Ergod. Th. & Dynam. Sys., (2022), **42**, 1057–1097 © The Author(s), 2021. Published by Cambridge University Press.
doi:10.1017/etds.2021.118

On the accumulation of separatrices by invariant circles

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(*Received* 14 *September* 2020 *and accepted in revised form* 5 *August* 2021)

Abstract. Let f be a smooth symplectic diffeomorphism of $\mathbb{R}^2$ admitting a (non-split) separatrix associated to a hyperbolic fixed point. We prove that if f is a perturbation of the time-1 map of a symplectic autonomous vector field, this separatrix is accumulated by a positive measure set of invariant circles. However, we provide examples of smooth symplectic diffeomorphisms with a Lyapunov unstable non-split separatrix that are not accumulated by invariant circles.

2020 Mathematics Subject Classification: 34C27, 34D20, 37C75 (Primary)

1. *Introduction*

A theorem by Herman,'Herman's last geometric theorem', cf. [9, 12], asserts that if a smooth orientation- and area-preserving diffeomorphism f of the 2-plane $\mathbb{R}^2$ (or the 2-cylinder $\mathbb{R}/\mathbb{Z}\times\mathbb{R}$) admits a Kolmogorov–Arnold–Moser (KAM) circle Σ (by definition, a smooth invariant curve, isotopic in $\mathbb{R}^2\setminus\{o\}$ to a circle centered at the origin in the case $f:\mathbb{R}^2\to\mathbb{R}^2$ or isotopic to $\mathbb{R}/\mathbb{Z}\times\{0\}$ in the cylinder case, on which the dynamics of f is conjugated to a Diophantine translation), then this KAM circle is accumulated by other KAM circles, the union of which has positive two-dimensional Lebesgue measure in any neighborhood of Σ. In this paper, we investigate whether such a phenomenon holds if, instead of being a KAM circle, the invariant set Σ is a *separatrix* of a hyperbolic fixed (or periodic) point of f.

More precisely, we consider the following situation (see Figure 1). Let $f:\mathbb{R}^2\to\mathbb{R}^2$, $f:(x,y)\mapsto f(x,y)$, $f(0,0)=(0,0)$ be a smooth diffeomorphism which is symplectic with respect to the usual symplectic form $\omega=dx\wedge dy$ ($f^*\omega=\omega$). We assume that

*A preliminary version of this paper was discussed by the authors some months before Anatole Katok passed away in April 2018.

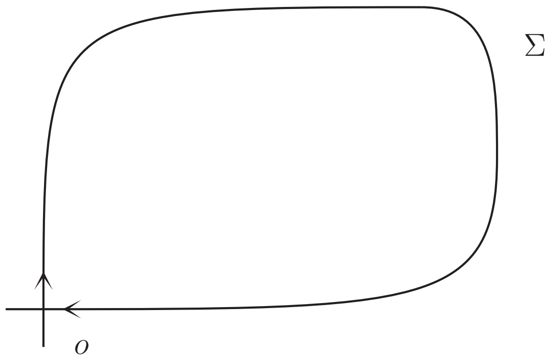

FIGURE 1. A (non-split) separatrix.

$o := (0, 0)$ is a *hyperbolic* fixed point of f (the matrix $Df(o) \in SL(2, \mathbb{R})$ has distinct real eigenvalues) and that there exists an f-invariant compact connected set $\Sigma \ni o$ such that $\Sigma \setminus \{o\}$ is a non-empty connected one-dimensional manifold included in both the stable and unstable manifolds $W^s_f(o)$, $W^u_f(o)$ associated to o:

$$\text{for all } (x, y) \in \Sigma, \quad \lim_{n \to \pm\infty} f^n(x, y) = o.$$

Note that because o is f-hyperbolic, Σ is homeomorphic to a circle and $\Sigma \setminus \{o\}$ coincides with one of the two connected components of $W^s_f(o) \setminus \{o\}$ (respectively $W^u_f(o) \setminus \{o\}$). We shall say that Σ is a *separatrix* of f associated to the hyperbolic fixed point o or, without referring to the hyperbolic fixed point o, that Σ is a separatrix of f.

Examples of such diffeomorphisms f can be obtained in the following way. Let X_0 be a smooth autonomous *Hamiltonian* vector field of the form

$$X_0 = J\nabla H_0, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{1.1}$$

where $H_0 : \mathbb{R}^2 \to \mathbb{R}$, of the form

$$H_0(x, y) = \lambda xy + O^3(x, y), \quad \lambda \in \mathbb{R}^*,$$

(we can assume without loss of generality $\lambda > 0$) is a smooth function. The time-1 map $f_0 := \phi^1_{X_0}$ of X_0 is a Hamiltonian (in particular, symplectic) diffeomorphism of $\mathbb{R}^2$ admitting o as a hyperbolic fixed point. We assume that it has a separatrix $\Sigma \ni o$ of the form

$$\Sigma \setminus \{o\} = \{\phi^t_{X_0}(p),\ t \in \mathbb{R}\} \quad \text{for some } p \in \mathbb{R}^2 \setminus \{o\} \text{ such that } \lim_{t \to \pm\infty} \phi^t_{X_0}(p) = o.$$

We now consider a smooth time-dependent Hamiltonian vector field $Y : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$, $(t, (x, y)) \mapsto Y(t, x, y)$ which is 1-periodic in t, symplectic with respect to (x, y), and tangent to $\Sigma \setminus \{o\}$:

$$\text{for all } t \in \mathbb{R}/\mathbb{Z}, \text{ for all } (x, y) \in \Sigma, \quad \det(X_0(x, y), Y(t, x, y)) = 0.$$

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One can for example choose $Y(t, x, y) = J\nabla F(t, x, y)$, where $F : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$ is a smooth time-dependent Hamiltonian that satisfies

$$\text{for all } t \in \mathbb{R}/\mathbb{Z}, \text{ for all } (x, y) \in \Sigma, \quad F(t, x, y) = F(t, 0, 0).$$

Note that because o is a hyperbolic fixed point of X_0, one has for all t, $Y(t, o) = 0$. For $\varepsilon \in \mathbb{R}$, define the 1-periodic in t symplectic vector field $\mathbb{R}^2 \to \mathbb{R}^2$ as

$$X_\varepsilon^t(x, y) := X_\varepsilon(t, x, y) = X_0(x, y) + \varepsilon Y(t, x, y). \tag{1.2}$$

For ε small enough, the time-0-to-1 map,

$$f_\varepsilon = \phi_{X_\varepsilon}^{1,0}, \tag{1.3}$$

of the symplectic vector field X_ε is a symplectic diffeomorphism of $\mathbb{R}^2$ admitting o as a hyperbolic fixed point and still Σ as a separatrix. (If $X(t, z)$ is a time dependent vector field, the time-s-to-t map of X is defined by $\phi_X^{t,s}(z(s)) = z(t)$ for any $z(\cdot)$ solution of $\dot{z}(t) = X(z(t))$. When X is time independent, the notation ϕ_X^t stands for $\phi_X^{t,0}$.) Note that f_ε is a *Hamiltonian* diffeomorphism (for more details on Hamiltonian diffeomorphisms, see [**16**]).

Here is the analogue of the aforementioned last geometric theorem of Herman.

THEOREM A. *For any $r \in \mathbb{N}^*$, there exists $\varepsilon_r > 0$ such that, for any $\varepsilon \in]-\varepsilon_r, \varepsilon_r[$, there exists a set of f_ε-invariant C^r KAM circles accumulating the separatix Σ and which covers a set of positive Lebesgue measure of $\mathbb{R}^2$ in any neighborhood of Σ.*

Let us clarify some points made in the preceding statement.

By a C^r *circle*, $r \geq 0$, we mean a C^r non-self-intersecting closed curve (or equivalently, if $r \geq 1$, a non-empty compact connected one-dimensional C^r submanifold of $\mathbb{R}^2$) which is isotopic in $\mathbb{R}^2 \setminus \{o\}$ to the separatrix Σ. Such a set Γ is invariant by f_ε if $f_\varepsilon(\Gamma) = \Gamma$.

We say that a set $\mathcal{G}$ of f_ε-invariant circles *accumulates* the set Σ if for any $\xi > 0$, the set of $\Gamma \in \mathcal{G}$ such that $\operatorname{dist}(\Gamma, \Sigma) < \xi$ is not empty, where dist denotes the Hausdorff distance,

$$\operatorname{dist}(A, B) = \max\left(\sup_{a\in A} d(a, B), \sup_{b\in B} d(b, A)\right)$$

(here $d(x, C) = \inf_{c\in C} \|x - c\|_{\mathbb{R}^2}$).

The f_ε-invariant circles obtained in Theorem A are *KAM circles*: the restrictions of f_ε on each of these curves are C^r circle diffeomorphisms that are conjugated to Diophantine translations. A real number α is Diophantine if there exist positive constants κ, τ such that, for any $(p, q) \in \mathbb{Z} \times \mathbb{N}^*$, $|\alpha - (p/q)| \geq \kappa/q^\tau$. The constants τ and κ are respectively called the *exponent* and the *constant* of the Diophantine condition. The set of Diophantine numbers with fixed exponent $\tau > 2$ has full Lebesgue measure if the constant is not specified and positive measure if the constant is also fixed (and small). In our case, the exponent of the Diophantine condition can be chosen to be independent of ε (it depends only on λ).

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Remark 1.1. However, and this is a difference with the situation of Herman's last geometric theorem, the *constants* of these Diophantine numbers are arbitrarily small. Moreover, as these circles accumulate the separatrix, their C^2-norm must explode.

Remark 1.2. The phase space $\mathbb{R}^2$ can be replaced by the cylinder $\mathbb{R}/\mathbb{Z} \times \mathbb{R}$ in the statement of the main theorem.

The smallness condition in Theorem A is indeed necessary as shown by the following theorem.

Let Δ_Σ be the bounded connected component of $\mathbb{R}^2 \setminus \Sigma$.

THEOREM B. *There exists a smooth symplectic diffeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ admitting a separatrix Σ which is included in an open set W of $\Sigma \cup \Delta_\Sigma$ that contains no f-invariant circle in $W \setminus \Sigma$.*

The situation described in Theorems A and B is *not generic*. Indeed, as Poincaré discovered, in general, the stable and unstable manifolds of a hyperbolic fixed or periodic point of a symplectic map intersect transversally (one usually refers to this phenomenon as the *splitting of separatrices*), a fact that forces the dynamics of f to be 'quite intricate'. This was Poincaré's key argument in his proof of the fact that the Three-body problem in Celestial Mechanics does not admit a complete set of independent commuting first integrals. Later, Smale [**18**] showed that this splitting of separatrices has an even more striking consequence on the dynamics of f, namely the existence of a *horseshoe*, that is, a uniformly hyperbolic f-invariant compact set (locally maximal) with positive topological entropy and on which the dynamics of f is 'chaotic' (isomorphic to a two-sided shift). By a result of the first author [**13**], in this situation, positive topological entropy is indeed equivalent to the existence of a horseshoe. A consequence of the splitting of a separatrix is thus the existence of a *Birkhoff instability zone* (open region without invariant circles) in the vicinity of this split separatrix (see [**11**] for a detailed exposition on the topic). In some sense, Theorem A shows that in the perturbative situation of equations (1.2)–(1.3) (ε small enough), the splitting of separatrices is essentially the only mechanism responsible for the creation of instability zones. However, in a 'non-perturbative' situation, Theorem B points in the opposite direction. Figures 4 and 7 illustrate the role that plays the smallness assumption in Theorem A (or its absence in Theorem B).

1.1. *On the proofs of Theorems A and B.*

As suggests Remark 1.1, the invariant circles of Theorem A cannot be obtained directly via a classical KAM approach. However, the existence of the (non-split) separatrix Σ allows to associate to each diffeomorphism f_ε a regular diffeomorphism $\mathring{f}_\varepsilon$, defined on a standard open annulus and preserving a finite probability measure, to which one can apply Moser's or Rüssmann's invariant (or translated) curve theorem [**15**, **17**] (see §6). The thus obtained invariant curves for $\mathring{f}_\varepsilon$ yield invariant curves for f_ε. The construction of the diffeomorphism $\mathring{f}_\varepsilon$ is done as follows. We first make preliminary reductions involving some Birkhoff and symplectic Sternberg-like normal forms (§2) to have a control on the dynamics in some neighborhood of the hyperbolic fixed point o (§3). This allows us to define in §4 a first return map $\hat{f}_\varepsilon$ for f_ε,

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in a fundamental domain $\mathcal{F}_\varepsilon$, the boundaries of which can be glued together to obtain an open *abstract* cylinder (or annulus). This abstract cylinder can be *uniformized* to become a standard annulus and the first return map $\hat{f}_\varepsilon$ then becomes a regular diffeomorphism $\bar{f}_\varepsilon$ of a standard annulus (preserving some probability measure). This is done in §5. We call *normalization* (see §5.3) the uniformization operation and we say that $\bar{f}_\varepsilon$ is the *renormalization* of f_ε. The term *renormalization* in this paper has the same acceptation as in the theories of circle diffeomorphisms, holomorphic germs, or quasi-periodic cocycles; cf. [**6**, **14**, **23**, **24**]. The dynamics of $\bar{f}_\varepsilon$ is closely related to that of f_ε in the sense that the existence of invariant curves for $\bar{f}_\varepsilon$ translates into a similar statement for f_ε (see §7). The renormalized diffeomorphism $\bar{f}_\varepsilon$ has a *large twist* (this is reminiscent of the hyperbolicity of f_ε at o) and we are thus led to *rescale* it to obtain the aforementioned diffeomorphism $\mathring{f}_\varepsilon$ which is now a small C^r-perturbation of an integrable twist map (this is where the smallness assumption of Theorem A appears) with a *controlled* twist (see §6). The proof of Theorem A is completed in §8.

To prove Theorem B (cf. §9), we construct a symplectic diffeomorphism f (named f_{pert} in that section) so that the associated renormalized diffeomorphism $\bar{f}$ has an orbit accumulating the boundary of the aforementioned annulus: this prevents the existence of $\bar{f}$-invariant curves close to this boundary and therefore of f-invariant curves close to the separatrix Σ.

We note that the authors of [**21**] introduce the 'separatrix map' constructed by a gluing construction to investigate the size of the instability zones. Our approach here, which is focused on a renormalization point of view, is different. The technique we use to prove Theorem A might be useful to study the dynamics of symplectic twist maps with zero topological entropy. That is, to which extent are they integrable? Angenent, [**1**], proves they are C^0-integrable in the sense that, for any rotation number, one can find a C^0-invariant curve with this rotation number. Can one prove C^k-integrability? The word 'integrable' is meant in a broad sense. Additionally, the construction of Theorem B might give a hint to provide examples of *smooth* twist maps admitting *isolated* invariant circles with *irrational* rotation number (if they exist, these curves bound two instability zones). A modification of the example of Theorem B yields examples of such isolated invariant curves with *rational* rotation numbers. For the existence of curves with *irrational* rotation number in *low regularity* and related results, see [**2**–**5**].

2. *Normal forms*

The main result of this section is the following Sternberg-like symplectic normal form theorem (Proposition 2.1) that will allow us in §3 to control the *long-time* dynamics of f_ε in a neighborhood of the hyperbolic point o. This will be useful when we shall define first return maps for f_ε in a convenient fundamental domain, see §4.

Let f_ε be defined by (1.2) and (1.3).

PROPOSITION 2.1. *For any $k \in \mathbb{N}^*$ large enough, there exists $\varepsilon_k > 0$ for which the following holds. There exist a smooth family $(q_{\varepsilon,k})_{\varepsilon\in I}$ ($I \ni 0$ some open interval of $\mathbb{R}$) of polynomials $q_{\varepsilon,k}(s) = \lambda s + O(s^2) \in \mathbb{R}[s]$ and a continuous family $(\Theta_{\varepsilon,k})_{\varepsilon\in I}$ of symplectic C^k-diffeomorphism of $\mathbb{R}^2$ such that $\Theta_{\varepsilon,k}(o) = o$, $D\Theta_{\varepsilon,k}(o) = \mathrm{id}$, and on a neighborhood*

V_k of o one has, provided $\varepsilon \in]-\varepsilon_k, \varepsilon_k[$:

$$\text{on } V_k, \quad f_{\varepsilon,k} \underset{\text{defin.}}{=} \Theta_{\varepsilon,k} \circ f_\varepsilon \circ \Theta_{\varepsilon,k}^{-1} \tag{2.4}$$

$$= \phi^1_{J\nabla Q_{\varepsilon,k}} \quad \text{where } Q_{\varepsilon,k}(x,y) = q_{\varepsilon,k}(xy) \tag{2.5}$$

and

$$\text{on } V_k, \quad (\Theta_{0,k})_* X_0 = J\nabla Q_{0,k}. \tag{2.6}$$

Note that o is still a hyperbolic fixed point of $f_{\varepsilon,k}$ and that

$$\Sigma_{\varepsilon,k} := \Theta_{\varepsilon,k}(\Sigma)$$

is still a separatrix for $f_{\varepsilon,k}$.

2.1. *Reduction of Theorem A to Theorem 2.1.* After applying Proposition 2.1, we are thus left with a family $(f_{\varepsilon,k})$ of C^k- symplectic diffeomorphisms, each $f_{\varepsilon,k}$ being conjugated to f_ε and admitting a separatrix $\Sigma_{\varepsilon,k}$. Because the conclusions of Theorem A are clearly invariant by conjugation, to prove Theorem A, we just need to prove that if $k \geq r$ and ε is small enough, each separatrix $\Sigma_{\varepsilon,k}$ is accumulated by a set of positive measure of KAM circles for $f_{\varepsilon,k}$. This is the content of Theorem 2.1 below that we shall apply to the family of C^k-diffeomorphisms $f_{\varepsilon,k}$ defined by (1.2), (1.3), and (2.4), but that holds for any family (that we still denote in what follows by $(f_\varepsilon)_{\varepsilon\in I}$ to alleviate the notations) of symplectic C^k-diffeomorphisms satisfying the following hypothesis.

Let $(f_\varepsilon)_{\varepsilon\in I}$, ($I \ni 0$ open interval of $\mathbb{R}$) be a family of C^k-symplectic diffeomorphisms of $\mathbb{R}^2$ that satisfies:

(H1) each f_ε has a (non-split) separatrix Σ_ε associated to the hyperbolic point o;

(H2) the map $I \ni \varepsilon \to f_\varepsilon - \mathrm{id} \in C^k(\mathbb{R}^2, \mathbb{R}^2)$ is continuous (the norm on C^k is the usual C^k-norm);

(H3) on some neighborhood V of o, each f_ε coincides with the time-1 map of a symplectic vector field $J\nabla Q_\varepsilon(x,y)$ where $Q_\varepsilon(x,y) = q_\varepsilon(xy)$, $q_\varepsilon \in C^{k+1}(\mathbb{R}^2)$

$$q_\varepsilon(t) = \lambda t + O^2(t), \quad \lambda > 0;$$

(H4) on $\mathbb{R}^2$, $f_0 = \phi^1_{X_0}$, where $X_0 = J\nabla H_0$ is a Hamiltonian vector field that coincides with $J\nabla Q_0$ on V.

Remark 2.1. On V, the orbits of $f_{\varepsilon|V} = \phi^1_{J\nabla Q_\varepsilon}$ are pieces of hyperbolae $\{xy = \text{constant}\}$ (condition (H3)).

When $\varepsilon = 0$, for any $z \in \{xy = c\} \cap V$, $N \in \mathbb{Z}$ such that $f_0^N(z) \in V$, one has $f_0^N(z) \in \{xy = c\} \cap V$ (condition (H4)).

