases} \left(\dfrac{n}{\sqrt{1-u^2}}\right)_t+\left(\dfrac{nu}{\sqrt{1-u^2}}\right)_r+\dfrac{(N-1)nu}{r\sqrt{1-u^2}}=0, \\ \left(\dfrac{iu}{1-u^2}\right)_t+\left(\dfrac{iu^2}{1-u^2}+p\right)_r+\dfrac{(N-1)iu^2}{(1-u^2)r}=0, \\ \left(\dfrac{iu^2}{1-u^2}+\rho\right)_t+\left(\dfrac{iu}{1-u^2}\right)_r+\dfrac{(N-1)iu}{(1-u^2)r}=0 \end{cases} \tag{1.7}$$

with the initial and boundary conditions

$$(u,p,S)(0,r)=(u_0,p_0,S_0), \quad (\rho u)(t,0)=0. \tag{1.8}$$

Here, the equation of state for (1.7) is given by

$$\rho=n+\frac{p}{\gamma-1}, \quad p=Sn^\gamma, \tag{1.9}$$

where $\gamma$ is a constant between 1 and $5/3$. The main result can be stated as the following theorem:

**Theorem 1.2** *For any datum $(u_0,p_0,S_0)$ with $p_0>0$, $S_0>0$, and $u_0\in(-1,1)$, the problem (1.7), (1.8) admits a self-similar entropy solution. Moreover,*

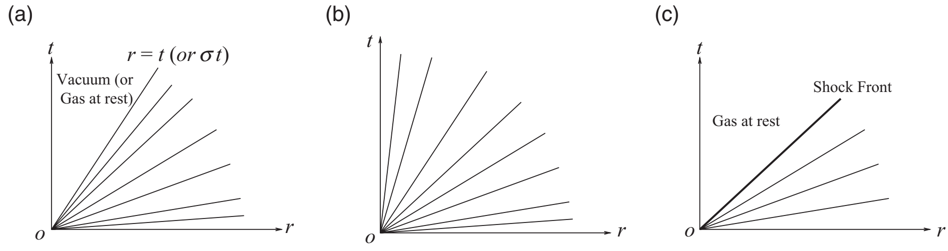


FIGURE 1. Self-similar solutions of the relativistic Euler equations with radial symmetry.

- *if* $u_0 \in (0, 1)$*, then the solution is continuous (see Figure 1(a) and (b));*
- *if* $u_0 \in (-1, 0)$*, then the solution contains a single shock followed by a constant state (see Figure 1(c)).*

The Riemann initial-boundary value problems can be seen as special multi-dimensional Riemann problems with data that possess a certain symmetry. Consider, e.g., (1.1) with the Riemann initial data

$$\big(n, S, u_1, u_2, u_3\big)(0, x_1, x_2, x_3) = \big(n_0, S_0, u_0 \sin\varphi \cos\theta, u_0 \sin\varphi \sin\theta, u_0 \cos\varphi\big), \tag{1.10}$$

where $(x_1, x_2, x_3) = (r \sin\varphi \cos\theta, r \sin\varphi \sin\theta, r \cos\varphi)$, $r > 0$ is the radial variable, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi)$, and $n_0 > 0$ and $S_0 > 0$ and $u_0 \in (-1, 1)$ are three constants. The problem (1.1), (1.10) allows us to look for radially symmetric solution

$$n = n(r, t), \quad S = S(r, t), \quad u_1 = u(t, r) \sin\varphi \cos\theta, \quad u_2 = u(t, r) \sin\varphi \sin\theta, \quad u_3 = u(t, r) \cos\varphi.$$

We can then reduce (1.1) to (1.7) with $N = 3$.

Another problem that admits self-similar solutions is the 'spherical piston' problem which describes the wave motion produced by a circle (sphere) which expands with constant speed into a quiet gas; see Coruant and Friedrichs [7]. This problem was first worked out by Taylor [22]. We also refer the reader to [9, 19] and the references cited therein for more related works. We intend to generalise the result about the spherical piston problem for the Euler equations to the relativistic Euler equations.

We consider (1.5) with the initial and boundary conditions

$$(u, p)(0, r) = (0, p_0), \quad u(t, \alpha t) = \alpha, \tag{1.11}$$

where $\alpha \in (0, 1)$ represents the speed of expansion of the circle (sphere). We obtain the following theorem:

**Theorem 1.3** *For any* $p_0 > 0$ *and* $\alpha \in (0, 1)$*, the spherical piston problem (1.5), (1.11) admits a unique self-similar entropy solution; see Figure 2.*

We also consider (1.7) with the initial and boundary conditions

$$(u, p, S)(0, r) = (0, p_0, S_0), \quad u(t, \alpha t) = \alpha, \tag{1.12}$$

where $\alpha \in (0, 1)$. We obtain the following theorem:

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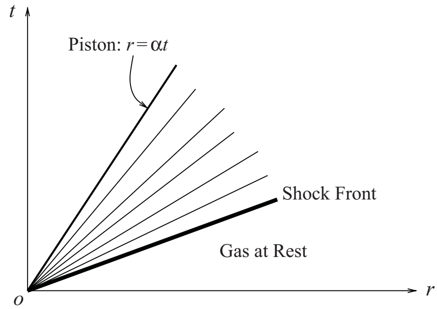


FIGURE 2. Self-similar solution of the spherical piston problem.

**Theorem 1.4** *For any $p_0 > 0$, $S_0 > 0$, and $\alpha \in (0, 1)$, the spherical piston problem (1.7), (1.12) admits a self-similar solution; see Figure 2.*

The self-similar solutions of the radially symmetric Euler equations for polytropic gases were first studied by Taylor, et al.; see [7] and the survey paper [12]. The readers can also see Zheng et al. [25, 26] for further examples of self-similar flows with swirl. They used the assumption of self-similarity to reduce the Euler equations for polytropic gases to a system of nonlinear autonomous ordinary differential equations. However, for the relativistic Euler equations, the ordinary differential equations derived by self-similar transformation are not autonomous and are quiet complex. That is the main difficulty of the present paper.

The spherical piston problem for (1.5) with the equation of state $p = \sigma^2\rho$ was first solved by Ding and Li [8]. In the present paper, we solve the spherical piston problem for (1.5) with general convex equations of state and for the full-relativistic Euler equations (1.7). Here, the convex equations of state are referred to the equations of state that satisfy $(p+\rho)p''(\rho) + 2p'(\rho)(1 - p'(\rho)) > 0$. There are also some other related work. For more general existence of weak solutions with radially symmetry for system (1.1) outside a core region, we refer the reader to [11, 18].

The rest of the paper is organised as follows. Section 2 is devoted to solve the problem (1.5), (1.6) for the ultra-relativistic fluid. Section 3 is devoted to solve the problem (1.7), (1.8). Actually, using the approach of Section 3, one can solve the problem (1.5), (1.6) for more general convex equations of state. Section 4 is devoted to solve the spherical piston problem for the relativistic Euler equations.

## 2 Self-similar solutions of the radially symmetric ultra-relativistic Euler equations

### 2.1 Ordinary differential equations

Since the problem (1.5), (1.6) is invariant under self-similar transformation, we look for self-similar solutions that depend only on $\xi = r/t$. By self-similar transformation, we have

$$\begin{cases} -\xi\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{iu}{1-u^2}\right) + \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{iu^2}{1-u^2}\right) + \dfrac{\mathrm{d}p}{\mathrm{d}\xi} + \dfrac{(N-1)iu^2}{(1-u^2)\xi} = 0, \\ -\xi\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{i}{1-u^2}\right) + \xi\dfrac{\mathrm{d}p}{\mathrm{d}\xi} + \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{iu}{1-u^2}\right) + \dfrac{(N-1)iu}{(1-u^2)\xi} = 0. \end{cases}$$

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By computations, we get

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}\xi} = -\dfrac{(N-1)iu(u-\xi)}{\xi\Big\{\big[\xi(1-u^2)+4(u-\xi)\big](u-\xi)+(1+u^2-2u\xi)(u\xi-1)\Big\}}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}\xi} = -\dfrac{(N-1)(u\xi-1)u(1-u^2)}{\xi\Big\{\big[\xi(1-u^2)+4(u-\xi)\big](u-\xi)+(1+u^2-2u\xi)(u\xi-1)\Big\}}, \end{cases} \tag{2.1}$$

where we have used the equation of state $p = \frac{1}{3}\rho$ of the ultra-relativistic fluid. The initial and boundary conditions (1.6) become

$$\lim_{\xi\to+\infty}(u,p)(\xi) = (u_0,p_0), \quad (\rho u)\,|_{\xi=0} = 0. \tag{2.2}$$

Let $s = 1/\xi$. We thus have the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{(N-1)(u-s)u(1-u^2)}{f}, \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} = \dfrac{4(N-1)up(us-1)}{f}, \end{cases} \tag{2.3}$$

$$(u,p)\,|_{s=0} = (u_0,p_0), \tag{2.4}$$

where

$$\begin{aligned} f &= \big[(1-u^2)+4(us-1)\big](us-1)+\big[(1+u^2)s-2u\big](u-s) \\ &= 3(us-1)^2-(u-s)^2. \end{aligned}$$

In order to solve the problem (2.3) and (2.4), we first study the initial value problem

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{(N-1)(u-s)u(1-u^2)}{f}, \quad u\,|_{s=0} = u_0, \tag{2.5}$$

and then $p = p(s)$ can be obtained by integrating the second equation of (2.3). The sign of $f$ is important in the following discussions.

**Remark 2.1** *We have the following conclusions about* $f$*:*

(1) *if* $0 < s \le 1$ *and* $-1 < u < \frac{\sqrt{3}+s}{\sqrt{3}s+1}$*, then* $f > 0$*;*

(2) *if* $s > 1$ *and* $u < \frac{\sqrt{3}-s}{s\sqrt{3}-1}$*, then* $f > 0$*;*

(3) *if* $s > 1$ *and* $\frac{\sqrt{3}-s}{\sqrt{3}s-1} < u < \frac{\sqrt{3}+s}{\sqrt{3}s+1}$*, then* $f < 0$*;*

*see Figure 3.*

**Lemma 2.1** *The initial value problem (2.3) has a unique solution in* $(0,1)$ *for any* $0 < u_0 < 1$*. Moreover, this solution satisfies*

$$0 < u(s) < 1 \quad \text{and} \quad \frac{\mathrm{d}u}{\mathrm{d}s}\begin{cases} > 0, & u > s; \\ = 0, & u = s; \\ < 0, & u < s \end{cases} \quad \text{as} \quad 0 < s < 1. \tag{2.6}$$

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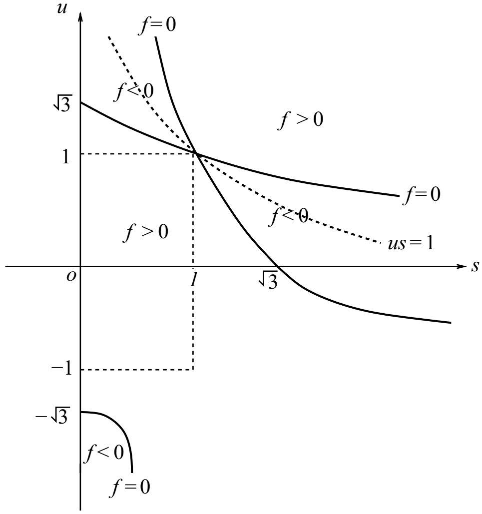


FIGURE 3. $f(u,s)=0$.

