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# Nodal solutions for the fractional Yamabe problem on Heisenberg groups

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Dedicated to Professor Patrizia Pucci on the occasion of her 65th birthday.

We prove that the fractional Yamabe equation $\mathcal{L}_\gamma u = |u|^{((4\gamma)/(Q-2\gamma))}u$ on the Heisenberg group $\mathbb{H}^n$ has $[n+1/2]$ sequences of nodal (sign-changing) weak solutions whose elements have mutually different nodal properties, where $\mathcal{L}_\gamma$ denotes the CR fractional sub-Laplacian operator on $\mathbb{H}^n$, $Q = 2n+2$ is the homogeneous dimension of $\mathbb{H}^n$, and $\gamma \in \bigcup_{k=1}^{n}[k, ((kQ)/Q-1)))$. Our argument is variational, based on a Ding-type conformal pulling-back transformation of the original problem into a problem on the CR sphere $S^{2n+1}$ combined with a suitable Hebey-Vaugon-type compactness result and group-theoretical constructions for special subgroups of the unitary group $\mathbf{U}(n+1)$.

*Keywords:* CR fractional sub-Laplacian; nodal solution; Heisenberg group

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## 1. Introduction

After the seminal paper of Caffarelli and Silvestre [8], considerable efforts have been made concerning the study of elliptic problems involving the fractional Laplace operator both in Euclidean and non-Euclidean settings. As expected, the Euclidean framework is much more developed; although many results concerning the pure Laplace operator can be transposed to the fractional setting in $\mathbb{R}^n$, there are also subtle differences which require a deep understanding of certain nonlinear phenomena, see for example, Cabré and Sire [6], Caffarelli [7], Caffarelli, Salsa and Silvestre [9], Di Nezza, Palatucci and Valdinoci [10], and references therein.

By exploring analytical and spectral theoretical arguments, important contributions have been obtained recently within the CR setting concerning the fractional Laplace operator with various applications in sub-elliptic PDEs, see Branson, Fontana and Morpurgo [5], Frank and Lieb [14], and Frank, del Mar González, Monticelli and Tan [15]. In particular, in the latter papers, sharp Sobolev and

Moser-Trudinger inequalities are established on the Heisenberg group $\mathbb{H}^n$, $n \geqslant 1$, the simplest non-trivial CR structure.

In the present paper, we shall consider the *fractional Yamabe problem* on the Heisenberg group $\mathbb{H}^n$, namely,

$$\begin{cases} \mathcal{L}_\gamma u = |u|^{((4\gamma)/(Q-2\gamma))} u \quad \text{on} \quad \mathbb{H}^n, \\ u \in D^\gamma(\mathbb{H}^n). \end{cases} \tag{$\mathbf{FYH}$)$_\gamma$}$$

Hereafter, $Q := Q_n = 2n + 2$ is the homogeneous dimension of $\mathbb{H}^n$, $\gamma > 0$ is a parameter specified later, $\mathcal{L}_\gamma$ denotes the CR fractional sub-Laplacian operator on $\mathbb{H}^n$, and the functional space $D^\gamma(\mathbb{H}^n)$ contains real-valued functions from $L^{((2Q)/(Q-2\gamma))}(\mathbb{H}^n)$ whose energy associated with the CR fractional sub-Laplacian operator $\mathcal{L}_\gamma$ is finite; see § 2.4 for details.

Due to the recent paper of Frank, del Mar González, Monticelli and Tan [**15**], we know the existence of positive solutions of $(\mathbf{FYH})_\gamma$ for $\gamma \in (0, Q/2)$, having the form

$$u(z,t) = c_0((1+|z|^2)^2 + t^2)^{((2\gamma - Q)/(4))}, \quad (z,t) \in \mathbb{H}^n, \tag{1.1}$$

for some $c_0 > 0$, allowing any left translation and dilation. In the special case $\gamma = 1$, when $\mathcal{L}_1 = \mathcal{L}$ is the usual sub-Laplacian operator on $\mathbb{H}^n$, the existence and uniqueness (up to left translation and dilation) of positive solutions of the form (1.1) for problem $(\mathbf{FYH})_1$ have been established by Jerison and Lee [**18**, **19**]; see also Garofalo and Vassilev [**16**] for generic Heisenberg-type groups (e.g. Iwasawa groups).

Our main result guarantees sign-changing solutions for the fractional Yamabe problem $(\mathbf{FYH})_\gamma$ as follows:

THEOREM 1.1. *Let* $\gamma \in \bigcup_{k=1}^{n}[k, ((kQ)/(Q-1)))$, *where* $Q = 2n+2$. *Then problem* $(\mathbf{FYH})_\gamma$ *admits at least* $[n + 1/2]$ *sequences of sign-changing weak solutions whose elements have mutually different nodal properties.* (*Hereafter,* $[r]$ *denotes the integer part of* $r \geqslant 0$.)

Before commenting on theorem 1.1, we recall that similar results are well known in the Euclidean setting; indeed, Bartsch, Schneider and Weth [**4**] proved the existence of infinitely many sign-changing weak solutions for the polyharmonic problem

$$\begin{cases} (-\Delta)^m u = |u|^{((4m)/(N-2m))} u \quad \text{in} \quad \mathbb{R}^N, \\ u \in \mathcal{D}^{m,2}(\mathbb{R}^N), \end{cases} \tag{$\mathbf{P}$)$_m$}$$

where $N > 2m$, $m \in \mathbb{N}$, and $\mathcal{D}^{m,2}(\mathbb{R}^N)$ denotes the usual higher order Sobolev space over $\mathbb{R}^N$. In fact, their proof is based on Ding's original idea, see [**11**], who considered the case $m = 1$, by pulling back the variational problem $(\mathbf{P})_m$ to the standard sphere $S^N$ by stereographic projection. In this manner, by exploring certain properties of suitable subgroups of the orthogonal group $\mathbf{O}(N+1)$, the authors are able to obtain compactness by exploring a suitable Sobolev embedding result of Hebey and Vaugon [**17**] which is indispensable in the application of the symmetric mountain pass theorem.

We notice that sign-changing solutions are already guaranteed to the usual CR-Yamabe problem $(\mathbf{FYH})_1$ by Maalaoui and Martino [**20**], and Maalaoui, Martino and Tralli [21] by exploring Ding's approach; their results are direct consequences of theorem 1.1 for $\gamma = 1$.

Coming back to theorem 1.1, we shall mimic Ding's original idea as well, emphasizing that our *CR fractional* setting requires a more delicate analysis than either the polyharmonic setting in the Euclidean case (see [4]) or the usual CR framework, that is, when $\gamma = 1$ (see [20, 21]). In the sequel, we sketch our strategy. As expected, we first consider the fractional Yamabe problem on the CR sphere $S^{2n+1}$, that is,

$$\begin{cases} \mathcal{A}_\gamma U = |U|^{((4\gamma)/(Q-2\gamma))} U \quad \text{on} \quad S^{2n+1}, \\ U \in H^\gamma(S^{2n+1}), \end{cases} \qquad (\mathbf{FYS})_\gamma$$

where the intertwining operator $\mathcal{A}_\gamma$ and Sobolev space $H^\gamma(S^{2n+1})$ are introduced in § 2.4. By using the Cayley transform between the Heisenberg group $\mathbb{H}^n$ and the CR sphere $S^{2n+1}$, we prove that there is an explicit correspondence between the weak solutions of $(\mathbf{FYH})_\gamma$ and $(\mathbf{FYS})_\gamma$, respectively, see proposition 3.1 (and remark 3.2 for an alternative proof). Being in the critical case, the energy functional associated with problem $(\mathbf{FYS})_\gamma$ does not satisfy the usual Palais-Smale condition due to the lack of compactness of the embedding $H^\gamma(S^{2n+1}) \hookrightarrow L^{\frac{2Q}{Q-2\gamma}}(S^{2n+1})$. In order to regain some compactness, we establish a CR fractional version of the Ding-Hebey-Vaugon compactness result on the CR sphere $S^{2n+1}$, see proposition 3.3. In fact, subgroups of the unitary group $\mathbf{U}(n+1)$ having the form $G = \mathbf{U}(n_1) \times ... \times \mathbf{U}(n_k)$ with $n_1 + \cdots + n_k = n+1$ will imply the compactness of the embedding of $G$-invariant functions of $H^\gamma(S^{2n+1})$ into $L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$. Here, we shall explore the compactness result of Maalaoui and Martino [20] combined with an iterative argument of Aubin [1] and the technical assumption $\gamma \in \bigcup_{k=1}^n [k, ((kQ)/(Q-1)))$; some comments on the necessity of the latter assumption are formulated in remark 3.4. Now, having such a compactness, the fountain theorem and the principle of symmetric criticality applied to the energy functional associated with $(\mathbf{FYS})_\gamma$ will guarantee the existence of a whole sequence of $G$-invariant weak solutions for $(\mathbf{FYS})_\gamma$, so for $(\mathbf{FYH})_\gamma$. The number of $[n+1/2]$ sequences of sign-changing weak solutions for $(\mathbf{FYH})_\gamma$ with mutually different nodal properties will follow by careful choices of the subgroups $G = \mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_k)$ of the unitary group $\mathbf{U}(n+1)$ with $n_1 + \cdots + n_k = n+1$, see proposition 3.6.