Remark 2.2. The intersection $\Sigma_\varepsilon \cap V$ is the union

$$\Sigma_\varepsilon \cap V = (W^s_{f_\varepsilon}(o) \cap V) \cup (W^u_{f_\varepsilon}(o) \cap V)$$

and

$$W^s_{f_\varepsilon}(o) \cap V = (\mathbb{R} \times \{0\}) \cap V, \quad W^u_{f_\varepsilon}(o) \cap V = (\{0\} \times \mathbb{R}) \cap V.$$

One then has the following.

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THEOREM 2.1. *There exists $k_0 \in \mathbb{N}$ for which the following holds. Let $k \geq k_0 + 2$ and let $(f_\varepsilon)_{\varepsilon \in I}$ be a family of C^k-symplectic diffeomorphisms of $\mathbb{R}^2$ satisfying the previous conditions (H1)–(H4). Then, there exists $\varepsilon_1 > 0$ such that, for any $\varepsilon \in]-\varepsilon_1, \varepsilon_1[$, the diffeomorphism f_ε admits a set of positive Lebesgue measure of invariant C^{k-k_0-2}-circles in any neighborhood of the separatrix Σ_ε.*

Moreover, if $k - k_0 - 2 \geq k_1$ (k_1 depending only on λ), these circles are KAM circles.

We shall give the proof of Theorem 2.1 in §8.

The proof of Proposition 2.1 occupies the rest of this section. It will be based on a first reduction obtained by performing some steps of Birkhoff normal forms (Proposition 2.3) and then the application of various Sternberg-like normal forms (Corollary 2.4 and Proposition 2.5).

2.2. *Birkhoff normal form for the time-periodic vector field X_ε^t.* A preliminary step in Sternberg's classical linearization theorem is to first conjugate the considered system (diffeomorphism or vector field) defined in the neighborhood of the hyperbolic fixed point o to a system which is tangent to an integrable model to some high enough order. This is what we do in this subsection and in a symplectic framework (see Proposition 2.3) by using Birkhoff normal form techniques.

2.2.1. *Periodically forced vector fields.* Let $X : \mathbb{R} \times \mathbb{R}^2 \ni (t, x) \mapsto X^t(z) := X(t, z) \in \mathbb{R}^2$ be a smooth time-dependent symplectic vector field: for each t, the 1-form $i_{X_t}\omega$ is closed (and hence locally exact). For $t, s \in \mathbb{R}$, we denote by $\phi_X^{t,s}$ the flow of X between times s and t when it is defined (see page 3 for the definition of $\phi_X^{t,s}$). If $t \mapsto g^t(\cdot)$ is a one-parameter family of symplectic diffeomorphisms, one has

$$g^t \circ \phi_X^{t,s} \circ (g^s)^{-1} = \phi_{\tilde{X}}^{t,s}, \tag{2.7}$$

where $\tilde{X} : (t, z) \mapsto \tilde{X}^t(z) := \tilde{X}(t, z)$ is the smooth time-dependent symplectic vector field

$$\tilde{X}^t = \partial_t g^t \circ (g^t)^{-1} + (g^t)_* X^t. \tag{2.8}$$

Conversely, if (2.8) is satisfied, then so is (2.7). Note that if g^t depends 1-periodically on t, then (2.7) yields the more classical conjugation equation

$$g \circ \phi_X^{1,0} \circ g^{-1} = \phi_{\tilde{X}}^{1,0},$$

where $g = g^0 = g^1$ (g^t is 1-periodic in t).

Assume now that X^t depends 1-periodically in t and, in a smooth way, on a small parameter $\varepsilon \in \mathbb{R}$; we furthermore assume that it is of the form

$$X_\varepsilon^t(z) = J \nabla H_\varepsilon^t(z), \tag{2.9}$$

where ($z = (z_1, z_2) \in \mathbb{R}^2$)

$$H_\varepsilon^t(z) = \lambda_\varepsilon(t) z_1 z_2 + O^3(z), \quad \int_{\mathbb{T}} \lambda_\varepsilon(t)\, dt > 0, \quad \lambda_0(t) = \lambda \in \mathbb{R}_+^*, \tag{2.10}$$

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$H_\varepsilon : \mathbb{R}/\mathbb{Z} \times \mathbb{R}^2 \to \mathbb{R}$, $H_\varepsilon : (t, z) \mapsto H_\varepsilon(t, z) := H_\varepsilon^t(z)$ being a smooth function. Assume also that, for some $j \in \mathbb{N}^*$,

$$g_\varepsilon^t(z) = \phi^1_{J\nabla G_\varepsilon^t}(z) = \mathrm{id} + O^j(z), \quad G_\varepsilon^t(z) = O^{j+1}(z),$$

where $G : I \times \mathbb{R}/\mathbb{Z} \times (\mathbb{R}^2, o) \ni (\varepsilon, t, z) \mapsto G_\varepsilon(t, z) := G_\varepsilon^t(z) \in \mathbb{R}$ is a smooth function. Then, one has

$$\partial_t g_\varepsilon^t \circ (g_\varepsilon^t)^{-1} = J\nabla \partial_t G_\varepsilon^t + O^{j+1}(z),$$
$$(g_\varepsilon^t)_* X_\varepsilon^t = J\nabla H_\varepsilon^t \circ (g_\varepsilon^t)^{-1} = J\nabla H_\varepsilon^t + J\nabla\{G_\varepsilon^t, H_\varepsilon^t\} + O^{j+1}(z),$$

(here $\{A, B\}$ denotes the Poisson bracket $\{A, B\} = \langle \nabla A, J\nabla B\rangle$) so that $\tilde{X}_\varepsilon^t$ defined by (2.8) is of the form

$$\tilde{X}_\varepsilon^t = J\nabla \tilde{H}_\varepsilon^t, \tag{2.11}$$

with

$$\tilde{H}_\varepsilon^t = H_\varepsilon^t + \partial_t G_\varepsilon^t + \{G_\varepsilon^t, H_\varepsilon^t\} + O^{j+2}(z) \tag{2.12}$$

$$= H_\varepsilon^t + \partial_t G_\varepsilon^t + \{G_\varepsilon^t, H_{2,\varepsilon}^t\} + O^{j+2}(z), \tag{2.13}$$

where we have denoted $H_{2,\varepsilon}^t(z_1, z_2) = \lambda_\varepsilon(t) z_1 z_2$.

If in the preceding equation, one chooses $G_\varepsilon^t = G_{\varepsilon,2}^t$ with $G_{\varepsilon,2}^t(z) = a_{\varepsilon,0}(t) z_1 z_2$, where $a_{\varepsilon,0}$ is the 1-periodic function defined by

$$a_{\varepsilon,0}(t) = -\int_0^t \left(\lambda_\varepsilon(s) - \int_{\mathbb{T}} \lambda_\varepsilon(u)\, du\right) ds,$$

one has

$$\tilde{H}_\varepsilon^t(z) = \bar{\lambda}_\varepsilon z_1 z_2 + O^3(z),$$

where $\bar{\lambda}_\varepsilon = \int_{\mathbb{T}} \lambda_\varepsilon(t)\, dt$. In other words, performing a change of coordinates (2.8) on X_ε^t with $g_\varepsilon^t = g_{\varepsilon,2}^t = \phi^1_{J\nabla G_{\varepsilon,2}^t}$, we can assume that in (2.10), $\lambda_\varepsilon(t)$ does not depend on t:

$$H_\varepsilon^t(z) = \lambda_\varepsilon z_1 z_2 + O^3(z), \quad \lambda_\varepsilon \in \mathbb{R}_+^* \tag{2.14}$$

(we write λ_ε in place of $\bar{\lambda}_\varepsilon$).

2.2.2. *Birkhoff normal form.* Having put H_ε^t under the form (2.14), we now eliminate by successive conjugations (2.8) *non-diagonal* higher-order terms in z from H_ε^t (note that they depend on t).

The following lemma describes this elimination procedure.

LEMMA 2.2. *Let $j \in \mathbb{N}$, $j \geq 2$. Assume that, for some polynomials $q_\varepsilon(s) = \lambda s + O(s^2) \in \mathbb{R}[s]$ of degree $\leq [j/2]$ depending smoothly on ε,*

$$H_\varepsilon^t(z) = q_\varepsilon(z_1 z_2) + O^{j+1}(z).$$

Then, there exist a smooth family $(\tilde{q}_\varepsilon)_\varepsilon$ of polynomials $\tilde{q}_\varepsilon(s) = \lambda s + O(s^2) \in \mathbb{R}[s]$ of degree $\leq [(j+1)/2]$ and a smooth family of smooth maps $G_\varepsilon : \mathbb{R}/\mathbb{Z} \times (\mathbb{R}^2, o) \ni (t, z) \to$

$G_\varepsilon(t,z) = G_\varepsilon^t(z) \in \mathbb{R}^2$ *such that on a neighborhood of o,*

$$\begin{cases} G_\varepsilon^t(z) = O^{j+1}(z), \\ H_\varepsilon^t(z) + \partial_t G_\varepsilon^t(z) + \{G_\varepsilon^t, H_\varepsilon^t\}(z) = \tilde{q}_\varepsilon(z_1 z_2) + O^{j+2}(z). \end{cases} \tag{2.15}$$

Moreover, if for $\varepsilon = 0$, H_0^t *does not depend on t, one can choose* G_0^t *to be independent of t.*

Proof. See Appendix A. □

Let now X_ε^t be the family of vector fields of (1.2).

PROPOSITION 2.3. *For any* $N \geq 1$ *there exist an open neighborhood* V_N *of o, a smooth two-parameters family* $(b_\varepsilon^t)_{\varepsilon \in I, t \in \mathbb{R}/\mathbb{Z}}$ *(I some open interval containing 0) of smooth symplectic diffeomorphisms* $b_\varepsilon^t : (\mathbb{R}^2, o) \circlearrowleft$ *satisfying* $b_\varepsilon^t(o) = o$, $Db_\varepsilon^t(o) = \mathrm{id}$ *and a smooth family of polynomials* $q_{\varepsilon,N}(s) = \lambda s + O(s^2)$ *of degree* $\leq [(N+1)/2]$*, such that, for any* $\varepsilon \in I$, $t \in \mathbb{R}/\mathbb{Z}$, $(x,y) \in V_N$ *one has*

$$\begin{aligned} X_\varepsilon^{(1),t} &\underset{\text{defin.}}{=} (\partial_t b_\varepsilon^t) \circ (b_\varepsilon^t)^{-1} + (b_\varepsilon^t)_* X_\varepsilon^t \\ &= J\nabla Q_{\varepsilon,N} + O^{N+1}(x,y) \quad \text{with } Q_{\varepsilon,N}(x,y) = q_{\varepsilon,N}(xy), \end{aligned}$$

and for $\varepsilon = 0$, b_0^t *is independent of t.*

Proof. Applying the preceding Lemma 2.2 and relations (2.8)–(2.12) inductively (starting from (2.14)), we thus construct polynomials $q_{\varepsilon,j}$ of degree $\leq [j/2]$ ($j \geq 2$) and functions $G_{\varepsilon,j}^t = O^{j+1}(z)$ such that if one defines

$$b_\varepsilon^t = g_{\varepsilon,N}^t \circ \cdots \circ g_{\varepsilon,2}^t = \mathrm{id} + O^2(z), \qquad g_{\varepsilon,j}^t = \phi^1_{J\nabla G_{\varepsilon,j}^t} = \mathrm{id} + O^j(z),$$

one has

$$\begin{aligned} \tilde{X}_\varepsilon^t &:= \partial_t b_\varepsilon^t \circ (b_\varepsilon^t)^{-1} + (b_\varepsilon^t)_* X_\varepsilon^t \\ &= J\nabla Q_{\varepsilon,N} + O^{N+1}(z) \quad \text{with } Q_{\varepsilon,N}(z) = q_{\varepsilon,N}(z_1 z_2), \end{aligned}$$

all depending on ε being smooth. Moreover, if X_0^t is independent of t, the diffeomorphism b_0^t is independent of t. □

Remark 2.3. Note that because $b_0^t \equiv b_0$ is independent of t, the vector field

$$X_0^{(1)} = (b_0)_* X_0$$

is autonomous.

2.3. *Symplectic Sternberg theorem for the autonomous vector field* $X_0^{(1)}$. We shall need a *symplectic* version of the famous theorem by S. Sternberg (on smooth linearization of hyperbolic germs of smooth vector fields, see [19]), as proved in [7] or [8] (see also [20]). We follow here the exposition of [7].

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Let $Z_i, i = 1, 2$, be two symplectic smooth autonomous vector fields such that, for some $\lambda \in \mathbb{R}^*$ and $N \in \mathbb{N}$, one has

$$\begin{cases} Z_i(x, y) = -\lambda x \dfrac{\partial}{\partial x} + \lambda y \dfrac{\partial}{\partial y} + O^2(x, y) \quad (i = 1, 2), \\ Z_1(x, y) - Z_2(x, y) = O^{N+1}(x, y). \end{cases} \tag{2.16}$$

THEOREM 2.2. [7, Theorem 1.2] *There exist positive constants A, B for which the following holds. Let $m \in \mathbb{N}^*$ large enough and $N = [(m + B)/A] + 1 \geq 1$. If (2.16) is satisfied, then there exists a C^m symplectic change of coordinates$S_0 : (\mathbb{R}^2, 0) \circlearrowleft$ such that on a neighborhood of o,*

$$\begin{cases} (S_0)_* Z_1 = Z_2, \\ S_0(o) = 0, \qquad DS_0(o) = \mathrm{id}. \end{cases} \tag{2.17}$$

We apply the preceding theorem to the case $Z_1 = X_0^{(1)}$ and $Z_2 = J\nabla Q_{0,N}$ ($X_0^{(1)}$, $Q_{0,N}$ given by Proposition 2.3 when $\varepsilon = 0$). In view of Proposition 2.3, the condition (2.16) is satisfied and we hence get a symplectic diffeomorphism S_0 satisfying $S_0(o) = 0$, $DS_0(o) = \mathrm{id}$, and such that on a neighborhood of o,

$$(S_0)_* X_0^{(1)} = J\nabla Q_{0,N}.$$

For each value of $t \in \mathbb{R}/\mathbb{Z}$ and $\varepsilon \in I$, the diffeomorphism $(S_0 \circ b_\varepsilon^t)$ fixes the origin and its derivative at the origin is the identity. It can thus be extended as a symplectic C^m-diffeomorphism R_ε^t of $\mathbb{R}^2$ (cf. Lemma B.1). Note that the dependence of R_ε^t with respect to t is smooth and 1-periodic ($t \in \mathbb{R}/\mathbb{Z}$). We now define on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$ the time-periodic vector field $X_\varepsilon^{(2)} : (t, (x, y)) \in (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}^2 \to \mathbb{R}^2$ by

$$X_\varepsilon^{(2),t} \underset{\text{defin.}}{=} (\partial_t R_\varepsilon^t) \circ (R_\varepsilon^t)^{-1} + (R_\varepsilon^t)_* X_\varepsilon^t, \tag{2.18}$$

and we observe that on a neighborhood of o,

$$(R_0^t)_* X_0 = X_0^{(2)} = J\nabla Q_{0,N}.$$

Because the conjugacy relation (2.18) is equivalent to (see §2.2.1)

$$\text{for all } t, s, \quad R_\varepsilon^t \circ \phi_{X_\varepsilon}^{t,s} \circ (R_\varepsilon^s)^{-1} = \phi_{X_\varepsilon^{(2)}}^{t,s},$$

we get by taking $t = 1$, $s = 0$, and setting $R_\varepsilon := R_\varepsilon^1 = R_\varepsilon^0$, the following corollary.

COROLLARY 2.4. *If $m \in \mathbb{N}^*$ is large enough and $N = [(m + B)/A] + 1$, there exists a smooth family (R_ε) of C^m symplectic diffeomorphisms of $\mathbb{R}^2$ such that $R_\varepsilon(o) = o$, $DR_\varepsilon(o) = \mathrm{id}$, and on a neighborhood of o,*

$$f_\varepsilon^{(1)} : \underset{\text{defin.}}{=} R_\varepsilon \circ f_\varepsilon \circ (R_\varepsilon)^{-1} \tag{2.19}$$

$$= \phi^1_{J\nabla Q_{\varepsilon,N}} + O^{N+1}(x, y). \tag{2.20}$$

Moreover,

$$(R_0)_* X_0 = J\nabla Q_{0,N}. \tag{2.21}$$

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Note that the last equation shows that

$$f_0^{(1)} = \phi^1_{J\nabla Q_{0,N}}. \tag{2.22}$$

2.4. *Symplectic Sternberg normal form for the diffeomorphism $f_\varepsilon^{(1)}$.* Theorem 2.2 has a version for smooth germs of symplectic diffeomorphisms which are hyperbolic at the origin. This is theorem 1.1 of [7]. In our paper, we shall need a parametric version of that result, which is not explicitly stated in [7] but that can be checked after close examination of the proof.

PROPOSITION 2.5. *There exist constants A_1, B_1 depending on $\lambda \in \mathbb{R}^*$ such that the following holds. Let $m \in \mathbb{N}^*$ large enough and $N = [m/2] - 3$. If $(g_{1,\varepsilon})_{\varepsilon\in I}$ and $(g_{2,\varepsilon})_{\varepsilon\in I}$ ($I \ni 0$ some open interval of $\mathbb{R}$) are two continuous (with respect to $\varepsilon \in I$) families of C^m symplectic diffeomorphisms$(\mathbb{R}^2, o) \circlearrowleft$ such that*

$$\begin{cases} \text{for all } \varepsilon, \quad g_{1,\varepsilon}(o) = g_{2,\varepsilon}(o) = o, \\ Dg_{1,0}(o) = \operatorname{diag}(\lambda, \lambda^{-1}) \quad (\text{is hyperbolic}), \\ g_{1,\varepsilon}(x, y) = g_{2,\varepsilon}(x, y) + O^{N+1}(x, y), \\ g_{1,0} = g_{2,0}, \end{cases} \tag{2.23}$$

then, there exists a continuous family $(S_\varepsilon^{(1)})_\varepsilon$ (with respect to $\varepsilon \in I$ small enough) of C^k symplectic diffeomorphisms such that $S_\varepsilon^{(1)}(o) = o$, $DS_\varepsilon^{(1)}(o) = \mathrm{id}$ with $k = [NA_1 - B_1] - 1$, and

$$\begin{cases} S_\varepsilon^{(1)} \circ g_{1,\varepsilon} \circ (S_\varepsilon^{(1)})^{-1} = g_{2,\varepsilon}, \\ S_0^{(1)} = \mathrm{id}. \end{cases}$$

2.5. *Proof of Proposition 2.1.* It will be a consequence of Corollary 2.4 and Proposition 2.5.

