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A reinterpretation of set differential equations as differential equations in a Banach space

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Set differential equations are usually formulated in terms of the Hukuhara differential. As a consequence, the theory of set differential equations is perceived as an independent subject, in which all results are proved within the framework of the Hukuhara calculus. We propose to reformulate set differential equations as ordinary differential equations in a Banach space by identifying the convex and compact subsets of $\mathbb{R}^d$ with their support functions. Using this representation, standard existence and uniqueness theorems for ordinary differential equations can be applied to set differential equations. We provide a geometric interpretation of the main result, and demonstrate that our approach overcomes the heavy restrictions that the use of the Hukuhara differential implies for the nature of a solution.

Keywords: set differential equations; ordinary differential equations in Banach spaces; existence and uniqueness

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1. Introduction

A set differential equation is an equation of the form

$$D_{\mathrm{H}}A(t) = f(t, A(t)), \quad A(0) = A_0, \tag{1.1}$$

where $t \mapsto A(t)$ is a curve in the space $\mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$ of non-empty convex and compact subsets of $\mathbb{R}^d$, the right-hand side is a mapping

$$f\colon [0, T] \times \mathcal{K}_{\mathrm{c}}(\mathbb{R}^d) \to \mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$$

and $D_{\mathrm{H}}A(t)$ is the so-called Hukuhara differential of the curve at $t \in (0, T)$. Set differential equations have been investigated in a considerable number of papers. For an overview of the literature we refer the reader to [11]. The use of the Hukuhara differential in (1.1) implies heavy restrictions on the nature of the solution, which can, for example, only grow in diameter, but not shrink (see [11, proposition 1.6.1]).

Recently, there have been attempts to modify the underlying Hukuhara difference to allow more flexibility of the solution curves (see [13, 14] and the references therein). The resulting differential is called the second-type Hukuhara differential. In this setting, solution curves of (1.1) can shrink, but not grow. There exist,

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however, curves in $\mathcal{K}_c(\mathbb{R}^d)$ with $d \geqslant 2$, which expand in some directions and contract in others simultaneously. Both Hukuhara-based approaches fail to capture this behaviour.

The Hukuhara differential is not the only approach to handling set evolutions. In particular, the abstract framework named *mutational analysis*, presented in [1] and further developed in [12] generalizes evolution equations from vector spaces to metric spaces and can handle evolutions not only in $\mathcal{K}_c(\mathbb{R}^d)$, but also in spaces of more general sets, such as the compact subsets of $\mathbb{R}^d$.

The aim of our paper is to show that a large family of evolutions in $\mathcal{K}_c(\mathbb{R}^d)$, containing the problems investigated in [11, 13, 14], can be written and treated as ordinary differential equations in a Banach space with the usual Fréchet derivative in time. We do not apply the apparatus from [1, 12], but obtain very satisfactory results by exploiting the intrinsic features of $\mathcal{K}_c(\mathbb{R}^d)$.

Identifying convex sets with their support functions yields an embedding of the space $\mathcal{K}_c(\mathbb{R}^d)$ into the Banach space $C(\mathbb{S}^{d-1})$ of continuous real-valued functions on the sphere (see [9]). As it is well known that any Hukuhara differentiable curve is Fréchet differentiable in support function representation (see [3, lemma 4.1]), it seems natural to consider set differential equations in a support function representation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{A(t)} = f(t, \sigma_{A(t)}), \quad \sigma_{A(0)} = \sigma_{A_0}, \tag{1.2}$$

where $t \mapsto A(t)$ is a curve in $\mathcal{K}_c(\mathbb{R}^d)$, $t \mapsto \sigma_{A(t)}$ is a curve in $C(\mathbb{S}^{d-1})$ and $\mathrm{d}\sigma_{A(t)}/\mathrm{d}t$ is the Fréchet differential of the curve at $t \in (0, T)$.

There are some technical difficulties when standard results on ordinary differential equations are applied to equations of the form (1.2). As we have to guarantee that solutions stay in the manifold $\Sigma \subset C(\mathbb{S}^{d-1})$ of all support functions associated with sets from $\mathcal{K}_c(\mathbb{R}^d)$, we need to understand the structure of the tangent cone $T_\Sigma(\sigma)$ to Σ at any $\sigma \in \Sigma$. To transfer the existence and uniqueness theorems for ordinary differential equations in Banach spaces with a non-Lipschitz right-hand side to (1.2), we need the compactness properties of Σ and a characterization of the semi-inner product on $(C(\mathbb{S}^{d-1}), \|\cdot\|_\infty)$. Some of these preliminary results can be taken from the literature; others are developed in this paper. In particular, we give a geometric interpretation of the one-sided Lipschitz condition in $\mathcal{K}_c(\mathbb{R}^d)$, which is a surprisingly mild condition on the behaviour of f.

This paper is organized as follows. In § 2, we collect the basic definitions and the preliminary results mentioned above, which we use in § 3 to transfer standard existence and uniqueness results to (1.2). In § 4, we briefly show that second-type Hukuhara differentiable curves are a special case of (1.2). The example discussed in § 5 illustrates that both Hukuhara approaches fail to capture very simple dynamics in $\mathcal{K}_c(\mathbb{R}^2)$, while the support function calculus is applicable and yields reasonable solutions.

2. Preliminaries

After introducing basic notation in § 2.1, we shall collect some known results about support functions and tangent cones in §§ 2.2 and 2.3. Section 2.4 investigates duality concepts, which are the ingredients for standard results on ordinary differential equations in Banach spaces in the particular case of set differential equations.

2.1. Basic definitions

Let $\mathbb{R}_0^+$ be the set of all non-negative real numbers. Throughout this paper, $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ will denote the sphere with respect to the Euclidean norm $\|\cdot\|\colon \mathbb{R}^d \to \mathbb{R}_0^+$, and the modulus will be denoted by $|\cdot|\colon \mathbb{R} \to \mathbb{R}_0^+$. Let $C(\mathbb{S}^{d-1})$ be the space of continuous real-valued functions on $\mathbb{S}^{d-1}$ equipped with the maximum norm $\|\cdot\|_\infty\colon C(\mathbb{S}^{d-1}) \to \mathbb{R}_0^+$. If $(X, \|\cdot\|_X)$ is a normed space, $x \in X$ and $r > 0$, then

$$\mathbb{B}_r(x) := \{x' \in X\colon \|x' - x\| \leqslant r\}$$

is the closed ball of radius r centred at x.