**Proof** The problem (2.3) is classically well-posed which has a unique local solution $u=u(s)$. We then prove that the solution satisfies $0<u(s)<1$. If there exists a $s_*$ such that $u(s)<1$ as $s<s_*<1$ and $u(s_*)=1$, then

$$\int_{u(s_*-\varepsilon)}^{1} \frac{1}{1-u}\,\mathrm{d}u = \int_{s_*-\varepsilon}^{s_*} \frac{(N-1)(u-s)u(1+u)}{f}\,\mathrm{d}s, \tag{2.7}$$

where $\varepsilon>0$ is sufficiently small. The left part of (2.7) is infinite, while the right part is finite. This leads to a contradiction. Similarly, if there exists a $s_*$ such that $u(s)>0$ as $s<s_*<1$ and $u(s_*)=0$, then

$$\int_{u(s_*-\varepsilon)}^{0} \frac{1}{u}\,\mathrm{d}u = \int_{s_*-\varepsilon}^{s_*} \frac{(N-1)(u-s)(1-u^2)}{f}\,\mathrm{d}s, \tag{2.8}$$

which leads to a contradiction.

By Remark 2.1(1), we know that the local solution can be extended to (0, 1) and satisfies $\frac{\mathrm{d}u}{\mathrm{d}s}>0$ as $u>s$; $=0$ as $u=s$; $<0$ as $u<s$. We then complete the proof of this lemma. □

## 2.2 Continuous solution for $u_0\in(0,1)$

We are going to show that the problem (1.5), (1.6) has a continuous self-similar solution for any $u_0\in(0,1)$. Interestingly, when $u_0$ is sufficiently large the wave structures for $N=2$ and $N=3$ are different.

### 2.2.1 $N=2$

**Lemma 2.2** *Assume $N=2$. Then, for any $u_0\in(0,1)$, there exists a $s_0\in(0,1)$ such that the solution of the problem (2.5) satisfies $s<u(s)<1$ as $0<s<s_0$ and $u(s_0)=s_0$; see Figure 4(left).*

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**Proof** If $0 < s < u < 1$, then we have

$$1 - us > 1 - s > 0 \quad \text{and} \quad 0 < u - s < 1 - s,$$

and hence

$$3(us-1)^2 - (u-s)^2 > 2(1-s)^2.$$

Thus, if $0 < s < u < 1$, then

$$\frac{\mathrm{d}u}{\mathrm{d}s} < \frac{(u-s)(1-u)}{(1-s)^2}. \tag{2.9}$$

We now consider the initial value problem

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}s} = \frac{(\bar{u}-s)(1-\bar{u})}{(1-s)^2}, \quad \bar{u}\,|_{s=0} = u_0. \tag{2.10}$$

It is obviously that the problem (2.10) admits a solution $u = \bar{u}(s)$ in $(0, 1)$ and $0 < \bar{u}(s) < 1$ as $0 < s < 1$, as shown in (2.7) and (2.8). By computation, we have

$$\frac{\mathrm{d}^2\bar{u}}{\mathrm{d}s^2} = \frac{(1-\bar{u})^2(2\bar{u}-s-1)}{(1-s)^4}. \tag{2.11}$$

If $u_0 \in (0, \frac{1}{2})$, then by (2.11) we know that $\bar{u}(s) < \frac{1+s}{2}$ and $\bar{u}''(s) < 0$ and $\bar{u}'(s) < u_0$ as $0 < s < 1$. Hence, there exists a $\bar{s}_0$ such that $\bar{u}(\bar{s}_0) = \bar{s}_0$.

We next prove that if $u_0 \in [\frac{1}{2}, 1)$ there also exist a $\bar{s}_0$ such that $\bar{u}(\bar{s}_0) = \bar{s}_0$. We shall prove this by contradiction. Suppose that there exists a $u_0 \in [\frac{1}{2}, 1)$ such that the solution of (2.10) satisfies $\bar{u}(s) > s$ as $0 < s < 1$. Then, by (2.11) we have

$$\bar{u}(s) > \frac{1+s}{2} \quad \text{as} \quad 0 < s < 1. \tag{2.12}$$

That is because if there exists a $s^0 \in (0, 1)$ such that $\bar{u}(s^0) = \frac{1+s^0}{2}$, then we have $\bar{u}'(s^0) = 1/4$ and $\bar{u}''(s) < 0$ as $s^0 < s < 1$. Consequently, $\bar{u}(s) < \frac{1+s^0}{2} + \frac{(s-s^0)}{4} < s$ as $s$ is sufficiently close to 1, which leads to a contradiction. From (2.11) and (2.12), we have

$$\bar{u}''(s) > 0 \quad \text{as} \quad s \in (0, 1). \tag{2.13}$$

Thus, we can define $d := \lim_{s\to 1} \frac{\mathrm{d}\bar{u}}{\mathrm{d}s}$. By (2.12), (2.13), and $0 < \bar{u}(s) < 1$ as $0 < s < 1$, we have

$$0 < u_0(1-u_0) < d \le \frac{1}{2}. \tag{2.14}$$

While, by a direct computation, we have

$$d = \lim_{s\to 1} \frac{(\bar{u}-s)(1-\bar{u})}{(1-s)^2} = \lim_{s\to 1} \frac{[(1-s)(1-d) + \mathrm{o}(1-s)][d(1-s) + \mathrm{o}(1-s)]}{(1-s)^2} = d(1-d),$$

and hence $d = 0$, which contradicts to (2.14). Therefore, when $u_0 \in [\frac{1}{2}, 1)$ there also exists a $\bar{s}_0$ such that $\bar{u}(\bar{s}_0) = \bar{s}_0$.

By comparison principle, we have $0 < u(s) < \bar{u}(s)$ as $0 < s < 1$. Hence, there exists a $s_0 \in (0, 1)$ such that $u(s_0) = s_0$. We then complete the proof of this lemma. □

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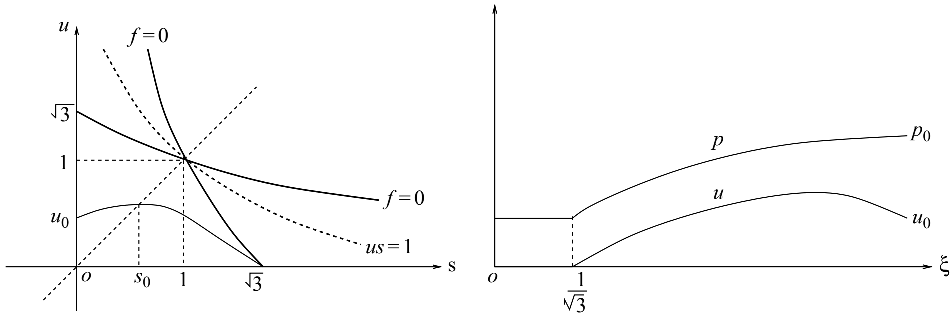


FIGURE 4. A continuous solution of the ultra-relativistic Euler equations with radial symmetry.

From Remark 2.1, we can see that $u'(s) < 0$ and $u(s) > 0$ as $s_0 < s \leq 1$. In what follows, we are going to discuss the solution of the problem (2.5) for $s > 1$.

**Lemma 2.3** *Assume $N = 2$. Then, for any $u_0 \in (0, 1)$, the solution of the problem (2.5) satisfies $u(s) > 0$ as $s < \sqrt{3}$ and $u(\sqrt{3}) = 0$; see Figure 4(left).*

**Proof** It suffices to prove that for any $u_0 \in (0, 1)$, the solution of the problem (2.5) satisfies

$$0 < u(s) < \frac{\sqrt{3} - s}{\sqrt{3}s - 1} \quad \text{as} \quad 1 \leq s < \sqrt{3}; \tag{2.15}$$

see Figure 4.

As shown in (2.8), we have $u(s) > 0$ as $1 \leq s < \sqrt{3}$.

Let $G(s) := u(s) - \frac{\sqrt{3}-s}{\sqrt{3}s-1}$. Then by $u(1) < 1$, we have $G(1) = u(1) - 1 < 0$. Suppose there exists a point $s' \in (1, \sqrt{3})$ such that $G(s') = 0$ and $G(s) < 0$ as $1 < s < s'$. Then we have $G'(s') \geq 0$. However, by a direct computation, we get

$$\lim_{s \to (s')^-} G'(s) = \lim_{s \to (s')^-} \frac{\mathrm{d}}{\mathrm{d}s}\left(u(s) - \frac{\sqrt{3} - s}{\sqrt{3}s - 1}\right) = -\infty,$$

which leads to a contradiction. We then have this lemma. □

From (2.3), we have

$$\frac{\mathrm{d}p}{\mathrm{d}u} = \frac{4p(us - 1)}{(u - s)(1 - u^2)}. \tag{2.16}$$

By integration, we obtain

$$\ln p(\sqrt{3}) - \ln p(1) = \int_{u(1)}^{0} \frac{4(us - 1)}{(u - s)(1 - u^2)}\, \mathrm{d}u, \tag{2.17}$$

where $s = s(u)$ can be seen as the inverse function of $u = u(s)$. Thus, we get $p(\sqrt{3}) > 0$.

We are now ready to construct the self-similar solution of the problem (1.5), (1.6) for $u_0 > 0$ and $N = 2$. When $\xi > 1/\sqrt{3}$, the solution is determined by the classical solution of the initial value problem (2.3), (2.4). We continue the solution by the constant state $(u, p) = (0, p(\sqrt{3}))$ where $p(\sqrt{3})$ is determined by (2.17). This is a continuous extension; see Figure 4(right).

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2.2.2 $N = 3$

**Lemma 2.4** *Assume $N = 3$. Then there exists a $u_* \in (0, 1)$, such that when $u_0 > u_*$ the solution of the problem (2.5) satisfies $u(1) = 1$; see Figure 5(left).*

**Proof** Let $\beta \in (0, 1)$ be satisfied $\beta^3 + 6\beta = 6$. Then we have

$$\begin{aligned}
&3(us-1)^2 - (u-s)^2\\
&< 3\big[(\beta(1-s)+s)s-1\big]^2 - \beta^2(1-s)^2\\
&= (1-s)^2\Big\{3\big[(\beta-1)s-1\big]^2 - \beta^2\Big\}\\
&< (1-s)^2\Big\{3\big[(\beta-1)-1\big]^2 - \beta^2\Big\}\\
&= 2\beta^2(1+\beta)(1-s)^2
\end{aligned}$$

as $(s, u) \in \{(s, u) \mid 0 < s < 1, \beta(1-s) + s \le u < 1\}$. Thus,

$$\frac{2(u-s)u(1-u^2)}{3(us-1)^2-(u-s)^2} > \frac{(u-s)u(1+u)(1-u)}{\beta^2(1+\beta)(1-s)^2} > \frac{(u-s)(1-u)}{\beta(1-s)^2}$$

as $(s, u) \in \{(s, u) \mid 0 < s < 1, \beta(1-s) + s \le u < 1\}$.

We now consider the initial value problem

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}s} = \frac{(\bar{u}-s)(1-\bar{u})}{\beta(1-s)^2}, \quad \bar{u}(0) = \beta. \tag{2.18}$$

It is easy to check that $\bar{u} = \beta(1-s) + s$ is the unique solution of the problem (2.18).