We first choose N so that $k = [NA_1 - B_1] - 1$ and we define m by $N = [(m + B)/A] + 1$. If k is large enough, m will satisfy the assumption of Corollary 2.4. We then apply Proposition 2.5 to $g_{1,\varepsilon} = f_\varepsilon^{(1)}$, $g_{2,\varepsilon} = \phi^1_{J\nabla Q_{\varepsilon,N}}$, which satisfies (2.23) (note that (2.20) is satisfied). This provides us with a continuous family $(S_\varepsilon^{(1)})_\varepsilon$ of C^k symplectic diffeomorphisms defined in a fixed neighborhood of o such that $S_\varepsilon^{(1)}(o) = o$, $DS_\varepsilon^{(1)}(o) = \mathrm{id}$, and on a neighborhood of o,

$$\begin{cases} S_\varepsilon^{(1)} \circ f_\varepsilon^{(1)} \circ (S_\varepsilon^{(1)})^{-1} = \phi^1_{J\nabla Q_{\varepsilon,N}}, \\ S_0^{(1)} = \mathrm{id}. \end{cases} \tag{2.24}$$

We can extend these $S_\varepsilon^{(1)}$ as symplectic C^k diffeomorphisms $S_\varepsilon^{(2)}$ of $\mathbb{R}^2$ which depend continuously on ε (cf. Lemma B.1). We then define

$$\Theta_{\varepsilon,k} = S_\varepsilon^{(2)} \circ R_\varepsilon$$

and we observe that on a neighborhood of o,

$$\begin{cases} \Theta_{\varepsilon,k} \circ f_\varepsilon \circ \Theta_{\varepsilon,k}^{-1} = \phi^1_{J\nabla Q_{\varepsilon,N}}, \\ (\Theta_{0,k})_* X_0 = J\nabla Q_{0,N}; \end{cases}$$

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indeed, the first equality comes from (2.19 and the first equation of (2.24), while the second is a consequence of (2.21) and the second equation of (2.24).

To conclude the proof, we rename $q_{\varepsilon,N}$, $Q_{\varepsilon,N}$ as $q_{\varepsilon,k}$, $Q_{\varepsilon,k}$.

Note: From now on, and until the end of §8, we shall work in the setting of Theorem 2.1 with a family of C^k symplectic diffeomorphisms satisfying conditions (H1)–(H4).

3. *Dynamics in a neighborhood of the origin*

The purpose of this section is to estimate the time spent by the orbits of the flow $\Phi^t_{J\nabla Q_\varepsilon}$ in the neighborhood V of the hyperbolic point o.

To do that, we perform one more change of coordinates.

Let us define the following diffeomorphisms Ξ_1, Ξ_2

$$\begin{cases}\text{for all } (x,y)\in \mathbb{R}_+^*\times\mathbb{R}, & \Xi_1(x,y)=(\ln x, xy),\\ \text{for all } (x,y)\in \mathbb{R}\times\mathbb{R}_+^*, & \Xi_2(x,y)=(-\ln y, xy).\end{cases} \tag{3.25}$$

Because $d(\ln x)\wedge d(xy)=d(-\ln y)\wedge d(xy)=dx\wedge dy$, we see that Ξ_i, $i=1,2$, are symplectic.

Let $I_1, I_2\subset\mathbb{R}_+^*$ be some open intervals such that $I_1\times\{0\}$ and $\{0\}\times I_2$ are both contained in V.

LEMMA 3.1. *Let* $(x_*,y_*)\in(I_1\times\mathbb{R})\cap V$ *and* $\bar t_{I_2}(x_*,y_*)=\inf\{t>0:\phi^t_{J\nabla Q_\varepsilon}(x_*,y_*)\in(\mathbb{R}\times I_2)\cap V\}$. *Then the following hold.*

(1) *There exists* $c(I_1,I_2)\geq 1$ *such that if* $0<x_*y_*\lesssim_{I_1,I_2,\lambda}1$, *one has*

$$c(I_1,I_2)^{-1}\frac{|\ln(x_*y_*)|}{\lambda}\leq \bar t_{I_2}(x_*,y_*)\leq c(I_1,I_2)\frac{|\ln(x_*y_*)|}{\lambda}. \tag{3.26}$$

(2) *For any* (x,y) *in a neighborhood of* (x_*,y_*) *and any* t *in a neighborhood of* $\bar t_{I_2}(x_*,y_*)$,

$$\Xi_2\circ\phi^t_{J\nabla Q_\varepsilon}\circ\Xi_1^{-1}:(u,v)\mapsto(u+\tau^t_\varepsilon(v),v), \tag{3.27}$$

with

$$\tau^t_\varepsilon(v)=tq'_\varepsilon(v)-\ln v. \tag{3.28}$$

Proof. (1) We evaluate $\bar t_{I_2}(x_*,y_*)$. Because

$$\phi^t_{J\nabla Q_\varepsilon}(x_*,y_*)=(e^{-tq'_\varepsilon(x_*y_*)}x_*, e^{tq'_\varepsilon(x_*y_*)}y_*),$$

we have $e^{tq'_\varepsilon(x_*y_*)}y_*\in I_2$ if and only if

$$t\in\left]\frac{\ln((x_*y_*)^{-1}\times x_*\min I_2)}{q'_\varepsilon(x_*y_*)},\frac{\ln((x_*y_*)^{-1}\times x_*\max I_2)}{q'_\varepsilon(x_*y_*)}\right[.$$

Hence for x_*y_* small enough,

$$\left|\bar t_{I_2}(x_*,y_*)-\frac{|\ln(x_*y_*)|}{q'_\varepsilon(x_*y_*)}\right|\leq\frac{\max(|\ln(x_*\min I_2)|,|\ln(x_*\max I_2)|)}{q'_\varepsilon(x_*y_*)}.$$

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Because for $0 < x_*y_* \lesssim 1$ one has $q'_\varepsilon(x_*y_*) \asymp \lambda$, there exists $c(I_1, I_2)$ such that if x_*y_* small enough (how small depends on I_1, I_2, λ), the inequality (3.26) is satisfied.

(2) We write

$$\Xi_2 \circ \phi^t_{J\nabla Q_\varepsilon} \circ \Xi_1^{-1} = \Xi_2 \circ \Xi_1^{-1} \circ \Xi_1 \circ \phi^t_{J\nabla Q_\varepsilon} \circ \Xi_1^{-1} = \Xi_2 \circ \Xi_1^{-1} \circ \phi^t_{J\nabla \tilde{Q}_\varepsilon},$$

with $\tilde{Q}_\varepsilon(u, v) = (Q_\varepsilon \circ \Xi_1^{-1})(u, v) = q_\varepsilon(v)$. Because $\phi^t_{J\nabla \tilde{Q}_\varepsilon}(u, v) = (u - tq'_\varepsilon(v), v)$ and $\Xi_2 \circ \Xi_1^{-1}(u, v) = (u - \ln v, v)$, we get (3.28). □

4. *Fundamental domains and first return maps*

We construct in this section adapted fundamental domains $\mathcal{F}_{\varepsilon,y_*}$ for the maps $(f_\varepsilon)_\varepsilon$ satisfying conditions (H1)–(H4) of Theorem 2.1 and define their first return maps $\hat{f}_\varepsilon$ in $\mathcal{F}_{\varepsilon,y_*}$.

4.1. *Fundamental domains.*

Let V be the domain of Theorem 2.1. One can choose $x_* > 0$ such that, for any ε small enough,

$$(x_*, 0) \in V \quad \text{and} \quad f_\varepsilon^{-1}(x_*, 0) \notin V.$$

For $y_* > 0$ small enough, we define the vertical segment

$$L_{x_*,y_*} := \{(x_*, ty_*),\ 0 < t < 1\}$$

and the domain

$$\mathcal{F}_{\varepsilon,x_*,y_*}$$

as the interior of the contour defined by (see Figure 2)

(a) the segment $[f_\varepsilon(x_*, 0), (x_*, 0)]$;
(b) the transversal $\overline{L_{x_*,y_*}}$;
(c) the piece of hyperbola joining (x_*, y_*) to $f_\varepsilon(x_*, y_*)$ (cf. Remark 2.1);
(d) the curve $f_\varepsilon(\overline{L_{x_*,y_*}})$.

We shall often drop the index x_* in the notations of $L_{x_*,y_*}, \mathcal{F}_{\varepsilon,x_*,y_*}$ and simply set

$$L_{y_*} := L_{x_*,y_*} \quad \text{and} \quad \mathcal{F}_{\varepsilon,y_*} = \mathcal{F}_{\varepsilon,x_*,y_*}.$$

If y_* is small enough, one has $\mathcal{F}_{\varepsilon,y_*}, \overline{L_{y_*}} \subset V$. We set

$$\tilde{\mathcal{F}}_{\varepsilon,y_*} = \mathcal{F}_{\varepsilon,y_*} \cup L_{y_*}.$$

4.2. *First return maps.*

Our aim in this subsection is to define the first return map of f_ε in $\tilde{\mathcal{F}}_{\varepsilon,y_*}$.

Because Σ_ε is a separatrix for f_ε, we can define (see Remark 2.2)

$$N(\varepsilon) \underset{\text{defn.}}{=} \min\{n \in \mathbb{N}^*,\ f_\varepsilon^{-n}(]f_\varepsilon(x_*), x_*]) \subset V\}.$$

We note that if ε is small enough, $N(\varepsilon)$ is independent of ε, so we shall denote it by N. Moreover, if ε and y_* are small enough,

$$N \underset{\text{defn.}}{=} \min\{n \in \mathbb{N}^*,\ f_\varepsilon^{-n}(\tilde{\mathcal{F}}_{\varepsilon,y_*}) \subset V\}. \tag{4.29}$$

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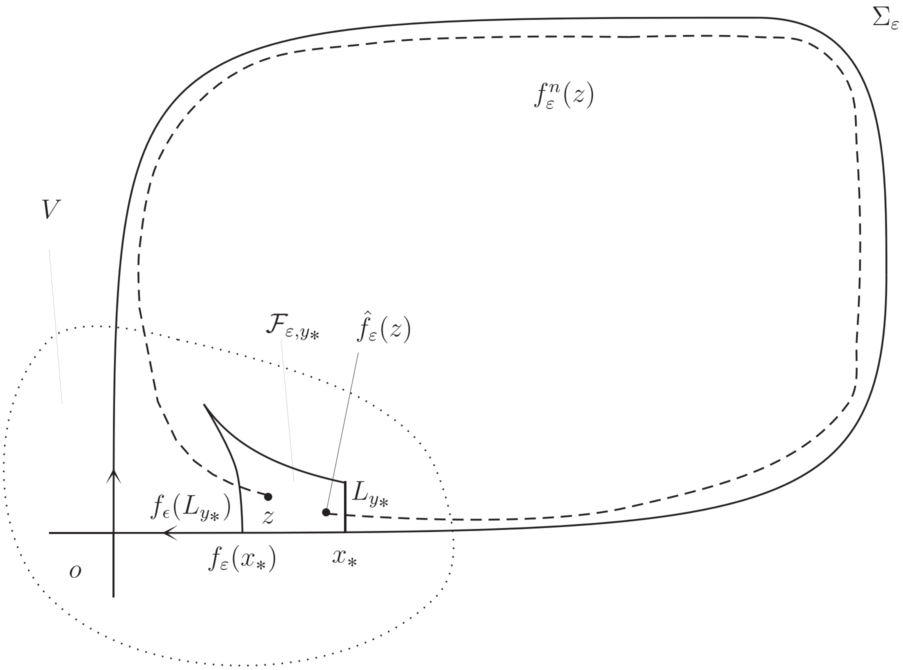

FIGURE 2. Fundamental domain $\mathcal{F}_{\varepsilon,y_*}$ for f_ε and the first return map $\hat{f}_\varepsilon$.

LEMMA 4.1. *There exists a constant* $0 < c_* < 1$ *such that, for* $(x, y) \in \mathcal{F}_{\varepsilon,c_* y_*}$,

$$\tilde{n}_\varepsilon(x, y) \underset{\text{defin.}}{:=} \min\{j \in \mathbb{N}^*,\ f_\varepsilon^j(x, y) \in f_\varepsilon^{-N}(\tilde{\mathcal{F}}_{\varepsilon,y_*})\} < \infty. \tag{4.30}$$

One has

$$\tilde{n}_\varepsilon(x, y) \asymp \ln(xy)/\lambda. \tag{4.31}$$

Proof. Note that the domain $f_\varepsilon^{-N}(\mathcal{F}_{\varepsilon,y_*}) \subset \tilde{V}$ is the interior of the contour defined by:

(a) the segment $[f_\varepsilon^{-(N-1)}(x_*, 0), f_\varepsilon^{-N}(x_*, 0)] \subset W^u_{f_\varepsilon}(o) \cap V \subset \{0\} \times \mathbb{R}$;

(b) the curve $f_\varepsilon^{-N}(\overline{L_{y_*}})$;

(c) a curve joining $f_\varepsilon^{-N}(x_*, y_*)$ to $f_\varepsilon^{-(N-1)}(x_*, y_*)$;

(d) the curve $f_\varepsilon^{-(N-1)}(\overline{L_{y_*}})$,

and

$$f_\varepsilon^{-N}(\tilde{\mathcal{F}}_{\varepsilon,y_*}) = f_\varepsilon^{-N}(\mathcal{F}_{\varepsilon,y_*}) \cup f_\varepsilon^{-N}(L_{y_*}).$$

Note that the lines $f_\varepsilon^{-N}(\overline{L_{y_*}})$, $f_\varepsilon^{-(N-1)}(\overline{L_{y_*}})$ are transversal to the segment $[f_\varepsilon^{-(N-1)}(x_*, 0), f_\varepsilon^{-N}(x_*, 0)]$.

Now let $(x, y) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$. We denote by $\mathcal{H}_{x,y}$ the hyperbola

$$\mathcal{H}_{x,y} := \{(x', y'),\ x'y' = xy\},$$

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and if $z, z' \in \mathcal{H}_{x,y}$, by $\mathcal{H}_{x,y}(z, z')$, the arc of hyperbola of $\mathcal{H}_{x,y}$ between z and z' which is open in z and closed in z'. If $y > 0$ is small enough, $\mathcal{H}_{x,y}$ intersects $f_\varepsilon^{-N}(\tilde{\mathcal{F}}_{\varepsilon,y_*}) \subset V$ in an arc of hyperbola of the form $\mathcal{H}_{x,y}(p, f_\varepsilon^{-1}(p))$ with $p \in f_\varepsilon^{-(N-1)}(L_{y_*})$ and $f_\varepsilon^{-1}(p) \in f_\varepsilon^{-N}(L_{y_*})$. The sets $f_\varepsilon^{-j}(\mathcal{H}_{x,y}(p, f_\varepsilon^{-1}(p)),\ j \geq 0$ form a partition of the semi-arc of parabola $\bigcup_{n\geq 0} \mathcal{H}_{x,y}(p, f_{\varepsilon,k}^{-n}(p))$ which contains (x, y). In particular, there exists $j \geq 0$ (in fact $j \geq 1$) such that $(x, y) \in f_\varepsilon^{-j}(\mathcal{H}_{x,y}(p, f_\varepsilon^{-1}(p))$ or equivalently

$$f_\varepsilon^{j}(x, y) \in \mathcal{H}_{x,y}(p, f_\varepsilon^{-1}(p)) \subset f_\varepsilon^{-N}(\tilde{\mathcal{F}}_{\varepsilon,y_*}).$$

This proves (4.30).

To prove (4.31), we note that there exists an interval I_2 not containing 0 and depending only on x_*, y_* such that $f_\varepsilon^{-N}(\tilde{\mathcal{F}}_{\varepsilon,y_*}) \subset \mathbb{R} \times I_2$. We then use Lemma 3.1 and the fact that $|\bar{t}_{I_2}(x, y) - \tilde{n}_\varepsilon(x, y)| \leq 1$. □

We now define

$$n_\varepsilon = N + \tilde{n}_\varepsilon. \tag{4.32}$$

By (4.31), one has

$$n_\varepsilon(x, y) \asymp \ln(xy)/\lambda. \tag{4.33}$$

The map $\hat{f}_\varepsilon : \tilde{\mathcal{F}}_{\varepsilon,c_* y_*} \to \tilde{\mathcal{F}}_{\varepsilon,y_*}$, defined by

$$\hat{f}_\varepsilon = f_\varepsilon^{n_\varepsilon}, \tag{4.34}$$

is the first return map of f_ε in $\tilde{\mathcal{F}}_{\varepsilon,y_*}$ (for points starting in $\tilde{\mathcal{F}}_{\varepsilon,c_* y_*}$). Note that $\hat{f}_\varepsilon$ is *not* C^k on the whole domain $\tilde{\mathcal{F}}_{\varepsilon,c_* y_*}$.

4.3. *Estimates on first return maps.*

We denote for $a \in \mathbb{R}$

$$T_a : (u, v) \mapsto (u + a, v)$$

and we recall the definition (3.25) of the symplectic diffeomorphisms Ξ_1, Ξ_2.

We observe that there exist open sets $W_1 \subset \mathbb{R}_+^* \cap \mathbb{R}$, $W_2 \subset \mathbb{R} \times \mathbb{R}_+^*$ such that, for any ε and $y_* > 0$ small enough,

$$\tilde{\mathcal{F}}_{\varepsilon,y_*} \subset W_1 \subset V, \quad f_0^{-N}(\tilde{\mathcal{F}}_{\varepsilon,y_*}) \subset W_2 \subset V.$$

LEMMA 4.2. *There exists a C^k function $\sigma_{0,N} \in C^k(\mathbb{R}_+^*, \mathbb{R})$ such that on $\Xi_2(W_2)$, one has*

$$\Xi_1 \circ f_0^N \circ \Xi_2^{-1} = T_{\sigma_{0,N}}. \tag{4.35}$$

Proof. From condition (H4), one can write on $\mathbb{R}^2$

$$f_0 = \phi^1_{J\nabla H_0}$$

and hence

$$f_0^N = \phi^N_{J\nabla H_0} \quad \text{where } H_0 \,|_V = Q_0.$$

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If $(u, v) \in \Sigma_2(W_2)$ and $(\tilde{u}, \tilde{v}) = \Xi_1(f_0^N(\Xi_2^{-1}(u, v)))$, one then has

$$Q_0(\Xi_1^{-1}(\tilde{u}, \tilde{v})) = Q_0(f_0^N(\Xi_2^{-1}(u, v))) = Q_0(\Xi_2^{-1}(u, v))$$

and hence

$$q_0(\tilde{v}) = q_0(v)$$

and thus $\tilde{v} = v$. Because the map $(u, v) \mapsto (\tilde{u}, \tilde{v})$ is symplectic, this forces $\tilde{u} = u + \tilde{\sigma}_{0,N}(v)$ for some C^k function $\tilde{\sigma}_{0,N}$; this function can be extended as a C^k function $\sigma_{0,N} : \mathbb{R} \to \mathbb{R}$. □

Recall the definition (4.34) of $\hat{f}_\varepsilon$.