The non-empty compact subsets of $\mathbb{R}^d$ will be denoted $\mathcal{K}(\mathbb{R}^d)$, and $\mathcal{K}_c(\mathbb{R}^d)$ will stand for the non-empty convex and compact subsets of $\mathbb{R}^d$. For any $\lambda \in \mathbb{R}$ and $A, B \in \mathcal{K}(\mathbb{R}^d)$, let

$$A + B := \{a + b\colon a \in A,\ b \in B\} \quad \text{and} \quad \lambda A := \{\lambda a\colon a \in A\}$$

denote Minkowski addition and multiplication. For any $A, B \in \mathcal{K}_c(\mathbb{R}^d)$, let

$$\begin{aligned} \operatorname{dist}(A, B) &:= \sup_{a \in A} \inf_{b \in B} \|a - b\|, \\ \operatorname{dist}_H(A, B) &:= \max\{\operatorname{dist}(A, B), \operatorname{dist}(B, A)\} \end{aligned}$$

denote the one-sided and the symmetric Hausdorff distance. For $a, b \in \mathbb{R}^d$, we write $\operatorname{dist}(a, B)$ and $\operatorname{dist}(A, b)$ instead of $\operatorname{dist}(\{a\}, B)$ and $\operatorname{dist}(A, \{b\})$. The projection of a point $a \in \mathbb{R}^d$ to a set $B \in \mathcal{K}(\mathbb{R}^d)$ is the non-empty set

$$\operatorname{proj}_B(a) := \{b \in B\colon \|a - b\| = \operatorname{dist}(a, B)\}.$$

When $B \in \mathcal{K}_c(\mathbb{R}^d)$, $a \mapsto \operatorname{proj}_B(a)$ is a single-valued mapping (see [5, lemma 7.3]), and it follows from [5, proposition 7.4] that this mapping is 1-Lipschitz.

We associate convex and compact subsets $A \in \mathcal{K}_c(\mathbb{R}^d)$ with their support functions

$$\sigma_A\colon \mathbb{S}^{d-1} \to \mathbb{R}, \qquad \sigma_A(p) := \sup_{a \in A} \langle p, a \rangle.$$

Sometimes, it is useful to consider their positive homogeneous extensions

$$\bar{\sigma}_A\colon \mathbb{R}^d \to \mathbb{R}, \qquad \sigma_A(p) := \sup_{a \in A} \langle p, a \rangle,$$

which obviously coincide with $\sigma_A(\cdot)$ on $\mathbb{S}^{d-1}$. We define

$$\varSigma(\mathbb{R}^d) := \{\sigma_A\colon A \in \mathcal{K}_c(\mathbb{R}^d)\}$$

to be the set of all support functions of convex and compact subsets of $\mathbb{R}^d$, and we set

$$\hat{\varSigma}(\mathbb{R}^d) := \varSigma(\mathbb{R}^d) - \varSigma(\mathbb{R}^d) = \{\sigma_A - \sigma_B\colon A, B \in \mathcal{K}_c(\mathbb{R}^d)\}.$$

2.2. Elementary facts about support functions

The following proposition is [15, corollary 13.2.2].

Proposition 2.1. *A bounded function $\sigma\colon \mathbb{S}^{d-1} \to \mathbb{R}$ is a support function of some $A \in \mathcal{K}_c(\mathbb{R}^d)$ if and only if its positive homogeneous extension $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}$ is convex.*

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Recall that every convex function $\bar{\sigma}\colon \mathbb{R}^d \to \mathbb{R}$ is continuous (see [15, theorem 10.1]). We may therefore interpret the set $\Sigma(\mathbb{R}^d)$ of all support functions as a subset of $C(\mathbb{S}^{d-1})$.

The following facts are well known (see [9, 10]).

Proposition 2.2. *If $A, B \in \mathcal{K}_c(\mathbb{R}^d)$ and $\lambda \geqslant 0$, then*

(a) $\sigma_{A+B} = \sigma_A + \sigma_B$ *and* $\sigma_{\lambda A} = \lambda\sigma_A$,

(b) $\operatorname{dist}(A, B) = \max_{p\in\mathbb{B}_1(0)}(\bar{\sigma}_A(p) - \bar{\sigma}_B(p))$,

(c) $\operatorname{dist}_H(A, B) = \max_{p\in\mathbb{S}^{d-1}} |\sigma_A(p) - \sigma_B(p)|$.

In particular, $\Sigma(\mathbb{R}^d)$ is a convex subcone of $C(\mathbb{S}^{d-1})$.

The cone $\Sigma(\mathbb{R}^d)$ is locally compact.

Proposition 2.3. *The cone $\Sigma(\mathbb{R}^d)$ is closed as a subset of $C(\mathbb{S}^{d-1})$, and for any $\sigma \in \Sigma(\mathbb{R}^d)$ and $r > 0$ the intersection $\Sigma(\mathbb{R}^d) \cap \mathbb{B}_r(\sigma) \subset C(\mathbb{S}^{d-1})$ is compact with respect to the maximum norm.*

Proof. Let $\sigma \in C(\mathbb{S}^{d-1})$, and let $(\sigma_n)_{n\in\mathbb{N}} \subset \Sigma(\mathbb{R}^d)$ be a sequence of support functions with $\|\sigma_n - \sigma\|_\infty \to 0$ as $n \to \infty$. By proposition 2.1, the extensions $\bar{\sigma}_n$ are convex. Hence, for any $\lambda \in [0, 1]$ and $x, y \in \mathbb{R}^d$ we have that

$$\begin{aligned}\bar{\sigma}(\lambda x + (1-\lambda)y) &\leftarrow \bar{\sigma}_n(\lambda x + (1-\lambda)y)\\ &\leqslant \lambda\bar{\sigma}_n(x) + (1-\lambda)\bar{\sigma}(y) \to \lambda\bar{\sigma}(x) + (1-\lambda)\bar{\sigma}(y)\end{aligned} \tag{2.1}$$

as $n \to \infty$, so that $\bar{\sigma}$ is convex. Therefore, proposition 2.1 implies that $\sigma \in \Sigma(\mathbb{R}^d)$.

By Blaschke's selection theorem (see [8, ch. 4]), the set $\Sigma(\mathbb{R}^d) \cap \mathbb{B}_{\|\sigma\|_\infty + r}(0)$ is compact. As $\Sigma(\mathbb{R}^d) \cap \mathbb{B}_r(\sigma)$ is the intersection of two closed sets, it is a closed subset of the compact set $\Sigma(\mathbb{R}^d) \cap \mathbb{B}_{\|\sigma\|_\infty + r}(0)$, and hence compact. □

2.3. Tangent cones

We are interested in $C(\mathbb{S}^{d-1})$-valued solutions of differential equations that do not leave $\Sigma(\mathbb{R}^d)$. The concept of tangency is central for existence theorems under state constraints.

Definition 2.4. Let X be a normed space, let $K \subset X$ be a set and let $x \in \bar{K}$. Then the tangent cone to K at x is given by

$$T_K(x) := \Big\{v \in X\colon \liminf_{h\searrow 0} h^{-1}\operatorname{dist}(x + hv, K) = 0\Big\}.$$

The following proposition is [2, lemma 4.2.5]. It will later be used to characterize tangency to the convex cone $\Sigma(\mathbb{R}^d)$.

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Proposition 2.5. *If X is a normed space and $K \subset X$ is a convex cone, then $T_K(x) = \overline{K + \mathbb{R}x}$ for all $x \in K$.*

2.4. The semi-inner product for support functions

In § 3, we shall apply a uniqueness theorem for ordinary differential equations in Banach spaces to set differential equations in a support function representation. Its main ingredient is a one-sided Lipschitz condition, which is given in terms of a so-called semi-inner product. Therefore, we investigate here how this product acts on $\hat{\Sigma}(\mathbb{R}^d) \subset C(\mathbb{S}^{d-1})$ and what this action means for the corresponding elements of $\mathcal{K}_c(\mathbb{R}^d)$.