Thus, by comparison principle we have that if $u_0 \in (\beta, 1)$ then the solution of the problem (2.5) satisfies

$$u(s) > \beta(1-s) + s \quad \text{as} \quad 0 < s < 1.$$

Combining this with (2.6) we can get this lemma. □

**Lemma 2.5** *Assume $N = 3$. If the solution of the initial value problem (2.3), (2.4) satisfies $u(1) = 1$, then we have $p(1) = 0$.*

**Proof** By the previous results, we know that if $u(1) = 1$ then $u(s) > s$ as $0 < s < 1$.

Along the integral curve of (2.3), (2.4) we have

$$\int_{p_0}^{p(y)} \frac{1}{4p}\,\mathrm{d}p = \int_{u_0}^{u(y)} \frac{(us-1)}{(u-s)(1-u^2)}\,\mathrm{d}u \quad \text{as} \quad 0 < y < 1.$$

(Remark: in the right integration, $s = s(u)$ is the inverse function of $u = u(s)$.)

By the previous results, we know that if $u(1) = 1$ then $u(s) > s$ as $0 < s < 1$. Thus, we have $\frac{us-1}{u-s} < -1$ along $u = u(s)$ $(0 < s < 1)$, and consequently

$$\int_{u_0}^{u(y)} \frac{(us-1)}{(u-s)(1-u^2)}\,\mathrm{d}u < -\int_{u_0}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u \quad \text{as} \quad 0 < y < 1.$$

Therefore, there must have $p(1) = 0$. We then have this lemma. □

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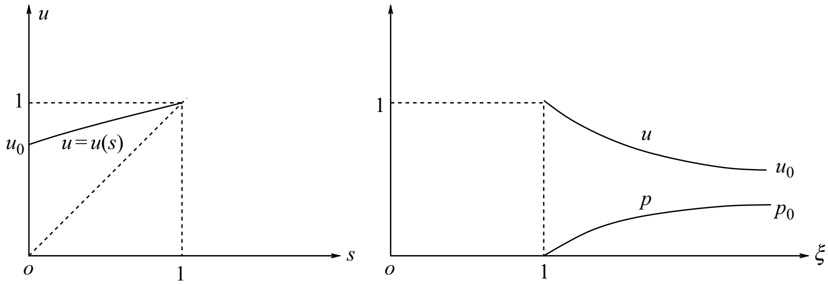


FIGURE 5. A continuous solution with a growing vacuum region.

**Remark 2.2** *Lemmas 2.4 and 2.5 imply that for* $N=3$*, if* $u_0<1$ *is sufficiently large, then the fluid will expand to vacuum and the speed of the fluid at* $\xi=1$ *is just the light speed.*

We are now ready to construct the self-similar solution to the problem (1.5), (1.6) in the case of $u(1)=1$. When $\xi>1$, the solution is determined by the classical solution of the initial value problem (2.3), (2.4) . We continue the solution by a vacuum state for $\xi<1$; see Figure 5(right).

### 2.3 Shock wave solution for $u_0 \in (-1, 0)$

By Remark 2.1, we know that the problem (2.5) has a solution in $(0, \sqrt{3})$. Moreover, this solution satisfies $u_0<u(s)<0$ and $\frac{\mathrm{d}u}{\mathrm{d}s}>0$ as $0<s<\sqrt{3}$. In what follows, we are going to show that there exists a $s_*>\sqrt{3}$ such that $\lim\limits_{s\to s_*}\frac{\mathrm{d}u}{\mathrm{d}s}=+\infty$. To start off with, we prove the following lemma.

**Lemma 2.6** *Let* $a>1$ *be a constant. Then the solution of the initial value problem*

$$\frac{\mathrm{d}v}{\mathrm{d}s}=\frac{(N-1)(v-s)v(1-v^2)}{a^2(vs-1)^2-(v-s)^2},\qquad v\,|_{s=0}=u_0\in(-1,0)$$

*satisfies* $v<0$ *as* $0<s\le a$.

**Proof** Since $a^2(vs-1)^2-(v-s)^2>0$ as $-1<v<0$ and $0<s<a$, we have $u_0<v<0$ as $0<s<a$.

In what follows we shall prove $v(a)<0$. Let

$$l=(N-1)(v-s)v(1-v^2)=(N-1)\big(-av+v^2-v(s-a)+av^3+v^3(s-a)-v^4\big)$$

and

$$\begin{aligned}m&=a^2(vs-1)^2-(v-s)^2\\&=a^2-v^2+a^2v^2s^2-s^2-(2a^2-2)vs\\&=-2a(s-a)-a(2a^2-2)v+(a^4-1)v^2-(s-a)^2-(2a^2-2)v(s-a)\\&\quad+2a^3(s-a)v^2+a^2v^2(s-a)^2.\end{aligned}$$

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Then, by a direct computation, we have

$$\frac{\mathrm{d}^2 v}{\mathrm{d}s^2} = \frac{(ml_v - lm_v)l + (ml_s - lm_s)m}{m^3}$$
$$= \frac{v\Big(4a^3(a^2-1)(N-1)v + 2a^3(N-1)(3-N)(s-a) + o(\sqrt{v^2+(s-a)^2})\Big)}{m^3}. \tag{2.19}$$

Assume that $v(a) = 0$. Then by (2.19), we know that there exists a $s' < a$ such that

$$\frac{\mathrm{d}^2 v}{\mathrm{d}s^2} > 0 \quad \text{as} \quad s \in (s', a). \tag{2.20}$$

Then by (2.20), we know that $\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} = +\infty$ or $\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} = d_0 > 0$ where $d_0$ is a finite number. Writing $v(s) = d(s)(s-a)$. Then by (2.20), we also have that $d(s)$ is monotonic increasing in $(s', a)$.

If $\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} = +\infty$, then we have $\lim_{s\to a} d(s) = +\infty$. Consequently, we have

$$\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} = \lim_{s\to a} \frac{(N-1)(v^2 - sv - v^4 + sv^3)}{a^2 - v^2 + a^2v^2s^2 - s^2 - (2a^2-2)vs}$$
$$= \lim_{s\to a} \frac{(N-1)(-sd + vd - v^3 d + v^2 sd)}{-(2a^2-2)sd - (a+s) - vd + a^2s^2vd} = \frac{N-1}{2a^2-2} < +\infty$$

which leads to a contradiction. So, $\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} \neq +\infty$. If $\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} = d_0$, then we have $\lim_{s\to a} d(s) = d_0$. Consequently, we have

$$\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s} = \lim_{s\to a} \frac{(N-1)(-sd + vd - v^3 d + v^2 sd)}{-(2a^2-2)sd - (a+s) - vd + a^2s^2vd} = \frac{d_0(N-1)}{(2a^2-2)d_0 + 2} \neq d_0,$$

since $a > 1$. This leads to a contradiction. So, $\lim_{s\to a} \frac{\mathrm{d}v}{\mathrm{d}s}$ can also not be a finite positive number. Therefore, $v(a) \neq 0$. We then complete the proof of this lemma. □

**Remark 2.3** *There is another way to prove Lemma 2.6. Let us consider the ordinary system*

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}t} = (N-1)(v-s)v(1-v^2), \\ \dfrac{\mathrm{d}s}{\mathrm{d}t} = a^2(vs-1)^2 - (v-s)^2. \end{cases} \tag{2.21}$$

*At the point $(v, s) = (0, a)$, we find that the linear part of the right-hand side of (2.21) is given by $M(v, s-a)^T$ where*

$$M = \begin{pmatrix} -a(N-1) & 0 \\ 2a - 2a^3 & -2a \end{pmatrix}.$$

*When $N = 2$, the matrix has two eigenvalues $\lambda_1 = -a$ and $\lambda_2 = -2a$ with associated eigenvectors $(1, 0)$ and $(2a^2-2, 1)$. So, along the integral cures of (2.21) we have $\frac{\mathrm{d}s}{\mathrm{d}v} \to 2 - 2a^2$ as $(v, s) \to (0, a)$ and $v \neq 0$; see Figure 6(left). When $N = 3$, the matrix has eigenvalues $\lambda_1 = \lambda_2 = -2a$. Since $2a - 2a^3 < 0$, along the integral curves of (2.21), we have $\frac{\mathrm{d}s}{\mathrm{d}v} \to -\infty$ as $(v, s) \to (0, a)$ and $v \neq 0$; see Figure 6(right).*

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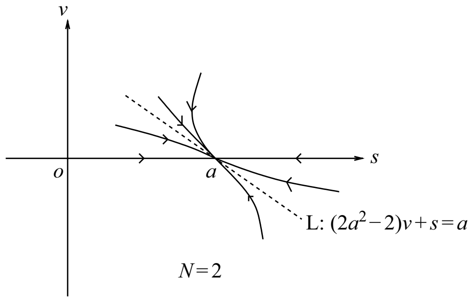


FIGURE 6. The integral curves of $\frac{\mathrm{d}v}{\mathrm{d}s} = \frac{(N-1)(v-s)v(1-v^2)}{a^2(vs-1)^2-(v-s)^2}$ near the point $(a, 0)$.

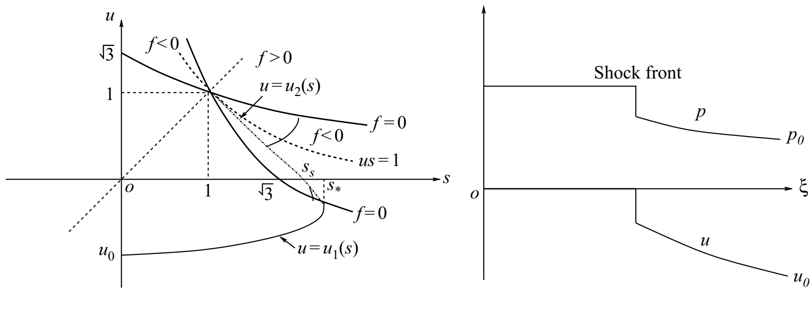


FIGURE 7. Shock wave solution for $u_0 \in (-1, 0)$.

**Lemma 2.7** *For any $u_0 \in (-1, 0)$, there exists a $s_* > \sqrt{3}$ such that the solution of the problem (2.5) satisfies $u(s) < \frac{s-\sqrt{3}}{1-\sqrt{3}s}$ as $\sqrt{3} < s < s_*$ and $u(s_*) = \frac{s_*-\sqrt{3}}{1-\sqrt{3}s_*}$; see Figure 7(left).*

**Proof** Suppose that the integral curve $u = u(s)$ and the curve $u = \frac{s-\sqrt{3}}{1-\sqrt{3}s}$ do not intersect. Then by $\frac{\mathrm{d}u}{\mathrm{d}s} > 0$ we know that there exists a $\hat{u} \in (u_0, -\sqrt{3}/3)$ such that $\lim_{s\to+\infty} u(s) = \hat{u}$. Thus, we have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{(N-1)(u-s)u(1-u^2)}{3(us-1)^2-(u-s)^2} > \frac{-(N-1)s\hat{u}(1-u_0^2)}{3(u_0 s-1)^2} \quad \text{as} \quad s > 0.$$

Integrating this from $s = 0$ to $s = +\infty$, we get

$$\lim_{s\to+\infty} u(s) = +\infty,$$

which leads to a contradiction. We then have this lemma. □

From the second equation of (2.3), we can obtain $p(s)$ $(0 < s < s_*)$. From (2.16) we also have that $\lim_{s\to s_*} p(s)$ exists.

From Lemma 2.7, we know that no continuous solutions exist in the case of $u_0 \in (-1, 0)$. We need to look for shock wave solutions. In the following discussions, we shall denote by $(u_1, p_1)(s)$ $(0 < s < s_*)$ the solution of the problem (2.3), (2.4) for $u_0 \in (-1, 0)$.