*Plan of the paper.* In § 2, we recall those notions and results that are indispensable to present our argument (e.g. basic facts about Heisenberg groups, the Cayley transform, spherical/zonal harmonics on $S^{2n+1}$, fractional Sobolev spaces on $S^{2n+1}$ and $\mathbb{H}^n$). Section 3 is devoted to the proof of theorem 1.1; in § 3.1, we prove the equivalence between the weak solutions of problems $(\mathbf{FYS})_\gamma$ and $(\mathbf{FYH})_\gamma$; in § 3.2, we establish the compactness result on the CR fractional setting for $S^{2n+1}$; in § 3.3, we treat the group-theoretical aspects of our problem concerning the choice of the subgroups $G = \mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_k)$ of the unitary group $\mathbf{U}(n+1)$ which is needed to produce $[n+1/2]$ sequences of sign-changing weak solutions for $(\mathbf{FYH})_\gamma$ with different nodal properties. Finally, in § 3.4, we assemble the aforementioned pieces in order to conclude the proof of Theorem 1.1.

## 2. Preliminaries

In order the paper to be self-contained, we recall in this section, some basic notions from [5, 13–15, 23] which are indispensable in the sequel.

### 2.1. Heisenberg groups

An element in the Heisenberg group $\mathbb{H}^n$ is denoted by $(x, y, t)$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, $t \in \mathbb{R}$, and we identify the pair $(x, y)$ with $z \in \mathbb{C}^n$ having coordinates $z_j = x_j + iy_j$ for all $j = 1, \ldots, n$. The correspondence with its Lie algebra via the exponential coordinates induces the group law

$$(z, t) \star (z', t') = \left(z + z', t + t' + 2\text{Im } z \cdot \overline{z'}\right), \quad \forall\ (z, t),\ (z', t') \in \mathbb{C}^n \times \mathbb{R},$$

where Im denotes the imaginary part of a complex number and $z \cdot \overline{z'} = \sum_{j=1}^n z_j \overline{z_j'}$ is the Hermitian inner product. In these coordinates, the neutral element of $\mathbb{H}^n$ is $0_{\mathbb{H}^n} = (0_{\mathbb{C}^n}, 0)$ and the inverse $(z, t)^{-1}$ of the element $(z, t)$ is $(-z, -t)$. Note that $(x, y, t) = (z, t)$ forms a real coordinate system for $\mathbb{H}^n$ and the system of vector fields given as differential operators

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad j \in \{1, \ldots n\}, \quad T = \frac{\partial}{\partial t},$$

forms a basis of the left-invariant vector fields on $\mathbb{H}^n$. The vectors $X_j, Y_j, j \in \{1, \ldots, n\}$ form the basis of the horizontal bundle. Let

$$N(z, t) = (|z|^4 + t^2)^{1/4}$$

be the homogeneous gauge norm on $\mathbb{H}^n$ and $d_{KC} : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{R}$ be the Korányi-Cygan metric given by

$$d_{KC}((z, t), (z', t')) = N((z', t')^{-1} \star (z, t)) = (|z - z'|^4 + (t - t' - 2\text{Im } z \cdot \overline{z'})^2)^{1/4}.$$

The Lebesgue measure of $\mathbb{R}^{2n+1}$ will be the Haar measure on $\mathbb{H}^n$ (uniquely defined up to a positive multiplicative constant).

### 2.2. Cayley transform

Let

$$S^{2n+1} = \{\zeta = (\zeta_1, \ldots, \zeta_{n+1}) \in \mathbb{C}^{n+1} : \langle \zeta, \overline{\zeta} \rangle = 1\}$$

be the unit sphere in $\mathbb{C}^{n+1}$, where $\langle \zeta, \overline{\eta} \rangle = \sum_{j=1}^{n+1} \zeta_j \overline{\eta_j}$. The distance function on $S^{2n+1}$ is given by

$$d_S(\zeta, \eta) = \sqrt{2|1 - \langle \zeta, \overline{\eta} \rangle|}, \quad \zeta, \eta \in S^{2n+1}.$$

The Cayley transform $\mathcal{C} : \mathbb{H}^n \to S^{2n+1} \setminus \{(0, \ldots, 0, -1)\}$ is defined by

$$\mathcal{C}(z, t) = \left(\frac{2z}{1 + |z|^2 + it}, \frac{1 - |z|^2 - it}{1 + |z|^2 + it}\right),$$

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whose Jacobian determinant is given by

$$\mathrm{Jac}_{\mathcal{C}}(z,t) = \frac{2^{2n+1}}{((1+|z|^2)^2+t^2)^{n+1}}, \quad (z,t) \in \mathbb{H}^n.$$

Accordingly, for any integrable function $f : S^{2n+1} \to \mathbb{R}$, we have

$$\int_{S^{2n+1}} f(\eta)\mathrm{d}\eta = \int_{\mathbb{H}^n} f(\mathcal{C}(z,t))\mathrm{Jac}_{\mathcal{C}}(z,t)\mathrm{d}z\mathrm{d}t. \tag{2.1}$$

If $w = (z,t)$, $v = (z',t')$ and $\zeta = \mathcal{C}(w)$, $\eta = \mathcal{C}(v)$, one has that

$$d_S(\zeta,\eta) = d_{KC}(w,v)\left(\frac{4}{((1+|z|^2)^2+t^2)}\right)^{1/4}\left(\frac{4}{((1+|z'|^2)^2+(t')^2)}\right)^{1/4}. \tag{2.2}$$

### 2.3. Spherical/zonal harmonics on $S^{2n+1}$

The Hilbert space $L^2(S^{2n+1})$, endowed by the inner product

$$(U,V) = \int_{S^{2n+1}} U\overline{V}\mathrm{d}\eta,$$

can be decomposed into $\mathbf{U}(n+1)$-irreducible components as

$$L^2(S^{2n+1}) = \bigoplus_{j,k\geqslant 0} \mathcal{H}_{j,k},$$

where $\mathcal{H}_{j,k}$ denotes the space of harmonic polynomials $p(z,\overline{z})$ on $\mathbb{C}^{n+1}$ restricted to $S^{2n+1}$ that are homogeneous of degree $j$ and $k$ in the variables $z$ and $\overline{z}$, respectively. We notice that the dimension of $\mathcal{H}_{j,k}$ is

$$m_{j,k} = \frac{(j+n-1)!(k+n-1)!(j+k+n)}{n!(n-1)!j!k!}.$$

Moreover, if $\{Y^l_{j,k}\}_{l=\overline{1,m_{j,k}}}$ is an orthonormal basis of $\mathcal{H}_{j,k}$, then the zonal harmonics are defined by

$$\Phi_{j,k}(\zeta,\eta) = \sum_{l=1}^{m_{j,k}} Y^l_{j,k}(\zeta)\overline{Y^l_{j,k}(\eta)}, \quad \zeta,\eta \in S^{2n+1}. \tag{2.3}$$

We recall that $\Phi_{j,k}$ can be represented as

$$\Phi_{j,k}(\zeta,\eta) = \frac{(\max\{j,k\}+n-1)!(j+k+n)}{\omega_{2n+1}n!(\max\{j,k\})!}\langle\zeta,\overline{\eta}\rangle^{|j-k|}P_k^{(n-1,|j-k|)}(2\langle\zeta,\overline{\eta}\rangle^2-1), \tag{2.4}$$

where $P_k^{(n,l)}$ denotes the Jacobi polynomials and $\omega_{2n+1}$ is the usual measure of $S^{2n+1}$.

