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TARIFF AND EQUILIBRIUM INDETERMINACY: A GLOBAL ANALYSIS

YAN ZHANG
Shanghai Jiao-Tong University

YAN CHEN
Arizona State University

Zhang [Tariff and Equilibrium Indeterminacy, available at http://mpra.ub.uni-muenchen.de/13099/ (2009)] shows that endogenous tariffs (or energy taxes) and endogenous labor income taxes are equivalent in generating local indeterminacy. Using methods developed by Stockman [*Journal of Economic Theory* 145 (2010), 1060–1085], we extend Zhang's analysis to prove that endogenous tariffs and endogenous labor income taxes are also equivalent in generating global indeterminacy (chaotic equilibria) under a balanced-budget rule. More precisely, we show that the existence of Euler equation branching in an arbitrarily small neighborhood of a steady state can imply topological chaos in the sense of Devaney. In addition, Euler equation branching occurs regardless of the local uniqueness of the equilibrium around the steady state.

Keywords: Endogenous Tariff Rate, Regime Switching, Chaos

1. INTRODUCTION

A large body of literature on trade taxes has recently suggested that a government can rely heavily on energy taxes (or tariffs) for its revenues.[1] Although much of the early research was concerned with cases of determinacy in dynamic stochastic general equilibrium (DSGE) models with energy in production [Rotemberg and Woodford (1994); de Miguel and Manzano (2006)], energy taxes on intermediate goods (e.g., imported energy)—which act like taxes on returns to factors of production—may generate indeterminacy in a way similar to factor income taxes.

Zhang (2009) shows that endogenous tariffs and endogenous labor income taxes [Schmitt-Grohe and Uribe (1997)] are equivalent in generating local indeterminacy. To be precise, local indeterminacy can emerge when the tariff rates levied on imported energy are endogenously determined by a balanced-budget rule with a

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constant level of government expenditures (or lump-sum transfers). In this paper, we extend Zhang's analysis to demonstrate that endogenous tariffs and endogenous labor income taxes are also equivalent in generating global indeterminacy (chaotic equilibria) under this balanced-budget rule. Our analysis shows that as in Stockman (2010), the existence of Euler equation branching in an arbitrarily small neighborhood of a steady state can imply topological chaos in the sense of Devaney. Euler equation branching here means that the dynamics going forward can be expressed by a differential inclusion of the form $\dot{x} \in F(x) := \{f(x), g(x)\}$.[2] In addition, multiple equilibria and chaos through deterministic regime-switching near a steady state can occur regardless of the local uniqueness of the equilibrium around the steady state. These results show that (1) global indeterminacy always exists in Zhang's model, no matter whether the (low tariff) steady state is locally indeterminate or not, and (2) tariffs and labor income taxes are equivalent in generating global indeterminacy because, as illustrated by Stockman (2010), the balanced-budget rule in Schmitt-Grohe and Uribe (1997) can lead to similar chaotic equilibria.

The sunspot equilibria discussed in our paper are deterministic. Roughly speaking, the equilibrium follows a deterministic sunspot process when the model economy switches between regimes [see Gardini et al. (2009)]. In discrete time, Christiano and Harrison (1999) analyze this type of regime-switching sunspot equilibria due to Euler equation branching in a one-sector economy with productive externalities. In continuous time, Stockman (2010) explores the same type of sunspot equilibria due to Euler equation branching in a one-sector economy with fiscal increasing returns. In continuous time, Stockman (2009) also investigates this type of sunspot equilibria due to Euler equation branching in a two-sector economy with productive externalities. This type of dynamics can also occur in the discrete-time cash-in-advance model [Michener and Ravikumar (1998); Kennedy et al. (2008)]. Medio and Raines (2007) call this type of indeterminacy in the cash-in-advance framework *backward dynamics* because the dynamical systems are multivalued moving forward, but single-valued moving backward.

We describe our model in Section 2 of this paper. In Section 3, we make a global analysis and explore the implications of Euler equation branching. In Section 4, other types of balanced-budget rules are discussed. In Section 5, we conclude the paper.

2. THE ONE-SECTOR ECONOMY WITH TARIFFS

This is the one-sector real business cycle model with imported oil studied by Zhang (2009). A representative agent maximizes the intertemporal utility function

$$\int_0^\infty e^{-\rho t}(\log c_t - bn_t)dt, \quad b > 0, \tag{1}$$

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where c_t and n_t are the individual household's consumption and hours worked, respectively, and $\rho \in (0, 1)$ is the subjective discount rate. We assume that there are no intrinsic uncertainties present in the model.

The budget constraint of the representative agent is given by

$$\dot{k}_t = (r_t - \delta)k_t + w_t n_t - c_t, \quad k_0 > 0 \text{ given}, \tag{2}$$

where $\dot{k}_t$ denotes net investment and the other variables are capital (k_t), rental rate (r_t), real wage (w_t), and depreciation rate (δ).

On the production side, a single good is produced with a Cobb–Douglas production technology by the representative firm,

$$y_t = k_t^{a_k} n_t^{a_n} o_t^{a_o}, \tag{3}$$

where y_t is total output, $a_k + a_n + a_o = 1$ (constant returns to scale), and the third factor in the production, nonreproducible natural resources—for example, oil (o_t)—is imported. Assuming that firms are price-takers in the factor markets and the international energy price (p^o) is exogenous to the economy, the profit of the firm is given by

$$\pi_t = y_t - w_t n_t - r_t k_t - p^o(1 + \tau_t)o_t,$$

where τ_t is the tariff rate levied on the imported oil and is uniform to all firms. Here we should emphasize that (1) in this standard neoclassical growth model, p^o is the relative price of the foreign input in terms of the single good, which is the numeraire and is tradable, and (2) in order to rule out the existence of import subsidies, we require that $\tau_t \geq 0$.[3]

Profit maximization by each firm leads to the following first–order conditions:

$$w_t = a_n \frac{y_t}{n_t},$$

$$r_t = a_k \frac{y_t}{k_t},$$

and

$$p^o(1 + \tau_t) = a_o \frac{y_t}{o_t}.$$

Because we assume that the foreign input is perfectly elastically supplied and the factor price (p^o) is independent of the factor demand for o_t, we can substitute out o_t in the production function using

$$o_t = a_o \frac{y_t}{p^o(1 + \tau_t)}$$

to obtain the production function

$$y_t = A_t k_t^{\frac{a_k}{1-a_o}} n_t^{\frac{a_n}{1-a_o}}, \tag{4}$$

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where $A_t = [\frac{a_o}{p^o(1+\tau_t)}]^{a_o/(1-a_o)}$ acts as the Solow residual in a neoclassical growth model, which is inversely related to the foreign factor price and τ_t.

The government must select $\{\tau_t\}$ to balance its budget each period:

$$p^o \tau_t o_t = G, \tag{5}$$

with $G > 0$ given. By combining equation (2), the first-order conditions of the firm, and the government budget constraint, we obtain the aggregate resource constraint $c_t + G + \dot{k}_t + \delta k_t + o_t p^o = y_t$.