Definition 2.6. For any Banach space X with dual space X^*, the duality map $J \colon X \rightrightarrows X^*$ is given by

$$J(x) = \{x^* \in X^* \colon x^*(x) = \|x\|_X^2 = \|x^*\|_{X^*}^2\}.$$

The mapping $\langle \cdot, \cdot \rangle_- \colon X \times X \to \mathbb{R}$ defined by

$$\langle x, y \rangle_- = \inf\{y^*(x) \colon y^* \in J(y)\}$$

is called a semi-inner product.

Consider the Banach space $X = C(M)$, where M is a compact metric space and $C(M)$ denotes the space of all continuous real-valued functions on M equipped with the maximum norm. Let $\mathcal{B}(M)$ denote the space of all signed Borel measures on M, and let $\mathcal{B}(M)^+$ denote the space of all positive Borel measures on M.

Proposition 2.7 (Jordan decomposition). *For any $\mu \in \mathcal{B}(M)$, there exists a unique pair $(\mu_P, \mu_N) \in \mathcal{B}^+(M) \times \mathcal{B}^+(M)$ supported on Borel sets $P, N \subset M$ such that $\mu = \mu_P - \mu_N$, and M is the disjoint union of P and N.*

For a proof, see [7, § III.4, theorem 10 and corollary 11].

As a consequence, the total variation of a signed Borel measure is well defined.

Definition 2.8. The total variation of a Borel measure $\mu \in \mathcal{B}(\mathbb{R}^d)$ with Jordan decomposition $\mu_P + \mu_N = \mu$ with associated Borel sets $P \cup N = M$ is defined by

$$|||\mu||| := \mu_P(P) + \mu_N(N).$$

It is well known that the dual space of $(C(M), \|\cdot\|_\infty)$ is $(\mathcal{B}(M), |||\cdot|||)$, which follows from the Riesz representation theorem (see [7, theorem IV.6.3]).

We shall now characterize the duality map on $C(M)$. For a given function $f \in C(M)$, we define the sets

$$E_f^P = \{x \in M \colon f(x) = \|f\|_\infty\}, \qquad E_f^N = \{x \in M \colon f(x) = -\|f\|_\infty\}$$

on which f attains its maximal modulus. Clearly, $E_f^P \cup E_f^N \neq \emptyset$. Note that either $E_f^P \cap E_f^N = \emptyset$ or $E_f^P \cap E_f^N = M$, which happens if and only if $f \equiv 0$.

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Proposition 2.9. *Let M be a compact metric space, let $f \in C(M)$ and let $\mu \in \mathcal{B}(M)$. Then $\mu \in J(f)$ if and only if*

$$|||\mu||| = \|f\|_\infty \tag{2.2}$$

and the Jordan decomposition of μ satisfies

$$\mu_P(M \setminus E_f^P) = 0 = \mu_N(M \setminus E_f^N). \tag{2.3}$$

Proof. Let $\mu \in J(f)$. Then, clearly, (2.2) holds. Moreover, if

$$\mu_P(M \setminus E_f^P) + \mu_N(M \setminus E_f^N) > 0,$$

then

$$\begin{aligned}\mu(f) &= \int_P f\,\mathrm{d}\mu_P - \int_N f\,\mathrm{d}\mu_N \\ &< (\mu_P(E_f^P) + \mu_P(M \setminus E_f^P) + \mu_N(E_f^N) + \mu_N(M \setminus E_f^N))\|f\|_\infty \\ &= |||\mu|||\|f\|_\infty = \|f\|_\infty^2,\end{aligned}$$

which contradicts $\mu(f) = \|f\|_\infty^2$. Hence, (2.3) holds.

On the other hand, if (2.2) and (2.3) hold, then

$$\begin{aligned}\mu(f) &= \int_{E_f^P} f\,\mathrm{d}\mu_P - \int_{E_f^N} f\,\mathrm{d}\mu_N \\ &= (\mu_P(E_f^P) + \mu_N(E_f^N))\|f\|_\infty \\ &= |||\mu|||\|f\|_\infty \\ &= \|f\|_\infty^2,\end{aligned}$$

so that $\mu \in J(f)$. □

The following proposition provides an explicit formula for the semi-inner product on $C(M)$.

Proposition 2.10. *Let M be a compact metric space and let $f, g \in C(M)$. Then*

$$\langle f, g\rangle_- = \|g\|_\infty \min\Big\{\min_{x \in E_g^P} f(x), \min_{x \in E_g^N} -f(x)\Big\}$$

with the convention $\min \emptyset = \infty$.

Note that $E_g^P = \emptyset = E_g^N$ is impossible, and that the right-hand side is therefore finite.

Proof. Since g is continuous, the sets E_g^P and E_g^N are non-empty and compact. Since f is continuous, it attains its minimum over E_g^P at some $x_g^P \in E_g^P$ and its maximum over E_g^N at some $x_g^N \in E_g^N$. As the Dirac measures $\delta_{x_g^P}$ and $\delta_{x_g^N}$ satisfy $\delta_{x_g^P} \in \mathcal{B}(M)^+$ and $\delta_{x_g^N} \in \mathcal{B}(M)^+$, and because

$$\delta_{x_g^P}(M \setminus E_g^P) = 0 = \delta_{x_g^N}(M \setminus E_g^N)$$

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and $\|\delta_{x_g^P}\| = \|\delta_{x_g^N}\| = 1$, proposition 2.9 implies $\|g\|_\infty \delta_{x_g^P} \in J(g)$ and $-\|g\|_\infty \delta_{x_g^N} \in J(g)$. Therefore, proposition 2.9 yields

$$\begin{aligned}\langle f, g\rangle_- &= \inf\{\mu(f) \colon \mu \in J(g)\}\\ &\leqslant \|g\|_\infty \min\{\delta_{x_g^P}(f), -\delta_{x_g^N}(f)\}\\ &= \|g\|_\infty \min\{f(x_g^P), -f(x_g^N)\}\\ &= \|g\|_\infty \min\Big\{\min_{x\in E_g^P} f(x), -\max_{x\in E_g^N} f(x)\Big\}.\end{aligned}$$

It is easy to see that no $\mu \in J(g)$ yields a lower value. □

When $X = C(\mathbb{S}^{d-1})$ and $A, B \in \mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$, explicit expressions for the sets $E^P_{\sigma_A-\sigma_B}$ and $E^N_{\sigma_A-\sigma_B}$ can be obtained using the following proposition about variational inequalities.