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### 2.4. Fractional Sobolev spaces on $S^{2n+1}$ and $\mathbb{H}^n$

The usual sub-Laplacian on $\mathbb{H}^n$ is defined as

$$\mathcal{L} = -\frac{1}{4}\sum_{j=1}^{n}(X_j^2 + Y_j^2).$$

If we introduce the differential operators

$$T_j = \frac{\partial}{\partial \eta_j} - \overline{\eta}_j \sum_{k=1}^{n+1} \eta_k \frac{\partial}{\partial \eta_k}, \quad \overline{T_j} = \frac{\partial}{\partial \overline{\eta_j}} - \eta_j \sum_{k=1}^{n+1} \overline{\eta_k} \frac{\partial}{\partial \overline{\eta_k}}, \quad j \in \{1, \ldots, n+1\},$$

the conformal sub-Laplacian on $S^{2n+1}$ is given by

$$\mathcal{D} = -\frac{1}{2}\sum_{j=1}^{n+1}(T_j\overline{T_j} + \overline{T_j}T_j) + \frac{n^2}{4}.$$

Note that for every $Y_{j,k} \in \mathcal{H}_{j,k}$, one has

$$\mathcal{D}Y_{j,k} = \lambda_j\lambda_k Y_{j,k},$$

where $\lambda_j = j + n/2$.

Let $0 < \gamma < Q/2 = n+1$ be fixed. Given $U \in L^2(S^{2n+1})$, its Fourier representation is

$$U = \sum_{j,k \geqslant 0} \sum_{l=1}^{m_{j,k}} c_{j,k}^l(U) Y_{j,k}^l$$

with Fourier coefficients $c_{j,k}^l(U) = \int_{S^{2n+1}} UY_{j,k}^l \mathrm{d}\eta$. Accordingly, we may define

$$\mathcal{D}^{\gamma/2}U = \sum_{j,k \geqslant 0} \sum_{l=1}^{m_{j,k}} (\lambda_j\lambda_k)^{\gamma/2} c_{j,k}^l(U) Y_{j,k}^l.$$

The fractional Sobolev space over $S^{2n+1}$ is defined as

$$H^\gamma(S^{2n+1}) = W^{\gamma,2}(S^{2n+1}) = \left\{U \in L^2(S^{2n+1}) : \mathcal{D}^{\gamma/2}U \in L^2(S^{2n+1})\right\},$$

endowed with the inner product and norm

$$(U,V)_{H^\gamma} = \int_{S^{2n+1}} \mathcal{D}^{\gamma/2}U\overline{\mathcal{D}^{\gamma/2}V}\mathrm{d}\eta \quad \text{and} \quad \|U\|_{H^\gamma} = (U,U)_{H^\gamma}^{1/2}$$
$$= \left(\sum_{j,k \geqslant 0} \sum_{l=1}^{m_{j,k}} (\lambda_j\lambda_k)^\gamma |c_{j,k}^l(U)|^2\right)^{1/2}.$$

The norm $\|\cdot\|_{H^\gamma}$ is equivalent to the norm coming from the inner product

$$(U,V)_\gamma = \sum_{j,k \geqslant 0} \sum_{l=1}^{m_{j,k}} \lambda_j(\gamma)\lambda_k(\gamma) c_{j,k}^l(U)\overline{c_{j,k}^l(V)},$$

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where

$$\lambda_j(\gamma) = \frac{\Gamma(((Q+2\gamma)/(4))+j)}{\Gamma(((Q-2\gamma)/(4))+j)}, \quad j \in \mathbb{N}_0 = \{0,1,2,\ldots\};$$

indeed, by asymptotic approximation of the Gamma function $\Gamma$, one has $\lambda_j(\gamma) \sim j^\gamma$.

The intertwining operator $\mathcal{A}_\gamma$ of order $2\gamma$ on $S^{2n+1}$ is given by

$$\mathrm{Jac}_\tau^{((Q+2\gamma)/(2Q))}(\mathcal{A}_\gamma U)\circ\tau = \mathcal{A}_\gamma(\mathrm{Jac}_\tau^{((Q-2\gamma)/(2Q))}(U\circ\tau)) \quad \text{for all } \tau \in \mathrm{Aut}(S^{2n+1}),$$
$$U \in C^\infty(S^{2n+1}),$$

where $\mathrm{Aut}(S^{2n+1})$ and $\mathrm{Jac}_\tau$ denote the group of automorphisms on $S^{2n+1}$ and the Jacobian of $\tau \in \mathrm{Aut}(S^{2n+1})$, respectively. In fact, the latter definition can be extended to every $U \in H^\gamma(S^{2n+1})$. Note that $\mathcal{A}_\gamma$ may by characterized (up to a constant) by its action on $\mathcal{H}_{j,k}$ as

$$\mathcal{A}_\gamma Y_{j,k} = \lambda_j(\gamma)\lambda_k(\gamma)Y_{j,k}, \quad Y_{j,k} \in \mathcal{H}_{j,k}. \tag{2.5}$$

Therefore,

$$(U,V)_\gamma = \int_{S^{2n+1}} \overline{V}\mathcal{A}_\gamma U \,\mathrm{d}\eta. \tag{2.6}$$

In particular, $\lambda_j(1) = \lambda_j$ for every $j \in \mathbb{N}_0$ and $\mathcal{A}_1 = \mathcal{D}$. Moreover, according to Frank and Lieb [**14**], for every real-valued function $U \in H^\gamma(S^{2n+1})$, one has the sharp fractional Sobolev inequality on the CR sphere $S^{2n+1}$, that is,

$$\left(\int_{S^{2n+1}} |U(\eta)|^{((2Q)/(Q-2\gamma))}\mathrm{d}\eta\right)^{((Q-2\gamma)/(Q))} \leqslant C(\gamma,n)\int_{S^{2n+1}} U(\eta)\mathcal{A}_\gamma U(\eta)\mathrm{d}\eta, \tag{2.7}$$

where

$$C(\gamma,n) = \frac{\Gamma((n+1-\gamma)/(2))^2}{\Gamma((n+1+\gamma)/(2))^2}\omega_{2n+1}^{-\gamma/n+1}.$$

The CR fractional sub-Laplacian operator on $\mathbb{H}^n$ is defined by

$$\mathcal{L}_\gamma = |2T|^\gamma \frac{\Gamma(\mathcal{L}|2T|^{-1} + ((1+\gamma)/(2)))}{\Gamma(\mathcal{L}|2T|^{-1} + 1 - \gamma/2)}.$$

Direct computation shows that $\mathcal{L}_1 = \mathcal{L}$, $\mathcal{L}_2 = \mathcal{L}^2 - |T|^2$. Moreover, the relationship between $\mathcal{L}_\gamma$ and $\mathcal{A}_\gamma$ is given by

$$\mathcal{L}_\gamma((2\mathrm{Jac}_{\mathcal{C}})^{((Q-2\gamma)/(2Q))}(U\circ\mathcal{C})) = (2\mathrm{Jac}_{\mathcal{C}})^{((Q+2\gamma)/(2Q))}(\mathcal{A}_\gamma U)\circ\mathcal{C},$$
$$\forall\, U \in H^\gamma(S^{2n+1}). \tag{2.8}$$

The fractional Sobolev space over $\mathbb{H}^n$ is defined by

$$D^\gamma(\mathbb{H}^n) = \left\{u \in L^{((2Q)/(Q-2\gamma))}(\mathbb{H}^n) : a_\gamma[u] < +\infty\right\},$$

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where the quadratic form $a_\gamma$ is associated with the operator $\mathcal{L}_\gamma$, that is,

$$a_\gamma[u] = \int_{\mathbb{H}^n} \overline{u}\mathcal{L}_\gamma u \mathrm{d}z\mathrm{d}t.$$

The form $a_\gamma$ can be equivalently represented by means of spectral decomposition, see [**15**, p. 126].