As in Stockman (2010), we consider a kind of global indeterminacy called Euler equation branching. Euler equation branching occurs in our model because multiple equilibria occur in the labor market. To be precise, we consider paths for prices $\{w_t, r_t\}$ and tariffs $\{\tau_t\}$ that are piecewise continuous, with the following property: there are, at most, a finite number of discontinuities for any finite time interval. That is, the control variables (c_t and n_t) in the agent's problem should be piecewise continuous and the state variable (k_t) should be continuous, with a piecewise continuous derivative with possible discontinuities that occur when the control variables and prices/tariffs are discontinuous. Following Stockman (2010), at these discontinuous points, we require that the left and right limits of these points exist and their number be finite (the first kind of discontinuity).

A competitive equilibrium (CE) is defined as follows: A set of prices $\{w_t, r_t\}$ and resource allocations $\{c_t, k_t, n_t\}$ and a fiscal policy $\{G, \tau_t\}$ are a CE if $\{c_t, k_t, n_t\}$ is a solution to the household maximization problem, $\{k_t, n_t\}$ is a solution to the firm profit-maximization problem, and $\{G, \tau_t\}$ satisfies the government budget constraint.

The current-value Hamiltonian for our problem is

$$V(k_t, c_t, n_t, \Lambda_t, t) := (\log c_t - bn_t) + \Lambda_t[(r_t - \delta)k_t + w_t n_t - c_t], \tag{6}$$

where Λ_t is the co-state variable. Using the same definitions of admissible trajectories and weak maximality as Stockman (2010), we have sufficient conditions for the weakly optimal solution to our problem.[4]

PROPOSITION 1. *Assume that prices $\{w_t, r_t\}$, tariffs $\{\tau_t\}$, and initial capital stock k_0 are given. The current value Hamiltonian $V(k_t, c_t, n_t, \Lambda_t, t)$ is concave in $\{k_t, c_t, n_t\}$ for any given Λ_t and t. Suppose there exist a continuous and piecewise continuously differentiable function $\Lambda_t^* : R_+ \longrightarrow R$ and an admissible interior plan $\{k_t^*, c_t^*, n_t^*\}$ that satisfies the following conditions:*

$$\frac{1}{c_t^*} = \Lambda_t^*, \tag{7}$$

$$b = \Lambda_t^* w_t^*, \tag{8}$$

$$\dot{\Lambda}_t^* = (\rho + \delta - r_t)\Lambda_t^* \text{ for almost every } t \in R_+, \tag{9}$$

$$\dot{k}_t^* = (r_t - \delta)k_t^* + w_t n_t^* - c_t^* \text{ for almost every } t \in R_+, \tag{10}$$

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$$\lim_{t\to\infty} e^{-\rho t}\Lambda_t^*(k_t - k_t^*) \geq 0 \textit{ for all admissible paths } k_t. \quad \textbf{(11)}$$

Then $\{k_t^*, c_t^*, n_t^*\}$ *is weakly optimal.*

Proof. The proof is similar to that of Proposition 1 in Stockman (2010). ∎

In this paper, weakly optimal trajectory and admissible trajectory are denoted by $\{k_t^*, c_t^*, n_t^*\}$ and $\{k_t, c_t, n_t\}$, respectively.

3. EULER EQUATION BRANCHING AND GLOBAL INDETERMINACY

We use the sufficient conditions given in the section above and the government budget constraint to show the existence of global indeterminacy. As in Zhang (2009), equilibrium conditions can be expressed as follows:

$$\dot{\Lambda}_t = \Lambda_t \left\{ \rho + \delta - a_k \left[\frac{a_o}{p^o(1+\tau_t)} \right]^{\frac{a_o}{1-a_o}} k_t^{\frac{a_k}{1-a_o}-1} n_t^{\frac{a_n}{1-a_o}} \right\}, \quad \textbf{(12)}$$

$$\dot{k_t} = \left(1 - \frac{a_o}{1+\tau_t}\right) y_t - \delta k_t - 1/\Lambda_t - G, \quad \textbf{(13)}$$

$$b/\Lambda_t = a_n \left[\frac{a_o}{p^o(1+\tau_t)} \right]^{\frac{a_o}{1-a_o}} k_t^{\frac{a_k}{1-a_o}} n_t^{\frac{a_n}{1-a_o}-1}, \text{ and} \quad \textbf{(14)}$$

$$\frac{\tau_t a_o y_t}{1+\tau_t} = G, \quad \textbf{(15)}$$

where $y_t = [a_o/p^o/(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{a_n/(1-a_o)}$. Equation (12) is the consumption Euler equation associated with the household problem ($1/c_t = \Lambda_t$ holds in equilibrium). Equation (13) is the aggregate resource constraint. Equation (14) is the optimality condition associated with the household problem with respect to the labor supply. Equation (15) is the government budget constraint.

In addition, any equilibrium path $\{\Lambda_t, k_t, n_t\}$ should also satisfy the following conditions:

(i) k_t and Λ_t are continuous and piecewise continuously differentiable;
(ii) n_t is piecewise continuous with restrictions stated in the section above; and
(iii) Λ_t, k_t, and n_t are bounded from above and not zero for any t.

Any path $\{\Lambda_t, k_t, n_t\}$ that satisfies the above conditions is a CE. Equations (14) and (15) will be used to show that multiple equilibria in the labor market are the key to Euler equation branching. To see this, first, from equation (14), we express n_t as a function of k_t, Λ_t, and τ_t: $n_t = \{a_n[a_o/p^o(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} \Lambda_t/b\}^{1/(1-\frac{a_n}{1-a_o})}$. Second, using $y_t = [a_o/p^o(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{a_n/(1-a_o)}$ and $n_t = \{a_n[a_o/p^o(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} \Lambda_t/b\}^{1/(1-\frac{a_n}{1-a_o})}$, equation (15) can be rewritten as follows:

$$G = \tau_t[a_o/(1+\tau_t)]^{1+a_o/a_k}(p^o)^{-a_o/a_k} k_t (a_n\Lambda_t/b)^{a_n/a}. \quad \textbf{(16)}$$

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We should emphasize that (1) from equations (7) and (8), the equilibrium labor-supply curve ($w_t = bc_t$) is horizontal in (n, w) space, and (2) the equilibrium labor-demand curve is given by $w_t = a_n[a_o/p^o/(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{-a_k/(1-a_o)}$, where τ_t can be expressed as a function of k_t and n_t using $G = [a_o\tau_t/(1+\tau_t)][a_o/p^o/(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{a_n/(1-a_o)}$. The equilibrium labor demand curve is not monotonic in (n, w) space. Therefore, the demand and supply curves can intersect twice.[5] As in Stockman (2010), we find that (1) the equilibrium labor demand curve is initially beneath and ultimately below the labor supply curve; and (2) this branching is global and exists in an arbitrarily small open neighborhood of a steady state (Λ^*, k^*).[6] Hereafter, we use the notation with a star [e.g., (Λ^*, k^*)] to denote the steady-state values of some variables.

Now, let us formally define Euler equation branching.