Proposition 2.11. *Let $A \in \mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$, $a^* \in A$ and $x \in \mathbb{R}^d$. Then*

$$\|x - a^*\| = \operatorname{dist}(x, A) \iff \langle x - a^*, a - a^*\rangle \leqslant 0 \quad \text{for all } a \in A, \tag{2.4}$$
$$\|a^* - x\| = \operatorname{dist}(A, x) \iff \langle x - a^*, a - a^*\rangle \geqslant 0 \quad \text{for all } a \in A. \tag{2.5}$$

Proof. Inequality (2.4) is standard (see, for example, [5, proposition 7.4]), and (2.5) can be obtained by an analogous proof. □

We are now in a position to characterize the sets $E^P_{\sigma_A-\sigma_B}$ and $E^N_{\sigma_A-\sigma_B}$.

Proposition 2.12. *Let $A, B \in \mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$, and let $\sigma_A, \sigma_B \in \Sigma(\mathbb{R}^d)$ be the corresponding support functions.*

(a) *If $A = B$, then $E^P_{\sigma_A-\sigma_B} = \mathbb{S}^{d-1}$.*

(b) *If $A \subsetneq B$, then $E^P_{\sigma_A-\sigma_B} = \emptyset$.*

(c) *Let $A \not\subset B$. Then, for any $p \in \mathbb{S}^{d-1}$, we have $p \in E^P_{\sigma_A-\sigma_B}$ if and only if there exist $a^* \in A$ and $b^* \in B$ such that $p = (a^* - b^*)/\|a^* - b^*\|$ and*

$$\|a^* - b^*\| = \operatorname{dist}(a^*, B) = \operatorname{dist}(A, B) = \operatorname{dist}_{\mathrm{H}}(A, B). \tag{2.6}$$

An analogous statement holds for the set $E^N_{\sigma_A-\sigma_B}$.

Proof. If $A = B$, then $\sigma_A = \sigma_B$, and hence

$$E^P_{\sigma_A-\sigma_B} = \{p \in \mathbb{S}^{d-1} \colon \sigma_A(p) - \sigma_B(p) = \|\sigma_A - \sigma_B\|_\infty\} = \mathbb{S}^{d-1},$$

which proves (a).

If $A \subsetneq B$, then $\|\sigma_A - \sigma_B\|_\infty > 0$ and $\sigma_A - \sigma_B \leqslant 0$, so that

$$E^P_{\sigma_A-\sigma_B} = \{p \in \mathbb{S}^{d-1} \colon \sigma_A(p) - \sigma_B(p) = \|\sigma_A - \sigma_B\|_\infty\} = \emptyset,$$

which is (b).

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Let us show the equivalence (c). Let $p \in E^P_{\sigma_A - \sigma_B}$. Using proposition 2.2, we find

$$\begin{aligned} \mathrm{dist}_{\mathrm{H}}(A,B) &= \|\sigma_A - \sigma_B\|_\infty \\ &= \sigma_A(p) - \sigma_B(p) \\ &= \sup_{a\in A}\langle p,a\rangle - \sup_{b\in B}\langle p,b\rangle \\ &= \sup_{a\in A}\inf_{b\in B}\langle p,a-b\rangle \\ &= \sup_{a\in A}\inf_{b\in B}\cos\angle(p,a-b)\|a-b\| \\ &\leqslant \sup_{a\in A}\cos\angle(p,a-\mathrm{proj}_B(a))\|a-\mathrm{proj}_B(a)\|. \end{aligned}$$

By the compactness of A and the continuity of the above expression, there exists $a^* \in A$ such that

$$\begin{aligned} \mathrm{dist}_{\mathrm{H}}(A,B) &\leqslant \sup_{a\in A}\cos\angle(p,a-\mathrm{proj}_B(a))\|a-\mathrm{proj}_B(a)\| \\ &= \cos\angle(p,a^*-\mathrm{proj}_B(a^*))\|a^*-\mathrm{proj}_B(a^*)\| \\ &= \cos\angle(p,a^*-\mathrm{proj}_B(a^*))\,\mathrm{dist}(a^*,B) \\ &\leqslant \cos\angle(p,a^*-\mathrm{proj}_B(a^*))\,\mathrm{dist}(A,B) \\ &\leqslant \mathrm{dist}(A,B). \end{aligned}$$

Hence, the above inequalities are, in fact, equalities, which enforces

$$\cos\angle(p,a^*-\mathrm{proj}_B(a^*)) = 1,$$
$$0 < \mathrm{dist}_{\mathrm{H}}(A,B) = \mathrm{dist}(A,B) = \mathrm{dist}(a^*,B).$$

Therefore, a^* and $b^* := \mathrm{proj}_B(a^*) \in B$ satisfy (2.6) and $p = (a^*-b^*)/\|a^*-b^*\|$.

To show the opposite implication, let $a^* \in A$ and $b^* \in B$ satisfy (2.6) and set $p = (a^*-b^*)/\|a^*-b^*\|$. Note that (2.6) and the assumption $A \not\subset B$ guarantee $a^* \neq b^*$. Using (2.5) and (2.4), we obtain

$$\langle a^*-b^*,a\rangle \leqslant \langle a^*-b^*,a^*\rangle \quad \text{for all } a\in A,$$
$$\langle a^*-b^*,b\rangle \leqslant \langle a^*-b^*,b^*\rangle \quad \text{for all } b\in B,$$

so that

$$\sup_{a\in A}\langle a^*-b^*,a\rangle = \langle a^*-b^*,a^*\rangle,$$
$$\sup_{b\in B}\langle a^*-b^*,b\rangle = \langle a^*-b^*,b^*\rangle.$$

Hence, using proposition 2.2, we find

$$\begin{aligned} \sigma_A(p) - \sigma_B(p) &= \sup_{a\in A}\langle p,a\rangle - \sup_{b\in B}\langle p,b\rangle \\ &= \frac{1}{\|a^*-b^*\|}\Big(\sup_{a\in A}\langle a^*-b^*,a\rangle - \sup_{b\in B}\langle a^*-b^*,b\rangle\Big) \end{aligned}$$

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$$= \frac{1}{\|a^* - b^*\|}(\langle a^* - b^*, a^*\rangle - \langle a^* - b^*, b^*\rangle)$$
$$= \|a^* - b^*\|$$
$$= \mathrm{dist}_{\mathrm{H}}(A, B)$$
$$= \|\sigma_A - \sigma_B\|_\infty,$$

so that $p \in E^P_{\sigma_A - \sigma_B}$. □

3. The existence and uniqueness of solutions

In this section we apply the standard existence and uniqueness results for the initial-value problem

$$x'(t) = f(t, x(t)), \qquad x(0) = x_0, \tag{3.1}$$

on a real Banach space X to the particular case of set differential equations in the support function representation (1.2). We first collect the necessary terminology and state a standard existence and uniqueness result from [6] for differential equations in Banach spaces.

Definition 3.1. Let X be a Banach space, and let $\mathcal{D}(X)$ be the family of all bounded subsets of X. The Kuratowski measure of non-compactness $\alpha\colon \mathcal{D}(X) \to \mathbb{R}$ is defined by

$$\alpha(A) = \inf\{d > 0\colon A \text{ admits a finite covering by sets of diameter} \leqslant d\}.$$

Definition 3.2 introduces standard classes of growth functions from [6]. The symbol D^- denotes the Dini derivative

$$D^-\rho(t) = \liminf_{h \searrow 0} h^{-1}(\rho(t+h) - \rho(t))$$

of functions $\rho\colon \mathbb{R} \to \mathbb{R}$.