Assume that there is shock wave with the speed $\xi = 1/s \in (1/s_*, 1)$ where $s_*$ is determined by Lemma 2.7. We make the Lorentz transformation

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$$\bar{r} = \frac{r - \xi t}{\sqrt{1-\xi^2}}, \quad \bar{t} = \frac{t - \xi r}{\sqrt{1-\xi^2}}. \tag{2.22}$$

Then in the coordinates $(\bar{r}, \bar{t})$, the shock is stationary. Thus, by Rankine–Hugoniot conditions of the shock waves of (1.5), we have

$$\begin{cases} \dfrac{\rho_1 \bar{u}_1}{1-\bar{u}_1^2} = \dfrac{\rho_2 \bar{u}_2}{1-\bar{u}_2^2}, \\ \dfrac{4}{3}\dfrac{\rho_1 \bar{u}_1^2}{1-\bar{u}_1^2} + p_1 = \dfrac{4}{3}\dfrac{\rho_2 \bar{u}_2^2}{1-\bar{u}_2^2} + p_2; \end{cases} \tag{2.23}$$

see [13, 20]. Here, '1' denotes the fluid in front of the shock, '2' denotes the fluid behind the shock,

$$\bar{u}_k = \frac{u_k - \xi}{1 - \xi u_k} \qquad (k = 1, 2) \tag{2.24}$$

denote the velocity of the fluid relatively to the shock.

In addition to the jump conditions (2.23), the admissible forward shock waves must satisfy the entropy condition

$$\bar{u}_1 < \bar{u}_2 < 0. \tag{2.25}$$

From (2.23), we immediately have

$$\bar{u}_1 \bar{u}_2 = \frac{1}{3}; \tag{2.26}$$

see, e.g., [20].

In what follows, we are going to seek an admissible forward shock wave with an appropriate speed $\xi = 1/s \in (1/s_*, 1)$ and the front side state $(u_1, p_1)(s)$ such that the backside state satisfies $u_2 = 0$.

**Lemma 2.8** *Assume that there is a shock with the speed $\xi = 1/s \in (1/s_*, 1)$ and the front side state $(u_1, p_1)(s)$. The back side state of the shock $(u_2, p_2)(s)$ can be determined by (2.23)–(2.26). Then we have*

$$3\big[u_2(s)s - 1\big]^2 - \big[u_2(s) - s\big]^2 < 0.$$

**Proof** This lemma can be obtained directly by (2.24), (2.26), and (2.25). □

**Lemma 2.9** *Assume that there is a shock with the speed $\xi = 1/s \in (1/s_*, 1)$ and the front side state $(u_1, p_1)(s)$. Then the back side state of the shock $(u_2, p_2)(s)$ satisfies*

$$u_2(s_*) = u_1(s_*) < 0, \qquad \lim_{s \to 1^+} u_2(s) = 1, \qquad \textit{and} \qquad u_2'(s) < 0.$$

**Proof** From (2.24) and (2.26), we have

$$\bar{u}_2(s) = \frac{1}{3}\left(\frac{1 - \xi u_1(s)}{u_1(s) - \xi}\right).$$

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Hence, we have

$$u_2(s) = \frac{\bar{u}_2(s) + \xi}{1 + \xi \bar{u}_2(s)} = \frac{1 + 2\xi u_1(s) - 3\xi^2}{-2\xi + 3u_1(s) - \xi^2 u_1(s)}. \tag{2.27}$$

Consequently, we get $\lim_{s \to 1^+} u_2(s) = 1$.

From Lemma 2.7, we have $u_1(s_*) = \frac{s_* - \sqrt{3}}{1 - \sqrt{3} s_*}$. Hence, we get $\bar{u}_1(s_*) = -\frac{1}{\sqrt{3}}$. Then by (2.26), we have $\bar{u}_2(s_*) = -\frac{1}{\sqrt{3}}$. And consequently, $\lim_{s \to s_*^-} u_2(s) = u_1(s_*)$.

By computation, we have

$$\bar{u}_2'(s) = \frac{-\xi^2 \left(1 - u_1'(s) - u_1^2(s) + \frac{1}{\xi^2} u_1'(s)\right)}{3[u_1(s) - \xi]^2} < 0.$$

Hence, by (2.27), we have

$$u_2'(s) = \frac{\xi^2(\bar{u}_2^2(s) - 1) + (1 - \xi^2)\bar{u}_2'(s)}{[1 + \xi \bar{u}_2(s)]^2} < 0.$$

We then complete the proof of the lemma. □

By Lemma 2.9, we know that there exists one and only one $s_s \in (1, s_*)$ such that $u_2(s_s) = 0$. Therefore, when $u_0 \in (-1, 0)$ the problem (1.5), (1.6) has a discontinuous solution with a single shock. The solution has the form

$$(u, p)(s) = \begin{cases} (u_1, p_1)(s), & s < s_s, \\ \big(0, p_2(s_s)\big), & s > s_s, \end{cases}$$

where $s = t/x$; see Figure 7(right).

One may ask whether the problem has another discontinuous solution with a shock located at $\xi_1 \neq 1/s_s$. If it is possible, then by Lemma 2.8 we know that $3\big(u_2(s_1)s_1 - 1\big)^2 - \big(u_2(s_1) - s_1\big)^2 < 0$. We then consider (2.3) with the data

$$(u, p)\,|_{s=s_1} = (u_2, p_2)(s_1). \tag{2.28}$$

By a method similar to Lemma 2.7, we know that there exists a $s^* > s_1$ such that the solution of the initial value problem (2.3), (2.28) satisfies $\lim_{s \to s^*} u(s) = \infty$; see Figure 7(left). Moreover, since this solution satisfies $3(us - 1)^2 - (u - s)^2 < 0$, for any $s \in (s_1, s^*)$ the state $(u, p)(s)$ cannot be the front side state of any admissible forward shock wave with the speed $1/s$.

At this point, we have completed the proof of Theorem 1.1.

## 3 Self-similar solutions of the radially symmetric full-relativistic Euler equations

### 3.1 Ordinary differential equations

From (1.1) and the law of thermodynamics

$$\mathrm{d}\left(\frac{e}{n}\right) = T\mathrm{d}S - p\mathrm{d}\left(\frac{1}{n}\right),$$

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we can deduce the entropy equation

$$S_{x_0} + \sum_{k=1}^{N} u_k S_{x_k} = 0, \tag{3.1}$$

see Li and Qin [14] for the details. Thus, for smooth flow system (1.7) is equivalent to

$$\begin{cases} \left(\dfrac{iu}{1-u^2}\right)_t + \left(\dfrac{iu^2}{1-u^2} + p\right)_r + \dfrac{(N-1)iu^2}{(1-u^2)r} = 0, \\ \left(\dfrac{i}{1-u^2} - p\right)_t + \left(\dfrac{iu}{1-u^2}\right)_r + \dfrac{(N-1)iu}{(1-u^2)r} = 0, \\ S_t + uS_r = 0. \end{cases} \tag{3.2}$$

Since the problem (1.7), (1.8) is invariant under self-similar transformation, we look for self-similar solutions that depend only on $\xi = r/t$. Then, by self-similar transformation system (3.2) can be changed into

$$\begin{cases} -\xi \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{iu}{1-u^2}\right) + \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{iu^2}{1-u^2}\right) + \dfrac{\mathrm{d}p}{\mathrm{d}\xi} + \dfrac{(N-1)iu^2}{(1-u^2)\xi} = 0, \\ -\xi \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{i}{1-u^2}\right) + \xi\dfrac{\mathrm{d}p}{\mathrm{d}\xi} + \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\dfrac{iu}{1-u^2}\right) + \dfrac{(N-1)iu}{(1-u^2)\xi} = 0, \\ (u-\xi)\dfrac{\mathrm{d}S}{\mathrm{d}\xi} = 0. \end{cases}$$

By tedious derivations, we get

$$\begin{cases} \dfrac{\mathrm{d}p}{\mathrm{d}\xi} = -\dfrac{(N-1)iu(u-\xi)}{\xi\left\{(u-\xi)\left[\xi(1-u^2) + i_p(p,S)(u-\xi)\right] + (1+u^2-2u\xi)(u\xi-1)\right\}}, \\ \dfrac{\mathrm{d}u}{\mathrm{d}\xi} = -\dfrac{(N-1)(u\xi-1)u(1-u^2)}{\xi\left\{(u-\xi)\left[\xi(1-u^2) + i_p(p,S)(u-\xi)\right] + (1+u^2-2u\xi)(u\xi-1)\right\}}, \\ (u-\xi)\dfrac{\mathrm{d}S}{\mathrm{d}\xi} = 0. \end{cases} \tag{3.3}$$

The initial and boundary conditions (1.8) become

$$\lim_{\xi\to+\infty}(u,n,S)(\xi) = (u_0,n_0,S_0), \quad (un)\,|_{\xi=0} = 0. \tag{3.4}$$

Let $s = 1/\xi$. We thus have the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{(N-1)(u-s)u(1-u^2)}{g}, \\ \dfrac{\mathrm{d}p}{\mathrm{d}s} = \dfrac{(N-1)iu(us-1)}{g}, \\ (us-1)\dfrac{\mathrm{d}S}{\mathrm{d}s} = 0, \end{cases} \tag{3.5}$$

$$(u,p,S)\,|_{s=0} = (u_0,p_0,S_0), \tag{3.6}$$

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where

$$
\begin{aligned}
g &= s^2\Big\{(u-\xi)\big[\xi(1-u^2)+i_p(p,S)(u-\xi)\big]+(1+u^2-2u\xi)(u\xi-1)\Big\}\\
&=(1-u^2)(us-1)+i_p(p,S)(us-1)^2+[(1+u^2)s-2u](u-s)\\
&=\rho_p(p,S)(us-1)^2-(u-s)^2.
\end{aligned}
$$

**Lemma 3.1** *The initial value problem (3.5), (3.6) has a unique local solution* $(u(s), p(s), S_0)$ *in* $(0,1)$ *for any* $u_0 \in (-1,1)$*. Moreover, this solution satisfies*

$$p(s)>0 \quad \textit{and} \quad \big(1-|u(s)|\big)|u(s)| \neq 0 \quad \textit{as} \quad 0<s<1.$$

**Proof** The problem (3.5), (3.6) is classically well-posed which has a unique local solution $(u(s), p(s), S_0)$. As shown in (2.7) and (2.8), we have that the solution satisfies $\big(1-|u(s)|\big)|u(s)| \neq 0$.

If there exists a $s_*$ such that $p(s)>0$ as $0<s<s_*$ and $p(s_*)=0$, then along the integral curve of (3.5), (3.6) we have

$$\int_{p(s_*-\varepsilon)}^{0} \frac{\big(\frac{1}{\gamma}S_0^{-\frac{1}{\gamma}}p^{\frac{1-\gamma}{\gamma}}+\frac{1}{\gamma-1}\big)(us-1)^2-(u-s)^2}{\frac{\gamma}{\gamma-1}p+(\frac{p}{S_0})^{\frac{1}{\gamma}}}\,\mathrm{d}p=\int_{s_*-\varepsilon}^{s_*}(N-1)u(us-1)\,\mathrm{d}s \tag{3.7}$$

where $\varepsilon>0$ is sufficiently small. Since $\gamma>1$, the left integration of (3.7) is infinite, while the right is finite. This leads to a contradiction. Thus, we have $p(s)>0$.