Suppose that $X \subset R^n$ is an open state space and $\Phi, \Psi : X \to R^n$ are continuous. Consider the differential inclusion that is given by $\dot{x} \in F(x) := \{\Phi(x), \Psi(x)\}$. Provided that $\Phi(x) \neq \Psi(x)$ holds at a point $x \in X$, we say that Euler equation branching occurs at this point. Notice that Φ and Ψ are continuous by assumption. Therefore, if $\Phi(x) \neq \Psi(x)$, then we have that $\Phi(y) \neq \Psi(y)$ holds for all $y \in N_x$, where N_x is a sufficiently small neighborhood of x and $N_x \subset X$.

PROPOSITION 2. *A steady state exists for small G, and Euler equation branching occurs in a small open neighborhood of the steady state. Moreover, we have the following results:*

(1) *In the steady state,* $\rho + \delta = a_k[a_o/p^o/(1+\tau^*)]^{a_o/(1-a_o)} k^{*a_k/(1-a_o)-1} n^{*a_n/(1-a_o)}$, $(1-a_o)\,[a_o/p^o/(1+\tau^*)]^{a_o/(1-a_o)} k^{*a_k/(1-a_o)} n^{*a_n/(1-a_o)} = \delta k^* + 1/\Lambda^*$, $n^* = \{a_n[a_o/p^o/(1+\tau^*)]^{a_o/(1-a_o)} k^{*a_k/(1-a_o)} \Lambda^*/b\}^{1/(1-\frac{a_n}{1-a_o})}$, *and* $G = \tau^*[a_o/(1+\tau^*)]^{1+a_o/a_k} (p^o)^{-a_o/a_k} k^*(a_n\Lambda^*/b)^{a_n/a_k}$ *hold. G is small, so that* $[1 - a_o/(1+\tau^*)][a_o/p^o/(1+\tau^*)]^{a_o/(1-a_o)} k^{*a_k/(1-a_o)} n^{*a_n/(1-a_o)} > \delta k^* + G$ *. There then exists a steady state* (Λ^*, k^*) *that is a solution to the first three equalities within result 1 (given* τ^**);* τ^* *is the solution to the last equality within this result (given that* k^* *and* Λ^* *are functions of* τ^**).*

(2) *In a small open neighborhood B of* (Λ^*, k^*)*, there are two solutions to equation (16), which are denoted by* $\tau_t = g^1(\Lambda_t, k_t)$ *and* $\tau_t = g^2(\Lambda_t, k_t)$*. Moreover,* $\tau^* = g^1(\Lambda^*, k^*)$ *and* $\tilde{\tau} = g^2(\Lambda^*, k^*) \neq \tau^*$*. Therefore, equations (12), (13), (14), and (16) define a multivalued dynamical system, which can be written as* $(\dot{\Lambda}_t, \dot{k}_t) \in \{\Phi(\Lambda_t, k_t), \Psi(\Lambda_t, k_t)\}$*, with* $0 = \Phi(\Lambda^*, k^*) \neq \Psi(\Lambda^*, k^*)$ *and* $\Phi(\Lambda_t, k_t) \neq \Psi(\Lambda_t, k_t)$ *for* $(\Lambda_t, k_t) \in B$*.* $\Phi(\Lambda_t, k_t)$ *and* $\Psi(\Lambda_t, k_t)$ *can be obtained from equations (12) and (13) by replacing* τ_t *with* $g^1(\Lambda_t, k_t)$ *and* $g^2(\Lambda_t, k_t)$*. In this case, Euler equation branching occurs on the set B.*

Proof. Verifying that (1) holds is trivial.

To prove the second result, first, let us rewrite equation (16) as follows:

$$\frac{G}{a_o^{1+a_o/a_k}\,(p^o)^{-a_o/a_k}\,k_t(a_n\Lambda_t/b)^{a_n/a_k}} = \frac{\tau_t}{(1+\tau_t)^{1+a_o/a_k}}.$$

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Second, let $\mathcal{H}(\tau_t) := \tau_t/(1+\tau_t)^{1+a_o/a_k}$. One sees that $\mathcal{H} : (0,+\infty) \to R$ and straightforward calculations give $\mathcal{H}'(\tau_t) = \frac{1-a_o\tau_t/a_k}{(1+\tau_t)^{2+a_o/a_k}}$.

It is easy to find that $\mathcal{H}(\tau_t)$ is single-peaked with $\mathcal{H}'(\tau_t) > 0$ for $\tau_t \in (0, a_k/a_o)$ and $\mathcal{H}'(\tau_t) < 0$ for $\tau_t \in (a_k/a_o, +\infty)$. When $\tau_t = a_k/a_o$, $\mathcal{H}(a_k/a_o) = (a_k/a_o)[a_o/(a_k+a_o)]^{1+a_o/a_k}/a_o$. Moreover, $\lim_{\tau_t\to 0}\mathcal{H}(\tau_t) = 0$ and $\lim_{\tau_t\to+\infty}\mathcal{H}(\tau_t) = 0$. Therefore, for any (Λ_t, k_t) satisfying

$$0 < \frac{G}{a_o^{1+\frac{a_o}{a_k}}(p^o)^{-\frac{a_o}{a_k}}k_t(a_n\Lambda_t/b)^{\frac{a_n}{a_k}}} < \frac{a_k}{a_o}\left(\frac{a_o}{a_k+a_o}\right)^{1+\frac{a_o}{a_k}},$$

two solutions exist for τ_t to

$$\frac{G}{a_o^{1+\frac{a_o}{a_k}}(p^o)^{-\frac{a_o}{a_k}}k_t(a_n\Lambda_t/b)^{\frac{a_n}{a_k}}} = \tau_t\left(\frac{1}{1+\tau_t}\right)^{1+\frac{a_o}{a_k}}.$$

When

$$\frac{G}{a_o^{1+\frac{a_o}{a_k}}(p^o)^{-\frac{a_o}{a_k}}k_t(a_n\Lambda_t/b)^{\frac{a_n}{a_k}}} > \frac{a_k}{a_o}\left(\frac{a_o}{a_k+a_o}\right)^{1+\frac{a_o}{a_k}},$$

there are no solutions. According to the implicit function theorem, when there are two solutions in a small open neighborhood B of (Λ^*, k^*), these two solutions are written as $\tau_t = g^1(\Lambda_t, k_t)$ and $\tau_t = g^2(\Lambda_t, k_t)$. In addition, both $g^1(\Lambda_t, k_t)$ and $g^2(\Lambda_t, k_t)$ are C^1 functions and $g^1(\Lambda_t, k_t) < g^2(\Lambda_t, k_t)$. From equation (12), let us define $m^\Lambda(\Lambda_t, k_t, \tau_t) = \Lambda_t\{\rho + \delta - a_k[a_o/p^o(1+\tau_t)]^{a_o/(1-a_o)}k_t^{a_k/(1-a_o)-1}n_t^{a_n/(1-a_o)}\}$. We find that m^Λ is monotonically increasing in τ_t because $m^\Lambda(\Lambda_t, k_t, \tau_t) = \Lambda_t\{\rho + \delta - a_k(a_n\Lambda_t/b)^{a_n/a_k}[a_o/p^o(1+\tau_t)]^{a_o/a_k}\}$. This implies that for a given $(\Lambda_t, k_t) \in B$, we have $m^\Lambda[\Lambda_t, k_t, g^1(\Lambda_t, k_t)] \neq m^\Lambda[\Lambda_t, k_t, g^2(\Lambda_t, k_t)]$. Hence, we have a multivalued dynamical system that can be written as $(\dot{\Lambda}_t, \dot{k}_t) \in \{\Phi(\Lambda_t, k_t), \Psi(\Lambda_t, k_t)\}$ with $0 = \Phi(\Lambda^*, k^*) \neq \Psi(\Lambda^*, k^*)$ and $\Phi(\Lambda_t, k_t) \neq \Psi(\Lambda_t, k_t)$ for $(\Lambda_t, k_t) \in B$. $\Phi(\Lambda_t, k_t)$ and $\Psi(\Lambda_t, k_t)$ can be obtained from equations (12) and (13) by replacing τ_t with $g^1(\Lambda_t, k_t)$ and $g^2(\Lambda_t, k_t)$. In this case, Euler equation branching occurs on the set B. ■