Definition 3.2. We distinguish the following classes of growth functions.

(U0) A continuous function $\omega\colon \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is said to be of class U_0 if the initial-value problem

$$\rho' = \omega(\rho), \qquad \rho(0) = 0,$$

possesses only the trivial solution.

(U1) Let $b > 0$. A function $\omega\colon (0, b] \times \mathbb{R}^+_0 \to \mathbb{R}$ is said to be of class U_1 if for each $\epsilon > 0$ there are a $\delta > 0$, a sequence $t_i \to 0^+$ and a sequence of continuous functions $\rho_i\colon [t_i, b] \to \mathbb{R}^+_0$ such that

(a) $\rho_i(t_i) \geqslant \delta t_i$ for all $i \in \mathbb{N}$,

(b) $0 < \rho_i(t) \leqslant \epsilon$ for all $i \in \mathbb{N}$ and $t \in (t_i, b]$,

(c) there exists a sequence $(\delta_i)_{i\in\mathbb{N}}$ with $\delta_i > 0$ such that

$$D^-\rho_i(t) \geqslant \omega(t, \rho_i(t)) + \delta_i$$

for all $i \in \mathbb{N}$ and $t \in (t_i, b]$.

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The following existence and uniqueness theorem is an excerpt of [6, theorem 4.1] applied in the present context.

Theorem 3.3. *Let* $(X, \|\cdot\|_X)$ *be a Banach space, and let* $D \subset X$, $x_0 \in D$ *and* $r > 0$ *be such that* $D_r := D \cap B_r(x_0)$ *is closed and convex. Let* $c > 0$, *let* $f\colon [0,T] \times D_r \to X$ *be a continuous function satisfying*

$$\|f(t,x)\|_X \leqslant c \quad \text{for all } t \in [0,T],\ x \in D_r,$$

and let $b := \min\{T, r/c\}$. *Suppose that the subtangent condition*

$$f(t,x) \in T_D(x) \quad \text{for all } t \in [0,b], x \in \partial D \cap B_r(x_0)$$

holds. Then the initial-value problem (3.1) has a solution $\varphi\colon [0,b] \to D_r$, *provided one of the following additional conditions is satisfied:*

(a) *there exists a function* $\omega\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *of class* U_0 *such that*

$$\alpha(f([0,b] \times A)) \leqslant \omega(\alpha(A)) \quad \text{for all } A \subset D_r;$$

(b) *there exists a function* $\omega\colon (0,b] \times \mathbb{R}_+ \to \mathbb{R}_+$ *of class* U_1 *such that*

$$\langle f(t,x) - f(t,y), x - y\rangle_- \leqslant \omega(t, \|x-y\|_X)\|x-y\|_X$$

for all $t \in [0,b]$ *and* $x, y \in D_r$.

In case (b), the solution is unique.

When adapting theorem 3.3 to set differential equations, we shall frequently use the version

$$f(t,\sigma) \in \overline{\Sigma(\mathbb{R}^d) - \mathbb{R}_0^+\sigma} \quad \text{for all } t \in [0,T],\ \sigma \in \Sigma(\mathbb{R}^d), \tag{3.2}$$

of the subtangent condition to ensure that solutions do not leave the cone $\Sigma(\mathbb{R}^d)$ associated with $\mathcal{K}_c(\mathbb{R}^d)$.

Our first result is a Peano-type theorem.

Theorem 3.4. *Let* $f\colon [0,T] \times \Sigma(\mathbb{R}^d) \to \hat{\Sigma}(\mathbb{R}^d)$ *be a continuous function, let* $A_0 \in \mathcal{K}_c(\mathbb{R}^d)$ *and let* $r > 0$. *Then there exists* $c > 0$ *such that*

$$\|f(t,\sigma)\|_\infty \leqslant c \quad \text{for all } t \in [0,T],\ \sigma \in D_r = \Sigma(\mathbb{R}^d) \cap B_r(\sigma_{A_0}). \tag{3.3}$$

Let $b := \min\{T, r/c\}$. *If, in addition, the subtangent condition (3.2) holds, then there exists a solution* $\sigma\colon [0,b] \to D_r$ *of the set differential equation (1.2) in the support function representation.*

Proof. Since balls defined in the maximum norm are always convex and $\Sigma(\mathbb{R}^d)$ is a convex cone, the intersection D_r is convex. By proposition 2.3, the set D_r is compact, and the existence of some $c > 0$ such that (3.3) holds is implied by the continuity of f. By proposition 2.5, condition (3.2) implies

$$f(t,\sigma) \in T_{\Sigma(\mathbb{R}^d)}(\sigma) \quad \text{for all } t \in [0,b],\ \sigma \in D_r.$$

By the compactness of D_r and continuity of f, the image $f([0,T]\times D_r)$ is compact, and hence we have

$$\alpha(f([0,T]\times A)) = 0 = \alpha(A) \quad \text{for all } A\subset D_r,$$

so that the compactness assumptions of theorem 3.3(a) are trivially satisfied with $\omega(\rho)=\rho$ of class U_0. □

The next result is a Picard–Lindelöf-type statement.

THEOREM 3.5. *Let $A_0\in\mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$, and let $f\colon [0,T]\times\Sigma(\mathbb{R}^d)\to\hat{\Sigma}(\mathbb{R}^d)$ be continuous and Lipschitz continuous in its second argument, i.e. we assume that there exists $L>0$ such that*

$$\|f(t,\sigma_A)-f(t,\sigma_B)\|_\infty \leqslant L\|\sigma_A-\sigma_B\|_\infty = L\,\mathrm{dist}_{\mathrm{H}}(A,B)$$

for all $A,B\in\mathcal{K}_{\mathrm{c}}(\mathbb{R}^d)$. If, in addition, f satisfies condition (3.2), then there exists a unique solution $\sigma\colon [0,T]\to\Sigma(\mathbb{R}^d)$ of (1.2).

Proof. As f is continuous and $[0,T]$ is compact, we have

$$\kappa := \sup_{t\in[0,T]}\|f(t,\sigma_{A_0})\|_\infty < \infty,$$

and Lipschitz continuity of f yields

$$c_r := \sup_{t\in[0,T],\sigma\in B_r(\sigma_{A_0})\cap\Sigma(\mathbb{R}^d)}\|f(t,\sigma)\|_\infty \leqslant Lr+\kappa.$$

Because

$$\begin{aligned}\langle f(t,\sigma)-f(t,\tilde{\sigma}),\sigma-\tilde{\sigma}\rangle_- &= \inf_{\mu\in J(\sigma-\tilde{\sigma})}\mu(f(t,\sigma)-f(t,\tilde{\sigma}))\\ &\leqslant \inf_{\mu\in J(\sigma-\tilde{\sigma})}|||\mu|||\|f(t,\sigma)-f(t,\tilde{\sigma})\|_\infty\\ &\leqslant L\|\sigma-\tilde{\sigma}\|_\infty^2\end{aligned}$$

for all $t\in[0,T]$ and $\sigma,\tilde{\sigma}\in\Sigma(\mathbb{R}^d)$, and by the arguments in the preceding proof, all the assumptions of theorem 3.3(b) are verified with $r=1$, $c=c_1$ and $\omega(t,s)=Ls$, so there exists a unique solution $\sigma_0(\cdot)\colon[0,b_0]\to\Sigma(\mathbb{R}^d)\cap B_1(\sigma_{A_0})$ of (1.2) with $b_0:=\min\{T,1/(L+\kappa)\}$. If $1/(L+\kappa)<T$, the same argument yields a unique solution $\sigma_1(\cdot)\colon[b_0,b_0+b_1]\to\Sigma(\mathbb{R}^d)\cap B_1(\sigma_0(b_0))$ of the set differential equation with $b_1:=\min\{T-b_0,1/(2L+\kappa)\}$.