Since

$$\rho_p(p,S_0)=\frac{1}{\gamma}S_0^{-\frac{1}{\gamma}}p^{\frac{1-\gamma}{\gamma}}+\frac{1}{\gamma-1}>\frac{1}{\gamma-1}>1, \tag{3.8}$$

we have

$$g=\rho_p(p,S_0)(us-1)^2-(u-s)^2>0 \quad \text{as } -1\le u\le 1 \text{ and } 0<s<1. \tag{3.9}$$

Thus, by $i(p,S_0)=S_0^{-\frac{1}{\gamma}}p^{\frac{1}{\gamma}}+\frac{\gamma p}{\gamma-1}$ we know that the local solution can be extended to $(0,1)$. We then complete the proof of this lemma. □

### 3.2 Continuous solution for $u_0 \in (0,1)$

**Lemma 3.2** *For any* $u_0 \in (0,1)$*, there exists a* $0<s_0<1$ *such that the solution of the initial value problem (3.5), (3.6) satisfies* $u(s_0)=s_0$*; see Figure 8(left).*

**Proof** From (3.9), we have

$$\frac{\mathrm{d}u}{\mathrm{d}s}\begin{cases}>0, & 0<s<u<1;\\ <0, & 0<u<s<1.\end{cases} \tag{3.10}$$

So, if $s_0$ does not exist, then we have $u(s)>s$ as $0<s<1$.

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Suppose that $u(s) > s$ as $0 < s < 1$. Then by $0 < u(s) < 1$ we know that $u(s) \to 1$ as $s \to 1$. Along the integral curve of (3.5) and (3.6), we have

$$\frac{\mathrm{d}p}{\mathrm{d}u} = \frac{i(p, S_0)(us-1)}{(u-s)(1-u^2)}.$$

Hence, using the equation of state (1.9), we have

$$\int_{p_0}^{p(y)} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p = \int_{u_0}^{u(y)} \frac{(us-1)}{(u-s)(1-u^2)}\,\mathrm{d}u \quad \text{for} \quad 0 < y < 1,$$

where $s = s(u)$ can be seen as the inverse function of $u = u(s)$. Since $\frac{us-1}{u-s} < -1$ along $u = u(s)$, we have

$$\int_{u_0}^{u(y)} \frac{(us-1)}{(u-s)(1-u^2)}\,\mathrm{d}u < -\int_{u_0}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u \quad \text{for} \quad 0 < y < 1.$$

Consequently, by $p'(s) < 0$, we get

$$\int_{p_0}^{0} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p \le \int_{p_0}^{p(y)} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p < -\int_{u_0}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u \tag{3.11}$$

for $0 < y < 1$.

Since $\gamma > 1$, the left integration of (3.11) is finite, while the right integration of (3.11) approaches $-\infty$ as $y \to 1$. This leads to a contradiction. We then have this lemma. □

From the last lemma we know that $0 < u(1) < 1$. In what follows, we are going to discuss the solution for $s > 1$. Let

$$h(s) = h(p(s), s) = \frac{s - \sqrt{\rho_p(p(s), S_0)}}{1 - s\sqrt{\rho_p(p(s), S_0)}} \quad \text{as} \quad s \ge 1.$$

We have the following conclusions about the solution for $s > 1$:

- if $p(s) > 0$, then by (3.8) we have

$$h(s) - \frac{1}{s} = \frac{s^2 - 1}{s(1 - s\sqrt{\rho_p(p(s), S_0)})} < 0; \tag{3.12}$$

- if $p(s) > 0$ and $u(s) < h(s)$, then we have

$$\rho_p(p(s), S_0)(u(s)s - 1)^2 - (u(s) - s)^2 > 0. \tag{3.13}$$

**Lemma 3.3** *For any $s > 1$, if $p(s) > 0$ and $u(s) > 0$, then we have $u(s) < h(s)$.*

**Proof** By a direct computation, we have

$$\frac{\mathrm{d}u(s)}{\mathrm{d}s} - \frac{\mathrm{d}h(s)}{\mathrm{d}s} = (N-1)\left(\frac{(u-s)u(1-u^2) - \frac{iu(us-1)(s^2-1)\rho_{pp}}{2\sqrt{\rho_p}(1-s\sqrt{\rho_p})^2}}{\rho_p(us-1)^2 - (u-s)^2}\right) - \frac{1-\rho_p}{(1-s\sqrt{\rho_p})^2}.$$

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By the last lemma, we have $u(1) - h(1) < 0$. Suppose there exists a $s' > 1$ such that $p(s') > 0$, $u(s') > 0$, $u(s') - h(s') = 0$, and $u(s) - h(s) < 0$ as $1 < s < s'$. Then, we have

$$\begin{aligned}
&(u-s)u(1-u^2) - \frac{iu(us-1)(s^2-1)\rho_{pp}}{2\sqrt{\rho_p}(1-s\sqrt{\rho_p})^2} \\
&= u(us-1)\left(\sqrt{\rho_p}(1-u^2) - \frac{i(s^2-1)\rho_{pp}}{2\sqrt{\rho_p}(1-s\sqrt{\rho_p})^2}\right) \\
&= u(us-1)\left(\sqrt{\rho_p}\left(1-\left(\frac{s-\sqrt{\rho_p}}{1-s\sqrt{\rho_p}}\right)^2\right) - \frac{i(s^2-1)\rho_{pp}}{2\sqrt{\rho_p}(1-s\sqrt{\rho_p})^2}\right) \\
&= \frac{u(us-1)(s^2-1)}{(1-s\sqrt{\rho_p})^2}\left(\sqrt{\rho_p}(\rho_p-1) - \frac{i\rho_{pp}}{2\sqrt{\rho_p}}\right) \\
&= \frac{u(us-1)(s^2-1)}{2(\rho_p)^{\frac{5}{2}}(1-s\sqrt{\rho_p})^2}\Big(ip_{\rho\rho} + 2p_\rho(1-p_\rho)\Big) < 0
\end{aligned}$$

at $s = s'$, since

$$u(s')s' - 1 = \frac{s'^2 - 1}{1 - s'\sqrt{\rho_p(p(s'), S_0)}} < 0 \quad \text{and} \quad ip_{\rho\rho} + 2p_\rho(1-p_\rho) > 0.$$

Consequently, by (3.13), we have

$$\lim_{s\to(s')^-}\left(\frac{\mathrm{d}u(s)}{\mathrm{d}s} - \frac{\mathrm{d}h(s)}{\mathrm{d}s}\right) = -\infty,$$

which contradicts to that $u(s) - h(s) < 0$ as $1 < s < s'$. So, $s'$ does not exist. We then have this lemma. □

As shown in (2.8), we have that for any $s > 1$, if $h(s) > 0$, then we have $u(s) > 0$. We are going to show the following three cases for the solution of the initial value problem (3.5) and (3.6):

I There exists a $s_* > 1$ such that $0 < u(s) < h(s) < \frac{1}{s}$ as $1 < s < s_*$, $u(s_*) = h(s_*) = \frac{1}{s_*}$, and $p(s_*) = 0$.
II There exists a $s_* > 1$ such that $0 < u(s) < h(s) < \frac{1}{s}$ as $1 < s \le s_*$ and $u(s_*) = h(s_*) = 0$.
III $0 < u(s) < h(s) < 1/s$ for all $s > 1$.

### 3.2.1 *Case I*

**Lemma 3.4** *If $u_0 > 0$ is sufficiently large, then there exists a $s_* > 1$ such that $0 < u(s) < h(s) < \frac{1}{s}$ as $1 < s < s_*$, $u(s_*) = h(s_*) = \frac{1}{s_*}$, and $p(s_*) = 0$; see Figure 8(left).*

**Proof** From (3.5), we have

$$\frac{(u-s)}{i(p,S_0)(us-1)}\,\mathrm{d}p = \frac{1}{(1-u^2)}\,\mathrm{d}u.$$

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Hence,

$$\int_{p(s_0)}^{p(y)} \frac{(u-s)}{i(p,S_0)(us-1)}\,\mathrm{d}p = \int_{u(s_0)}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u \quad \text{as} \quad y > s_0, \tag{3.14}$$

where $s_0$ is given in Lemma 3.2.

By (3.13), we know that if $0 < u(s) < h(s)$ and $p(s) > 0$ then

$$\frac{u-s}{us-1} < \frac{1}{\sqrt{p_\rho(\rho,S_0)}}.$$

Inserting this into (3.14), we have that for any $y > s_0$, if $0 < u(y) < h(y)$ and $p(y) > 0$, then

$$\begin{aligned}\int_{p_0}^{p(y)} \frac{1}{i(p,S_0)\sqrt{p_\rho(\rho,S_0)}}\,\mathrm{d}p &< \int_{p(s_0)}^{p(y)} \frac{1}{i(p,S_0)\sqrt{p_\rho(\rho,S_0)}}\,\mathrm{d}p \\ &< \int_{p(s_0)}^{p(y)} \frac{(u-s)}{i(p,S_0)(us-1)}\,\mathrm{d}p = \int_{u(s_0)}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u \\ &< \int_{u_0}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u.\end{aligned} \tag{3.15}$$

Let $\chi$ be defined so that

$$\begin{aligned}\int_{\chi}^{0} \frac{1}{1-u^2}\,\mathrm{d}u &= \int_{p_0}^{0} \frac{1}{i(p,S_0)\sqrt{p_\rho(\rho,S_0)}}\,\mathrm{d}p \\ &= \int_{p_0}^{0} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma-1}\right]^{-1}\left[\frac{1}{\gamma-1} + \frac{1}{\gamma S_0}\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}-1}\right]^{\frac{1}{2}}\mathrm{d}p.\end{aligned}$$

Then by (3.15), we know that if $u_0 > \chi$ then we have

$$\int_{u_0}^{0} \frac{1}{1-u^2}\,\mathrm{d}u < \int_{\chi}^{0} \frac{1}{1-u^2}\,\mathrm{d}u < \int_{p_0}^{p(y)} \frac{1}{i(p,S_0)\sqrt{p_\rho(\rho,S_0)}}\,\mathrm{d}p.$$

Hence, there exists a $s_*$ such that $p(s_*) = 0$. That is because if this is not true then by (3.12) and Lemma 3.3, we have that $u(y)$ approaches 0 as $y$ increases, and hence (3.15) will not be satisfied when $y$ is sufficiently large. By the definition of $h(s)$ we also have $h(s_*) = 1/s_*$.

Next, we shall show $u(s_*) = 1/s_*$. If $u(s_*) < 1/s_*$, then by $p_\rho\big(\rho(s_*), S_0\big) = 0$ there exists a sufficiently small $\varepsilon > 0$ such that

$$(us-1)^2 - (u-s)^2 p_\rho\big(\rho(s), S_0\big) > \frac{1}{2}(us-1)^2 \quad \text{as} \quad s_* - \varepsilon < s < s_*.$$

Therefore, from the second equation of (3.5), we have

$$\begin{aligned}&\int_{p(s_*-\varepsilon)}^{p(s)} \frac{1}{i(p,S_0)p_\rho(\rho(p),S_0)}\,\mathrm{d}p \\ &= \int_{s_*-\varepsilon}^{s} \frac{(N-1)u(us-1)}{(us-1)^2 - (u-s)^2 p_\rho(\rho,S_0)}\,\mathrm{d}s \\ &> \int_{s_*-\varepsilon}^{s} \frac{2(N-1)u}{us-1}\,\mathrm{d}s \qquad \text{as} \quad s_* - \varepsilon < s < s_*.\end{aligned}$$

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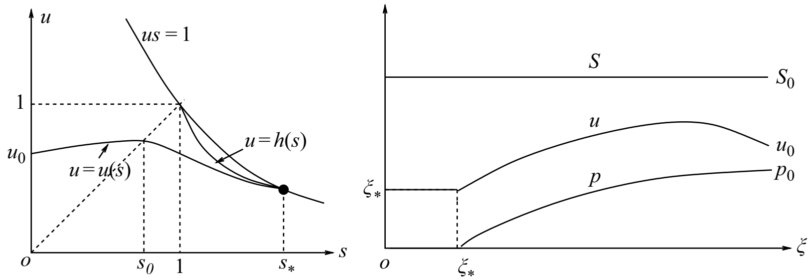


FIGURE 8. Continuous solution with a growing vacuum region for case I.