## 3. Proof of Main Theorem

### 3.1. Equivalent critical problems on $\mathbb{H}^n$ and $S^{2n+1}$.

Let $\gamma \in (0, n+1)$ be fixed. We consider the fractional Yamabe problem on the CR sphere, that is,

$$\begin{cases} \mathcal{A}_\gamma U = |U|^{((4\gamma)/(Q-2\gamma))}U \quad \text{on} \quad S^{2n+1}, \\ U \in H^\gamma(S^{2n+1}). \end{cases} \qquad (\mathbf{FYS})_\gamma$$

Hereafter, we are considering real-valued functions both in $H^\gamma(S^{2n+1})$ and $D^\gamma(\mathbb{H}^n)$, respectively. The main result of this subsection constitutes the bridge between $(\mathbf{FYS})_\gamma$ and $(\mathbf{FYH})_\gamma$ as follows:

Proposition 3.1. *Let $0 < \gamma < Q/2 = n+1$. Then $U \in H^\gamma(S^{2n+1})$ is a weak solution of $(\mathbf{FYS})_\gamma$ if and only if $u = (2\mathrm{Jac}_\mathcal{C})^{((Q-2\gamma)/(2Q))}U \circ \mathcal{C} \in D^\gamma(\mathbb{H}^n)$ is a weak solution of $(\mathbf{FYH})_\gamma$.*

*Proof.* We first prove the following

Claim: *Let $U : S^{2n+1} \to \mathbb{R}$ and $u : \mathbb{H}^n \to \mathbb{R}$ be two functions such that $u = \mathrm{Jac}_\mathcal{C}^{((Q-2\gamma)/(2Q))}U \circ \mathcal{C}$. Then $U \in H^\gamma(S^{2n+1})$ if and only if $u \in D^\gamma(\mathbb{H}^n)$.*

Fix $U \in H^\gamma(S^{2n+1})$; we shall prove first that $(z,t) \mapsto u(z,t) = \mathrm{Jac}_\mathcal{C}(z,t)^{((Q-2\gamma)/(2Q))}U(\mathcal{C}(z,t))$ belongs to $D^\gamma(\mathbb{H}^n)$. By (2.1) one has

$$\begin{aligned}\int_{\mathbb{H}^n} |u(z,t)|^{((2Q)/(Q-2\gamma))}\mathrm{d}z\mathrm{d}t &= \int_{\mathbb{H}^n} \mathrm{Jac}_\mathcal{C}(z,t)|U(\mathcal{C}(z,t))|^{((2Q)/(Q-2\gamma))}\mathrm{d}z\mathrm{d}t \\ &= \int_{S^{2n+1}} |U(\eta)|^{((2Q)/(Q-2\gamma))}\mathrm{d}\eta. \end{aligned} \qquad (3.1)$$

Furthermore, by the fractional Sobolev inequality (2.7) and relation (2.5), one has that

$$\begin{aligned}\left(\int_{S^{2n+1}} |U(\eta)|^{((2Q)/(Q-2\gamma))}\mathrm{d}\eta\right)^{((Q-2\gamma)/(Q))} &\leqslant C(\gamma,n)\int_{S^{2n+1}} U(\eta)\mathcal{A}_\gamma U(\eta)\mathrm{d}\eta \\ &= C(\gamma,n)\sum_{j,k\geqslant 0}\sum_{l=1}^{m_{j,k}} \lambda_j(\gamma)\lambda_k(\gamma)|c_{j,k}^l(U)|^2 \\ &\leqslant C'(\gamma,n)\|U\|_{H^\gamma}^2 \\ &< +\infty,\end{aligned}$$

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where $C'(\gamma,n)=C_\gamma C(\gamma,n)$ and $C_\gamma>0$ is such that $(V,V)_\gamma \leqslant C_\gamma\|V\|^2_{H^\gamma}$ for every $V\in H^\gamma(S^{2n+1})$; thus $u\in L^{((2Q)/(Q-2\gamma))}(\mathbb{H}^n)$. Moreover, by (2.8) and (2.1) one has

$$
\begin{aligned}
a_\gamma[u] &= \int_{\mathbb{H}^n} u\mathcal{L}_\gamma u\mathrm{d}z\mathrm{d}t = \\
&= 2^{\alpha'}\int_{\mathbb{H}^n} \mathrm{Jac}_{\mathcal{C}}(z,t)^{((Q-2\gamma)/(2Q))}U(\mathcal{C}(z,t)) \\
&\quad \mathcal{L}_\gamma((2\mathrm{Jac}_{\mathcal{C}}(z,t))^{((Q-2\gamma)/(2Q))}U(\mathcal{C}(z,t)))\mathrm{d}z\mathrm{d}t \\
&= 2^{\alpha'}\int_{\mathbb{H}^n} \mathrm{Jac}_{\mathcal{C}}(z,t)^{((Q-2\gamma)/(2Q))}U(\mathcal{C}(z,t)) \\
&\quad (2\mathrm{Jac}_{\mathcal{C}}(z,t))^{((Q+2\gamma)/(2Q))}(\mathcal{A}_\gamma U)(\mathcal{C}(z,t))\mathrm{d}z\mathrm{d}t \\
&= 2^{\alpha''}\int_{\mathbb{H}^n} U(\mathcal{C}(z,t))(\mathcal{A}_\gamma U)(\mathcal{C}(z,t))\mathrm{Jac}_{\mathcal{C}}(z,t)\mathrm{d}z\mathrm{d}t \\
&= 2^{\alpha''}\int_{S^{2n+1}} U(\eta)\mathcal{A}_\gamma U(\eta)\mathrm{d}\eta \qquad (3.2)\\
&< +\infty,
\end{aligned}
$$

where $\alpha'=-((Q-2\gamma)/(2Q))$ and $\alpha''=\alpha'+((Q+2\gamma)/(2Q))=((2\gamma)/(Q))$. Therefore, $u\in D^\gamma(\mathbb{H}^n)$.

Conversely, let us assume that $u\in D^\gamma(\mathbb{H}^n)$. In particular, we have that $u\in L^{((2Q)/(Q-2\gamma))}(\mathbb{H}^n)$, thus by relation (3.1) it turns out that $U\in L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$; therefore, $U\in L^2(S^{2n+1})$. Furthermore, by (3.2) we also have that

$$
\int_{S^{2n+1}} U(\eta)\mathcal{A}_\gamma U(\eta)\mathrm{d}\eta = 2^{-\alpha''}a_\gamma[u] < +\infty,
$$

that is, $U\in H^\gamma(S^{2n+1})$, which concludes the proof of *Claim*.

Let $U\in H^\gamma(S^{2n+1})$ be a weak solution of $(\mathbf{FYS})_\gamma$; then we have

$$
\int_{S^{2n+1}} \mathcal{A}_\gamma UV\mathrm{d}\eta = \int_{S^{2n+1}} |U|^{((4\gamma)/(Q-2\gamma))}UV\mathrm{d}\eta \quad \text{for every } V\in H^\gamma(S^{2n+1}). \tag{3.3}
$$

Let $v\in D^\gamma(\mathbb{H}^n)$ be arbitrarily fixed and define $V=(\mathrm{Jac}_{\mathcal{C}}\circ\mathcal{C}^{-1})^{((2\gamma-Q)/(2Q))}v\circ\mathcal{C}^{-1}$. Since $v=\mathrm{Jac}_{\mathcal{C}}^{((Q-2\gamma)/(2Q))}V\circ\mathcal{C}$, by the *Claim* we have that $V\in H^\gamma(S^{2n+1})$. Accordingly, the function $V$ can be used as a test-function in (3.3), obtaining

$$
\begin{aligned}
&\int_{S^{2n+1}} \mathcal{A}_\gamma U(\mathrm{Jac}_{\mathcal{C}}\circ\mathcal{C}^{-1})^{((2\gamma-Q)/(2Q))}v\circ\mathcal{C}^{-1}\mathrm{d}\eta \\
&\quad = \int_{S^{2n+1}} |U|^{((4\gamma)/(Q-2\gamma))}U(\mathrm{Jac}_{\mathcal{C}}\circ\mathcal{C}^{-1})^{((2\gamma-Q)/(2Q))}v\circ\mathcal{C}^{-1}\mathrm{d}\eta.
\end{aligned}
$$