Assume that $b_0+b_1<T$ and that this construction can be repeated indefinitely with $\sum_{k=0}^N b_k<T$ for all $N\in\mathbb{N}$. But then

$$T\geqslant\sum_{k=0}^\infty b_k = \sum_{k=0}^\infty\frac{1}{kL+\kappa}=\infty,$$

which is a contradiction. Hence, there exists a smallest index $N\in\mathbb{N}$ such that $b_N=T$. Concatenating the unique solutions $\sigma_0,\ldots,\sigma_N$ yields a unique solution $\sigma\colon[0,T]\to\Sigma(\mathbb{R}^d)$ of (3.1) on the entire interval $[0,T]$. □

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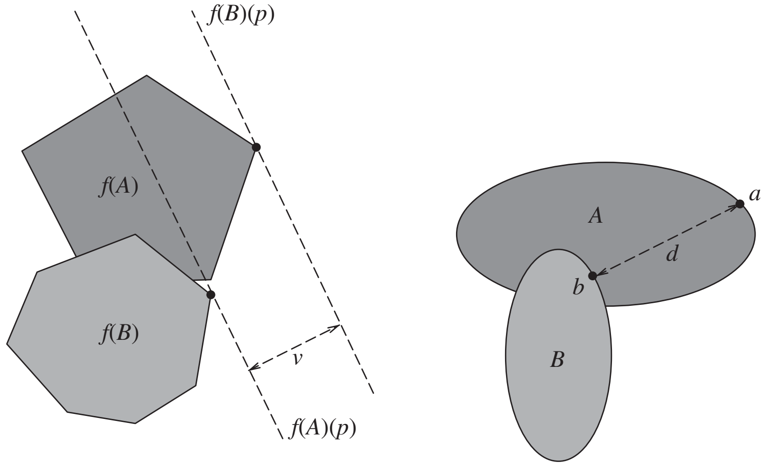

Figure 1. The geometric condition in theorem 3.6 in an important special case. Let $t \in (0, T)$ and assume that there exist $\tilde{A}, \tilde{B} \in \mathcal{K}_c(\mathbb{R}^d)$ such that $\sigma_{\tilde{A}} = f(t, \sigma_A)$ and $\sigma_{\tilde{B}} = f(t, \sigma_B)$. Then, as shown, the relative velocity is $v_r(p) = f(t, \sigma_A)(p) - f(t, \sigma_B)(p)$ in the critical direction $p = (a - b)/\|a - b\|$.

In contrast to the Picard–Lindelöf-type result above, the following statement fully exploits theorem 3.3(b) and the considerations from § 2.4. Roughly speaking, it states that the uniqueness of the solution can be guaranteed by controlling the relative velocity $f(t, \sigma_A) - f(t, \sigma_B)$ for two sets $A, B \in \mathcal{K}_c(\mathbb{R}^d)$ in only one critical direction, which is given by a pair $(a, b) \in A \times B$ that realizes the Hausdorff distance of A and B.

Theorem 3.6. *Let $f\colon [0, T] \times \Sigma(\mathbb{R}^d) \to \hat{\Sigma}(\mathbb{R}^d)$ be continuous, let $A_0 \in \mathcal{K}_c(\mathbb{R}^d)$ and let $r > 0$. Let $b, c > 0$ and D_r be as in theorem 3.4, let*

$$D'_r := \{A \in \mathcal{K}_c(\mathbb{R}^d)\colon \operatorname{dist}_H(A, A_0) \leqslant r\},$$

let $\omega\colon (0, T] \times \mathbb{R}_+ \to \mathbb{R}$ be of class U_1 and assume that the subtangent condition (3.2) holds. If, in addition, for any $t \in [0, b]$ and $A, B \in D'_r$ with $A \neq B$, there exist $a \in A$ and $b \in B$ such that $p := (a - b)/\|a - b\|$ is well defined and one of the conditions

$$\left.\begin{aligned} \|a - b\| = \operatorname{dist}(a, B) = \operatorname{dist}(A, B) = \operatorname{dist}_H(A, B), \\ f(t, \sigma_A)(p) - f(t, \sigma_B)(p) \leqslant \omega(t, \operatorname{dist}_H(A, B)) \end{aligned}\right\} \tag{3.4}$$

and

$$\left.\begin{aligned} \|a - b\| = \operatorname{dist}(b, A) = \operatorname{dist}(B, A) = \operatorname{dist}_H(A, B), \\ f(t, \sigma_B)(-p) - f(t, \sigma_A)(-p) \leqslant \omega(t, \operatorname{dist}_H(A, B)) \end{aligned}\right\} \tag{3.5}$$

is satisfied, then there exists a unique solution $\sigma\colon [0, b] \to D_r$ of (1.2).

The geometric principle behind conditions (3.4) and (3.5) is depicted in figure 1 for the case when $f(t, \sigma_A), f(t, \sigma_B) \in \Sigma$.

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Proof. By theorem 3.4, we know that the desired solution exists. According to theorem 3.3(b), to ensure uniqueness, we need to verify that

$$\langle f(t,\sigma_A) - f(t,\sigma_B), \sigma_A - \sigma_B\rangle_- \leqslant \omega(t, \|\sigma_A - \sigma_B\|_\infty)\|\sigma_A - \sigma_B\|_\infty$$

for any $t \in (0,b]$ and $\sigma_A, \sigma_B \in D_r$. By proposition 2.10 this is true if and only if, for any $t \in (0,b]$ and $\sigma_A, \sigma_B \in D_r$, at least one of the following inequalities is satisfied:

$$\min_{p\in E^P_{\sigma_A-\sigma_B}} (f(t,\sigma_A)(p) - f(t,\sigma_B)(p)) \leqslant \omega(t, \mathrm{dist_H}(A,B)),$$
$$\min_{p\in E^P_{\sigma_B-\sigma_A}} (f(t,\sigma_B)(p) - f(t,\sigma_A)(p)) \leqslant \omega(t, \mathrm{dist_H}(A,B)).$$

If $\sigma_A \neq \sigma_B$, this is, according to proposition 2.12, ensured by conditions (3.4) and (3.5). This can be checked by addressing all possible relations $A \subsetneq B$, $B \subsetneq A$ and $A \not\subset B \wedge B \not\subset A$ between the sets A and B separately. If $\sigma_A = \sigma_B$, both inequalities are obviously valid. □

4. Hukuhara-type differentials

In this section, we clarify that curves $A\colon [0,T] \to \mathcal{K}_\mathrm{c}(\mathbb{R}^d)$, which are second-type Hukuhara differentiable, are time-reversed Hukuhara differentiable curves with the same derivative up to sign change. This insight has some important consequences.