LEMMA 4.3. *There exists a continuous family $(\hat{\eta}_\varepsilon)_\varepsilon$ of C^k symplectic diffeomorphisms defined on $\mathbb{R}^2$ and a neighborhood W of $f_\varepsilon^{-1}(\tilde{\mathcal{F}}_{\varepsilon,y_*}) \cup \tilde{\mathcal{F}}_{\varepsilon,y_*} \cup f_\varepsilon(\tilde{\mathcal{F}}_{\varepsilon,y_*})$ such that*

$$\begin{cases} \lim_{\varepsilon\to 0} \|\hat{\eta}_\varepsilon - \mathrm{id}\|_k = 0, \\ \hat{\eta}_\varepsilon(W \cap (\mathbb{R} \times \{0\}) \subset \mathbb{R} \times \{0\} \end{cases} \tag{4.36}$$

and on a neighborhood of $\tilde{\mathcal{F}}_{\varepsilon,c_* y_*}$, one has

$$\Xi_1 \circ \hat{f}_\varepsilon \circ \Xi_1^{-1} = \hat{\eta}_\varepsilon \circ T_{\hat{l}_\varepsilon}, \tag{4.37}$$

where

$$\hat{l}_\varepsilon(v) = \sigma_{0,N}(v) + \hat{n}_\varepsilon(u, v) q_\varepsilon'(v) - \ln v \quad \textit{with } \hat{n}_\varepsilon = \tilde{n}_\varepsilon \circ \Xi_1^{-1}. \tag{4.38}$$

Proof. We write (we use (4.34), (4.32), (H3)):

$$\begin{aligned} \hat{f}_\varepsilon &= f_\varepsilon^{N+\tilde{n}_\varepsilon} \\ &= f_\varepsilon^N \circ \phi_{J\nabla Q_\varepsilon}^{\tilde{n}_\varepsilon} \\ &= \eta_\varepsilon \circ f_0^N \circ \phi_{J\nabla Q_\varepsilon}^{\tilde{n}_\varepsilon}, \end{aligned}$$

with

$$\eta_\varepsilon = f_\varepsilon^N \circ f_0^{-N}. \tag{4.39}$$

As a consequence, if we set

$$\hat{\eta}_\varepsilon = \Xi_1 \circ \eta_\varepsilon \circ \Xi_1^{-1} \quad \text{and} \quad \hat{n}_\varepsilon = \tilde{n}_\varepsilon \circ \Xi_1^{-1}, \tag{4.40}$$

we have, using (4.35),

$$\begin{aligned} \Xi_1 \circ \hat{f}_\varepsilon \circ \Xi_1^{-1} &= \hat{\eta}_\varepsilon \circ (\Xi_1 \circ f_0^N \circ \Xi_2^{-1}) \circ (\Xi_2 \circ \phi_{J\nabla Q_\varepsilon}^{\tilde{n}_\varepsilon} \circ \Xi_2^{-1}) \circ (\Xi_2 \circ \Xi_1^{-1}) \\ &= \hat{\eta}_\varepsilon \circ T_{\sigma_{0,N}} \circ \phi_{J\nabla(Q_\varepsilon \circ \Xi_2^{-1})}^{\tilde{n}_\varepsilon \circ \Xi_1^{-1}} \circ T_{-\ln v} \\ &= \hat{\eta}_\varepsilon \circ T_{\sigma_{0,N}} \circ T_{q_\varepsilon'}^{\hat{n}_\varepsilon} \circ T_{-\ln v} \\ &= \hat{\eta}_\varepsilon \circ T_{\sigma_{0,N} + \hat{n}_\varepsilon q_\varepsilon' - \ln v}, \end{aligned}$$

which is (4.37) together with (4.38).

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Note that by (4.39), (4.40), Remark 2.2, and the fact that $\mathbb{R} \ni \varepsilon \mapsto f_\varepsilon \in C^k(V_k, \mathbb{R}^2)$ is continuous, one has

$$\begin{cases} \lim_{\varepsilon\to 0} \|\hat{\eta}_\varepsilon - \mathrm{id}\|_{C^k} = 0, \\ \hat{\eta}_\varepsilon(W \cap (\mathbb{R} \times \{0\})) \subset \mathbb{R} \times \{0\}. \end{cases}$$

□

5. *Renormalization*

We define in this section a *renormalization* $\bar{f}_\varepsilon$ of the map f_ε. The first return map $\hat{f}_\varepsilon$ of f_ε in the fundamental domain $\mathcal{F}_{\varepsilon,y_*}$ that we have constructed in §4 is not differentiable at every point (see (4.37), (4.38), and the fact that the integer valued function $\hat{n}_\varepsilon$ has, in general, discontinuity points). However, if one *glues* the 'vertical' boundaries of $\mathcal{F}_{\varepsilon,y_*}$ by f_ε, we obtain an *abstract* open annulus $\tilde{F}_{\varepsilon,y_*}/f_\varepsilon$ (see §§5.1 and 5.2) and the map $\hat{f}_\varepsilon$ is now C^k on it. We can *uniformize* this abstract annulus so that it becomes the standard (with the usual topology) open annulus $\mathbb{R}/\mathbb{Z} \times]0, c[$ (some $c > 0$), see §5.3, and the map $\hat{f}_\varepsilon$ in these new coordinates turns into a C^k diffeomorphism $\bar{f}_\varepsilon$ defined on (part of) this standard annulus. This is the (one should say 'a' instead of 'the' since the uniformizing/normalizing procedure is not unique) *renormalized* diffeomorphism associated to f_ε. Uniformizing the annulus is equivalent to conjugating f_ε to $(x, y) \mapsto (x + 1, y)$ on a domain containing $\mathcal{F}_{\varepsilon,y_*}$. This procedure is, in a different context, the one described in [**24**]. We shall often call the uniformization operation *normalization* in reference to the corresponding renormalization procedure defined for quasi-periodic cocycles, cf. [**6**, **14**].

5.1. *Gluing.*

Let $\mathcal{F}$ be an open set of $\mathbb{R}^2$, L a one-dimensional submanifold of $\mathbb{R}^2$ and f an orientation preserving smooth diffeomorphism from a neighborhood of $\mathcal{F} \cup L$ to a neighborhood of $f(\mathcal{F} \cup L)$. We assume that:

(1) $f(\mathcal{F} \cup L) \cap (\mathcal{F} \cup L) = \emptyset$;

(2) $\mathcal{F} \cup L$ is a two-dimensional submanifold of $\mathbb{R}^2$ with boundary and this boundary is $\partial(\mathcal{F} \cup L) = L$; in particular, for any point $p \in L$, there exists an open set U_p, $p \in U_p \subset \mathbb{R}^2$, and a smooth diffeomorphism $\varphi_p : U_p \to \varphi_p(U_p) \subset \mathbb{R}^2$ such that $\varphi_p(U_p \cap L) = \varphi_p(U_p) \cap (\mathbb{R} \times \{0\})$ and $\varphi_p(U_p \cap \mathcal{F}) = \varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}^*_+)$;

(3) for any $p \in \mathcal{F} \cup L$ and U_p, one has $U_p \cap f(\mathcal{F} \cup L) = \emptyset$;

(4) for any $p \in L$ one has $f^{-1}(f(U_p) \cap \mathcal{F}) = \varphi_p^{-1}(\varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}^*_-))$, for any of the previous chart (U_p, φ_p) at p.

We define the *topological space* $(\mathcal{F} \cup L, \mathcal{T})$ as being the set $\mathcal{F} \cup L$ endowed with the following topology $\mathcal{T}$: a subset S of $\mathcal{F} \cup L$ is an element of $\mathcal{T}$ (that is an open set) if for every $p \in S$, there exists an open set $V \subset \mathbb{R}^2$ (contained in a neighborhood of $\mathcal{F} \cup L$ where f is defined) such that $V \cap f(\mathcal{F} \cup L) = \emptyset$ and $p \in (V \cup f(V)) \cap (\mathcal{F} \cup L) \subset S$.

We can then define the following *differentiable structure* on $(\mathcal{F} \cup L, \mathcal{T})$ as follows: (a) if $p \in \mathcal{F}$, we define the local chart $C_p := (W_p, \mathrm{id})$, where W_p is an open set of $\mathbb{R}^2$ such that $p \in W_p \subset \mathcal{F}$; and (b) if $p \in L$, we define the local chart $C_p := (W_p, \psi_p)$ where W_p is the open set of $\mathcal{F} \cup L$ (see condition (3)), $W_p = (\mathcal{F} \cup L) \cap (U_p \cup f(U_p))$ (here (U_p, φ_p) is the local chart for $p \in L$ as defined in (2)), and where ψ_p is defined by

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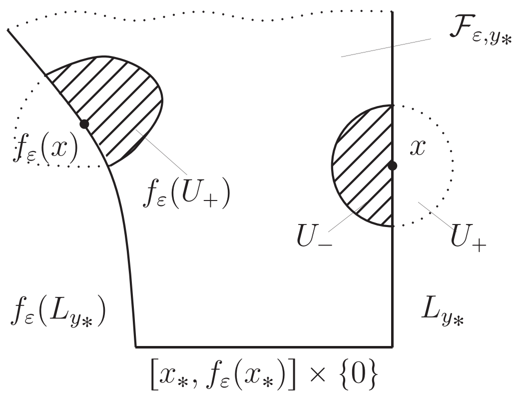

FIGURE 3. Gluing: $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_\varepsilon$.

(we use condition (4))

$$\begin{cases} \psi_p = \varphi_p \quad \text{on } U_p \cap (\mathcal{F} \cup L) = \varphi_p^{-1}((\varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}_+)), \\ \psi_p = \varphi_p \circ f^{-1} \quad \text{on } f(U_p) \cap \mathcal{F} = f \circ \varphi_p^{-1}((\varphi_p(U_p) \cap (\mathbb{R} \times \mathbb{R}_-^*)). \end{cases}$$

We denote by $\mathcal{A}$ the collection of all these local charts C_p and we set $(\mathcal{F} \cup L)/f = (\mathcal{F} \cup L, \mathcal{T}, \mathcal{A})$.

Remark 5.1. If we assume in addition that f preserves the standard symplectic form $dx \wedge dy$ on $\mathbb{R}^2$, we can endow $(\mathcal{F} \cup L)/f$ with a symplectic form ω.

Remark 5.2. If $g : \mathcal{F} \to g(\mathcal{F})$ is a smooth diffeomorphism defined in a neighborhood of $\mathcal{F}$, it induces a smooth diffeomorphism (that we still denote g) $g : (\mathcal{F} \cup L)/f \to (g(\mathcal{F}) \cup g(L))/(g \circ f \circ g^{-1})$.

Remark 5.3. If $\mathcal{F} = [0, 1[\times]0, 1[$, $L =]0, 1[$ and $f = T_1 : (x, y) \mapsto (x + 1, y)$, one sees that $(\mathcal{F} \cup L)/T_1$ is (diffeomorphic to) the standard open annulus $(\mathbb{R}/\mathbb{Z} \times]0, 1[$, canonical) endowed with its canonical differentiable structure.

5.2. *The space* $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_\varepsilon$. If ε and y_* are small enough, item (1) is satisfied and we can find charts (p, U_p) such that items (2)–(4) are satisfied. See Figure 3. We can then define the manifold $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_\varepsilon$. We shall see that it is an annulus without boundary, cf. Lemma 5.3.

Note that if $0 < c_* < 1$, the smaller set $\tilde{\mathcal{F}}_{\varepsilon,c_*y_*} = \mathcal{F}_{\varepsilon,c_*y_*} \cup L_{c_*y_*}$ is an open subset of $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_\varepsilon$ (which means that it belongs to $\mathcal{T}$) and it can be endowed with the topology and differentiable structure induced by the inclusion. We denote $(\mathcal{F}_{\varepsilon,c_*y_*} \cup L_{c_*y_*})/f_\varepsilon$ the thus obtained submanifold of $(\mathcal{F}_{\varepsilon,y_*} \cup L_{y_*})/f_\varepsilon$. The following lemma is then tautological.

LEMMA 5.1. *The map* $\hat{f}_\varepsilon$ *induces a* C^k *map* $\tilde{\mathcal{F}}_{\varepsilon,c_*y_*}/f_\varepsilon \to \tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$.

We shall need in §6 the following lemma.

LEMMA 5.2. *There exists a probability measure with positive density π_{ε,y_*} on $\tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$ which is $\hat{f}_\varepsilon$ invariant: for any measurable set $A \in \tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$ such that $\hat{f}_\varepsilon^{-1}(A) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$, one has $\pi_{\varepsilon,y_*}(A) = \pi_{\varepsilon,y_*}(\hat{f}_\varepsilon^{-1}(A))$.*

Proof. We shall in fact construct this measure π_{ε,y_*} on the bigger set $\hat{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$,

$$\hat{\mathcal{F}}_{\varepsilon,y_*} = \tilde{\mathcal{F}}_{\varepsilon,y_*} \cup \sigma(\tilde{\mathcal{F}}_{\varepsilon,y_*}),$$

where $\sigma : \mathbb{R}^2 \to \mathbb{R}^2$ is the reflection $(x, y) \mapsto (x, -y)$ (it commutes with f_ε in V, see condition (H3)). From Remark 5.1, there exists a symplectic form ω_ε on $\hat{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$. Note that the first return map $\hat{f}_\varepsilon$ is not defined on the whole set $\hat{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$ but nevertheless

$$(\hat{f}_\varepsilon)^*\omega_\varepsilon = \omega_\varepsilon$$

whenever this formula makes sense. The probability measure π_{ε,y_*} defined by

$$\pi_{\varepsilon,y_*}(A) = \int_A |\omega_\varepsilon| \Big/ \int_{\mathcal{F}_{\varepsilon,y_*}} |\omega_\varepsilon|$$

is $\hat{f}_\varepsilon$ invariant. □

5.3. *Normalization of f_ε.* We now uniformize the abstract annulus $\tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$. To do that, it is enough to normalize f_ε in the sense of item 2 of the following lemma.

LEMMA 5.3. (Normalization Lemma) *There exists a continuous family $(h_\varepsilon)_\varepsilon$ of (not necessarily symplectic) C^k-diffeomorphisms defined on a neighborhood of $\tilde{\mathcal{F}}_{\varepsilon,y_*}$ such that for some $c > 0$:*

(1) *h_ε sends $\tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$ to the standard open annulus* $((\mathbb{R}/\mathbb{Z}) \times]0, c[$, canonical)*;*
(2) *$h_\varepsilon \circ f_\varepsilon \circ h_\varepsilon^{-1} = T_1 : (x, y) \mapsto (x + 1, y)$;*
(3) *$h_\varepsilon([x_*, f_\varepsilon(x_*)] \times \{0\} = [0, 1[\times \{0\}$.*

Proof. Using condition (H3) and the change of coordinates (3.25) of §3, we see that on a neighborhood of $\tilde{\mathcal{F}}_{\varepsilon,y_*}$, one has (we use the notation (x, y) for (u, v))

$$\Xi_1 \circ f_\varepsilon \circ \Xi_1^{-1} = T_{q'_\varepsilon} : (x, y) \mapsto (x + q'_\varepsilon(y), y).$$

If g_ε is the (not necessarily symplectic) smooth diffeomorphism

$$g_\varepsilon : (x, y) \mapsto \left(\frac{x}{q'_\varepsilon(y)}, y\right), \tag{5.41}$$

one has

$$g_\varepsilon \circ \Xi_1 \circ f_\varepsilon \circ \Xi_1^{-1} \circ g_\varepsilon^{-1} = T_1. \tag{5.42}$$

The set $\overline{(g_\varepsilon \circ \Xi_1)(\mathcal{F}_{\varepsilon,y_*})}$ is of the form

$$\overline{(g_\varepsilon \circ \Xi_1)(\mathcal{F}_{\varepsilon,y_*})} = \{(x, y),\ y \in [0, c],\ \gamma_\varepsilon(y) \le x \le \gamma_\varepsilon(y) + 1\},$$

where $c > 0$, $\gamma_\varepsilon : [0, c] \to \mathbb{R}_+$ is C^k, $\gamma_\varepsilon(0) = 0$, and the map $\mathbb{R} \ni \varepsilon \mapsto \gamma_\varepsilon \in C^k([0, c], \mathbb{R})$ is continuous. This indeed follows from the definition of $\tilde{\mathcal{F}}_{\varepsilon,y_*}$ in §4.1,

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the definition of Ξ_1 (3.25), and (5.41), (5.42). As a consequence, if we denote

$$j_\varepsilon : (x, y) \mapsto (x - \gamma_\varepsilon(y), y), \tag{5.43}$$

we have

$$\begin{cases} j_\varepsilon \circ T_1 = T_1 \circ j_\varepsilon, \\ j_\varepsilon((g_\varepsilon \circ \Sigma_1)(\mathcal{F}_{\varepsilon,y_*})) =]0, 1[\times]0, c[, \\ j_\varepsilon((g_\varepsilon \circ \Sigma_1)(L_{y_*})) = \{0\} \times]0, c[. \end{cases} \tag{5.44}$$

By Remarks 5.2 and 5.3, the map

$$h_\varepsilon = j_\varepsilon \circ g_\varepsilon \circ \Xi_1 \tag{5.45}$$

is a diffeomorphism that sends $\tilde{\mathcal{F}}_{\varepsilon,y_*}/f_\varepsilon$ to the standard annulus $([0, 1[\times]0, c[)/T_1 \simeq (\mathbb{R}/\mathbb{Z}) \times]0, c[$ and such that

$$h_\varepsilon \circ f_\varepsilon \circ h_\varepsilon^{-1} = T_1.$$

To conclude the proof, we notice (2) is an immediate consequence of the definition (5.45) of h_ε. □

Remark 5.4. Note that if $T_a(x, y) = (x + a(y), y)$, one has

$$(h_\varepsilon \circ \Xi_1^{-1}) \circ T_a \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} = T_{\tilde{a}}, \quad \tilde{a}(y) = a(y)/q'_\varepsilon(y).$$

5.4. *The renormalization $\bar{f}_\varepsilon$ of f_ε.* There exists $\delta \in\ ]0, c[$ such that the map

$$\bar{f}_\varepsilon \underset{\text{defin.}}{=} h_\varepsilon \circ \hat{f}_\varepsilon \circ h_\varepsilon^{-1} : \mathbb{R}/\mathbb{Z} \times]0, \delta[\ \to \mathbb{R}/\mathbb{Z} \times]0, c[\tag{5.46}$$

is well defined and is a C^k diffeomorphism onto its image.

PROPOSITION 5.4. *One has*

$$\bar{f}_\varepsilon = \bar{\eta}_\varepsilon \circ T_{l_\varepsilon}, \tag{5.47}$$

where $\bar{\eta}_\varepsilon$ is a C^k diffeomorphism defined on $\mathbb{R}/\mathbb{Z} \times]0, \delta[$ and $l_\varepsilon \in C^k(]0, c[, \mathbb{R}/\mathbb{Z})$; they satisfy

$$l_\varepsilon(y) = \frac{\sigma_{0,N}(y)}{q'_\varepsilon(y)} - \frac{\ln y}{q'_\varepsilon(y)} \mod \mathbb{Z}, \tag{5.48}$$

$$\lim_{\varepsilon \to 0} \|\bar{\eta}_\varepsilon - \mathrm{id}\|_{C^k} = 0, \tag{5.49}$$

$$\bar{\eta}_\varepsilon : (x, y) \mapsto (x + a_\varepsilon(x, y), y + y b_\varepsilon(x, y)), \tag{5.50}$$

where $a_\varepsilon \in C^k$, $b_\varepsilon \in C^{k-1}$ are functions defined on $\mathbb{R}/\mathbb{Z} \times (0, \delta)$.

Moreover, the map $\bar{f}_\varepsilon$ preserves a probability measure $\bar{\pi}_{\varepsilon,y_*}$ with positive density defined on $\mathbb{R}/\mathbb{Z} \times]0, c[$.

Proof. By (4.37) and Remark 5.4 after Lemma 5.3,

$$\begin{aligned} \bar{f}_\varepsilon &= (h_\varepsilon \circ \Xi_1^{-1}) \circ \hat{\eta}_\varepsilon \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} \circ (h_\varepsilon \circ \Xi_1^{-1}) \circ T_{\hat{l}_\varepsilon} \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} \\ &= \bar{\eta}_\varepsilon \circ T_{l_\varepsilon}, \end{aligned}$$

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where

$$\bar{\eta}_\varepsilon = (h_\varepsilon \circ \Xi_1^{-1}) \circ \hat{\eta}_\varepsilon \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1} \quad \text{and} \quad l_\varepsilon(y) = (1/q'_\varepsilon(y))\hat{l}_\varepsilon(y). \tag{5.51}$$

Because $\bar{\eta}_\varepsilon := (h_\varepsilon \circ \Xi_1^{-1}) \circ \hat{\eta}_\varepsilon \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1}$ and $\bar{f}_\varepsilon$ are C^k, the function $l_\varepsilon : \,]0, c[\, \to \mathbb{R}/\mathbb{Z}$ is also C^k and

$$l_\varepsilon(y) = (1/q'_\varepsilon(y))\hat{l}_\varepsilon(y).$$

By (4.38) (remember that $\hat{n}_\varepsilon$ takes its value in $\mathbb{Z}$),

$$\begin{aligned} l_\varepsilon(y) &= \frac{\sigma_{0,N}(y)}{q'_\varepsilon(y)} + \hat{n}_\varepsilon(x, y) - \frac{\ln y}{q'_\varepsilon(y)} \\ &= \frac{\sigma_{0,N}(y)}{q'_\varepsilon(y)} - \frac{\ln y}{q'_\varepsilon(y)} \mod \mathbb{Z}, \end{aligned}$$

which is (5.48).