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By a change of variables, it follows that

$$\int_{\mathbb{H}^n} (\mathcal{A}_\gamma U \circ \mathcal{C})(\mathrm{Jac}_{\mathcal{C}})^{((2\gamma-Q)/(2Q))+1} v\mathrm{d}z\mathrm{d}t$$
$$= \int_{\mathbb{H}^n} |U \circ \mathcal{C}|^{((4\gamma)/(Q-2\gamma))}(U \circ \mathcal{C})(\mathrm{Jac}_{\mathcal{C}})^{((2\gamma-Q)/(2Q))+1} v\mathrm{d}z\mathrm{d}t.$$

This relation and (2.8) imply that

$$2^{-((Q+2\gamma)/2Q))} \int_{\mathbb{H}^n} \mathcal{L}_\gamma((2\mathrm{Jac}_{\mathcal{C}})^{((Q-2\gamma)/(2Q))}(U \circ \mathcal{C}))v\mathrm{d}z\mathrm{d}t$$
$$= \int_{\mathbb{H}^n} |U \circ \mathcal{C}|^{((4\gamma)/(Q-2\gamma))} U \circ \mathcal{C}(\mathrm{Jac}_{\mathcal{C}})^{((2\gamma+Q)/(2Q))} v\mathrm{d}z\mathrm{d}t.$$

Since $u = (2\mathrm{Jac}_{\mathcal{C}})^{((Q-2\gamma)/(2Q))} U \circ \mathcal{C}$, the latter equality is equivalent to

$$\int_{\mathbb{H}^n} \mathcal{L}_\gamma u v\mathrm{d}z\mathrm{d}t = \int_{\mathbb{H}^n} |u|^{((4\gamma)/(Q-2\gamma))} u v\mathrm{d}z\mathrm{d}t,$$

which means precisely that $u \in D^\gamma(\mathbb{H}^n)$ is a weak solution of $(\mathbf{FYH})_\gamma$. The converse argument works in a similar way. □

REMARK 3.2. One can provide an alternative proof to proposition 3.1 by exploring the explicit form of the fundamental solution of $\mathcal{L}_\gamma$; a similar approach is due to Bartsch, Schneider and Weth [4] for the polyharmonic operator $(-\Delta)^m$ in $\mathbb{R}^N$, where $m \in \mathbb{N}$ and $N > 2m$. For completeness, we sketch the proof.

We recall that the fundamental solution of $\mathcal{L}_\gamma$ is

$$\mathcal{L}_\gamma^{-1}((z,t),(z',t')) = \frac{c_\gamma}{2} d_{KC}^{2\gamma-Q}((z,t),(z',t')), \tag{3.4}$$

where

$$c_\gamma = \frac{2^{n-\gamma}\Gamma((Q-2\gamma)/(4))^2}{\pi^{n+1}\Gamma(\gamma)},$$

see Branson, Fontana and Morpurgo [5, p. 21]. For every $\psi \in L^{((2Q)/(Q+2\gamma))}(S^{2n+1})$ we introduce the function

$$[\mathcal{K}_\gamma \psi](\zeta) = c_\gamma \int_{S^{2n+1}} \psi(\eta)|1 - \langle \zeta, \overline{\eta} \rangle|^{((2\gamma-Q)/(2))} \mathrm{d}\eta. \tag{3.5}$$

One can prove that $\mathcal{K}_\gamma \psi \in H^\gamma(S^{2n+1})$ for every $\psi \in L^{((2Q)/(Q+2\gamma))}(S^{2n+1})$. Moreover, the Funk-Hecke theorem on the CR sphere $S^{2n+1}$ gives

$$[\mathcal{K}_\gamma Y_{j,k}](\zeta) = \frac{2^{Q/2-\gamma}}{\lambda_j(\gamma)\lambda_k(\gamma)} Y_{j,k}(\zeta),$$

see Frank and Lieb [14, corollary 5.3]. Thus, a direct computation yields that

$$(\mathcal{K}_\gamma \psi, V)_\gamma = 2^{Q/2-\gamma} \int_{S^{2n+1}} \psi V \mathrm{d}\eta \quad \text{for every } V \in H^\gamma(S^{2n+1}).$$

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Note that if $U \in H^\gamma(S^{2n+1})$ is a weak solution of $(\mathbf{FYS})_\gamma$, the latter relation implies that

$$\mathcal{K}_\gamma(|U|^{((4\gamma)/(Q-2\gamma))}U) = 2^{Q/2-\gamma}U \quad \text{on } S^{2n+1}. \tag{3.6}$$

Accordingly, by relations (3.6), (3.5), (2.1) and (2.2), it turns out that

$$\begin{aligned} u(z,t) &= (2\mathrm{Jac}_{\mathcal{C}}(z,t))^{((Q-2\gamma)/(2Q))}U(\mathcal{C}(z,t)) \\ &= 2^{-Q/2+\gamma}(2\mathrm{Jac}_{\mathcal{C}}(z,t))^{((Q-2\gamma)/(2Q))}\mathcal{K}_\gamma(|U(\mathcal{C}(z,t))|^{((4\gamma)/(Q-2\gamma))}U(\mathcal{C}(z,t))) \\ &= \frac{c_\gamma}{2}\int_{\mathbb{H}^n} d_{KC}^{2\gamma-Q}((z,t),(z',t')|u(z',t')|^{((4\gamma)/(Q-\gamma))}u(z',t')\mathrm{d}z'\mathrm{d}t', \quad (z,t) \in \mathbb{H}^n. \end{aligned}$$

The latter relation is equivalent to the fact that

$$u(z,t) = \frac{c_\gamma}{2}(|u|^{((4\gamma)/(Q-\gamma))}u) * d_{KC}^{2\gamma-Q}((z,t),\cdot), \quad (z,t) \in \mathbb{H}^n, \tag{3.7}$$

where $'*'$ denotes the (noncommutative) convolution operation on the Heisenberg group $\mathbb{H}^n$. By (3.4), a similar argument as in Folland [**12**, theorem 2] gives that $\mathcal{L}_\gamma u = |u|^{((4\gamma)/(Q-\gamma))}u$ on $\mathbb{H}^n$, which concludes the claim.

### 3.2. Compactness

According to Frank and Lieb [**14**], see also (2.7), the embedding $H^\gamma(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ is continuous, but not compact. This subsection is devoted to regain certain compactness by using suitable group actions on the CR sphere $S^{2n+1}$.

To complete this purpose, let $n_j \in \mathbb{N}$, $j = 1,\ldots,k$, with $n_1 + \cdots + n_k = n+1$. Associated with these numbers, let

$$G = \mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_k) \tag{3.8}$$

be the subgroup of the unitary group $\mathbf{U}(n+1) = \{g \in \mathbf{O}(2n+2) : gJ = Jg\}$, where

$$J = \begin{bmatrix} 0 & I_{\mathbb{R}^{n+1}} \\ -I_{\mathbb{R}^{n+1}} & 0 \end{bmatrix}.$$

Let

$$H_G^\gamma(S^{2n+1}) = \{U \in H^\gamma(S^{2n+1}) : g \circ U = U \quad \text{for every } g \in G\}$$

be the subspace of $G$-invariant functions of $H^\gamma(S^{2n+1})$, where

$$(g \circ U)(\eta) = U(g^{-1}\eta), \quad \eta \in S^{2n+1}. \tag{3.9}$$

It is clear that $H_G^\gamma(S^{2n+1})$ is an infinite-dimensional closed subspace of $H^\gamma(S^{2n+1})$, whenever $k \geqslant 2$ in the splitting (3.8).

With the above notations in our mind, a Ding-Hebey-Vaugon-type compactness result reads as follows:

Proposition 3.3. *Let $\gamma \in \bigcup_{k=1}^n [k,((kQ)/(Q-1)))$. The embedding $H_G^\gamma(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ is compact, where $G = \mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_k)$ is any choice with $n_j \in \mathbb{N}$, $j = 1,\ldots,k$, and $n_1 + \ldots + n_k = n+1$.*

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*Proof.* First, when $G = \mathbf{U}(n+1)$, the space $H_G^{\gamma}(S^{2n+1})$ contains precisely the constant functions defined on $S^{2n+1}$; in this case, the proof is trivial.