(i) As Hukuhara differentiable curves can only grow in diameter (see [11, proposition 1.6.1]), second-type Hukuhara differentiable curves can only shrink in diameter, as claimed in § 1.

(ii) As the support function representation of Hukuhara differentiable curves is Fréchet differentiable [3, lemma 4.1], this also holds for second-type Hukuhara differentiable curves. Furthermore, by the same lemma, the Hukuhara and second-type Hukuhara differentials of a curve coincide with its Fréchet differential (up to a sign change) whenever the Hukuhara-type differentials exist. Therefore, set differential equations based on both types of Hukuhara derivatives are special cases of our support function approach.

The notions of Hukuhara difference and Hukuhara differential are standard. The concept of generalized or second-type Hukuhara differentials goes back to [4]. Their use for set differential equations was investigated in [13, 14].

Definition 4.1 (Hukuhara differences and differentials).

(a) Let $A, B \in \mathcal{K}_\mathrm{c}(\mathbb{R}^d)$. If there exists $C \in \mathcal{K}_\mathrm{c}(\mathbb{R}^d)$ such that $A = B + C$, then C is called the Hukuhara difference between A and B, and we define $C = A \ominus_\mathrm{H} B$.

(b) A curve $A\colon [0,T] \to \mathcal{K}_\mathrm{c}(\mathbb{R}^d)$ is called Hukuhara differentiable at $t \in (0,T)$ with Hukuhara differential $D_\mathrm{H}A(t) \in \mathcal{K}_\mathrm{c}(\mathbb{R}^d)$ if the limits

$$\lim_{h\searrow 0} h^{-1}(A(t+h) \ominus_\mathrm{H} A(t)), \qquad \lim_{h\searrow 0} h^{-1}(A(t) \ominus_\mathrm{H} A(t-h))$$

with respect to Hausdorff distance exist and equal $D_\mathrm{H}A(t)$.

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(c) A curve $A\colon [0,T] \to \mathcal{K}_c(\mathbb{R}^d)$ is called second-type Hukuhara differentiable at $t \in (0,T)$ with differential $D_H^* A(t) \in \mathcal{K}_c(\mathbb{R}^d)$ if the limits

$$\lim_{h\searrow 0}(-h)^{-1}(A(t) \ominus_H A(t+h)), \qquad \lim_{h\searrow 0}(-h)^{-1}(A(t-h) \ominus_H A(t))$$

with respect to Hausdorff distance exist and equal $D_H^* A(t)$.

The following proposition shows that second-type Hukuhara differentiable curves are precisely those curves that are Hukuhara differentiable in the ordinary sense after time reversal.

PROPOSITION 4.2. *Let $A\colon [0,T] \to \mathcal{K}_c(\mathbb{R}^d)$ be a curve, and let $B\colon [-T,0] \to \mathcal{K}_c(\mathbb{R}^d)$ be given by $B(t) = A(-t)$. Then A is second-type Hukuhara differentiable at $t \in (0,T)$ if and only if B is Hukuhara differentiable at $-t$ in the usual sense. In that case, the respective differentials satisfy*

$$D_H^* A(t) = -D_H B(-t).$$

Proof. The statement follows immediately from the identities

$$\lim_{h\searrow 0}(-h)^{-1}(A(t) \ominus_H A(t+h)) = -\lim_{h\searrow 0} h^{-1}(B(-t) \ominus_H (B(-t-h))),$$
$$\lim_{h\searrow 0}(-h)^{-1}(A(t-h) \ominus_H A(t)) = -\lim_{h\searrow 0} h^{-1}(B(-t+h) \ominus_H B(-t))$$

for the Hausdorff limits. □

5. Example

We conclude with a simple but instructive example, which illustrates that the usefulness of both types of Hukuhara derivative depends not only on the equation, but also on the initial value. Consider the set differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{A(t)} = \sigma_Q - \sigma_{A(t)}, \qquad \sigma_{A(0)} = \sigma_{A_0} \tag{5.1}$$

in $\mathcal{K}_c(\mathbb{R}^2)$ with $Q = [-1,1]^2$ and $A_0 = [a_1,b_1] \times [a_2,b_2] \subset \mathbb{R}^2$. The curve

$$A(t) = \mathrm{e}^{-t}A_0 + (1-\mathrm{e}^{-t})Q \tag{5.2}$$

is a solution of (5.1) because

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma_{A(t)} = \mathrm{e}^{-t}(\sigma_Q - \sigma_{A_0}) = \sigma_Q - \sigma_{A(t)}.$$

By theorem 3.5, the solution is unique. Clearly, the set Q is a globally asymptotically stable fixed point.

By [3, lemma 4.1], any Hukuhara differentiable solution of the reformulation

$$D_H A(t) = Q \ominus_H A(t), \qquad A(0) = A_0 \tag{5.3}$$

of (5.1) in set notation must coincide with this curve. Note that, for many $A \in \mathcal{K}_c(\mathbb{R}^2)$, the right-hand side $Q \ominus_H A$ of (5.3) is not well defined. Since $\sigma_Q - \sigma_{A_0} \in \Sigma(\mathbb{R}^d)$ if and only if

$$\max\{b_1 - a_1, b_2 - a_2\} \leqslant 2, \tag{5.4}$$

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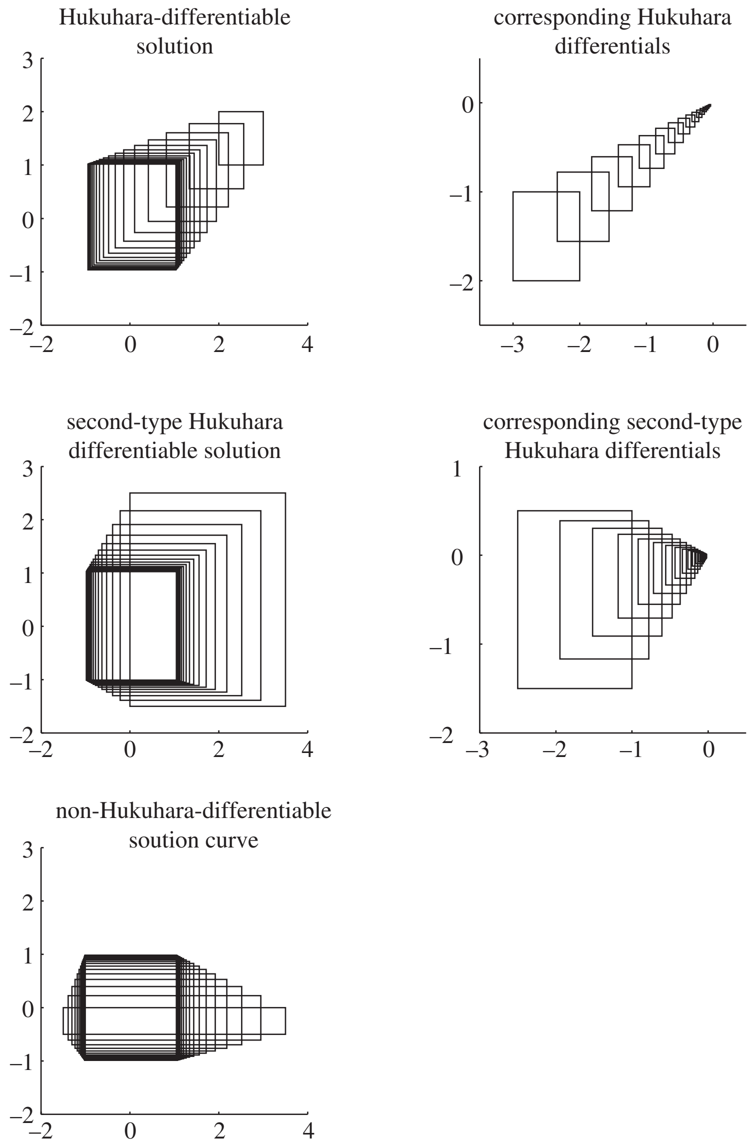