Equation (5.49) is a consequence of the definition of $\bar{\eta}_\varepsilon$, cf. (5.51), the first equation of (4.36), and of the fact that $\mathbb{R} \ni \varepsilon \mapsto h_\varepsilon \in C^k$ is continuous (Lemma 5.3).

We now claim that if $\bar{\eta}_\varepsilon(x, y) = (x + a_\varepsilon(x, y), y + \bar{b}_\varepsilon(x, y))$, one has for any y,

$$\bar{b}_\varepsilon(x, 0) = 0. \tag{5.52}$$

Indeed, because

$$\bar{\eta}_\varepsilon := (h_\varepsilon \circ \Xi_1^{-1}) \circ \hat{\eta}_\varepsilon \circ (h_\varepsilon \circ \Xi_1^{-1})^{-1},$$

equality (5.52) is a consequence of the second equation of (4.36), of item (3) of Lemma 5.3, and of the fact that $\Xi_1(\mathbb{R}_+^* \times \{0\}) = \mathbb{R}_+^* \times \{0\}$.

To prove (5.50), we thus notice that equality (5.52) gives us for $\bar{b}_\varepsilon$ a decomposition

$$\begin{cases} \bar{b}_\varepsilon(x, y) = y b_\varepsilon(x, y), \\ b_\varepsilon \in C^{k-1}. \end{cases}$$

Finally, to conclude the proof of the proposition, we observe that because the map $\hat{f}_\varepsilon : \tilde{\mathcal{F}}_{\varepsilon, c_* y_*}/f_\varepsilon \to \tilde{\mathcal{F}}_{\varepsilon, y_*}/f_\varepsilon$ preserves the probability measure π_{ε, y_*}, cf. Lemma 5.2, the diffeomorphism $\bar{f}_\varepsilon : \mathbb{R}/\mathbb{Z} \times \,]0, \delta[\, \to \mathbb{R}/\mathbb{Z} \times \,]0, c[$ preserves the probability measure $\bar{\pi}_{\varepsilon, y_*} = (h_\varepsilon)_* \pi_{\varepsilon, y_*}$ defined on $\mathbb{R}/\mathbb{Z} \times \,]0, c[$ (in the sense that if $A \subset \mathbb{R}/\mathbb{Z} \times \,]0, c[$ is a Borelian set such that $\bar{f}_\varepsilon^{-1}(A) \subset \mathbb{R}/\mathbb{Z} \times \,]0, c[$, one has $\bar{\pi}_{\varepsilon, y_*}(A) = \bar{\pi}_{\varepsilon, y_*}(\bar{f}_\varepsilon^{-1}(A))$). □

6. *Applying the translated curve theorem*

We apply in this section Rüssmann's (or Moser's) translated curve theorem to some *rescaled* version $\mathring{f}_{\varepsilon, n}$ of the renormalization $\bar{f}_\varepsilon$ of f_ε defined in §5.4.

6.1. *The translated curve theorem.*

Let $\psi : \mathbb{R}/\mathbb{Z} \times \,]e^{-1}, 1[\, \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}$ ($\ln e = 1$) be a C^k diffeomorphism defined on the annulus (or cylinder) $\mathbb{R}/\mathbb{Z} \times \,]e^{-1}, 1[$. We say that the graph $\mathrm{Gr}_\gamma := \{(x, \gamma(x)) : x \in \mathbb{R}/\mathbb{Z}\}$ of a continuous map $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times \,]e^{-1}, 1[$ is *translated* by ψ if for some $t \in \mathbb{R}$,

$$\psi(\mathrm{Gr}_\gamma) = \mathrm{Gr}_{t+\gamma} \tag{6.53}$$

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and *invariant* if $t = 0$. If Gr_γ satisfies (6.53), there exists an orientation preserving homeomorphism of the circle $g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $\psi(x, \gamma(x)) = \psi(g(x), t + \gamma(g(x))$. If $t = 0$ (respectively $t \neq 0$), we define (respectively with a clear abuse of language) the rotation number of (ψ on) the invariant (respectively translated) graph Gr_γ as the rotation number of the circle diffeomorphism g. We say that ψ has the *intersection property* if for any continuous $\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \times]e^{-1}, 1[$, the curve $\mathrm{Gr}_\gamma := \{(x, \gamma(x)) : x \in \mathbb{R}/\mathbb{Z}\}$ intersects its image $\psi(\mathrm{Gr}_\gamma)$. Note the following important fact: If ψ has the intersection property, any translated graph by ψ is *invariant.*

We state the translated curve theorem by Rüssmann [17] (which implies the invariant curve theorem by Moser [15]):

THEOREM 6.1. (Rüssmann, [17]) *There exists $k_0 \in \mathbb{N}$ for which the following holds. Let $k \geq k_0$, $C, \mu > 0$, and $l : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ a C^k map satisfying the twist condition,*

$$\min_y |\partial_y l(y)| > \mu > 0 \quad \text{and} \quad \|l\|_{C^{k_0}} \leq C, \tag{6.54}$$

and define

$$\psi_0 : (x, y) \mapsto (x + l(y), y).$$

There exists $\varepsilon_0 = \varepsilon_0(C, \mu) > 0$ such that for any C^k diffeomorphism

$$\psi : \mathbb{R}/\mathbb{Z} \times]e^{-1}, 1[\ \to \mathbb{R}/\mathbb{Z} \times \mathbb{R}$$

satisfying

$$\|\psi - \psi_0\|_{C^{k_0}} < \varepsilon_0, \tag{6.55}$$

the diffeomorphism ψ admits a set of positive Lebesgue measure of C^{k-k_0} translated graphs contained in $(\mathbb{R}/\mathbb{Z}) \times]e^{-3/4}, e^{-1/4}[$. Moreover, all these translated graphs have Diophantine rotation numbers (they are in a fixed Diophantine class $DC(\kappa, \tau)$ (the exponent is τ and the constant κ) that can be prescribed in advance once μ is fixed (k_0 then depends on τ and ε_0 on κ and τ)).

6.2. *The rescaled diffeomorphism $\mathring{f}_{\varepsilon,n}$.* Let $\bar{f}_\varepsilon$ be the renormalized map defined in §5.4 and define $u_\varepsilon, v_\varepsilon$ by

$$\bar{f}_\varepsilon(x, y) = (x + u_\varepsilon(x, y), y + v_\varepsilon(x, y)).$$

Because $\bar{f}_\varepsilon = \bar{\eta}_\varepsilon \circ T_{l_\varepsilon}$ (cf. (5.47)), one has using (5.50):

$$u_\varepsilon(x, y) = l_\varepsilon(y) + a_\varepsilon(x + l_\varepsilon(y), y),$$
$$v_\varepsilon(x, y) = y b_\varepsilon(x + l_\varepsilon(y), y).$$

Now, let $n \in \mathbb{N}^*$ large enough so that

$$]e^{-(n+1)}, e^{-n}[\subset]0, \delta[\tag{6.56}$$

(the δ of (5.46)) and introduce the rescaled C^k diffeomorphism $\mathring{f}_{\varepsilon,n}$ defined on the annulus $\mathbb{R}/\mathbb{Z} \times]e^{-1}, 1[$ by

$$\mathring{f}_{\varepsilon,n} \underset{\text{defin.}}{=} \Lambda_{e^n} \circ \bar{f}_\varepsilon \circ \Lambda_{e^n}^{-1}, \tag{6.57}$$

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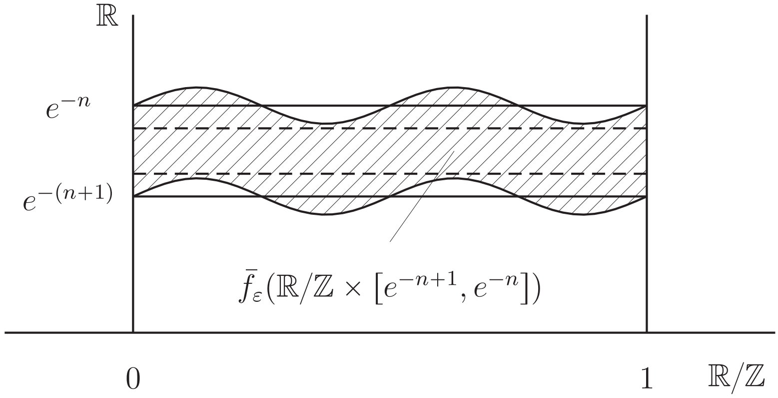

FIGURE 4. The diffeomorphism $\bar{f}_\varepsilon$ on $\mathbb{R}/\mathbb{Z}\times[e^{-(n+1)}, e^{-n}]$.

where $\Lambda_{e^n} : (x, y) \mapsto (x, e^n y)$. Let us denote

$$\mathring{f}_{\varepsilon,n}(x, y) = (x + u_{\varepsilon,n}(x, y), y + v_{\varepsilon,n}(x, y)).$$

A computation shows that:

$$\begin{cases} u_{\varepsilon,n}(x, y) &= l_{\varepsilon,n}(y) + a_\varepsilon(x + l_{\varepsilon,n}(y), e^{-n}y), \\ v_{\varepsilon,n}(x, y) &= yb_\varepsilon(x + l_{\varepsilon,n}(y), e^{-n}y), \end{cases} \tag{6.58}$$

where

$$l_{\varepsilon,n}(y) = l_\varepsilon(e^{-n}y). \tag{6.59}$$

We can now state the following important proposition the proof of which occupies the next subsection.

PROPOSITION 6.1. *Assume that $k \geq k_0 + 2$ (k is the regularity in conditions (H1)–(H4) and k_0 is the one of Theorem 6.1). There exists $\varepsilon_1 > 0$ such that the following holds. If $|\varepsilon| \leq \varepsilon_1$ and $n \gg 1$, $\mathring{f}_{\varepsilon,n}$ admits a set of positive Lebesgue measure of invariant C^{k-k_0-2}-graphs in $\mathbb{R}/\mathbb{Z} \times]e^{-1}, 1[$.*

6.3. *Proof of Proposition 6.1.*

6.3.1. *Twist condition for $l_{\varepsilon,n}$.*

LEMMA 6.2. *There exist $C, \mu > 0$ such that, for any ε small enough and any $n \gg 1$, the map $l_{\varepsilon,n}$ satisfies the twist condition (6.54) provided $k \geq k_0 + 1$.*

Proof. Using (5.48), (6.59), we have

$$\begin{aligned} l_{\varepsilon,n}(y) &= l_\varepsilon(e^{-n}y) \\ &= \frac{\sigma_{0,N}(e^{-n}y)}{q'_\varepsilon(e^{-n}y)} + \frac{n}{q'_\varepsilon(e^{-n}y)} - \frac{\ln y}{q'_\varepsilon(e^{-n}y)} \mod \mathbb{Z} \\ &= \frac{\sigma_{0,N}(0) + n}{\lambda} - \frac{\ln y}{\lambda} + \theta_{\varepsilon,n}(y) \mod \mathbb{Z}, \end{aligned}$$

where

$$\|\theta_{\varepsilon,n}\|_{C^{k-1}([e^{-1},1])} = O(e^{-n});$$

this last inequality is a consequence of the fact that $q_\varepsilon(s) = \lambda s + O(s^2)$ is continuous with respect to ε (cf. condition H2) and of the fact that $\sigma_{0,N}$ is C^k (cf. Lemma 4.2). In particular, for some $C_k > 0$ (depending on λ),

$$\|l_{\varepsilon,n}\|_{C^{k-1}} \le C_k$$

and because $\partial_y l_{\varepsilon,n}(y) = -1/(\lambda y) + \partial_y \theta_{\varepsilon,n}(y)$ and $y \in\ ]e^{-1}, 1[$,

$$|\partial_y l_{\varepsilon,n}(y)| \ge 1/(2\lambda).$$

Hence (6.54) holds uniformly in ε, n with $C = C_{k_0+1}$ and $\mu = 1/(2\lambda)$ as soon as n is large enough. □

6.3.2. *$\mathring{f}_{\varepsilon,n}$ is close to a twist.* We observe that from (5.49), (5.50), (6.58), and Lemma 6.2, one has uniformly in n,

$$\lim_{\varepsilon\to 0} \max(\|u_{\varepsilon,n} - l_{\varepsilon,n}\|_{C^{k-2}}, \|v_{\varepsilon,n}\|_{C^{k-2}}) = 0. \tag{6.60}$$

In particular, if n is large enough, inequality (6.55) is satisfied if $k \ge k_0 + 2$ with $\psi = \mathring{f}_{\varepsilon,n}$ and $\psi_0 : (x, y) \mapsto (x + l_{\varepsilon,n}(y), y)$.

We see from §§6.3.1 and 6.3.2 that, if

$$|\varepsilon| \le \varepsilon_1 \underset{\text{defin.}}{=} \varepsilon_0(C_{k_0+1}, 1/(2\lambda))$$

and $n \gg 1$, the assumptions of Theorem 6.1 are then satisfied by $\mathring{f}_{\varepsilon,n}$ with $k-2$ in place of k. Under these conditions, there thus exists a set $\mathring{\mathcal{G}}_{\varepsilon,n}$ of C^{k-k_0-2} $\mathring{f}_{\varepsilon,n}$-translated graphs, the union of which covers a set of positive Lebesgue measure in $(\mathbb{R}/\mathbb{T}) \times\ ]e^{-3/4}, e^{-1/4}[$. We just have to check that these translated graphs are indeed invariant.

6.3.3. *$\mathring{f}_{\varepsilon,n}$-translated graphs are invariant.* Let $\mathring{\gamma} \subset (\mathbb{R}/\mathbb{T}) \times\ ]e^{-3/4}, e^{-1/4}[$ be a $\mathring{f}_{\varepsilon,n}$-translated graph: $\mathring{f}_{\varepsilon,n}(\mathring{\gamma}) = \mathring{\gamma} + (0, t)$ for some $t \in \mathbb{R}$. We shall prove that $t = 0$. We can without loss of generality assume that $t \ge 0$ (the case $t \le 0$ is treated in a similar way).

Formula (6.60) shows that if $n \gg 1$, one has $\mathring{f}_{\varepsilon,n}(\mathring{\gamma}) \subset (\mathbb{R}/\mathbb{T}) \times\ ]e^{-1}, 1[$. From the conjugation relation (6.57), we see that (cf. (6.56))

$$\bar{\gamma} := \Lambda_{e^n}^{-1}(\mathring{\gamma}) \subset (\mathbb{R}/\mathbb{Z}) \times\ ]e^{-n-3/4}, e^{-n-1/4}[\subset (\mathbb{R}/\mathbb{Z}) \times\ ]0, \delta[$$

is a $\bar{f}_\varepsilon$-translated graph such that

$$\bar{f}_\varepsilon(\bar{\gamma}) = \bar{\gamma} + (0, e^{-n}t) \subset (\mathbb{R}/\mathbb{Z}) \times\ ]e^{-(n+1)}, e^{-n}[\subset (\mathbb{R}/\mathbb{Z}) \times\ ]0, \delta[.$$

Let A be the open domain of $(\mathbb{R}/\mathbb{Z}) \times\ ]0, c[$ between $(\mathbb{R}/\mathbb{Z}) \times \{0\}$ and $\bar{\gamma}$. Because $t \ge 0$, one has $A \subset \bar{f}_\varepsilon(A) \subset (\mathbb{R}/\mathbb{Z}) \times\ ]0, c[$.

Assume by contradiction that $t > 0$; then the set $\bar{f}_\varepsilon(A) \setminus A$ contains a non-empty open set. We have seen (cf. Proposition 5.4) that $\bar{f}_\varepsilon$ preserves a probability measure $\bar{\pi}_{\varepsilon,y_*}$

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with positive density defined on $(\mathbb{R}/\mathbb{Z}) \times]0, c[$, so $\bar{\pi}_{\varepsilon,y_*}(\bar{f}_\varepsilon(A) \setminus A) > 0$. However, this contradicts the invariance of $\bar{\pi}_{\varepsilon,y_*}$ by $\bar{f}_\varepsilon$.

The proof of Proposition 6.1 is complete. $\square$

6.4. *Invariant curves for $\bar{f}_\varepsilon$.* We can now state the following.

THEOREM 6.2. *Let $k \geq k_0 + 2$ and $|\varepsilon| \leq \varepsilon_1$. There exists $\nu \in]0, \delta[$ such that, for any $\nu \in]0, \nu[$, there exists a set $\bar{\mathcal{G}}_{\varepsilon,\nu}$ of C^{k-k_0-2}, $\bar{f}_\varepsilon$-invariant graphs contained in $(\mathbb{R}/\mathbb{Z}) \times]e^{-1}\nu, \nu[$ such that*

$$\mathrm{Leb}_{\mathbb{R}^2}\Bigg(\bigcup_{\bar{\gamma}\in\bar{\mathcal{G}}_{\varepsilon,\nu}} \bar{\gamma}\Bigg) > 0.$$

Proof. We choose n so that

$$]e^{-(n+1)}, e^{-n}[\subset]0, \nu[\tag{6.61}$$

and we observe that when $\nu \to 0$, one has $n \to \infty$. Define

$$\mathring{f}_{\varepsilon,n} = \Lambda_{e^n} \circ \bar{f}_\varepsilon \circ \Lambda_{e^n}^{-1}.$$

By Proposition 6.1, there exists $\nu_1 > 0$ such that if $\nu \in]0, \nu_1[$ (n satisfying (6.61) is then large enough), the diffeomorphism $\mathring{f}_{\varepsilon,n}$ admits C^{k-k_0-2}-invariant curves in $\mathbb{T} \times]e^{-1}, 1[$ covering a set of positive Lebesgue measure; hence, $\bar{f}_{\varepsilon,k}$ has C^{k-k_0-2}-invariant curves in $\mathbb{T} \times]e^{-1}\nu, \nu[$ covering a set of positive Lebesgue measure. $\square$

We shall denote

$$\bar{\mathcal{G}}_\varepsilon = \bigcup_{\nu\in]0,\nu_1[} \bar{\mathcal{G}}_{\varepsilon,\nu}.$$

Remark 6.1. For all $\bar{\gamma} \in \bar{\mathcal{G}}_{\varepsilon,\nu}$, the rotation number of the circle diffeomorphism $\bar{f}_\varepsilon |_{\bar{\gamma}}$ is Diophantine in a fixed Diophantine class $DC(\kappa, \tau)$ (see the comment at the end of the statement of Theorem 6.1).

7. *Invariant curves for f_ε*

We define

$$r = k - k_0 - 2$$

and assume that $|\varepsilon| \leq \varepsilon_1$.

Let $\bar{\gamma} \subset (\mathbb{R}/\mathbb{Z}) \times]0, \delta[$, $\bar{\gamma} \in \bar{\mathcal{G}}_\varepsilon$ be a C^r invariant graph for $\bar{f}_\varepsilon : (\mathbb{R}/\mathbb{Z}) \times]0, \delta[\to (\mathbb{R}/\mathbb{Z}) \times]0, c[$. Note that there exists $\delta_1 > 0$ such that $\bar{\gamma} \subset (\mathbb{R}/\mathbb{Z}) \times]\delta_1, \delta[$.