Next, let us formally define Devaney chaos: Suppose that those conditions in Proposition 2 are satisfied. For $(\Lambda, k) \in B$, the model dynamics is described as follows:

$$\begin{bmatrix}\dot{\Lambda}_t \\ \dot{k}_t\end{bmatrix} \in F(\Lambda_t, k_t) =: \{\Phi(\Lambda_t, k_t), \Psi(\Lambda_t, k_t)\}.$$

We then provide several definitions of our dynamical system generated by a differential inclusion. Assume that the state space $B \subseteq R^2$ (open and nonempty) has the Euclidean metric d and $T := R_+$ is the time index. The space of all continuous functions from T into B is denoted by $W := \{\gamma \mid \gamma : T \to B\}$.

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Let Z be those functions (in W) that are continuous and piecewise continuously differentiable. In addition, the functions in Z satisfy the following condition: for any time interval $[t_1, t_2] \subset T$, $\dot{\gamma}$ has at most a finite number of discontinuities of the first kind. In this paper, we only consider this class of set-valued functions: $F : B \to 2^{R^2}$, where $F(x) := \{\Phi(x), \Psi(x)\}$ and $\Phi, \Psi : B \to R^2$ are C^r $(r \geq 1)$ with $\Phi(x) \neq \Psi(x)$ for all $x \in B$.

The following definitions are from Stockman (2010):

DEFINITION 1. *A dynamical system on B generated by F is a subset of Z. It can be written as follows:*

$$D := \{\gamma \in Z | \ \dot{\gamma} \in F[\gamma(t)] \text{ almost everywhere}\},$$

Such γ in D are called solutions to the differential inclusion F.

DEFINITION 2. *A set $K \subset B$ is (forward) invariant under the dynamical system D generated by F, if for each $x \in K$, there exists a solution $\gamma \in D$, such that $\gamma(0) = x$ and $\gamma(t) \in K$ for all $t \in T$; i.e., for each $x \in K$, there exists a γ that starts at x and stays in K forever.*

DEFINITION 3. *Let $K \subset B$ be a compact (forward) invariant set under the dynamical system D. D has sensitive dependence on initial conditions on K if there exists a constant $\epsilon > 0$ such that for any given $x \in K$ and its neighborhood $N(x) \subset K$, there exist solutions γ, $\theta \in D$ and $m \geq 0$ such that $\gamma(0) = x$, $\theta(0) \in N(x)$, $\gamma(t)$, $\theta(t) \in K$ for all $t \in T$, and $d(\gamma(m), \theta(m)) > \epsilon$.*

DEFINITION 4. *A closed (forward) invariant set K is said to be topologically transitive under the dynamical system D generated by F if, for any two nonempty open sets $U, V \subset K$, there exist a solution $\gamma \in D$ and $s \in T$ with $\gamma(t) \in K$ for all $t \in T$, $\gamma(0) \in U$ and $\gamma(s) \in V$.*

DEFINITION 5. *D has a periodic solution of length $m(> 0)$ if there exists a $\gamma \in D$ with $\gamma(t) = \gamma(t + m)$ for all $t \in T$ and there does not exist an $n \in (0, m)$ with $\gamma(t) = \gamma(t + n)$ for all $t \in T$. A point $x \in B$ is periodic if there exists a periodic solution $\gamma \in D$ with $\gamma(t) = x$ for some t. D has a periodic solution of length $m = 0$ if there exists a solution $\gamma \in D$ with $\gamma(t) = \gamma^*$ for all $t \in T$.*

DEFINITION 6. *Let $K \subset B$ be a compact (forward) invariant set under the dynamical system D. D has a dense set of periodic points in K if, for any given $x \in K$ and its neighborhood N_x, there exists a periodic solution $\gamma \in D$ with $\gamma(0) \in N_x$ and $\gamma(t) \in K$ for all $t \in T$.*

DEFINITION 7. *K is said to be a chaotic invariant set if K is a compact invariant set under the dynamical system D generated by F such that*

(1) *D has sensitive dependence on initial conditions on K, and*
(2) *K is topologically transitive under D.*

K is a Devaney-chaotic invariant set if in addition

(3) *D has a dense set of periodic points in K.*

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We will see that the dynamical system D generated by F is chaotic on K if K is a chaotic invariant set. We will say that the dynamical system D generated by F is Devaney-chaotic on K if K is a Devaney-chaotic invariant set.

The key theorem in this paper is Theorem 1 in Section 4 of Stockman (2010).

THEOREM 1. *Let $X \subseteq R^2$ be an open set containing x^* and consider the multivalued dynamical system (MVDS) defined by $\dot{x} \in \{\Phi(x), \Psi(x)\}$ for all $x \in X$, where $\Phi, \Psi : X \to R^2$ are C^r functions as in Definition 1. Suppose that x^* is a steady state of the single-valued differential equation $\dot{x} = \Phi(x)$, i.e., $\Phi(x^*) = 0$, and assume that $\Psi(x^*) = \kappa \neq 0$ is not collinear with any of the eigenvectors of the Jacobian matrix $E = D\Phi(x^*)$ evaluated at the steady state x^*. The MVDS is then Devaney-chaotic on an invariant compact set with a nonempty interior in each of the following three cases:*

(1) *Saddle: The steady state x^* is a saddle under Φ; i.e., $E = D\Phi(x^*)$ has real eigenvalues λ_1, λ_2 with $\lambda_1 < 0 < \lambda_2$.*
(2) *Sink or source with distinct real roots: The steady state x^* is a sink or source under Φ with distinct real roots; i.e., $E = D\Phi(x^*)$ has distinct real eigenvalues with $0 < \lambda_1 < \lambda_2$ or $\lambda_2 < \lambda_1 < 0$.*
(3) *Sink or source with complex roots: The steady state x^* is a sink or source under Φ with complex roots; i.e., $E = D\Phi(x^*)$ has complex eigenvalues $u \pm vi$ with $u \neq 0$.*