Figure 2. Solutions to set differential equation (5.1) with three different initial values. The rectangles in the graphs on the left are the values $A(t)$, $t = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \dots,$ of the solutions. The rectangles in the top-right graph are the Hukuhara differentials $D_{\mathrm{H}}A(t)$, and the rectangles in the second graph on the right are the second-type Hukuhara differentials $D^{*}_{\mathrm{H}}A(t)$ at the same time points. There is no graph in the bottom-right position, because the third solution curve is neither Hukuhara nor second-type Hukuhara differentiable.

a Hukuhara differentiable solution does not exist if this condition is violated. A computation shows that (5.4) is sufficient for (5.2) being a solution of first Hukuhara type.

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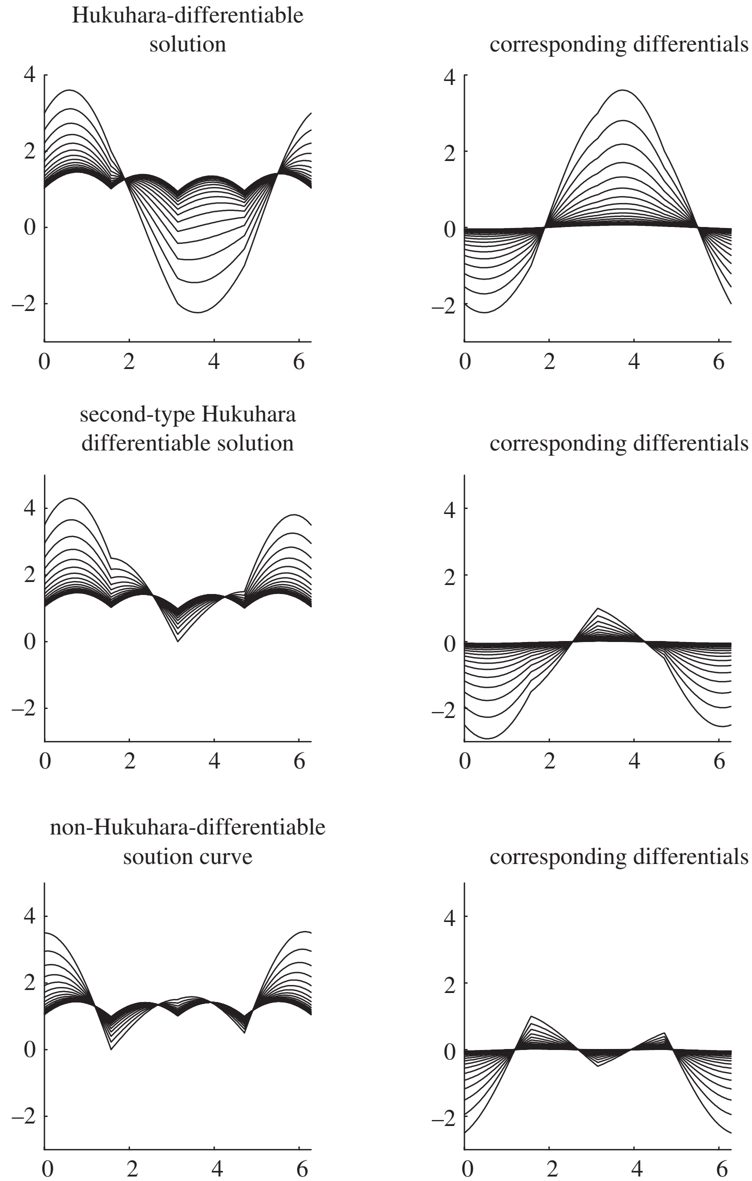

Figure 3. Solutions from figure 2 and the corresponding differentials in the support function representation. The differentials of the third curve cannot be interpreted as sets, but are well defined as elements of $\hat{\Sigma}(\mathbb{R}^d)$.

Proposition 4.2, however, shows that the curve (5.2) can only be a second-type Hukuhara solution if $\sigma_Q - \sigma_{A_0} \in -\Sigma(\mathbb{R}^d)$, which is equivalent to

$$\min\{b_1 - a_1, b_2 - a_2\} \geqslant 2, \tag{5.5}$$

and condition (5.5) is sufficient for (5.2) being a solution of second Hukuhara type.

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Figures 2 and 3 display solutions of (5.1) with three different initial values: $A_0^1 = [2,3] \times [1,2]$, $A_0^2 = [0,3.5] \times [-1.5,2.5]$ and $A_0^3 = [-1.5,3.5] \times [-0.5,0]$.

The left-hand column of Figure 2 depicts the evolution of the solutions. It can clearly be seen that $\mathrm{dist}_{\mathrm{H}}(A(t),Q) \to 0$ as $t \to \infty$. The first curve is Hukuhara differentiable but not second-type Hukuhara differentiable, and the Hukuhara differentials are plotted in the top right graph. The second curve is second-type Hukuhara, but not Hukuhara differentiable, and the second-type Hukuhara differentials are plotted in the middle of the right-hand column. In both cases, the differentials converge to $\{0\}$ when the state approaches Q. The third curve is neither Hukuhara differentiable nor second-type Hukuhara differentiable, because it shrinks in the direction of the first axis and grows in the direction of the second axis.

Figure 3 depicts the same three curves in a support function representation. The left-hand column shows the evolution of the support functions, while the right-hand column shows the Fréchet differentials along that curve. In this representation, the third curve can be treated like any other; the fact that its differentials are elements of $\hat{\Sigma}(\mathbb{R}^d) \setminus \Sigma(\mathbb{R}^d)$ causes no problems. In all three cases, the derivatives converge to the zero function as the state approaches equilibrium.

We conclude that both types of Hukuhara differentiability only yield solutions for very special initial conditions, while the support function approach yields a solution that exhibits the expected behaviour for any initial condition without any technical complications.

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