In the general case, we recall by Maalaoui and Martino [**20**, lemma 3.3] that the embedding $W_G^{1,2}(S^{2n+1}) = H_G^1(S^{2n+1}) \hookrightarrow L^q(S^{2n+1})$ is compact for every $1 \leqslant q < q_1^*$, where $q_1^* = ((2(Q-1))/(Q-3))$ is the Riemannian critical exponent on the $(Q-1)-$dimensional sphere $S^{2n+1}$.

By our assumption $\gamma \in \bigcup_{k=1}^{n} [k, ((kQ)/(Q-1)))$ we have that $l := [\gamma] \geqslant 1$ and

$$\gamma\left(1 - \frac{1}{Q}\right) < l \leqslant \gamma. \tag{3.10}$$

The iterative argument developed by Aubin [**1**, proposition 2.11], applied for $l$ times, gives that the embedding $W_G^{l,2}(S^{2n+1}) = H_G^l(S^{2n+1}) \hookrightarrow L^q(S^{2n+1})$ is compact for every $1 \leqslant q < q_l^*$, where $q_l^* = ((2(Q-1))/(Q-1-2l))$. On one hand, since $l \leqslant \gamma$, we have that $H_G^{\gamma}(S^{2n+1}) = W_G^{\gamma,2}(S^{2n+1}) \subset W_G^{l,2}(S^{2n+1})$. On the other hand, the left-hand side of (3.10) is equivalent to $q_l^* > ((2Q)/(Q-2\gamma))$. Combining these facts, we have the chain of inclusions

$$H_G^{\gamma}(S^{2n+1}) \subset W_G^{l,2}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1}),$$

where the latter embedding is compact. □

Remark 3.4. Our assumption $\gamma \in \bigcup_{k=1}^{n} [k, ((kQ)/(Q-1))$ is important to guarantee the left-hand side of (3.10), which in turn, implies that $((2Q)/(Q-2\gamma))$ is within the range $[1, q_l^*)$ where the embedding $W_G^{l,2}(S^{2n+1}) \hookrightarrow L^q(S^{2n+1})$ is compact, $q \in [1, q_l^*)$. We are wondering if this assumption can be removed in order to prove the compactness of the above embedding for the whole spectrum $(0, Q/2)$ of the parameter $\gamma$.

### 3.3. Special group actions

The goal of this subsection is to describe symmetrically different functions belonging to $H^{\gamma}(S^{2n+1})$ via subgroups of the form $G = \mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_k)$ with $n_1 + \cdots + n_k = n+1$. To handle this problem, we explore a Rubik-cube technique, described in a slightly different manner in Balogh and Kristály [**2**]; roughly speaking, $n+1$ corresponds to the number of total sides of the cube, while the sides of the cube are certain blocks in the decomposition subgroup $G = \mathbf{U}(n_1) \times \cdots \times \mathbf{U}(n_k)$.

To be more precise, let $n \geqslant 1$ and for $i \in \{1, \ldots, [n+1/2]\}$, we consider the subgroup of the unitary group $\mathbf{U}(n+1)$ as

$$G_i = \begin{cases} \begin{bmatrix} \mathbf{U}\left(\dfrac{n+1}{2}\right) & 0 \\ 0 & \mathbf{U}\left(\dfrac{n+1}{2}\right) \end{bmatrix}, & \text{if} \quad n+1 = 2i, \\ \\ \begin{bmatrix} \mathbf{U}(i) & 0 & 0 \\ 0 & \mathbf{U}(n+1-2i) & 0 \\ 0 & 0 & \mathbf{U}(i) \end{bmatrix}, & \text{if} \quad n+1 \neq 2i. \end{cases}$$

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It is clear that a particular $G_i$ does not act transitively on the sphere $S^{2n+1}$. However, to recover the transitivity, we shall combine different groups of the type $G_i$; for further use, let $[G_i; G_j]$ be the group generated by $G_i$ and $G_j$.

LEMMA 3.5. *Let $i, j \in \{1, \ldots, [n+1/2]\}$ with $i \neq j$. Then the group $[G_i; G_j]$ acts transitively on the CR sphere $S^{2n+1}$.*

*Proof.* Without loss of generality, we assume that $j > i$. For further use, let $0_k = (0, \ldots, 0) \in \mathbb{C}^k = \mathbb{R}^{2k}$, $k \in \{1, \ldots, n\}$. Let us fix $\eta = (\eta_1, \eta_2, \eta_3) \in S^{2n+1}$ arbitrarily with $\eta_1, \eta_3 \in \mathbb{C}^j$ and $\eta_2 \in \mathbb{C}^{n+1-2j}$; clearly, $\eta_2$ disappears from $\eta$ whenever $2j = n+1$. Taking into account the fact that $\mathbf{U}(j)$ acts transitively on $S^{2j-1}$, there are $g_j^1, g_j^2 \in \mathbf{U}(j)$ such that if $g_j = g_j^1 \times I_{\mathbb{C}^{n+1-2j}} \times g_j^2 \in G_j$, then $g_j\eta = (0_{j-1}, 0, |\eta_1|, \eta_2, |\eta_3|, 0, 0_{j-1})$. Since $j - 1 \geqslant i$, the transitive action of $\mathbf{U}(n+1-2i)$ on $S^{2n+1-4i}$ implies the existence of $g_i^1 \in \mathbf{U}(n+1-2i)$ such that $g_i^1(0_{j-i-1}, 0, |\eta_1|, \eta_2, |\eta_3|, 0, 0_{j-i-1}) = (1, 0, 0_{n-2i})$. Therefore, if $g_i = I_{\mathbb{C}^i} \times g_i^1 \times I_{\mathbb{C}^i} \in G_i$ then $g_i g_j \eta = (0_i, 1, 0, 0_{n-i}) \in S^{2n+1}$.

By repeating the same procedure for another element $\tilde{\eta} \in S^{2n+1}$, there exists $\tilde{g}_i \in G_i$ and $\tilde{g}_j \in G_j$ such that $\tilde{g}_i\tilde{g}_j\tilde{\eta} = (0_i, 1, 0, 0_{n-i}) \in S^{2n+1}$. Accordingly,

$$\eta = g_j^{-1} g_i^{-1} \tilde{g}_i \tilde{g}_j \tilde{\eta} = g_j^{-1} \overline{g}_i \tilde{g}_j \tilde{\eta},$$

where $\overline{g}_i = g_i^{-1}\tilde{g}_i \in G_i$, which concludes the proof. □

For every fixed $i \in \{1, \ldots, [n+1/2]\}$, we introduce the matrix $A_i$ as

$$A_i = \begin{cases} \begin{bmatrix} 0 & I_{\mathbb{C}((n+1)/(2))} \\ I_{\mathbb{C}((n+1)/(2))} & 0 \end{bmatrix}, & \text{if} \quad n+1 = 2i, \\[2ex] \begin{bmatrix} 0 & 0 & I_{\mathbb{C}^i} \\ 0 & I_{\mathbb{C}^{n+1-2i}} & 0 \\ I_{\mathbb{C}^i} & 0 & 0 \end{bmatrix}, & \text{if} \quad n+1 \neq 2i. \end{cases}$$

The following construction is inspired by Bartsch and Willem [3]. Since one has $A_i \in \mathbf{U}(n+1) \setminus G_i$, $A_i^2 = I_{\mathbb{C}^{n+1}}$ and $A_i G_i = G_i A_i$, the group generated by $G_i$ and $A_i$ is $\hat{G}_i = [G_i; A_i] = G_i \cup A_i G_i$, that is,

$$\hat{G}_i = \begin{cases} \begin{bmatrix} \mathbf{U}(n+1/2) & 0 \\ 0 & \mathbf{U}(n+1/2) \end{bmatrix} \cup \begin{bmatrix} 0 & \mathbf{U}(n+1/2) \\ \mathbf{U}(n+1/2) & 0 \end{bmatrix}, & \text{if} \quad n+1=2i, \\[2ex] \begin{bmatrix} \mathbf{U}(i) & 0 & 0 \\ 0 & \mathbf{U}(n+1-2i) & 0 \\ 0 & 0 & \mathbf{U}(i) \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & \mathbf{U}(i) \\ 0 & \mathbf{U}(n+1-2i) & 0 \\ \mathbf{U}(i) & 0 & 0 \end{bmatrix}, & \text{if} \quad n+1 \neq 2i. \end{cases} \tag{3.11}$$

In fact, in $\hat{G}_i$, there are only two types of elements: either of the form $g \in G_i$, or $A_i g \in \hat{G}_i \setminus G_i$ (with $g \in G_i$), respectively.