This theorem states that a steady state associated with Euler equation branching implies chaos. To see this in our model, we consider two numerical examples and find that no matter whether the low-tariff steady state is locally indeterminate or not, there always exist numerous Devaney-chaotic invariant sets with nonempty interiors. Remember that rearranging terms in equation (16) gives

$$\frac{G}{a_o^{1+\frac{a_o}{a_k}}(p^o)^{-a_o/a_k}k_t(a_n\Lambda_t/b)^{a_n/a_k}} = \tau_t\left(\frac{1}{1+\tau_t}\right)^{1+a_o/a_k}, \tag{17}$$

where $\mathcal{H}(\tau_t) = \tau_t[1/(1+\tau_t)]^{1+a_o/a_k}$ was defined in Proposition 2. We already have the following results:

(1) $\mathcal{H}(\tau_t)$ is single-caved with $\mathcal{H}'(\tau_t) > 0$ for $\tau_t < a_k/a_o$ and $\mathcal{H}'(\tau_t) < 0$ for $\tau_t > a_k/a_o$.
(2) A unique equilibrium exists in the labor market with $\tau_t = a_k/a_o$ for

$$\frac{G}{a_o^{1+\frac{a_o}{a_k}}(p^o)^{-\frac{a_o}{a_k}}k_t(a_n\Lambda_t/b)^{\frac{a_n}{a_k}}} = \mathcal{H}\left(\frac{a_k}{a_o}\right).$$

(3) Two equilibria exist in the labor market for

$$\frac{G}{a_o^{1+\frac{a_o}{a_k}}(p^o)^{-\frac{a_o}{a_k}}k_t(a_n\Lambda_t/b)^{\frac{a_n}{a_k}}} < \mathcal{H}\left(\frac{a_k}{a_o}\right).$$

We call these two equilibria τ_{1t} and τ_{2t}, with

$$0 < \tau_{1t} < \frac{a_k}{a_o} < \tau_{2t} < \infty.$$

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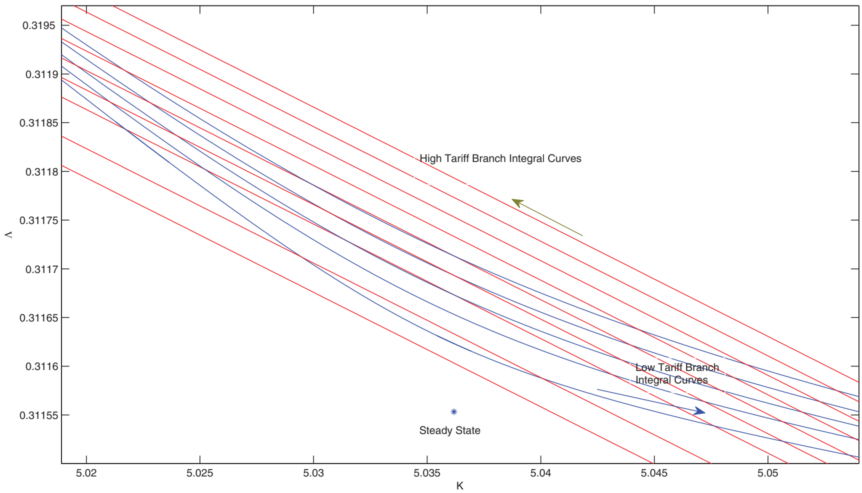

FIGURE 1. The low-tariff steady state is locally a saddle. The plotted trajectories from the high-tariff branch are flowing from the bottom right to the top left. However, the plotted trajectories from the low-tariff branch are flowing down and to the right.

In the following numerical exercises, the parameter value of a_o is taken from Aguiar-Conraria and Wen (2005), and the parameter values of a_n, ρ, and δ are taken from Schmitt-Grohe and Uribe (1997).

Example 1 (Local determinacy)

In the first numerical exercise, we use the following parameter values to demonstrate our results: $\rho = 0.04$, $a_o = 0.21$, $a_n = 0.64$, $p^o = 0.01$, $b = 0.5$, $\delta = 0.1$, and $G = 0.25$. We calculate the two steady states and eigenvalues from the linearization at these two steady states. We have the following results (see Figure 1):

(1) Low-tariff steady state values: $\tau^* = 0.3392$, $k^* = 5.0362$, $\Lambda^* = 0.31155$, $n^* = 1.8745$, $c^* = 3.2097$, and $y^* = 4.7004$; eigenvalues: $\mu_1 = -0.7237$ and $\mu_2 = 0.8903$.

(2) High-tariff steady state values: $\tau^* = 83.8794$, $k^* = 1.2907$, $\Lambda^* = 1.2156$, $n^* = 1.8745$, $c^* = 0.8226$, and $y^* = 1.2047$; eigenvalues: $\mu_1 = 0.2482$ and $\mu_2 = -0.3494$.

It is obvious that these two steady states are locally determinate. We then draw the trajectories from both branches near the low tariff steady state and find that numerous Devaney-chaotic invariant sets with nonempty interiors appear.

To provide some intuition for why the model dynamics is chaotic on these Devaney-chaotic invariant sets, we consider the dynamical system that is generated by a constant function and a linear function. Let $X := R^2$ and $H(x) := \{Ex, \kappa\}$, where E is a 2×2 matrix with no purely imaginary eigenvalues and $\kappa \in X$. D is the dynamical system generated by $\dot{x} \in H(x)$. Consider Example 1. We assume that the horizontal axis and the vertical axis are the unstable and stable

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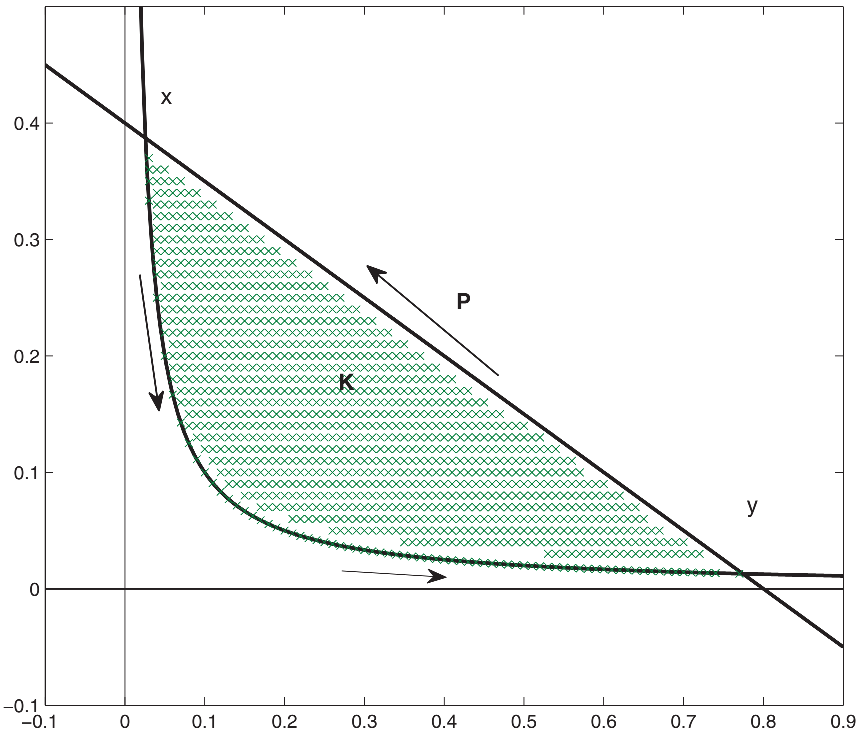