If $u(s_*) < 1/s_*$, then $\int_{s_*-\varepsilon}^{s_*} \frac{2(N-1)u}{us-1} \mathrm{d}s$ is finite. However,

$$\int_{p(s_*-\varepsilon)}^{p(s_*)} \frac{1}{i(p,S_0)p_\rho(\rho(p),S_0)}\,\mathrm{d}p$$
$$= \int_{p(s_*-\varepsilon)}^{0} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma-1}\right]^{-1}\left[\frac{1}{\gamma-1} + \frac{1}{\gamma S_0}\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}-1}\right]\mathrm{d}p = -\infty.$$

This leads to a contradiction. We then have $u(s_*) = 1/s_*$.

We then complete the proof of this lemma. □

We are now ready to construct the self-similar solution of the problem (1.7), (1.8) for case I. When $\xi > \xi_* = 1/s_*$, the solution is determined by the classical solution of the initial value problem (3.5), (3.6). We continue the solution by a vacuum state for $\xi < \xi_*$. This is a continuous extension; see Figure 8(right).

### 3.2.2 *Case II*

**Lemma 3.5** *If $u_0 > 0$ is sufficiently small, then there exists a $s_* > 1$ such that $0 < u(s) < h(s) < 1/s$ as $1 < s \le s_*$ and $u(s_*) = h(s_*) = 0$; see Figure 9(left).*

**Proof** It is easy to prove by (3.5) that for any small $\delta > 0$ there exists a $\varepsilon > 0$ such that if $0 < u_0 < \varepsilon$ then

$$0 < u(1) < \delta \quad \text{and} \quad p(1) > p_0 - \delta. \tag{3.16}$$

From (3.5), we have

$$\frac{\mathrm{d}p}{\mathrm{d}u} = \frac{i(p,S_0)(1-us)}{(s-u)(1-u^2)} \le \frac{i(p,S_0)}{1-u^2} \quad \text{as} \quad s > 1.$$

Thus, by integration, we have

$$\begin{aligned}\int_{p(s)}^{p_0-\delta} \frac{1}{i(p,S_0)}\,\mathrm{d}p &\le \int_{p(s)}^{p(1)} \frac{1}{i(p,S_0)}\,\mathrm{d}p \le \int_{u(s)}^{u(1)} \frac{1}{1-u^2}\,\mathrm{d}u \\ &< \int_0^{u(1)} \frac{1}{1-u^2}\,\mathrm{d}u < \int_0^{\delta} \frac{1}{1-u^2}\,\mathrm{d}u \quad \text{as} \quad s > 1.\end{aligned} \tag{3.17}$$

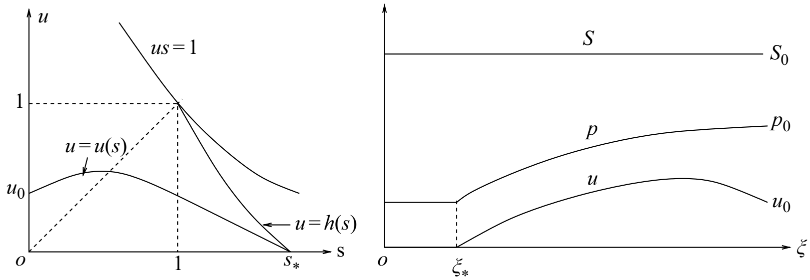


FIGURE 9. A continuous solution for case II.

Therefore, from (3.16) and (3.17), we have that when $\delta$ is sufficiently small there exists a $p_m > 0$ such that

$$p_m < p(s) < p(1) \quad \text{as} \quad s > 1.$$

Consequently, by the definition of $h(s)$ we know that there exists a $s_* > 1$ such that $s_* = \sqrt{\rho_p\big(p(s_*), S_0\big)}$ and $h(s_*) = 0$.

As shown in (2.8), we can prove $u(s) > 0$ as $1 < s < s_*$. Therefore, by Lemma 3.3, we also have $u(s_*) = 0$. We then have this lemma. □

We are now ready to construct the self-similar solution of the problem (1.7), (1.8) for case II. When $\xi > \xi_* = 1/s_*$, the solution is determined by the classical solution of the initial value problem (3.5), (3.6). We continue the solution by a constant state $(u, p, S) = (0, p(s_*), S_0)$ for $\xi < \xi_*$. This is a continuous extension; see Figure 9(right).

### 3.2.3 *Case III*

In what follows, we are going to show that case III can be happened.

**Lemma 3.6** *If case I happens as* $(u, p, S)(0) = (u_0, p_0, S_0)$*, then there exists a sufficiently small* $\varepsilon > 0$ *such that case I will happen for any* $(u, p, S)(0) \in (u_0 - \varepsilon, u_0 + \varepsilon) \times \{p_0\} \times \{S_0\}$.

**Proof** Denote by $(\bar{u}, \bar{p}, \bar{S})(s)$ the solution of (3.5) with data $(u, p, S)(0) = (u_0, p_0, S_0)$. Then there exists a $\bar{s}_* > 1$ such that $(\bar{u}, \bar{p}, \bar{S})(s)$ satisfies $\bar{p}(\bar{s}_*) = 0$ and $\bar{u}(\bar{s}_*) = h(\bar{p}(\bar{s}_*), \bar{s}_*) = 1/\bar{s}_*$.

Let $\mathcal{M} = \int_{\frac{1}{\bar{s}_*}}^{0} \frac{1}{1-u^2}\, \mathrm{d}u$. Then there exists a sufficiently small $\delta > 0$ such that

$$\int_{\delta}^{0} \Big[\Big(\frac{p}{S_0}\Big)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\Big]^{-1} \Big[\frac{1}{\gamma - 1} + \frac{1}{\gamma S_0}\Big(\frac{p}{S_0}\Big)^{\frac{1}{\gamma} - 1}\Big]^{\frac{1}{2}} \mathrm{d}p > \frac{\mathcal{M}}{4}, \tag{3.18}$$

and

$$\int_{\frac{1}{\bar{s}_*} - \delta}^{0} \frac{1}{1 - u^2}\, \mathrm{d}u < \frac{3}{4}\mathcal{M}. \tag{3.19}$$

Since $\bar{p}(s)$ is continuous on $[0, \bar{s}_*]$, there exists a sufficiently small $\eta > 0$ such that

$$\bar{p}(\bar{s}_* - \eta) < \frac{\delta}{2}. \tag{3.20}$$

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When $\varepsilon > 0$ is sufficiently small, the solution $(u, p, S)(s)$ of the initial value problem for (3.5) with the initial data $(u, p, S)(0) \in (u_0 - \varepsilon, u_0 + \varepsilon) \times \{p_0\} \times \{S_0\}$ satisfies

$$|p(\bar{s}_* - \eta) - \bar{p}(\bar{s}_* - \eta)| < \frac{\delta}{4} \quad \text{and} \quad |u(\bar{s}_* - \eta) - \bar{u}(\bar{s}_* - \eta)| < \frac{\delta}{4}. \tag{3.21}$$

By (3.15), we have

$$\begin{aligned} &\int_{p(\bar{s}_*-\eta)}^{p(y)} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \left[\frac{1}{\gamma - 1} + \frac{1}{\gamma S_0}\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}-1}\right]^{\frac{1}{2}} \mathrm{d}p \\ &\qquad < \int_{u(\bar{s}_*-\eta)}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u \qquad \text{as} \quad y > \bar{s}_* - \eta. \end{aligned} \tag{3.22}$$

Combining with (3.18)–(3.22), we have

$$\frac{\mathcal{M}}{4} < \int_{\delta}^{p(y)} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \left[\frac{1}{\gamma - 1} + \frac{1}{\gamma S_0}\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}-1}\right]^{\frac{1}{2}} \mathrm{d}p < \int_{\frac{1}{\bar{s}_*}-\delta}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u, \tag{3.23}$$

since $u(\bar{s}_* - \eta) > \bar{u}(\bar{s}_* - \eta) - \frac{\delta}{4} > 1/s_* - \frac{\delta}{4}$ and $p(\bar{s}_* - \eta) < \bar{p}(\bar{s}_* - \eta) + \frac{\delta}{4} < \delta$. Thus, by (3.19), we know that there exists a $s_* > 1$ such that $h(p(s), s) < 1/s$ as $1 < s < s_*$ and $u(s_*) = h(p(s_*), s_*) = 1/s_*$. Or else, there exists a $y$ such that $\int_{\frac{1}{\bar{s}_*}-\delta}^{u(y)} \frac{1}{1-u^2}\,\mathrm{d}u < \frac{\mathcal{M}}{2}$, which contradicts to (3.23). We then have this lemma. □

**Lemma 3.7** *If case II happens as* $(u, p, S)(0) = (u_0, p_0, S_0)$*, then there exists a sufficiently small* $\varepsilon > 0$ *such that case II will happen for any* $(u, p, S)(0) \in (u_0 - \varepsilon, u_0 + \varepsilon) \times \{p_0\} \times \{S_0\}$.

**Proof** Denote by $(\bar{u}, \bar{p}, \bar{S})(s)$ the solution of (3.5) with data $(u, p, S)(0) = (u_0, p_0, S_0)$. Then there exists a $\bar{s}_* > 1$ such that $\bar{u}(\bar{s}_*) = 0$ and $\bar{p}(\bar{s}_*) = p_* > 0$.

Let

$$\mathcal{N} = \int_0^{\frac{p_*}{2}} \frac{1}{i(p, S_0)}\,\mathrm{d}p = \int_0^{\frac{p_*}{2}} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p.$$

Then there exists a $\delta > 0$ such that $\int_0^{\delta} \frac{1}{1-u^2}\mathrm{d}u < \mathcal{N}$. There exists a sufficiently small $\eta > 0$ such that

$$0 < \bar{u}(\bar{s}_* - \eta) < \frac{\delta}{4}. \tag{3.24}$$

When $\varepsilon > 0$ is sufficiently small, the solution $(u, p, S)(s)$ of (3.5) with data $(u, p, S)(0) \in (u_0 - \varepsilon, u_0 + \varepsilon) \times \{p_0\} \times \{S_0\}$ satisfies

$$|p(\bar{s}_* - \eta) - \bar{p}(\bar{s}_* - \eta)| < \frac{p_*}{4} \quad \text{and} \quad |u(\bar{s}_* - \eta) - \bar{u}(\bar{s}_* - \eta)| < \frac{\delta}{4}. \tag{3.25}$$

By (3.17), we have

$$\int_{p(s)}^{p(\bar{s}_*-\eta)} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p < \int_{u(s)}^{u(\bar{s}_*-\eta)} \frac{1}{1-u^2}\,\mathrm{d}u \quad \text{as} \quad s > \bar{s}_* - \eta.$$

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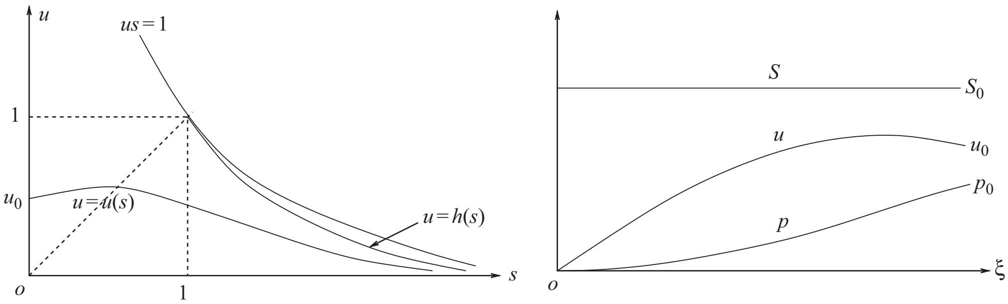


FIGURE 10. A smooth solution for case III.