We can view $\bar{\gamma}$ as an invariant graph sitting in $([0, 1[\times]0, c[)/T_1$ (recall $T_1(x, y) = (x + 1, y)$). In particular, one can find a C^r, 1-periodic function

$$\bar{z} : \mathbb{R} \to ([0, 1[\times]0, \delta[)/T_1$$

such that, for all t, $(d/dt)\bar{z}(t) \neq 0$ and

$$\bar{\gamma} = \bar{z}([0, 1[), \quad \bar{z}(0) \in \{0\} \times]0, c[, \quad \lim_{t\to 1-} \bar{z}(t) = T_1(\bar{z}(0)) \in \{1\} \times]0, c[.$$

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Let

$$\hat{\gamma} = h_\varepsilon^{-1}(\bar{\gamma}),$$

where h_ε was defined in Lemma 5.3. Because $\bar{f}_\varepsilon = h_\varepsilon \circ \hat{f}_\varepsilon \circ h_\varepsilon^{-1}$ (cf. (5.46)), we see that

$$\hat{\gamma} \subset h_\varepsilon^{-1}((\mathbb{R}/\mathbb{Z}) \times]\delta_1, \delta[) \subset \tilde{\mathcal{F}}_{\varepsilon, c_* y_*}$$

is a C^r compact, connected, one-dimensional submanifold (without boundary) of $\tilde{\mathcal{F}}_{\varepsilon, c_* y_*}/f_\varepsilon$, which is invariant by $\hat{f}_\varepsilon : \tilde{\mathcal{F}}_{\varepsilon, c_* y_*}/f_\varepsilon \to \tilde{\mathcal{F}}_{\varepsilon, y_*}/f_\varepsilon$. Moreover, the function

$$\hat{z} \underset{\text{defin.}}{=} h_\varepsilon^{-1} \circ \bar{z} : \mathbb{R} \to \tilde{\mathcal{F}}_{\varepsilon, c_* y_*}/f_\varepsilon$$

is a C^r, 1-periodic function and

$$\hat{\gamma} = \hat{z}([0, 1[), \quad \hat{z}(0) \in L_{y_*}, \quad \lim_{t \to 1-} \hat{z}(t) = f_\varepsilon(\hat{z}(0)) \in f_\varepsilon(L_{y_*}).$$

The main result of this section is the following proposition.

PROPOSITION 7.1. *The set*

$$\hat{\Gamma} = \bigcup_{n \in \mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \subset \mathbb{R}^2$$

is an invariant C^r curve for f_ε: it is a compact, connected, one-dimensional C^r submanifold of $\mathbb{R}^2$ which is invariant by f_ε.

We give the proof of this proposition in §7.2.

7.1. *Preliminary results.* We define the function $\hat{Z} : \mathbb{R} \to \mathbb{R}^2$

$$\text{for all } t \in \mathbb{R}, \quad \hat{Z}(t) = f_{\varepsilon,k}^{[t]}(\hat{z}(t - [t]))$$

($[t]$ denotes the integer part of t *that is* the unique integer such that $[t] \leq t < [t] + 1$).

LEMMA 7.2. *The function $\hat{Z} : \mathbb{R} \to \mathbb{R}^2$ is C^r.*

Proof. Note that, for $t \in [0, 1[$, $\hat{Z}(t) = \hat{z}(t)$. Also, the very definition of $\tilde{\mathcal{F}}_{\varepsilon, y_*}/f_{\varepsilon,k}$ shows that the function $\hat{Z}$ is C^r on a neighborhood of $t = 1$. It is hence C^r on $[0, 2[$ and because for $j \in \mathbb{Z}$, $\hat{Z}(t + j) = f_\varepsilon^j(\hat{Z}(t))$, it is C^r on $\mathbb{R}$. □

Let us set

$$\tau \underset{\text{defin.}}{=} \inf\{t \geq 1,\ \hat{Z}(t) \in \tilde{\mathcal{F}}_{\varepsilon, y_*}\}.$$

Note that

$$2 \leq \tau < \infty. \tag{7.62}$$

Indeed, the left-hand side inequality is a consequence of the fact that $\tilde{\mathcal{F}}_{\varepsilon, y_*} \cap f_\varepsilon(\tilde{\mathcal{F}}_{\varepsilon, y_*}) = \emptyset$. For the right-hand side, we observe that because $\hat{Z}(0) = \hat{z}(0) \in \tilde{\mathcal{F}}_{\varepsilon, c_* y_*}$, one has (see (4.34)), $\hat{Z}(n_\varepsilon(\hat{z}(0))) = f_\varepsilon^{n_\varepsilon(\hat{z}(0))}(\hat{z}(0)) \in \tilde{\mathcal{F}}_{\varepsilon, y_*}$, hence $\tau \leq n_\varepsilon(\hat{z}(0)) < \infty$.

LEMMA 7.3. *The map $\hat{Z} : [0, \tau[\ \to \mathbb{R}^2$ is injective.*

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Proof. Assume by contradiction that $\hat{Z} : [0, \tau[\to \mathbb{R}^2$ is not injective; then, there exists $m_i \in \mathbb{N}, 0 \le s_i < 1$,

$$0 \le m_i + s_i < \tau, \quad i = 1, 2, \quad \hat{Z}(s_1 + m_1) = \hat{Z}(s_2 + m_2). \tag{7.63}$$

Hence, $f_\varepsilon^{m_1}(\hat{\gamma}) \cap f_\varepsilon^{m_2}(\hat{\gamma}) \neq \emptyset$ and if $m := m_2 - m_1 \ge 0$, $f_\varepsilon^m(\hat{\gamma}) \cap \hat{\gamma} \neq \emptyset$. In particular, there exists $t \in [m, m+1[$ such that $\hat{Z}(t) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$ and then $t \ge \tau$. As a consequence, $m > \tau - 1$ and because $0 \le m < \tau$ (m_1, m_2 are both in the interval $[0, \tau[$), one has $m = [\tau]$, and hence $m_2 = m = [\tau]$ and $m_1 = 0$. We then have from (7.63), $\hat{Z}(s_2 + [\tau]) = \hat{Z}(s_1) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$ (because $s_1 \in [0, 1[$) and hence by the definition of τ, $s_2 + [\tau] \ge \tau$, which contradicts $m_2 + s_2 < \tau$. □

LEMMA 7.4. *If, for some $t \ge 1$, $\hat{Z}(t) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$, then $\hat{Z}(t) \in \hat{\gamma}$.*

Proof. Indeed, writing $t = s + n$, $s \in [0, 1[$, $n \in \mathbb{N}^*$, one has $\hat{Z}(t) = f_\varepsilon^n(\hat{z}(s))$. The integer $n \ge 1$ is thus a m^{th} return time of $\hat{z}(s)$ in $\tilde{\mathcal{F}}_{\varepsilon,y_*}$, $\hat{Z}(t) = \hat{f}_\varepsilon^m(\hat{z}(s))$, and because $\hat{\gamma}$ is invariant by $\hat{f}_\varepsilon$, it is readily seen by induction on m that $f_\varepsilon^n(\hat{z}(s)) \in \hat{\gamma}$. □

LEMMA 7.5. *One has $\hat{Z}(\tau) = \hat{z}(0)$.*

Proof. From the definition of τ and Lemma 7.4, we have $\hat{Z}(\tau) \in \text{closure}(\hat{\gamma}) \cap \text{closure}(L_{y_*} \cup f_\varepsilon(L_{y_*}))$ and hence $\hat{Z}(\tau) \in \{\hat{z}(0), f_\varepsilon(\hat{z}(0))\}$. To conclude, we observe that one cannot have $\hat{Z}(\tau) = f_\varepsilon(\hat{z}(0))$ because otherwise, one would have $\hat{Z}(\tau - 1) = \hat{z}(0) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$, which contradicts the definition of τ (from (7.62) $\tau - 1 \ge 1$). □

LEMMA 7.6. *The derivative of $\hat{Z}$ at τ is transverse to L_{y_*}.*

Proof. (1) If there exists a sequence $t_n \in \mathbb{R}$, $\lim t_n = \tau$, such that $Z(t_n) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$, then from Lemma 7.4, one has $Z(t_n) \in \hat{\gamma}$ and consequently $(d\hat{Z}/dt)(\tau)$ is tangent to $\hat{\gamma}$, thus transverse to L_{y_*}.

(2) Otherwise, there exists an open interval $I \subset \mathbb{R}$, $I \ni \tau$, such that, for all $t \in I \setminus \{\tau\}$, $\hat{Z}(t) \notin \tilde{\mathcal{F}}_{\varepsilon,y_*}$ and $f_\varepsilon(\hat{Z}(t)) \in \mathcal{F}_{\varepsilon,y_*}$. From Lemma 7.4, one then has for all $t \in I \setminus \{\tau\}$, $\hat{Z}(t+1) = f_\varepsilon(\hat{Z}(t)) \in \hat{\gamma}$ (see item (2) of §5.1), and hence $Df_\varepsilon(f_\varepsilon(\hat{Z}(\tau))) \cdot (d\hat{Z}/dt)(\tau)$ is tangent to $\hat{\gamma}$ and in particular transverse to $f_\varepsilon(L_{y_*})$. This implies that $(d\hat{Z}/dt)(\tau)$ is transverse to L_{y_*}. □

LEMMA 7.7. *One has $\hat{Z}([\tau, \tau + 1[) = \hat{Z}([0, 1[)$.*

Proof. We define $s_* = \sup\{s \ge 0 : \text{for all } t \in [\tau, \tau + s[, \ \hat{Z}(t) \in \mathcal{F}_{\varepsilon,y_*}\}$. From Lemmata 7.4 and 7.6, one has: (a) $s_* > 0$; (b) for any $t \in [\tau, \tau + s_*[$, $\hat{Z}(t) \in \hat{\gamma}$; and (c) $\hat{Z}(\tau + s_*) \in f_\varepsilon(L_{y_*}) \cap \text{closure}(\hat{\gamma}) = f_\varepsilon(\hat{z}(0)) = \hat{Z}(1)$. In particular, $\hat{Z}(\tau + s_* - 1) = f_\varepsilon^{-1}(\hat{Z}(1)) = \hat{Z}(0) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}$ and by the definition of τ, this implies $s_* \ge 1$. Now we notice that one cannot have $s_* > 1$ because otherwise $\tau + 1 \in [\tau, \tau + s_*[$ and by definition of s_*, $\hat{Z}(\tau + 1) \in \mathcal{F}_{\varepsilon,y_*}$; however, $\hat{Z}(\tau + 1) = f_\varepsilon(\hat{Z}(\tau))$ and because $\hat{Z}(\tau) = \hat{z}(0)$ (Lemma 7.5), one has $\hat{Z}(\tau + 1) = f_\varepsilon(\hat{z}(0)) \notin \mathcal{F}_{\varepsilon,y_*}$. We have thus proven that $s_* = 1$. This implies that $\hat{Z}([\tau, \tau + 1[) = \hat{Z}([0, 1[)$. □

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7.2. *Proof of Proposition 7.1.* We first observe that

$$\hat{\Gamma} = \bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) = \hat{Z}(\mathbb{R}) = \bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{Z}([0, \tau+1[)). \tag{7.64}$$

Next we note the following.

(1) One has $\hat{Z}([0, \tau+1[) = \hat{Z}([0, \tau])$.

(2) The set $\hat{Z}([0, \tau+1[)$ is f_ε-invariant.

Item (1) is a consequence of

$$\begin{aligned}\hat{Z}([0, \tau+1[) &= \hat{Z}([0, \tau]) \cup \hat{Z}([\tau, \tau+1[)\\ &= \hat{Z}([0, \tau]) \cup \hat{Z}([0, 1[) \quad \text{(Lemma 7.7)}\\ &= \hat{Z}([0, \tau]) \quad (1 \le \tau).\end{aligned}$$

Item (2) follows from item (1) and

$$\begin{aligned}f_\varepsilon(\hat{Z}([0, \tau+1[)) &= f_\varepsilon(\hat{Z}([0, \tau]))\\ &= \hat{Z}([1, \tau+1])\\ &= \hat{Z}([1, \tau]) \cup \hat{Z}([\tau, \tau+1])\\ &= \hat{Z}([1, \tau]) \cup \hat{Z}([0, 1]) \quad \text{(Lemma 7.7)}\\ &= \hat{Z}([0, \tau]) \quad (1 \le \tau)\\ &= \hat{Z}([0, \tau+1[).\end{aligned}$$

Item (2) and (7.64) yield

$$\hat{\Gamma} = \hat{Z}([0, \tau+1[).$$

This last identity shows that $\hat{\Gamma}$ is a connected, compact (cf. item (1)) subset of $\mathbb{R}^2$ which is f_ε-invariant.

Let us prove that $\hat{\Gamma}$ is a one-dimensional submanifold of $\mathbb{R}^2$. Because $\hat{Z}(\tau) = \hat{Z}(0)$ (Lemma 7.5), one has

$$\hat{Z}([0, \tau+1[) = \hat{Z}(]0, \tau+1[) = \hat{Z}(]0, \tau[) \cup \hat{Z}(]\tau-1, \tau+1[).$$

From Lemmata 7.2 and 7.3, the set $\hat{Z}(]0, \tau[)$ is a one-dimensional submanifold of $\mathbb{R}^2$ as well as the set $\hat{Z}(]\tau-1, \tau+1[)$ (note that $\hat{Z}(]\tau-1, \tau+1[) = f_\varepsilon(\hat{Z}([\tau-2, \tau[))$. The intersection of these two sets is $\hat{Z}(]\tau, \tau+1[)$ and from Lemma 7.7, it is equal to $\hat{Z}(]0, 1[)$ which is a one-dimensional submanifold of $\mathbb{R}^2$. As a consequence, the union $\hat{Z}(]0, \tau[) \cup \hat{Z}(]\tau-1, \tau+1[)$ is one-dimensional submanifold of $\mathbb{R}^2$.

This concludes the proof of Proposition 7.1 □

8. *Proof of Theorem 2.1 (and hence of Theorem A)*

As we have mentioned in §2.5, Theorem A follows from Theorem 2.1; we describe the proof of this latter result in this section.

Let $r = k - k_0 - 2$, $|\varepsilon| \le \varepsilon_1$, and $\nu \le \nu_1$. Theorem 6.2 yields a set $\bar{\mathcal{G}}_{\varepsilon,\nu}$ of C^r, $\bar{f}_\varepsilon$-invariant graphs contained in $(\mathbb{R}/\mathbb{Z}) \times]e^{-1}\nu, \nu[$, the union of which covers a set of positive Lebesgue measure.

In the previous section (cf. Proposition 7.1), for all $\nu \in \,]0, \nu_1[$, we have associated to each $\bar{f}_\varepsilon$-invariant graph $\bar{\gamma} \in \bar{\mathcal{G}}_{\varepsilon,\nu}$ an f_ε-invariant C^r-curve:

$$\hat{\Gamma} = \bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \quad \text{where } \hat{\gamma} = h_\varepsilon^{-1}(\bar{\gamma}). \tag{8.65}$$

We denote by $\hat{\mathcal{G}}_{\varepsilon,\nu}$ the set of all such curves $\hat{\Gamma}$.

To prove Theorem 2.1, we just have to prove that, for all $\nu \in \,]0, \nu_1[$,

$$\text{(Positive measure)} \quad \mathrm{Leb}_{\mathbb{R}^2}\bigg(\bigcup_{\hat{\Gamma}\in\hat{\mathcal{G}}_{\varepsilon,\nu}} \hat{\Gamma}\bigg) > 0 \tag{8.66}$$

and

$$\text{(Accumulation)} \quad \lim_{\nu\to 0} \sup_{\hat{\Gamma}\in\hat{\mathcal{G}}_{\varepsilon,\nu}} \mathrm{dist}(\hat{\Gamma}, \Sigma_\varepsilon) = 0. \tag{8.67}$$

8.1. *Proof of (8.66) (positive measure).* This is a consequence of the inclusion (cf. (8.65))

$$h_\varepsilon^{-1}\bigg(\bigcup_{\bar{\gamma}\in\bar{\mathcal{G}}_{\varepsilon,\nu}} \bar{\gamma}\bigg) \subset \bigcup_{\hat{\Gamma}\in\hat{\mathcal{G}}_{\varepsilon,\nu}} \hat{\Gamma}$$

and of the fact that $\mathrm{Leb}_2(\bigcup_{\bar{\gamma}\in\bar{\mathcal{G}}_{\varepsilon,\nu}} \bar{\gamma}) > 0$ (this is the content of Theorem 6.2).

8.2. *Proof of (8.67) (accumulation).* Let $\bar{\gamma} \in \bar{\mathcal{G}}_{\varepsilon,\nu}$, $\bar{\gamma} \subset (\mathbb{R}/\mathbb{Z}) \times \,]0, \nu[$. From the definition (5.45) of the diffeomorphism h_ε, we see that, for some positive constant C_λ depending on λ (cf. condition (H3)),

$$\hat{\gamma} = h_\varepsilon^{-1}(\bar{\gamma}) \subset \{(x, y) \in \tilde{\mathcal{F}}_{\varepsilon,y_*}, \quad xy \in \,]0, C_\lambda \nu[\}.$$

However,

$$\hat{\Gamma} = \bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) = \bigg(\bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \cap V\bigg) \cup \bigcup_{\substack{n\in\mathbb{Z} \\ f_\varepsilon^n(\hat{\gamma})\not\subset V}} f_\varepsilon^n(\hat{\gamma}).$$

From condition (H3), one has

$$\bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \cap V \subset V \cap \{(x, y), \quad xy \in \,]0, C_\lambda \nu[\},$$

and hence, using Remark 2.2,

$$\mathrm{dist}\bigg(\bigcup_{n\in\mathbb{Z}} f_\varepsilon^n(\hat{\gamma}) \cap V, \Sigma_\varepsilon \cap V\bigg) = o_\nu(1) \quad (\text{uniform in } \hat{\gamma}). \tag{8.68}$$

Now, recalling the definition (4.29) of the integer N of §4.2, one has

$$\bigcup_{\substack{n\in\mathbb{Z} \\ f_\varepsilon^n(\hat{\gamma})\not\subset V}} f_\varepsilon^n(\hat{\gamma}) \subset \bigcup_{n=1}^{N} f_\varepsilon^{-n}(\hat{\gamma}),$$

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and using the fact that $\mathrm{dist}(\hat{\gamma}, \Sigma_\varepsilon \cap [(x_*, 0), f_\varepsilon(x_*, 0)[) = o_\nu(1)$, one can see that ($N$ is fixed)

$$\mathrm{dist}\bigg(\bigcup_{\substack{n\in\mathbb{Z}\\ f_\varepsilon^n(\hat{\gamma})\not\subset V}} f_\varepsilon^n(\hat{\gamma}), \Sigma_\varepsilon \cap \bigcup_{j=1}^{N} f_\varepsilon^{-1}([(x_*, 0), f_\varepsilon(x_*, 0)[)\bigg) = o_\nu(1), \tag{8.69}$$

where the previous limit is uniform in $\hat{\gamma}$.

Equations (8.68) and (8.69) give

$$\mathrm{dist}(\hat{\Gamma}, \Sigma_\varepsilon) = o_\nu(1).$$

8.3. *KAM circles for* f_ε. Let $\hat{\Gamma}$ be a C^r invariant curve for f_ε of the form (8.65) and $g_{\hat{\Gamma}}$ the restriction of f_ε to $\hat{\Gamma}$. The map $g_{\hat{\Gamma}}$ can be identified with a circle diffeomorphism. Similarly, the restriction of $\bar{f}_\varepsilon$ to the invariant curve $\bar{\gamma}$ yields a circle diffeomorphism $g_{\bar{\gamma}}$.