FIGURE 2. Saddle: chaotic region K.

manifolds of E, and the flow from κ is going from "southeast" to "northwest" (see Figure 2). We further assume that κ is not a scalar multiple of either eigenvector, because the integral curves generated by κ can typically intersect twice with those generated by E under this assumption. The region K in Figure 2 is one of those compact invariant sets on which F is Devaney-chaotic. Let us consider the flow from Ex. We can have a path that starts from x and moves to y; using the flow from κ, we can get back from y to x. We call this periodic path from x to y and then back to xP. The region K includes the periodic path P and its interior. Now, let us demonstrate why the system has topological transitivity. Selecting any two points $u, v \in K$, we can obtain a solution γ contained in K with $\gamma(0) = u$ and $\gamma(M) = v$ for some $M > 0$. Figure 3 illustrates how this happens. First, from u follow the solution generated by κ to $u_2 \in P$. Then follow the path P to $v_1 \in P$. Finally, from v_1 follow the flow generated by κ to v. Figure 4 shows how the system is sensitively dependent on initial conditions. Let γ_x be the periodic orbit that follows the path P from x to y and then back to x. Let γ_u be the periodic orbit from u to v, and then back to u. Notice that $\gamma_x(s)$ and $\gamma_u(s)$ are always more than some positive scalar δ^* apart whenever $\gamma_x(s) = x$. Let $w_1 \in K$ and $\varepsilon > 0$. Without loss of generality, we assume that w_1 lies in the path vu generated by κ. For any $z \in B_\varepsilon(w_1)$, there is a solution γ_z contained in K with $\gamma_z(0) = z$ and $\gamma_z(M_z) = u$ for some $M_z > 0$. In Figure 4, we show that γ_z follows the path

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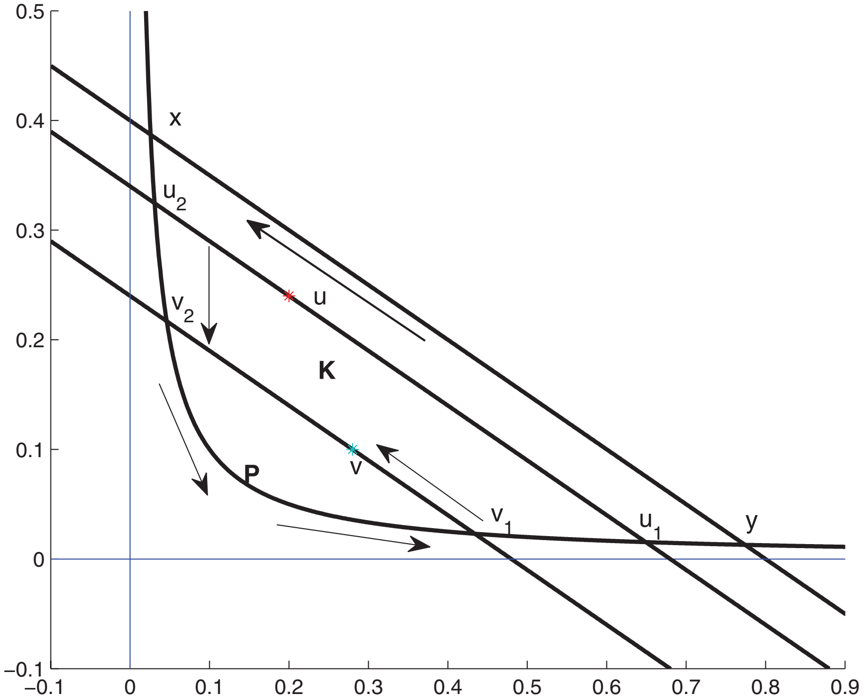

FIGURE 3. Saddle: a solution from any u to any v in K.

$v'u'$ generated by κ, and then moves along the path $u'u$. There exists a solution γ_{w_1} contained in K with $\gamma_{w_1}(0) = w_1$ and $\gamma_{w_1}(M_{w_1}) = x$ for some $M_{w_1} > 0$. We then let $\gamma(t) = \gamma_{w_1}(t)$ for $0 \leq t \leq M_{w_1}$ and $\gamma(t) = \gamma_x(t - M_{w_1})$ for $t \geq M_{w_1}$. Let $\theta(t) = \gamma_z(t)$ for $0 \leq t \leq M_z$ and $\theta(t) = \gamma_u(t - M_z)$ for $t \geq M_z$. Then for $m > \max\{M_{w_1}, M_z\}$ sufficiently large with $\gamma(m) = x$, $d(\gamma(m), \theta(m)) > \delta^*$ holds.

Example 2 (Local indeterminacy)

In the second numerical exercise, we use the following parameter values to show our results: $\rho = 0.04$, $a_o = 0.21$, $a_n = 0.64$, $p^o = 0.01$, $b = 0.5$, $\delta = 0.1$, and $G = 0.4$. We calculate the two steady states and eigenvalues from the linearization at these two steady states. We have the following results:

(1) Low-tariff steady state values: $\tau^* = 0.8092$, $k^* = 4.5628$, $\Lambda^* = 0.3439$, $n^* = 1.8745$, $c^* = 2.9080$, and $y^* = 4.2586$; eigenvalues $\mu_1 = -0.5767 + 1.3309i$ and $\mu_2 = -0.5767 - 1.3309i$.
(2) High-tariff steady state values: $\tau^* = 16.5738$, $k^* = 2.1640$, $\Lambda^* = 0.7251$, $n^* = 1.8745$, $c^* = 1.3792$, and $y^* = 2.0197$; eigenvalues $\mu_1 = 0.2278$ and $\mu_2 = -0.3341$.

It is obvious that the low-tariff steady state is locally indeterminate and the high-tariff steady state is locally determinate. We then draw the trajectories from

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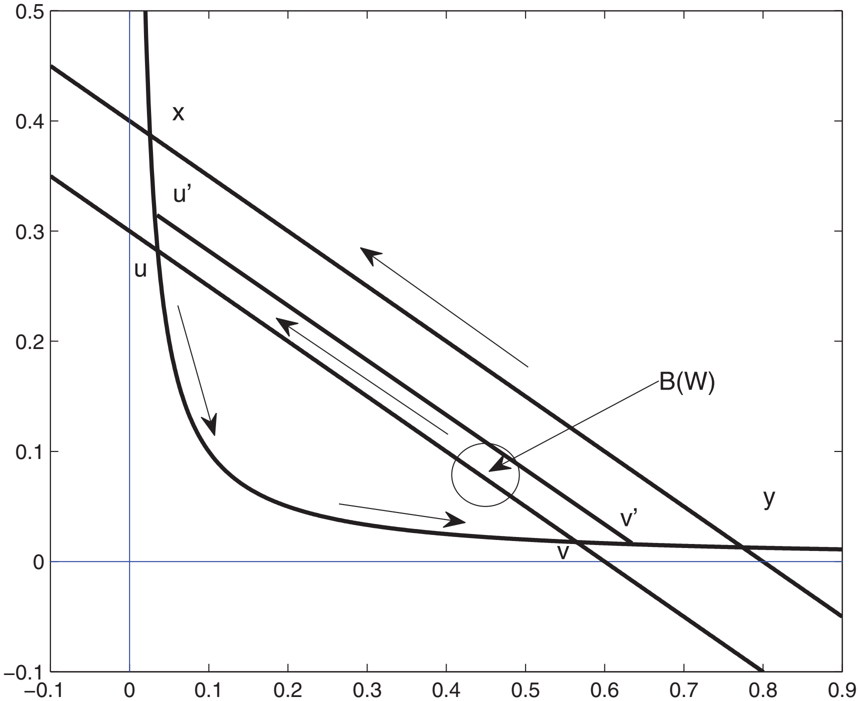