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The action $\hat{G}_i \circledast H^\gamma(S^{2n+1}) \mapsto H^\gamma(S^{2n+1})$ of the group $\hat{G}_i$ on $H^\gamma(S^{2n+1})$ is defined by

$$(\hat{g} \circledast U)(\eta) = \begin{cases} U(g^{-1}\eta), & \text{if} \quad \hat{g} = g \in G_i, \\ -U(g^{-1}A_i^{-1}\eta), & \text{if} \quad \hat{g} = A_i g \in \hat{G}_i \setminus G_i, \end{cases} \tag{3.12}$$

for every $\hat{g} \in \hat{G}_i$, $U \in H^\gamma(S^{2n+1})$ and $\eta \in S^{2n+1}$. We notice that this action is well-defined, continuous and linear. Similarly, as in (3.9), we introduce the space of $G_i$-invariant functions of $H^\gamma(S^{2n+1})$ as

$$H^\gamma_{G_i}(S^{2n+1}) = \{U \in H^\gamma(S^{2n+1}) : g \circ U = U \quad \text{for every } g \in G_i\},$$

where the action $'\circ'$ corresponds to the first relation in (3.12). Furthermore, let

$$H^\gamma_{\hat{G}_i}(S^{2n+1}) = \left\{U \in H^\gamma(S^{2n+1}) : \hat{g} \circledast U = U \quad \text{for every } \hat{g} \in \hat{G}_i\right\}$$

be the space of $\hat{G}_i$-invariant functions of $H^\gamma(S^{2n+1})$.

The following result summarizes the constructions in this subsection.

PROPOSITION 3.6. *Let* $\gamma \in \bigcup_{k=1}^n [k, kQ/Q-1)$, *and fix* $i, j \in \{1, \ldots, [n+1/2]\}$ *such that* $i \neq j$. *The following statements hold*:

(i) *The embedding* $H^\gamma_{\hat{G}_i}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ *is compact*;

(ii) $H^\gamma_{G_i}(S^{2n+1}) \cap H^\gamma_{G_j}(S^{2n+1}) = \{constant\ functions\ on\ S^{2n+1}\}$;

(iii) $H^\gamma_{\hat{G}_i}(S^{2n+1}) \cap H^\gamma_{\hat{G}_j}(S^{2n+1}) = \{0\}$.

*Proof.*

(i) It is clear that $H^\gamma_{\hat{G}_i}(S^{2n+1}) \subset H^\gamma_{G_i}(S^{2n+1})$. Moreover, by proposition 3.3, we have that the embedding $H^\gamma_{G_i}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ is compact.

(ii) Let us fix $U \in H^\gamma_{G_i}(S^{2n+1}) \cap H^\gamma_{G_j}(S^{2n+1})$. Since $U$ is both $G_i$- and $G_j$- invariant, it is also $[G_i; G_j]$-invariant, that is, $U(g\eta) = U(\eta)$ for every $g \in [G_i; G_j]$ and $\eta \in S^{2n+1}$. According to lemma 3.5, the group $[G_i; G_j]$ acts transitively on the CR sphere $S^{2n+1}$, that is, the orbit of every element $\eta \in S^{2n+1}$ by the group $[G_i; G_j]$ is the whole sphere $S^{2n+1}$. Thus, $U$ is a constant function.

(iii) Let $U \in H^\gamma_{\hat{G}_i}(S^{2n+1}) \cap H^\gamma_{\hat{G}_j}(S^{2n+1})$. On one hand, by (ii), we first have that $U$ is constant. On the other hand, the second relation from (3.12) implies that $U(\eta) = -U(A_i\eta)$ for every $\eta \in S^{2n+1}$. Therefore, we necessarily have that $U = 0$.

□

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### 3.4. Proof of Theorem 1.1.

We associate to problem $(\mathbf{FYS})_\gamma$ the energy functional $E : H^\gamma(S^{2n+1}) \to \mathbb{R}$ defined by

$$E(U) = \frac{1}{2}\int_{S^{2n+1}} U\mathcal{A}_\gamma U \mathrm{d}\eta - \frac{Q-2\gamma}{2Q}\int_{S^{2n+1}} |U|^{((2Q)/(Q-2\gamma))}\mathrm{d}\eta, \quad U \in H^\gamma(S^{2n+1}).$$

Due to (2.7), the functional $E$ is well-defined, belonging to $C^1(H^\gamma(S^{2n+1}), \mathbb{R})$. Moreover, $U \in H^\gamma(S^{2n+1})$ is a critical point of $E$ if and only if $U$ is a weak solution of $(\mathbf{FYS})_\gamma$.

Let us fix $i \in \{1, \ldots, [n+1/2]\}$. In order to guarantee critical points for $E$, we first consider the functional $E_i : H^\gamma_{\hat{G}_i}(S^{2n+1}) \to \mathbb{R}$, the restriction of $E$ to the space $H^\gamma_{\hat{G}_i}(S^{2n+1})$. It is clear that $E_i$ is an even functional and it has the mountain pass geometry. Since the embedding $H^\gamma_{\hat{G}_i}(S^{2n+1}) \hookrightarrow L^{((2Q)/(Q-2\gamma))}(S^{2n+1})$ is compact, see proposition 3.6 (i), we may apply the fountain theorem, see for example, Bartsch and Willem [**3**, theorem 3.1], guaranteeing a sequence $\{U_i^k\}_{k\in\mathbb{N}} \subset H^\gamma_{\hat{G}_i}(S^{2n+1})$ of critical points for $E_i$ with the additional property that $\|U_i^k\|_{H^\gamma} \to \infty$ as $k \to \infty$.

By using the principle of symmetric criticality of Palais [**22**], we are going to prove that $\{U_i^k\}_{k\in\mathbb{N}} \subset H^\gamma_{\hat{G}_i}(S^{2n+1})$ are in fact critical points for the original energy functional $E$, thus weak solutions of $(\mathbf{FYS})_\gamma$. To do this, it suffices to verify that $E$ is a $\hat{G}_i$-invariant functional, that is,

$$E(\hat{g} \circledast U) = E(U) \quad \text{for every } \hat{g} \in \hat{G}_i,\ U \in H^\gamma(S^{2n+1}).$$

On one hand, according to relation (2.6), for the quadratic term in $E$, it is enough to prove that $\hat{G}_i$ acts isometrically on $H^\gamma(S^{2n+1})$, that is,

$$(\hat{g} \circledast U, \hat{g} \circledast U)_\gamma = (U, U)_\gamma \quad \text{for every } \hat{g} \in \hat{G}_i,\ U \in H^\gamma(S^{2n+1}). \tag{3.13}$$

To see this, let us fix $\hat{g} \in \hat{G}_i$ and $U \in H^\gamma(S^{2n+1})$ arbitrarily. We recall that by definition

$$(\hat{g} \circledast U, \hat{g} \circledast U)_\gamma = \sum_{j,k\geqslant 0} \lambda_j(\gamma)\lambda_k(\gamma) \sum_{l=1}^{m_{j,k}} |c^l_{j,k}(\hat{g} \circledast U)|^2.$$

By using (2.3), one has

$$\begin{aligned}\sum_{l=1}^{m_{j,k}} |c^l_{j,k}(\hat{g} \circledast U)|^2 &= \int_{S^{2n+1}}\int_{S^{2n+1}} (\hat{g} \circledast U)(\zeta)(\hat{g} \circledast U)(\eta) \sum_{l=1}^{m_{j,k}} Y^l_{j,k}(\zeta)\overline{Y^l_{j,k}(\eta)}\mathrm{d}\zeta\mathrm{d}\eta \\ &= \int_{S^{2n+1}}\int_{S^{2n+1}} (\hat{g} \circledast U)(\zeta)(\hat{g} \circledast U)(\eta)\Phi_{j,k}(\zeta, \eta)\mathrm{d}\zeta\mathrm{d}\eta. \end{aligned} \tag{3.14}$$

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Note that for every $g \in G_i \subset \mathbf{U}(n+1)$ and $\zeta, \eta \in S^{2n+1}$, we have

$$\langle g\zeta, \overline{g\eta}\rangle = \langle A_i g\zeta, \overline{A_i g\eta}\rangle = \langle \zeta, \overline{\eta}\rangle;$$

therefore, by the representation (2.4) of the zonal harmonics we also have that

$$\Phi_{j,k}(g\zeta, g\eta) = \Phi_{j,k}(A_i g\zeta, A_i g\eta) = \Phi_{j,k}(\zeta, \eta).$$

Thus, relation (3.12) and suitable changes of variables in (3.14) imply that

$$\sum_{l=1}^{m_{j,k}} |c_{j,k}^l(\hat{g} \circledast U)|^2 = \int_{S^{2n+1}} \int_{S^{2n+1}} U(\zeta)U(\eta)\Phi_{j,k}(\zeta,\eta)\mathrm{d}\zeta\mathrm{d}\eta = \sum_{l=1}^{m_{j,k}} |c_{j,k}^l(U)|^2,$$

which proves (3.13).