Using (3.24)–(3.25), we have

$$\int_{p(s)}^{\frac{p_*}{2}} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p < \int_{p(s)}^{p(\bar{s}_*-\eta)} \left[\left(\frac{p}{S_0}\right)^{\frac{1}{\gamma}} + \frac{\gamma p}{\gamma - 1}\right]^{-1} \mathrm{d}p$$
$$< \int_{u(s)}^{u(\bar{s}_*-\eta)} \frac{1}{1-u^2}\mathrm{d}u < \int_0^{\frac{\delta}{2}} \frac{1}{1-u^2}\mathrm{d}u < \mathcal{N} \quad \text{as} \quad s > \bar{s}_* - \eta.$$

Thus, there exists a $p_m > 0$ such that $p(s) > p_m$ as $s > \bar{s}_* - \eta$. Consequently, there exists a $s_* > \bar{s}_* - \eta$ such that $s_* = \sqrt{\rho_p\big(p(s_*), S_0\big)}$. We then have this lemma. □

Using Lemmas 3.2–3.7 and the argument of continuity, we know that for any $p_0 > 0$ and $S_0 > 0$ there exists a $u_0 \in (0, 1)$ such that the solution of the initial value problem (3.5), (3.6) satisfies $0 < u(s) < h(s) < 1/s$ for all $s > 1$. That is to say, the problem (1.7), (1.8) admits a global smooth solution; see Figure 10(right).

### 3.3 Shock wave solution for $u_0 \in (-1, 0)$

**Lemma 3.8** *There exists a $s_1 > 1$ such that the initial value problem (3.5), (3.6) admits a solution on $[0, s_1]$. Moreover, this solution satisfies $u_0 < u(s) < 0 < h(s)$ as $0 < s \le s_1$ and $h(s_1) = 0$.*

**Proof** Lemma 3.1 has obtained that the problem (3.5), (3.6) has a solution on [0, 1]. For $s > 1$, it is easy to see that if $h(s) > 0$ then $u(s) < 0$, $p(s) > 0$, and $\frac{\mathrm{d}p}{\mathrm{d}s} > 0$ . Hence, by $\rho_{pp}(p, S_0) < 0$, we know that there exists a $s_1 > 1$ such that $u(s) < 0 < h(s)$ as $1 < s < s_1$ and $h(s_1) = 0$. In what follows, we are going to prove that $u(s_1) < 0$. We shall prove this by contradiction.

We now consider the initial value problem

$$\frac{\mathrm{d}\bar{u}}{\mathrm{d}s} = \frac{(N-1)(\bar{u}-s)\bar{u}(1-\bar{u}^2)}{s_1^2(\bar{u}s-1)^2 - (\bar{u}-s)^2}, \qquad \bar{u}(0) = u_0. \tag{3.26}$$

By Lemma 2.6, we have that the solution of the problem (3.26) satisfies $\bar{u}(s_1) < 0$.

From $\rho_{pp}(p, S_0) < 0$ and $\frac{\mathrm{d}p}{\mathrm{d}s} > 0$, we have $s_1 < \sqrt{\rho_p(p(s), S_0)}$ as $0 < s < s_1$. Hence, we have

$$\frac{(N-1)(u-s)u(1-u^2)}{\rho_p(p, S_0)(us-1)^2 - (u-s)^2} < \frac{(N-1)(u-s)u(1-u^2)}{s_1^2(us-1)^2 - (u-s)^2} \quad \text{as} \quad 0 < s < s_1.$$

Therefore, by comparison principle, we have $u(s_1) < \bar{u}(s_1) < 0$. We then complete the proof of this lemma. □

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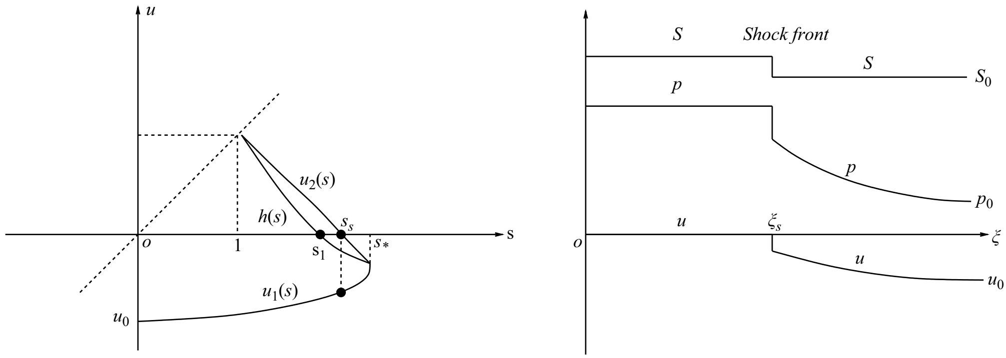


FIGURE 11. Shock wave solution for $u_0 \in (-1,0)$.

**Lemma 3.9** *For any $u_0 \in (-1,0)$ there exists a $s_* > s_1$ such that the solution of the initial value problem (3.5), (3.6) satisfies $u(s) < h(s)$ as $1 < s < s_*$ and $u(s_*) = h(s_*) < 0$; see Figure 11(left).*

**Proof** By computation, we have

$$\frac{\mathrm{d}h(s)}{\mathrm{d}s} = \frac{1}{(1-s\sqrt{\rho_p(p,S_0)})^2}\left(1-\rho_p(p,S_0)+\frac{\rho_{pp}(p,S_0)}{2\sqrt{\rho_p(p,S_0)}}\frac{\mathrm{d}p}{\mathrm{d}s}(s^2-1)\right) < 0 \tag{3.27}$$

as $s > 1$. Thus, if the curve $u = u(s)$ and the curve $u = h(s)$ do not intersect, then there exists a $\hat{u} \in (-1,0)$ such that $\lim_{s\to+\infty} u(s) = \hat{u}$. Thus, we have

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{(N-1)(u-s)u(1-u^2)}{\rho_p(p,S_0)(us-1)^2-(u-s)^2} > \frac{-(N-1)(1-u_0^2)\hat{u}s}{\rho_p(p_0,S_0)(u_0s-1)^2}$$

as $s > 0$. Thus, by integration, we have

$$\hat{u}-u_0 = \int_0^{+\infty}\frac{(N-1)(u-s)u(1-u^2)}{\rho_p(p,S_0)(us-1)^2-(u-s)^2}\,\mathrm{d}s > \int_0^{+\infty}\frac{-(N-1)(1-u_0^2)\hat{u}s}{\rho_p(p_0,S_0)(u_0s-1)^2}\,\mathrm{d}s = +\infty,$$

which leads to a contradiction. We then have this lemma. □

Lemma 3.9 implies that when $u_0 \in (-1,0)$ the problem (1.7), (1.8) does not have a global continuous solution. So, we need to look for a shock wave solution. In the following discussions, we denote by $(u_1,p_1,S_1)(s)$ $(0 < s < s_*)$ the solution of the initial value problem (3.5), (3.6) for $u_0 < 0$.

The Rankine–Hugoniot conditions of shock waves for (1.7) are

$$\begin{cases} \xi\left(\dfrac{n_1}{\sqrt{1-u_1^2}}-\dfrac{n_2}{\sqrt{1-u_2^2}}\right) = \left(\dfrac{n_1u_1}{\sqrt{1-u_1^2}}-\dfrac{n_2u_2}{\sqrt{1-u_2^2}}\right), \\ \xi\left(\dfrac{i_1u_1}{1-u_1^2}-\dfrac{i_2u_2}{1-u_2^2}\right) = \left(\dfrac{i_1u_1^2}{1-u_1^2}+p_1-\dfrac{i_2u_2^2}{1-u_2^2}-p_2\right), \\ \xi\left(\dfrac{i_1}{1-u_1^2}-p_1-\dfrac{i_2}{1-u_2^2}+p_2\right) = s\left(\dfrac{i_1u_1}{1-u_1^2}-\dfrac{i_2u_2}{1-u_2^2}\right), \end{cases} \tag{3.28}$$

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where '1' denotes the fluid in front of the shock, '2' denotes the fluid behind the shock, and $\xi$ denotes the speed of the shock front.

Admissible forward shock waves must satisfy the entropy inequality

$$S_2 > S_1, \tag{3.29}$$

and the stability condition

$$\frac{u_1 + \sqrt{p_\rho(\rho_1, S_1)}}{1 + u_1\sqrt{p_\rho(\rho_1, S_1)}} < \xi < \frac{u_2 + \sqrt{p_\rho(\rho_2, S_2)}}{1 + u_2\sqrt{p_\rho(\rho_2, S_2)}}. \tag{3.30}$$

**Lemma 3.10** *For any $s \in (1, s_*)$, there exists an unique $(u_2, p_2, S_2)(s)$ such that $(u_1, p_1, S_1)(s)$ and $(u_2, p_2, S_2)(s)$ can be connected by an admissible forward shock with the speed $\xi = 1/s$.*

**Proof** From $\rho_p\big(p_1(s), S_0\big)\big(u_1(s)s - 1\big)^2 - \big(u_1(s) - s\big)^2 > 0$, $u_1(s)s < 1$, and $s > u_1(s)$ we immediately have

$$\frac{1}{s} > \frac{u_1(s) + \sqrt{p_\rho(\rho_1(s), S_0)}}{1 + u_1(s)\sqrt{p_\rho(\rho_1(s), S_0)}} \quad \text{for any } s \in (1, s_*).$$

Then using the result of Chen $\big($cf. Theorems 4.1 and 4.2 of [6]$\big)$ we can get this lemma. □

We are going to look for an admissible forward shock wave with an appropriate speed $\xi = 1/s \in (1/s_*, 1)$ and the front side state $(u_1, \rho_1, S_1)(s)$ such that the backside state of the shock satisfies $u_2 = 0$.

**Lemma 3.11** *Let $(u_2, p_2, S_2)(s)$ $(1 < s < s_*)$ be determined by the R-H conditions (3.28), the entropy condition (3.29), and the stability condition (3.30). Then there exists a $s_s \in (1, s_*)$ such that $u_2(s_s) = 0$.*

**Proof** By a direction computation, we have

$$p_\rho(\rho, S) = \frac{1}{\frac{1}{\gamma-1} + \frac{1}{\gamma}\left(\frac{1}{S}\right)^{\frac{1}{\gamma}} p^{\frac{1}{\gamma}-1}} < \gamma - 1 < 1, \tag{3.31}$$

since $\gamma \in (1, 5/3)$. By (3.30), we have

$$u_2(s) > \frac{1 - s\sqrt{p_\rho(\rho_2, S_2)}}{s - \sqrt{p_\rho(\rho_2, S_2)}}. \tag{3.32}$$

Combining with this and (3.31), we have that if $s$ is sufficiently close to 1 then $u_2(s) > 0$.