FIGURE 4. Saddle: sensitive dependence on initial conditions.

both branches near the low-tariff steady state (see Figure 5) and find that numerous Devaney-chaotic invariant sets with nonempty interiors appear (see Figure 6).[7]

In this section, we showed that global indeterminacy can occur near the low-tariff steady state regardless of the local stability properties of that steady state, and the high-tariff steady state is always a saddle. We illustrated that the existence of Euler equation branching crucially depends on how the balanced-budget rule is set up and that endogenous changes in the tariff rate are crucial for generating global indeterminacy.

4. OTHER BALANCED-BUDGET RULES

In this section, following Schmitt-Grohe and Uribe (1997) and Stockman (2010), we consider two other types of fiscal policies:

(P1) fixed tariff rates and endogenous spending, and
(P2) endogenous tariff rates with income-elastic government spending.

In the following proposition, we show that when tariff rates are fixed under a balanced-budget rule (P1), Euler equation branching cannot occur.

PROPOSITION 3 [Policy (P1)]. *Let us consider the model described in Section 2, but with a different balanced-budget rule. In particular, we let the tariff rate be fixed. We find that the labor supply and demand curves can intersect at most once.*

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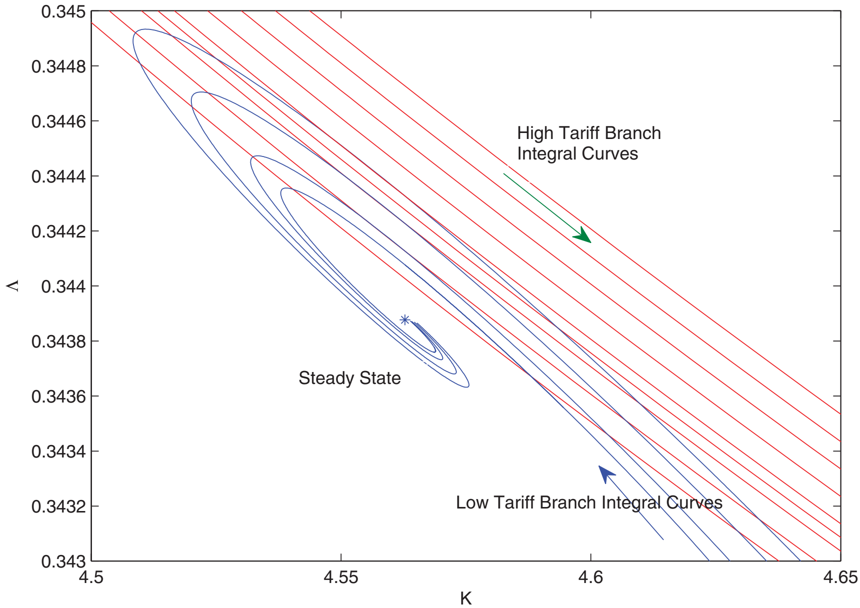

FIGURE 5. The low-tariff steady state is locally a sink. The plotted trajectories from the high-tariff branch are flowing from the top left to the bottom right. The plotted trajectories for the low-tariff branch are flowing counterclockwise.

Proof. Under this fiscal policy, the labor supply curve is horizontal. The labor demand curve is $w_t = a_n[a_o/p^o(1+\bar{\tau})]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{-a_k/(1-a_o)}$, which is strictly downward-sloping. Therefore, the two curves can intersect at most once and Euler equation branching cannot occur. ■

In the following proposition, we show that under policy (P2), the labor demand and supply curves can intersect twice and Euler equation branching can occur in an arbitrarily small neighborhood of a steady state.

PROPOSITION 4 [Policy (P2)]. *Consider the model described in Section 2 with a different balanced-budget rule (P2)—income-elastic government spending and endogenous tariffs. $G_t = \bar{G}(y_t/\bar{y})^{\Delta}$ holds, where $0 \leq \Delta < 1$ and $\bar{G}$ and $\bar{y}$ denote the steady state values of G_t and y_t. The labor demand and supply curves can then intersect twice and Euler equation branching can occur in an arbitrarily small neighborhood of a steady state.*

Proof. $G_t = \bar{G}(y_t/\bar{y})^{\Delta}$ holds for all t with $0 \leq \Delta < 1$. $\bar{y}$ is the steady state value of output and $\bar{G}$ is the steady state value of government spending. Let us consider equations (14), (15), and $G_t = \bar{G}(y_t/\bar{y})^{\Delta}$. After tedious algebra, we have the following labor market–equilibrium condition:

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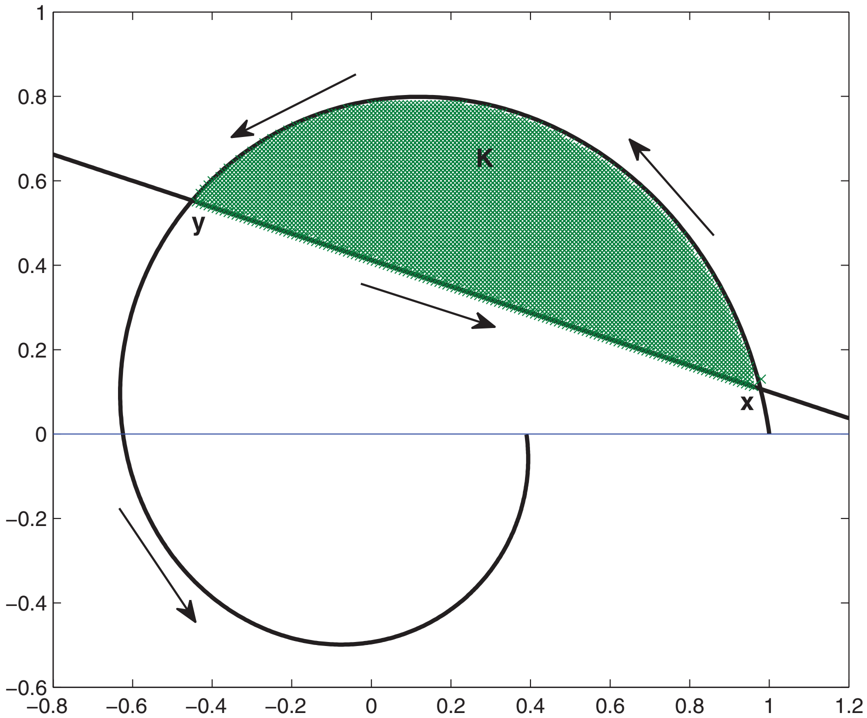