Let $\hat{\alpha}$ and $\bar{\alpha}$ be the rotation numbers of $g_{\hat{\Gamma}}$ and $g_{\bar{\gamma}}$.

LEMMA 8.1. *One has* $\{1/\hat{\alpha}\} = \bar{\alpha}$ *(here* $\{\cdot\}$ *denotes the fractional part).*

Proof. We refer to the renormalization procedure defined in §§4 and 5. Let $\hat{J}$ be the arc $\tilde{\mathcal{F}}_{\varepsilon,y_*} \cap \hat{\Gamma}$. The restriction on $\hat{J}$ of $\hat{f}_\varepsilon$, the first return map of f_ε in $\tilde{\mathcal{F}}_{\varepsilon,y_*}$, defines a C^r diffeomorphism of the abstract circle $\hat{J}/f_\varepsilon$. Classical arguments show that the rotation number of this circle diffeomorphism is equal to $\{1/\hat{\alpha}\}$. However, after normalization of f_ε by h_ε (cf. formula (5.46)), $\hat{\Gamma}$ is transported to $\bar{\gamma}$ and the C^r diffeomorphism $\hat{f}_\varepsilon : \hat{J}/f_\varepsilon \to \hat{J}/f_\varepsilon$ to the circle diffeomorphism $\bar{f}_\varepsilon : \bar{J}/T_1 \to \bar{J}/T_1$, where $\bar{J} = h_\varepsilon(\hat{J}) \subset \bar{\gamma}$ is a fundamental domain of $\bar{f} \mid \bar{\gamma}$. The rotation numbers of $\hat{f}_\varepsilon : \hat{J}/f_\varepsilon \to \hat{J}/f_\varepsilon$ and $\bar{f}_\varepsilon : \bar{J}/T_1 \to \bar{J}/T_1$ are hence equal. However, the rotation number of $\bar{f}_\varepsilon : \bar{J}/T_1 \to \bar{J}/T_1$ is (same argument as before) equal to $\{1/\bar{\alpha}\}$. □

Because $\bar{\alpha}$ can be chosen in a fixed Diophantine class $DC(\kappa, \tau)$ (see Remark 6.1), the rotation number $\hat{\alpha}$ is Diophantine with the *same* exponent τ. By the Herman–Yoccoz theorem on linearization of C^r-circle diffeomorphisms [**10**, **22**], this implies that if r is large enough (depending on τ which is fixed), the diffeomorphism $g_{\hat{\Gamma}}$ is *linearizable*; in other words, $\hat{\Gamma}$ is a *KAM* curve. However, one has *a priori* no control on the Diophantine constant of $\hat{\alpha}$.

This concludes the proof of Theorem 2.1, whence of Theorem A. □

9. *Proof of Theorem B*

We construct in §9.1 a symplectic diffeomorphism f_{pert} admitting a separatrix Σ (see Figure 5) and depending on a ('large') parameter M. We renormalize f_{pert} like in §§4 and 5 to get a diffeomorphism $\bar{f}_{\mathrm{pert}}$ of an open annulus $\mathbb{R}/\mathbb{Z} \times]0, c[$. We prove in Proposition 9.3 of §9.2 that this renormalized diffeomorphism $\bar{f}_{\mathrm{pert}}$ sends some graphs projecting on a fixed interval J_M (see (9.81) on graphs which project on the whole circle and which are below the initial graphs we have started from, see Figure 6. We then iterate this procedure in §9.3 to find an orbit of $\bar{f}_{\mathrm{pert}}$ accumulating the boundary $\mathbb{R}/\mathbb{Z} \times \{0\}$ of the aforementioned annulus: this prevents the existence of $\bar{f}_{\mathrm{pert}}$-invariant curves close

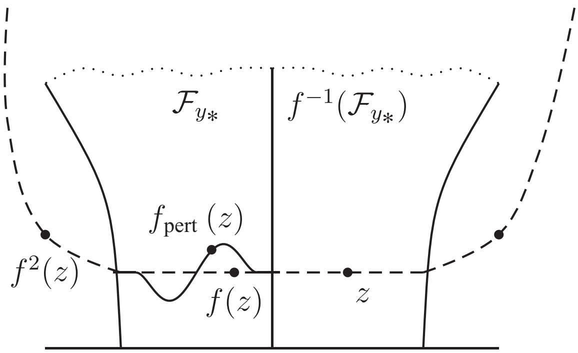

FIGURE 5. The perturbed map f_{pert}.

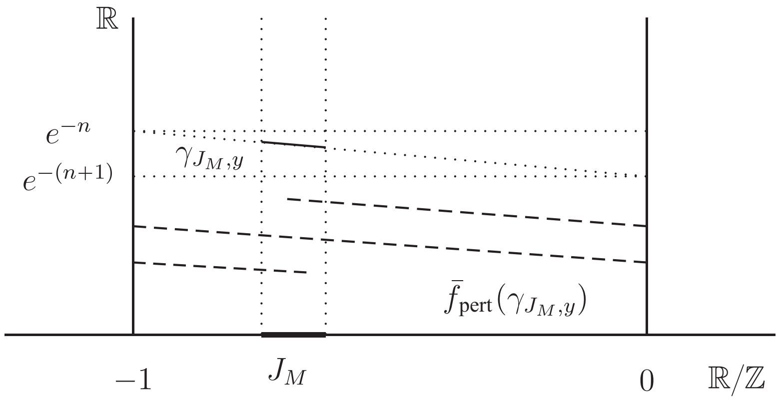

FIGURE 6. The image of the graph $\gamma_{J_M,y}$ by the diffeomorphism $\bar{f}_{\text{pert}}$.

to this boundary and therefore of f_{pert}-invariant curves close to the separatrix Σ. The diffeomorphism f_{pert} is the searched for example of Theorem B.

9.1. *Construction of the example.* We start with a smooth autonomous symplectic vector field of the form $X_0 = J\nabla H_0$, where $H_0 : \mathbb{R}^2 \to \mathbb{R}$ satisfies on some neighborhood V of $o = (0, 0)$

$$H_0(x, y) = xy \quad \text{on } V$$

and has the property that $\Sigma = H_0^{-1}(H_0(0, 0))$ is compact and connected. The set Σ is a separatrix of

$$f \underset{\text{defin.}}{=} \phi^1_{J\nabla H_0}$$

associated to the hyperbolic fixed point o.

Fixing $x_* > 0$ small enough, we can define like in §4, for $y_* > 0$ small enough, a fundamental domain $\tilde{\mathcal{F}}_{y_*} = \mathcal{F}_{y_*} \cup L_{y_*} \subset V$, where $\mathcal{F}_{y_*}$ is defined by $(a) - (d)$ (§4.1) with $\phi^1_{J\nabla H_0}$ in place of f_ε. We can even assume that $\phi^{-j}_{J\nabla H_0}(\tilde{\mathcal{F}}_{y_*}) \subset V$, $j = 1, 2$. There exists

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$c_* > 0$ such that the first return map,

$$\hat{f} : \tilde{\mathcal{F}}_{c_* y_*} \to \mathcal{F}_{y_*},$$

is well defined. We can renormalize $f = \phi^1_{J\nabla H_0}$ like in §5 by first normalizing f (cf. Lemma 5.3):

$$h \circ f \circ h^{-1} = T_1, \tag{9.70}$$

where

$$h : \mathcal{F}_{y_*} \to [0, 1[\times]0, c[\tag{9.71}$$

is symplectic (see (5.45), (5.41), and the fact that we choose $q(s) = s$) and then setting (cf. (5.46)):

$$\bar{f} \underset{\text{defin.}}{=} h \circ \hat{f} \circ h^{-1} : \mathbb{R}/\mathbb{Z} \times]0, \delta[\to \mathbb{R}/\mathbb{Z} \times]0, c[. \tag{9.72}$$

By (5.48) of Proposition 5.4, we have

$$\bar{f} = T_l, \quad l(y) = \sigma(y) - \ln y \tag{9.73}$$

for some smooth function σ.

We can assume that $h(\tilde{\mathcal{F}}_{y_*}) = [0, 1[\times]0, c[$ and that $T^{-j}([0, 1[\times]0, c[) \subset h^{-1}(V)$, $j = 1, 2$.

We now construct a symplectic perturbation $f_{\text{pert}} : \mathbb{R}^2 \to \mathbb{R}^2$ of f which admits Σ as a separatrix. We shall need first the following lemma.

LEMMA 9.1. *There exist $b \in (0, 1)$ and a non-empty compact interval $I \subset]0, 1[$ such that, for any $M > 0$, there exists a smooth function $\varphi_M : \mathbb{R} \to \mathbb{R}$ satisfying:*

(1) $\varphi_M |_I \le -bM$;

(2) $(b^{-1}M/|I|) \ge -\varphi'_M |_I \ge (M/|I|)$;

(3) *the map $s_M : \mathbb{R} \to \mathbb{R}$, defined by*

$$s_M(t) = \int_0^t e^{\varphi_M(u)} du,$$

is an increasing smooth diffeomorphism of $\mathbb{R}$ that coincides with the identity on $\mathbb{R} \setminus [0, 1]$.

Proof. See Appendix C. □

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function equal to 1 on $[-c/2, c/2]$ and to 0 on $\mathbb{R} \setminus [-(3/4)c, (3/4)c]$, and define

$$S_M(x, y) = (s_M(x)y)\chi(y) + xy(1 - \chi(y)). \tag{9.74}$$

The canonical (hence symplectic) mapping g_M associated to S_M:

$$g_M(x, y) = (\tilde{x}, \tilde{y}) \iff \begin{cases} x = \dfrac{\partial S_M}{\partial y}(\tilde{x}, y), \\ \tilde{y} = \dfrac{\partial S_M}{\partial \tilde{x}}(\tilde{x}, y), \end{cases} \tag{9.75}$$

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is equal to the identity on $(\mathbb{R}\setminus[0,1])\times[-c,c]$ and satisfies for $(x,y)\in[0,1[\times]0,c/2[$

$$\begin{cases}\tilde{x}=s_M^{-1}(x),\\ \tilde{y}=s'_M\circ s_M^{-1}(x)y.\end{cases}\tag{9.76}$$

The following symplectic perturbation of f:

$$\begin{aligned}f_{\mathrm{pert}}:\underset{\text{defin.}}{=}&\ h^{-1}\circ(g_M\circ T_1)\circ h\\ =&\ (h^{-1}\circ g_M\circ h)\circ f\end{aligned}$$

(recall h satisfies (9.70)) is thus defined on $\mathbb{R}^2$ and coincides with f outside $f^{-1}(\tilde{\mathcal{F}}_{y_*})$. Moreover, because $g_M(\mathbb{R}\times\{0\})=\mathbb{R}\times\{0\}$,

$$\Sigma\ \text{ is a separatrix for } f_{\mathrm{pert}}.$$

Now, because $\tilde{\mathcal{F}}_{y_*}$ is a fundamental domain for f_{pert} (f_{pert} coincide with f on $\tilde{\mathcal{F}}_{y_*}$), for some $c_{\mathrm{pert}}>0$ small enough, the first return map

$$\hat{f}_{\mathrm{pert}}:f_{\mathrm{pert}}^{-1}(\tilde{\mathcal{F}}_{c_{\mathrm{pert}}y_*})\to f_{\mathrm{pert}}^{-1}(\tilde{\mathcal{F}}_{y_*}),$$

is well defined and satisfies

$$\hat{f}_{\mathrm{pert}}=(\hat{f}\circ f^{-1})\circ f_{\mathrm{pert}}.$$

In particular, on

$$[-1,0[\times]0,c/2[,$$

one has (cf. (9.72), (9.73))

$$\bar{f}_{\mathrm{pert}}\underset{\text{defin.}}{=}h\circ\hat{f}_{\mathrm{pert}}\circ h^{-1}=\bar{f}\circ T_{-1}\circ g_M\circ T_1\tag{9.77}$$

$$=T_{l-1}\circ g_M\circ T_1\tag{9.78}$$

and

$$\bar{f}_{\mathrm{pert}}:\mathbb{R}\times]0,c/2[\ \to\mathbb{R}\times]0,c[\quad\text{satisfies}\quad\bar{f}_{\mathrm{pert}}\circ T_1=T_1\circ\bar{f}_{\mathrm{pert}};$$

in particular, it defines a smooth map $(\mathbb{R}/\mathbb{Z})\times]0,c/2[\ \to(\mathbb{R}/\mathbb{Z})\times]0,c[$.

Note that because g_M is the identity outside $[0,1]\times[-c,c]$, it admits a T_1-periodization $\tilde{g}_M:\mathbb{R}\times[-c,c]\to\mathbb{R}\times[-c,c]$ (which means that $\tilde{g}_M$ and g_M coincide on $[0,1]\times[-c,c]$ and $\tilde{g}_M$ commutes with T_1). This $\tilde{g}_M$ is defined by the same formula (9.75) as g_M, where now the new function $\tilde{s}_M$ involved in (9.74) is the $\mathbb{Z}$-periodization of s_M. To simplify the notation, we shall continue to denote $\tilde{g}_M$ and $\tilde{s}_M$ by g_M and s_M.

Let

$$t:=t(x):=s_M^{-1}(x+1).\tag{9.79}$$

LEMMA 9.2. *For $(x,y)\in[-1,0[\times]0,c/2[$, the point $(\bar{x},\bar{y}):=\bar{f}_{\mathrm{pert}}(x,y)$ satisfies with the notation (9.79)*

$$\begin{cases}\bar{x}=t-1+\sigma(s'_M(t)\times y)-\ln(s'_M(t))-\ln y,\\ \ln\bar{y}=\ln(s'_M(t))+\ln y.\end{cases}\tag{9.80}$$

Proof. Let $(x, y) \in [-1, 0[\times]0, c/2[$; with the notation $(x_1, y_1) = (g_M \circ T_1)(x, y) = g_M(x + 1, y)$, one has from (9.78), $(\bar{x}, \bar{y}) = T_{l-1}(x_1, y_1)$ and from (9.76), (9.73),

$$\begin{cases} x_1 = s_M^{-1}(x+1), \\ y_1 = s_M' \circ s_M^{-1}(x+1) \times y, \end{cases} \quad \text{and} \quad \begin{cases} \bar{x} = x_1 - 1 + \sigma(y_1) - \ln y_1, \\ \bar{y} = y_1, \end{cases}$$

and hence (9.80). □

9.2. *Image of a piece of graph by $\bar{f}_{\mathrm{pert}}$.* We take $M > 0$ (from Lemma 9.1) large enough and we define

$$J_M = s_M(I) - 1 \subset [-1, 0[, \tag{9.81}$$

where I is the interval introduced in Lemma 9.1.

If $y : J \to \mathbb{R}_+^*$, $x \to y(x)$ is a differentiable function, we denote by $\gamma_{J,y}$ its graph:

$$\gamma_{J,y} = \{(x, y(x)),\ x \in J\} \subset [-1, 0[\times]0, c/2[.$$

PROPOSITION 9.3. *There exists a constant $y_{\mathrm{pert}} > 0$ for which the following holds. Assume that $y : J_M \to]0, y_{\mathrm{pert}}[$, $x \mapsto y(x)$ is a differentiable function such that*

$$\text{for all } x \in J_M, \quad \left| \frac{d \ln y}{dx} + 1 \right| \le 1/2.$$

Then, $\bar{f}_{\mathrm{pert}}(\gamma_{J_M,y}) + (\mathbb{Z}, 0)$ contains the graph $\gamma_{[-1,0[,\bar{y}}$ of a differentiable function $\bar{y} : [-1, 0[\to \mathbb{R}_+^*$ (see Figure 6)

$$\gamma_{[-1,0[,\bar{y}} = \{(\bar{x}, \bar{y}(\bar{x})),\ \bar{x} \in [-1, 0[\},$$

such that

$$\text{for all } \bar{x} \in [-1, 0[, \quad \left| \frac{d \ln \bar{y}}{d\bar{x}} + 1 \right| \le 1/2; \tag{9.82}$$

$$\sup_{\bar{x} \in [-1,0[} \ln \bar{y}(\bar{x}) \le \sup_{x \in J_M} \ln y(x) - bM. \tag{9.83}$$

Moreover, for some interval $J_M^1 \subset J_M$, one has

$$\gamma_{[-1,0[,\bar{y}} = \bar{f}_{\mathrm{pert}}(\gamma_{J_1,y}). \tag{9.84}$$

We prove this proposition in §9.2.2.

9.2.1. *Preliminary results.* If we introduce the variable

$$\varphi = \ln(s_M'(t)) = \ln s_M' \circ s_M^{-1}(x+1) \quad (\text{recall } t = s_M^{-1}(x+1)),$$

we can write (9.80) as

$$(\bar{x}, \bar{y}) = f_{\mathrm{pert}}(x, y(x)) \iff \begin{cases} \bar{x} = t - 1 + \sigma(e^{\varphi} \times y(x)) - \varphi - \ln y(x), \\ \ln \bar{y} = \varphi + \ln y(x). \end{cases} \tag{9.85}$$

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Note that the maps $I \ni t \mapsto \varphi = \ln s'_M(t) \in \varphi(I)$ and $J_M \ni x \mapsto \varphi = \ln s'_M \circ s_M^{-1}(x+1) \in \varphi_M(I)$ are smooth diffeomorphisms. In particular, the maps $\varphi_M(I) \ni \varphi \mapsto \bar{x}$ and $\varphi_M(I) \ni \varphi \mapsto \ln y$, $\varphi_M(I) \ni \varphi \mapsto \ln \bar{y}$ are well defined and smooth.

LEMMA 9.4. *For any φ such that $t \in I$, one has*

$$\left|\frac{dt}{d\varphi}\right| \le |I|/M \le 1/4.$$

Proof. This follows from the identity (recall $\varphi = \ln(s'_M(t))$, $s'_M = e^{\varphi_M}$)

$$\frac{dt}{d\varphi} = \frac{1}{d\varphi/dt} = \frac{1}{\varphi'_M(t)}$$

and the estimates given by the second item of Lemma 9.1 (M is assumed to be large enough). □

LEMMA 9.5. *One has*

$$\sup_{\varphi_M(I)} \left|\frac{d\bar{x}}{d\varphi} + 1\right| \le 1/4, \tag{9.86}$$

$$\sup_{\varphi_M(I)} \left|\frac{d \ln \bar{y}}{d\varphi} - 1\right| \le 1/4. \tag{9.87}$$

Proof. Indeed, from (9.85),

$$\begin{aligned}\frac{d\bar{x}}{d\varphi} &= \frac{dt}{d\varphi} + e^{\varphi}\sigma'(e^{\varphi} \times y)\frac{dy}{d\varphi} - 1 - \frac{d \ln y}{d\varphi} \\ &= \frac{dt}{d\varphi} + ye^{\varphi}\sigma'(e^{\varphi} \times y)\frac{d \ln y}{d\varphi} - 1 - \frac{d \ln y}{d\varphi} \\ &= -1 + A\end{aligned}$$

with

$$A = \frac{dt}{d\varphi} + ye^{\varphi}\sigma'(e^{\varphi} \times y)\frac{d \ln y}{d\varphi} - \frac{d \ln y}{d\varphi}.$$

Note that (recall $x = s_M(t) - 1$, $s'_M = e^{\varphi_M}$)

$$\frac{d \ln y}{d\varphi} = \frac{d \ln y}{dx}\frac{dx}{dt}\frac{dt}{d\varphi} = \frac{d \ln y}{dx}e^{\varphi}\frac{dt}{d\varphi}$$

so, by Lemma 9.4,

$$|A| \le (|I|/M) + (ye^{-bM}\|\sigma'\|_0 + 1)e^{-bM}(|I|/M)\left|\frac{d \ln y}{dx}\right|$$

and if M is large enough,

$$|A| \le 1/4. \tag{9.88}$$

In a similar way,

$$\frac{d \ln \bar{y}}{d\varphi} = 1 + \frac{d \ln y}{dx}\frac{dx}{dt}\frac{dt}{d\varphi} = 1 + \frac{d \ln y}{dx}e^{\varphi}\frac{dt}{d\varphi} = 1 + B$$

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with

$$|B| \le 2e^{-bM} \times (1/4) \le 1/4 \quad (M \gg 1). \tag{9.89}$$

□

9.2.2. *Proof of Proposition 9.3.* From (9.86) of Lemma 9.5, we see that the map $\varphi_M(I) \ni \varphi \mapsto \bar{x} \in \mathbb{R}$ is a diffeomorphism onto its image $\bar{J}_M \subset \mathbb{R}$, and hence the maps $J_M \ni x \mapsto \bar{x} \in \bar{J}_M$ and $I \ni t \mapsto \bar{x} \in \bar{J}_M$ are diffeomorphisms. Note that from (9.86), one has

$$|\bar{J}_M| \ge (3/4)|\varphi_M(I)|$$

and from item (2) of Lemma 9.1, one has

$$|\bar{J}_M| \ge (3/4)(M/|I|) \times |I| > 2; \tag{9.90}$$

there thus exists an interval $J_M^1 \subset J_M$ such that the map $J_M^1 \ni x \mapsto \bar{x} \in n + [-1, 0[$ (for some $n \in \mathbb{Z}$) is a differentiable homeomorphism. Replacing $\bar{y}(\bar{x})$ by $\bar{y}(\bar{x} + n)$ shows (9.84).