On the other hand, the $\hat{G}_i$-invariance of the nonlinear term $U \mapsto \int_{S^{2n+1}} |U|^{((2Q)/(Q-2\gamma))}$ trivially follows by a change of variable, by using the isometric character of the group $\mathbf{U}(n+1)$ on $S^{2n+1}$.

Accordingly, for every $i \in \{1, \ldots, [n+1/2]\}$, the functions $\{U_i^k\}_{k\in\mathbb{N}} \subset H^{\gamma}_{\hat{G}_i}(S^{2n+1})$ are non-trivial weak solutions of $(\mathbf{FYS})_\gamma$. Due to proposition 3.1, $u_i^k = (2\mathrm{Jac}_{\mathcal{C}})^{((Q-2\gamma)/(2Q))} U_i^k \circ \mathcal{C} \in D^\gamma(\mathbb{H}^n)$ are non-trivial weak solutions of the original fractional Yamabe problem $(\mathbf{FYH})_\gamma$; by construction, $u_i^k$ are sign-changing functions.

Due to proposition 3.6 (iii), we state that the sequences $\{U_i^k\}_{k\in\mathbb{N}} \subset H^{\gamma}_{\hat{G}_i}(S^{2n+1})$ and $\{U_j^k\}_{k\in\mathbb{N}} \subset H^{\gamma}_{\hat{G}_j}(S^{2n+1})$ with $i, j \in \{1, \ldots, [n+1/2]\}$, $i \neq j$, cannot be compared from symmetrical point of view. Therefore, the sequences $\{u_i^k\} \subset D^\gamma(\mathbb{H}^n)$ and $\{u_j^k\} \subset D^\gamma(\mathbb{H}^n)$ have mutually different nodal properties for every $i, j \in \{1, \ldots, [n+1/2]\}$, $i \neq j$, which concludes the proof.

Remark 3.7. Consider a nonzero solution $u_i^k = (2\mathrm{Jac}_{\mathcal{C}})^{((Q-2\gamma)/(2Q))} U_i^k \circ \mathcal{C} \in D^\gamma(\mathbb{H}^n)$ of $(\mathbf{FYH})_\gamma$, with $\{U_i^k\}_{k\in\mathbb{N}} \subset H^{\gamma}_{\hat{G}_i}(S^{2n+1}) \setminus \{0\}$. For simplicity, we consider the case $n+1 = 2i$. Let us introduce the nodal domain of $U_i^k$ (or $u_i^k$) as the connected components of $C_i^k = S^{2n+1} \setminus N_i^k$, where $N_i^k = \overline{\{\eta \in S^{2n+1} : U_i^k(\eta) = 0\}}$. Since $U_i^k \in H^{\gamma}_{\hat{G}_i}(S^{2n+1})$, by relation (3.12) it follows that $U_i^k$ has the form $U_i^k(\eta) = U_i^k(|\eta_1|, |\eta_2|)$ with the property that $U_i^k(|\eta_1|, |\eta_2|) = -U_i^k(|\eta_2|, |\eta_1|)$, $\eta = (\eta_1, \eta_2) \in S^{2n+1}$, $\eta_1, \eta_2 \in \mathbb{C}^i$. Accordingly, since $U_i^k(|\eta_1|, |\eta_2|) = U_i^k(|\pm\eta_1|, |\pm\eta_2|)$, $U_i^k$ is sign-changing with at least four non-degenerate nodal domains in $C_i^k$; in two of them the function $U_i^k$ is negative, while in the other two it is positive, respectively. When $n+1 \neq 2i$, a similar discussion can be performed.

We conclude the paper by the following table providing explicit forms of subgroups of the unitary group $\mathbf{U}(n+1)$ and admissible intervals for the parameter $\gamma$, depending on the dimension $n$, where our main theorem applies; we only consider the cases when $n \in \{1, \ldots, 8\}$:

| $n$ | $Q=2n+2$ | $G_i,\ i\in\{1,\ldots,[n+1/2]\}$ | Admissible domains for $\gamma\in(0,Q/2)$ | Number of symmetrically distinct sequences of solution of $(\mathbf{FYH})_\gamma$ |
|---|---|---|---|---|
| 1 | 4 | $G_1=\mathbf{U}(1)\times\mathbf{U}(1)$ | $[1,4/3)$ | 1 |
| 2 | 6 | $G_1=\mathbf{U}(1)\times\mathbf{U}(1)\times\mathbf{U}(1)$ | $[1,6/5)\cup[2,12/5)$ | 1 |
| 3 | 8 | $G_1=\mathbf{U}(1)\times\mathbf{U}(2)\times\mathbf{U}(1)$<br>$G_2=\mathbf{U}(2)\times\mathbf{U}(2)$ | $[1,8/7)\cup[2,16/7)\cup$<br>$\cup[3,24/7)$ | 2 |
| 4 | 10 | $G_1=\mathbf{U}(1)\times\mathbf{U}(3)\times\mathbf{U}(1)$<br>$G_2=\mathbf{U}(2)\times\mathbf{U}(1)\times\mathbf{U}(2)$ | $\bigcup_{k=1}^{4}[k,10k/9)$ | 2 |
| 5 | 12 | $G_1=\mathbf{U}(1)\times\mathbf{U}(4)\times\mathbf{U}(1)$<br>$G_2=\mathbf{U}(2)\times\mathbf{U}(2)\times\mathbf{U}(2)$<br>$G_3=\mathbf{U}(3)\times\mathbf{U}(3)$ | $\bigcup_{k=1}^{5}[k,12k/11)$ | 3 |
| 6 | 14 | $G_1=\mathbf{U}(1)\times\mathbf{U}(5)\times\mathbf{U}(1)$<br>$G_2=\mathbf{U}(2)\times\mathbf{U}(3)\times\mathbf{U}(2)$<br>$G_3=\mathbf{U}(3)\times\mathbf{U}(1)\times\mathbf{U}(3)$ | $\bigcup_{k=1}^{6}[k,14k/13)$ | 3 |
| 7 | 16 | $G_1=\mathbf{U}(1)\times\mathbf{U}(6)\times\mathbf{U}(1)$<br>$G_2=\mathbf{U}(2)\times\mathbf{U}(4)\times\mathbf{U}(2)$<br>$G_3=\mathbf{U}(3)\times\mathbf{U}(2)\times\mathbf{U}(3)$<br>$G_4=\mathbf{U}(4)\times\mathbf{U}(4)$ | $\bigcup_{k=1}^{7}[k,16k/15)$ | 4 |
| 8 | 18 | $G_1=\mathbf{U}(1)\times\mathbf{U}(7)\times\mathbf{U}(1)$<br>$G_2=\mathbf{U}(2)\times\mathbf{U}(5)\times\mathbf{U}(2)$<br>$G_3=\mathbf{U}(3)\times\mathbf{U}(3)\times\mathbf{U}(3)$<br>$G_4=\mathbf{U}(4)\times\mathbf{U}(1)\times\mathbf{U}(4)$ | $\bigcup_{k=1}^{8}[k,18k/17)$ | 4 |

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