FIGURE 6. Sink with complex roots: chaotic region K.

$$\frac{a_o^{\frac{a_o}{a_k}\Delta}\bar{G}}{\bar{y}^{\Delta}}(a_n\Lambda_t/b)^{(\Delta-1)\frac{a_n}{a_k}} = k_t^{1-\frac{a_k+a_n}{1-a_o}\Delta}\Omega^{\mathrm{n}}(\tau_t),$$

where $\Omega^{\mathrm{n}}(\tau_t) = a_o^{a_o/a_k+1}(p^o)^{(\Delta-1)a_o/a_k}\,\tau_t\,(1+\tau_t)^{(a_o/a_k)(\Delta-1)-1}$. One can see that $\Omega^{\mathrm{n}}(\tau_t)$ is single-caved with $\Omega^{\mathrm{n}\prime}(\tau_t) > 0$ for $\tau_t < a_k/a_o(1-\Delta)$ and $\Omega^{\mathrm{n}\prime}(\tau_t) < 0$ for $\tau_t > a_k/a_o(1-\Delta)$. Therefore, we have the following results:

(1) A unique equilibrium exists in the labor market with $\tau_t = \frac{a_k}{a_o(1-\Delta)}$ when

$$\frac{a_o^{\frac{a_o}{a_k}\Delta}\bar{G}}{\bar{y}^{\Delta}}(a_n\Lambda_t/b)^{(\Delta-1)\frac{a_n}{a_k}} = k_t^{1-\frac{a_k+a_n}{1-a_o}\Delta}\Omega^{\mathrm{n}}\left(\frac{a_k}{a_o(1-\Delta)}\right).$$

(2) Two equilibria exist in the labor market when

$$k_t^{1-\frac{a_k+a_n}{1-a_o}\Delta}\Omega^{\mathrm{n}}\left(\frac{a_k}{a_o(1-\Delta)}\right) > \frac{a_o^{\frac{a_o}{a_k}\Delta}\bar{G}}{\bar{y}^{\Delta}}(a_n\Lambda_t/b)^{(\Delta-1)\frac{a_n}{a_k}}.$$

We call these two equilibria τ_{1t} and τ_{2t}, with

$$0 < \tau_{1t} < \frac{a_k}{a_o(1-\Delta)} < \tau_{2t} < \infty.$$

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Similarly to Proposition 2, we say that the labor demand and supply curves can intersect twice and Euler equation branching can occur in an arbitrarily small neighborhood of a steady state if government spending is income-elastic. ■

5. CONCLUSION

We show that under a balanced–budget rule, endogenous tariffs and endogenous labor-income taxes are equivalent in generating global indeterminacy in the form of Euler equation branching. The methodology in our paper comes from Stockman (2010). Similarly to Stockman (2010), the existence of Euler equation branching crucially depends on an endogenous tariff rate. These findings show that those multiple equilibria caused by the balanced-budget rule studied by Zhang can always exist and extend beyond local indeterminacy.

NOTES

1. See, for example, Bizer and Stuart (1987), Rotemberg and Woodford (1994), and de Miguel and Manzano (2006).

2. We often consider differential inclusions $\dot{x} \in F(x)$, where F is a set-valued map that associates with any point $x \in R^n$ a set $F(x) \subset R^n$. Euler equation branching is a special type of differential inclusion. Differential inclusions play a crucial role in studying ordinary differential equations with an inaccurately known right-hand side. If the right-hand side of a differential equation is in an ϵ_1-neighborhood of a given function $\mathfrak{f}(x)$, any solution of the differential equation is a solution to the differential inclusion $\dot{x} \in \mathfrak{f}(x) + \epsilon_1 B_n$, where B_n is a unit ball in R^n centered at zero [see Smirnov (2002)].

3. The model is based on the standard DSGE models that incorporate foreign energy as a third production factor. This class of models [such as those of Rotemberg and Woodford (1994) and Aguiar-Conraria and Wen (2005, 2007, 2008)] have been widely used to study the business-cycle effects of oil price shocks.

4. A trajectory $P := (c, n, k)$ is admissible if (a) $c(t), n(t), k(t) \geq 0$ and $k(0) = k_0 > 0$ is given; (b) c and n are piecewise continuous with, at most, a countable number of discontinuities and they satisfy the property that at most a finite number of discontinuities occur during any finite time interval $[a, b]$; and (c) k is continuous and piecewise continuously differentiable and $\dot{k}_t = (r_t - \delta)k_t + w_t n_t - c_t$ holds for almost every t. Two admissible paths P^* and P are comparable if we define the following function: $D(P^*, P, \text{Time}) = \int_0^{\text{Time}} e^{-\rho t}(\log c_t^* - bn_t^*)dt - \int_0^{\text{Time}} e^{-\rho t}(\log c_t - bn_t)dt$. The path P^* is weakly optimal if for every admissible path P, $\underline{\lim}_{\text{Time}\to\infty} D(P^*, P, \text{Time}) \geq 0$ holds.

5. Note that for $k_t, \Lambda_t > 0$, the number of equilibria in the labor market can be either zero or two. There is also some possibility of the demand and supply curves being tangent. In this special case, the number of equilibria is one. To remain comparable to the analysis of Stockman (2010), we only consider the case with two equilibria in the labor market.

6. Our results hold under general preferences. For example, preferences are given by $U(c, n) := \mathfrak{u}(c) - b\mathfrak{v}(n)$, where $\mathfrak{u}$ and $\mathfrak{v}$ are $\mathcal{C}^2$ functions with $\mathfrak{u}', \mathfrak{v}', \mathfrak{v}'' > 0$ and $\mathfrak{u}'' < 0$ with the Inada properties satisfied. We find that if the labor demand and supply curves intersect, there will be an even number of such intersections. To see this, let w_t be the after-tariff real wage. The labor-supply curve is defined from the first-order conditions: $b\mathfrak{v}'(n_t) = \Lambda_t w_t$. Because $\mathfrak{v}'' > 0$, the labor supply curve is increasing in n_t. The equilibrium labor demand curve is given by $w_t = a_n[a_o/p^o(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{-a_k/(1-a_o)}$, where τ_t can be expressed as a function of k_t and n_t using $G = [\tau_t a_o/(1+\tau_t)][a_o/p^o(1+\tau_t)]^{a_o/(1-a_o)} k_t^{a_k/(1-a_o)} n_t^{a_n/(1-a_o)}$. The equilibrium labor demand curve can be initially beneath and ultimately below the labor supply curve. Therefore, these two curves intersect twice.

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7. From Figures 5 and 6, one sees that as in the saddle case, the chaotic region K for the sink case (with complex roots) is the periodic path, which is generated by one branch from x to y and then moves back from y to x along the other branch, and its interior.

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