We now prove (9.82): for $\bar{x} \in [-1, 0[$,

$$\left| \frac{d \ln \bar{y}}{d\bar{x}} + 1 \right| \le 1/2. \tag{9.91}$$

Indeed, let $I_1 \subset I$ be the image of $[0, 1[$ by $\bar{J}_M \ni \bar{x} \mapsto t \in I$; from Lemma 9.5, for any $\varphi \in \varphi_M(I_1)$, one has for some $A, B \in [0, 1/4]$

$$\frac{d\bar{x}}{d\varphi} = -1 + A, \quad \frac{d \ln \bar{y}}{d\varphi} = 1 + B,$$

so that

$$\left| \frac{d \ln \bar{y}}{d\bar{x}} + 1 \right| = \left| \left(\frac{d \ln \bar{y}}{d\varphi} \Big/ \frac{d\bar{x}}{d\varphi} \right) + 1 \right| = \left| \frac{1 + B}{-1 + A} + 1 \right| \le 1/2.$$

The preceding discussion shows that the map $\bar{y} : [-1, 0[\ni \bar{x} \mapsto \bar{y}(\bar{x})$ is a well-defined differentiable function, that its graph is included in $f_{\text{pert}}(\gamma_{J_M,y}) + (\mathbb{Z}, 0)$, and that (9.82) holds.

There remains to prove (9.83). By the second equality of (9.85), if $(\bar{x}, \bar{y}(\bar{x})) = f_{\text{pert}}(x, y)$, one has

$$\ln \bar{y}(\bar{x}) \le \ln y(x) - bM \le \sup_{x \in J_M} \ln y - bM$$

and as a consequence, because the map $J_M \supset J_M^1 \ni x \mapsto \bar{x} \in [-1, 0[$ is a bijection, (9.83) holds. □

9.3. *End of the proof of Theorem B.* We shall prove that if M is large enough, the diffeomorphism f_{pert} constructed in §9.1 provides the searched for example of Theorem B.

Let M be large enough and $y_0 \in \,]0, y_{\text{pert}}[$; we define the function

$$y_0 : [-1, 0[\,\to \mathbb{R}, \quad x \mapsto y_0 e^{-x}.$$

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Using inductively Proposition 9.3, we construct differentiable functions

$$y_n : [-1, 0[\ \to \mathbb{R}$$

such that, for every $n \in \mathbb{N}^*$,

$$\text{for all } x \in J_M, \quad \left| \frac{d \ln y_n}{dx} + 1 \right| \leq 1/2, \tag{9.92}$$

$$\gamma_{[-1,0[,y_n} \subset \bar{f}_{\text{pert}}(\gamma_{J_M,y_{n-1}}) + (\mathbb{Z}, 0), \tag{9.93}$$

$$\sup_{x\in[-1,0[} \ln y_n(x) \leq \sup_{x\in J_M} \ln y_{n-1}(x) - bM. \tag{9.94}$$

Inclusion (9.93) implies the existence of a decreasing sequence of non-empty compact intervals $K_n \subset J_M$ such that

$$\gamma_{[-3/4,-1/4],y_n} = \bar{f}^n_{\text{pert}}(\gamma_{K_n,y_0}) \mod (\mathbb{Z}, 0).$$

In particular, if $x_\infty \subset \bigcap_{n\in\mathbb{N}^*} K_n$, one has

$$\text{for all } n \in \mathbb{N}^*, \quad \bar{f}^n_{\text{pert}}((x_\infty, y_0)) \in \gamma_{[-3/4,-1/4],y_n} \subset \gamma_{[-1,0[,y_n} \mod (\mathbb{Z}, 0). \tag{9.95}$$

From (9.94),

$$\sup_{x\in[-1,0[} y_n(x) \leq e^{-nbM} y_0,$$

and hence, using (9.95), we see that $\bar{f}^n_{\text{pert}}((x_\infty, y_0))$ accumulates $\mathbb{R} \times \{0\}$:

$$\bar{f}^n_{\text{pert}}((x_\infty, y_0)) \in [-1, 0[\times]0, e^{-nbM} y_0[\mod (\mathbb{Z}, 0). \tag{9.96}$$

As a consequence of (9.77) and of the fact that, for some constant $C > 0$

$$\text{for all } \nu \in \ ]0, c[, \quad h^{-1}([-1, 0[\times]0, \nu[) \subset f^{-1}_{\text{pert}}(\hat{\mathcal{F}}_{C\nu})$$

(this is owing to the fact that the diffeomorphism h given by (9.71) is indeed defined on a neighborhood of $\tilde{\mathcal{F}}_{y_*}$), one has

$$\hat{f}^n_{\text{pert}}(h^{-1}(x_\infty, y_0)) \in f^{-1}_{\text{pert}}(\hat{\mathcal{F}}_{Ce^{-nbM} y_0}).$$

Because $\hat{f}_{\text{pert}}$ is the first return map of f_{pert} in $f^{-1}_{\text{pert}}(\hat{\mathcal{F}}_{y_*})$, there exists a sequence $(p_n)_{n\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$, $\lim_{n\to\infty} p_n = \infty$ such that

$$f^{p_n}_{\text{pert}}(h^{-1}(x_\infty, y_0)) \in f^{-1}_{\text{pert}}(\hat{\mathcal{F}}_{Ce^{-nbM} y_0}). \tag{9.97}$$

However, this last fact prevents the existence of invariant circles in Δ_Σ accumulating the separatrix Σ of f_{pert}. More precisely, let W be a neighborhood of Σ in $\Sigma \cup \Delta_\Sigma$ (we recall that Δ_Σ is the bounded connected component of $\mathbb{R}^2 \setminus \Sigma$), such that

$$h^{-1}(x_\infty, y_0) \notin W.$$

We claim that $W \setminus \Sigma$ does not contain any f_{pert}-invariant circle Γ. Indeed, if this were not the case, the topological annulus $\mathcal{A} \subset W$ having Σ and Γ for boundaries would be

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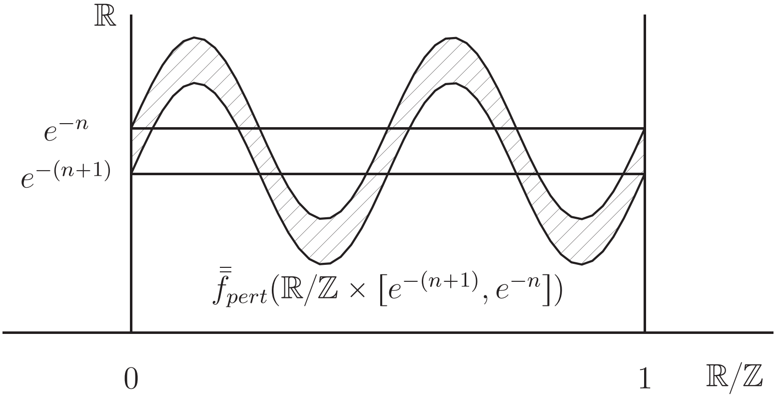

FIGURE 7. The diffeomorphism $\bar{\bar{f}}_{\text{pert}}$ on $\mathbb{R}/\mathbb{Z} \times [e^{-(n+1)}, e^{-n}]$. Compare with Figures 4 and 6.

f_{pert}-invariant (by topological degree theory). However, this is impossible because one would have at the same time

$$h^{-1}(x_\infty, y_0) \notin \mathcal{A} \quad \text{and} \quad f_{\text{pert}}^{p_n}(h^{-1}(x_\infty, y_0)) \in \mathcal{A}$$

for some large p_n (see (9.97)).

Remark 9.1. If we define the renormalization $\bar{\bar{f}}_{\text{pert}}$ of f_{pert} by considering the first return map of f_{pert} in $\mathcal{F}_{y_*}$ instead of $f_{\text{pert}}^{-1}(\mathcal{F}_{y_*})$, as we have done to construct $\bar{f}_{\text{pert}}$, the dynamics of $\bar{\bar{f}}_{\text{pert}}$ looks more like the one pictured in Figure 7. The comparison of this picture and that of Figure 4 illustrates the effect of the perturbative assumption in Theorem A.

Acknowledgments. R.K. wishes to thank Bassam Fayad and the referee for their thorough reading of a preliminary version of this paper and their very useful comments. A.K. was partially supported by NSF grant DMS 1602409. R.K. was supported by a Chaire d'Excellence LABEX MME-DII, the project ANR BEKAM: ANR-15-CE40-0001 and an AAP project from CY Cergy Paris Université.

A. *Appendix. Proof of Lemma 2.2*

We write for $j \geq 2$

$$H_\varepsilon^t(z) = \lambda_\varepsilon z_1 z_2 + \sum_{2\leq i\leq[j/2]} a_{\varepsilon,i} \times (z_1 z_2)^i + \sum_{\substack{i_1,i_2\in\mathbb{N}\\ i_1+i_2=j+1}} h_{\varepsilon,i_1,i_2}(t) z_1^{i_1} z_2^{i_2} + O^{j+2}(z),$$

where $a_{\varepsilon,i} \in \mathbb{R}$ and the $h_{\varepsilon,i_1,i_2}(\cdot)$ are smooth 1-periodic functions. We define

$$H_{\varepsilon,2}(z) = \lambda_\varepsilon z_1 z_2.$$

We first observe that if G_ε^t is a solution of

$$\begin{cases} G_\varepsilon^t(z) = O^{j+1}(z), \\ H_\varepsilon^t(z) + \partial_t G_\varepsilon^t(z) + \{G_\varepsilon^t, H_{\varepsilon,2}^t\}(z) = \tilde{q}_\varepsilon(z_1 z_2), \end{cases} \tag{A.98}$$

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for some $\tilde{q}_\varepsilon(u) = \lambda_\varepsilon u + \sum_{2\le i\le[(j+1)]/2} \tilde{a}_{\varepsilon,i} \times u^i$, $\tilde{a}_{\varepsilon,i} \in \mathbb{R}$, then G^t_ε solves (2.15). We then have to solve (A.98) for some $\tilde{q}_\varepsilon$ and some G^t_ε of the form

$$\tilde{q}_\varepsilon(u) = \lambda_\varepsilon u + \sum_{2\le i\le[(j+1)]/2} \tilde{a}_{\varepsilon,i} \times u^i$$
$$G^t_\varepsilon(z) = \sum_{i_1+i_2=j+1} g_{\varepsilon,i_1,i_2}(t) z_1^{i_1} z_2^{i_2} = O^{j+1}(z),$$

where the $g_{\varepsilon,i_1,i_2}(\cdot)$ are 1-periodic. This amounts to finding 1-periodic solutions to the equations

$$h_{\varepsilon,i_1,i_2}(t) + \partial_t g_{\varepsilon,i_1,i_2}(t) - \lambda_\varepsilon(i_1 - i_2) g_{\varepsilon,i_1,i_2}(t) = 0 \quad \text{if } i_1 \ne i_2, \tag{A.99}$$
$$h_{\varepsilon,i,i}(t) + \partial_t g_{\varepsilon,i,i}(t) = \tilde{a}_{\varepsilon,i} \quad \text{if } i_1 = i_2 = i, \tag{A.100}$$

for each couple $(i_1, i_2) \in \mathbb{N}^2$ such that $i_1 + i_2 = j + 1$. Note that in (A.100), this last equality occurs only if $j + 1$ is even and $i = (j + 1)/2$. Equation (A.100) is then easily solved by setting

$$\tilde{a}_{\varepsilon,i} = \int_{\mathbb{R}/\mathbb{Z}} h_{\varepsilon,i,i}(t)\, dt, \quad g_{\varepsilon,i,i}(t) = -\int_0^t (h_{\varepsilon,i,i}(s) - \tilde{a}_{\varepsilon,i})\, ds.$$

Equation (A.99) always admits unique 1-periodic solutions of the form

$$\begin{cases} g_{\varepsilon,i_1,i_2}(t) = e^{\lambda_\varepsilon(i_1-i_2)t} c_{\varepsilon,i_1,i_2} - \int_0^t e^{(t-s)\lambda_\varepsilon(i_1-i_2)} h_{\varepsilon,i_1,i_2}(s)\, ds, \\ \text{where} \quad c_{\varepsilon,i_1,i_2} = (e^{\lambda_\varepsilon(i_1-i_2)} - 1)^{-1} \int_0^1 e^{(1-s)\lambda_\varepsilon(i_1-i_2)} h_{\varepsilon,i_1,i_2}(s)\, ds. \end{cases}$$

In the preceding solutions, the dependence on ε is smooth and if, for $\varepsilon = 0$, the functions h_{0,i_1,i_2} do not depend on t, we see that g_{0,t_1,t_2} is a constant.

This concludes the proof of Lemma 2.2. □

B. *Appendix. Extension of symplectic diffeomorphisms*

LEMMA B.1. *Let $(\Theta_\varepsilon)_{\varepsilon\in\,]-\varepsilon_0,\varepsilon_0[}$ be a smooth (or continuous) family of C^k symplectic diffeomorphisms C^1-close to the identity, defined on some open disk $D(o, \delta)$ of $\mathbb{R}^2$, and such that $\Theta_\varepsilon(o) = o$. Then, there exists $(\tilde{\Theta}_\varepsilon)_{\varepsilon\in\,]-\varepsilon_0,\varepsilon_0[}$, a smooth (or continuous) family of C^k symplectic diffeomorphisms of $\mathbb{R}^2$ such that on $D(o, \delta/2)$, one has $\tilde{\Theta}_\varepsilon = \Theta_\varepsilon$.*

Proof. We use the notation $\Theta_\varepsilon(x, y) = (\tilde{x}, \tilde{y})$. Because Θ_ε is symplectic, the 1-form $\tilde{y}d\tilde{x} - ydx$ is closed and defined on a disk $D(o, 4\delta/5)$ of center o and radius $4\delta/5$ (we assume Θ_ε C^1-close to the identity so that we can use the implicit function theorem). It is hence locally exact and there exists a function $S_\varepsilon(y, \tilde{y})$ such that $\tilde{y}d\tilde{x} - ydx = dS_\varepsilon$. Now the function $F_\varepsilon(x, \tilde{y}) = -S_\varepsilon(y, \tilde{y}) + (\tilde{x} - x)\tilde{y}$ is defined on $D(0, 3\delta/4)$ and satisfies $(y - \tilde{y})dx + (\tilde{x} - x)d\tilde{y} = dF_\varepsilon$ or equivalently,

$$\Theta_\varepsilon(x, y) = (\tilde{x}, \tilde{y}) \iff \begin{cases} \tilde{x} = x + \partial_{\tilde{y}} F_\varepsilon(x, \tilde{y}), \\ y = \tilde{y} + \partial_x F_\varepsilon(x, \tilde{y}). \end{cases} \tag{B.101}$$

Note that we can choose $(F_\varepsilon)_\varepsilon$ as a C^k-family of C^{k+1}-functions such that $F_\varepsilon(o) = 0$, $DF_\varepsilon(o) = 0$.

We can then choose $\chi : \mathbb{R}^2 \to \mathbb{R}$ as a smooth function which is equal to 1 on $D(o, 2\delta/3)$ and 0 outside $D(o, 3\delta/4)$, set

$$\tilde{F}_\varepsilon = \chi \times F_\varepsilon,$$

and define $\tilde{\Theta}_\varepsilon$ by (B.101) with F_ε replaced by $\tilde{F}_\varepsilon$. The family of diffeomorphisms $(\tilde{\Theta}_\varepsilon)_\varepsilon$ is a smooth (or continuous) family of exact symplectic C^k-diffeomorphisms. □

C. *Appendix. Proof of Lemma 9.1*

Let $\chi : \mathbb{R} \to [0, 1]$ be a smooth even function with support in $[-1/2, 1/2]$ such that $\chi(0) = 1$ and which is increasing on $[-1/2, 0]$. There exists $\alpha \in\ ]0, 1/4[$ such that, for all $x \in\ ]-2\alpha, 2\alpha[$, one has $\chi(x) > 1/2$ and

$$\beta_{\min} := \min_{[-2\alpha,-\alpha]} \chi' > 0, \quad \beta_{\max} := \max_{[-2\alpha,-\alpha]} \chi' > 0.$$

We define for $\rho \in\ ]0, 1/12]$ and $C_M > 0$,

$$\varphi_M(x) = a(\rho, C_M)\chi\left(\frac{x - 1/3}{1/12}\right) - C_M \chi\left(\frac{x - 2/3}{\rho}\right),$$

where $a(\rho, C_M) > 0$ is chosen so that

$$\int_0^1 e^{\varphi_M(u)} du = 1.$$

Let $I = (2/3) +\ ]-2\alpha\rho, -\alpha\rho[$. For $x \in I$, one has

$$\varphi_M(x) \leq -C_M/2 = -C_M\alpha\beta_{\min}/(2\alpha\beta_{\min}),$$
$$\varphi'_M(x) \leq -(C_M/\rho)\beta_{\min} = -(C_M\alpha\beta_{\min})/(\alpha\rho) = -C_M\alpha\beta_{\min}/|I|,$$
$$\varphi'_M(x) \geq -(C_M/\rho)\beta_{\max} = -(C_M\alpha\beta_{\max})/(\alpha\rho) = -(\beta_{\max}/\beta_{\min})C_M\alpha\beta_{\min}/|I|.$$

Fixing ρ (for example $\rho = 1/12$) and taking

$$b^{-1} = \max\left(\frac{\beta_{\max}}{\beta_{\min}}, 2\alpha\beta_{\min}\right), \quad C_M = \frac{M}{\alpha\beta_{\min}},$$

provides the first two items of Lemma 9.1.

Let us check the third item is satisfied. From the definition of s_M, one has $s'_M(x) = e^{\varphi_M(x)} = 1$ for $x \notin [0, 1]$. Because $s_M(0) = 0$, one has $s_M(x) = x$ for $x \leq 0$. Similarly, because

$$s_M(1) = \int_0^1 e^{\varphi_M(u)} du = 1,$$

we have $s_M(x) = x$ for $x \geq 1$.

Because in any case $s'(x) > 0$, this concludes the proof of Lemma 9.1. □

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