It is easy to see that $\lim\limits_{s \to s_*^-} u_2(s) = u_1(s_*) < 0$. Therefore, there exists a $s_s \in (1, s_*)$ such that $u_2(s_s) = 0$. We then have this lemma. □

**Remark 3.1** *The uniqueness of $s_s$ is still a problem, since it is difficult to prove that $u_2(s)$ is monotonically decreasing in $(1, s_*)$.*

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Therefore, when $u_0 \in (-1, 0)$ the initial-boundary value problem (1.7), (1.8) has a discontinuous solution with a single shock. The solution has the form

$$(u, p, S)(s) = \begin{cases} \big(u_1(s), p_1(s), S_0\big), & s < s_s, \\ \big(0, p_2(s_s), S_2(s_s)\big), & s > s_s, \end{cases}$$

where $s = t/x$; see Figure 11(right).

At this point, we have completed the proof of Theorem 1.2.

## 4 The spherical piston problem

Motivated by the result of the spherical piston problem for the compressible Euler equations, we shall look for self-similar discontinuous solutions with a single shock of the problem.

### 4.1 The spherical piston problem for the ultra-relativistic Euler equations

Assume that the speed of the single shock is $1/s_p$, where $s_p \in (1, \sqrt{3})$. Then by (2.24), (2.26), and (2.27) we know that the velocity of the backside state of the shock is $u_2 = \frac{3-s_p^2}{2s_p}$. Thus, in order to solve the spherical piston problem, we only need to find a $s_p \in (1, \sqrt{3})$ such that the solution of the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{(N-1)(u-s)u(1-u^2)}{3(us-1)^2-(u-s)^2}, & s > s_p; \\ u(s_p) = \dfrac{3-s_p^2}{2s_p} \end{cases} \tag{4.1}$$

satisfies $u\big(\frac{1}{\alpha}\big) = \alpha$.

**Lemma 4.1** *For any $\alpha \in (0, 1)$, there exists one and only one $s_p \in (1, \sqrt{3})$ such that the solution of the initial value problem (4.1) satisfies $u\big(\frac{1}{\alpha}\big) = \alpha$; see Figure 12(right).*

**Proof** For any $s_p \in (1, \sqrt{3})$, the solution $u = u(s)$ of the problem (4.1) is monotonically increasing. Hence, there exists a $\tilde{s}(s_p) > s_p$ such that $\tilde{s}(s_p)u(\tilde{s}(s_p)) = 1$. It is easy to see that $\tilde{s}(s_p)$ is continuous and strictly monotonically increasing with respect to the variable $s_p \in (1, \sqrt{3})$.

Consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}s} = \dfrac{(N-1)(u-s)u(1-u^2)}{3(us-1)^2-(u-s)^2}, s < \beta; \\ u\,|_{s=\beta} = \frac{1}{\beta}. \end{cases} \tag{4.2}$$

Then by Remark 2.3, we know that for any $\beta > 1$, there exists a $\hat{s}(\beta)$ such that the solution of (4.2) satisfies $u(\hat{s}(\beta)) = \frac{3-\hat{s}^2(\beta)}{2\hat{s}(\beta)}$. Moreover, $\hat{s}(\beta)$ is a monotonically increasing function of $\beta \in (1, +\infty)$.

Therefore, by the argument of continuity, we can get this lemma. □

From Lemma 4.1, we immediately have Theorem 1.3. The structure of the solution of the spherical piston problem (1.5), (1.11) can be illustrated in Figure 12(right).

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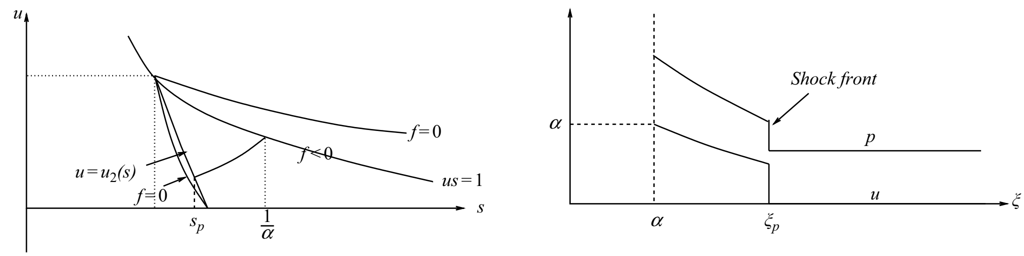


FIGURE 12. The solution of the spherical piston problem for the ultra-relativistic Euler equations.

### 4.2 The spherical piston problem for the full-relativistic Euler equations

Let $s_* = \sqrt{\rho_p(p_0, S_0)}$. Assume that the speed of the single shock is $1/s_p$ where $s_p \in (1, s_*)$. Then by (3.28) we can get the back side state $(u_2, p_2, S_2)(s_p)$. Thus, we only need to find a $s_p \in (1, s_*)$ such that the solution of (3.5) with data

$$(u, p, S)\,|_{s=s_p} = (u_2, p_2, S_2)(s_p) \tag{4.3}$$

satisfies $u\left(\frac{1}{\alpha}\right) = \alpha$.

**Lemma 4.2** *For any $\alpha \in (0, 1)$, there exists a $s_p \in (1, s_*)$ such that the solution of the initial value problem (3.5), (4.3) satisfies $u\left(\frac{1}{\alpha}\right) = \alpha$.*

**Proof** By (3.30), we have

$$\frac{1}{s_p} < \frac{u_2(s_p) + \sqrt{p_\rho(\rho_2(s_p), S_2(s_p))}}{1 + u_2(s_p)\sqrt{p_\rho(\rho_2(s_p), S_2(s_p))}} \tag{4.4}$$

For forward shock waves, we have

$$u_2(s_p) < \frac{1}{s_p}. \tag{4.5}$$

Combining with this and (4.4), we have

$$\rho_p\big(p_2(s_p), S_2(s_p)\big)(u_2(s_p)s_p - 1)^2 - (u_2(s_p) - s_p)^2 < 0. \tag{4.6}$$

Hence, the initial value problem (3.5), (4.3) is well-posed and has a local solution $(u, p, S)(s)$. Moreover, this solution satisfies

$$\frac{\mathrm{d}u}{\mathrm{d}s} > 0 \quad \text{and} \quad \frac{\mathrm{d}p}{\mathrm{d}s} \begin{cases} > 0, & us < 1; \\ < 0, & us > 1. \end{cases} \tag{4.7}$$

By computation, we have

$$\frac{s + \sqrt{\rho_p(p, S)}}{1 + s\sqrt{\rho_p(p, S)}} > \frac{1}{s} \quad \text{as} \quad s > 1. \tag{4.8}$$

Thus, by (3.27), (4.5), (4.7), and (4.8), we know that there exists a $\tilde{s}(s_p) > s_p$ such that $u(\tilde{s}(s_p)) = 1/\tilde{s}(s_p)$. It is easy to see that $\tilde{s}(s_p)$ is a continuous function of $s_p \in (1, s_*)$. Thus, in order to prove this lemma, we only need to prove

$$\inf_{s_p \in (1, s_*)} \tilde{s}(s_p) = 1 \quad \text{and} \quad \sup_{s_p \in (1, s_*)} \tilde{s}(s_p) = +\infty.$$

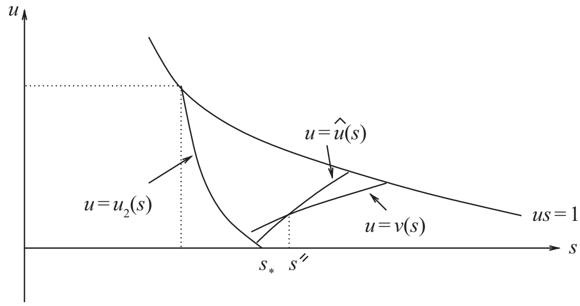


FIGURE 13. $\hat{u}(s)$ and $v(s)$.

Since $u=u(s)$ is monotonically increasing in $(s_p, \tilde{s}(s_p))$, we have $\frac{1}{\tilde{s}(s_p)} > u_2(s_p)$. From (3.31) and (3.32), we know that $u_2(s_p) \to 1$ as $s_p \to 1^+$. Therefore, we get $\inf\limits_{s_p \in (1,s_*)} \tilde{s}(s_p) = 1$.

We next prove $\sup\limits_{s_p \in (1,s_*)} \tilde{s}(s_p) = +\infty$. We shall prove this by contradiction. Suppose that

$$\sup_{s_p \in (1,s_*)} \tilde{s}(s_p) = \mathcal{R}, \tag{4.9}$$

where $\mathcal{R} > 1$ is a finite number. We consider the following initial value problem:

$$\begin{cases} \dfrac{\mathrm{d}v}{\mathrm{d}s} = \dfrac{(N-1)(v-s)v(1-v^2)}{\rho_p(p_0,S_0)(vs-1)^2-(v-s)^2}, & s < \mathcal{R}+1; \\ v(\mathcal{R}+1) = \dfrac{1}{\mathcal{R}+1}. \end{cases} \tag{4.10}$$

By the result of Remark 2.3, we know that the integral curve $u=v(s)$ of the problem (4.10) and the curve $u = \frac{s-s_*}{1-ss_*}$ $(s>1)$ interact at some point $(s', u')$ with $s' < s_*$ and $u' > 0$. Since $u_2(s_p) \to 0$ as $s_p \to s_*^-$, there exists a $s'_p$ sufficiently close to $s_*$, such that $0 < u_2(s'_p) < u'$ and $s'_p > s'$. Let $(\hat{u}, \hat{p}, \hat{S})(s)$ be the solution of the initial value problem for (3.5) with the initial data $(u,p,S)\,|_{s=s'_p} = (u_2, p_2, S_2)(s'_p)$. Then by the assumption (4.9) we know that there exists $s'' \in (s'_p, \mathcal{R}+1)$ such that $v(s'') = \hat{u}(s'')$ and $v(s) > \hat{u}(s)$ as $s \in (s'_p, s'')$; see Figure 13. Hence, we have

$$v'(s'') - \hat{u}'(s'') \le 0 \tag{4.11}$$

By $\rho_{pp}(p,S) < 0$, $S_2(s'_p) > S_0$, $\hat{p}(s'') > \hat{p}(s_p) > p_0$, and $\rho_{pS}(p,S) < 0$ we have $\rho_p(\hat{p}, S_2(s'_p)) < \rho_p(p_0, S_0)$, and consequently

$$\underbrace{\frac{(N-1)(v-s)v(1-v^2)}{\rho_p(p_0,S_0)(vs-1)^2-(v-s)^2}}_{v'(s)} > \underbrace{\frac{(N-1)(\hat{u}-s)\hat{u}(1-\hat{u}^2)}{\rho_p(\hat{p},S_2(s'_p))(\hat{u}s-1)^2-(\hat{u}-s)^2}}_{\hat{u}'(s)} > 0 \quad \text{at} \quad s = s'',$$

which contradicts to (4.11). Thus, we have $\sup\limits_{s_p \in (1,s_*)} \tilde{s}(s_p) = +\infty$. Then we complete the proof of this lemma. □

From Lemma 4.2, we immediately have Theorem 1.4.

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## Conflicts of interest

The author has no conflicts of interest.

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