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Distributed adaptive control strategy for flexible link manipulators

Fareh Raouf†*, Saad Mohamad‡, Saad Maarouf§, and Bettayeb Maamar†

†*Department of Electrical and Computer Engineering, University of Sharjah, P.O. Box 27272, Sharjah, United Arab Emirates. E-mail: maamar@sharjah.ac.ae*
‡*School of Engineering, Université du Québec en Abitibi-Témiscamingue, 445, boul. de l'Université, Rouyn-Noranda, Quebec J9X 5E4, Canada. E-mail: mohamad.saad@uqat.ca*
§*Department of Electrical Engineering, École de technologie supérieure, Université du Québec, 1100 Notre-Dame West, Montreal, Quebec H3C 1K3, Canada. E-mail: maarouf.saad@etsmtl.ca*

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SUMMARY
This paper presents an adaptive distributed control strategy for n-serial-flexible-link manipulators. The proposed adaptive controller is used for flexible-link-manipulators: (1) to solve the tracking control problem in the joint space, and (2) to reduce vibrations of the links. The dynamical model of flexible link manipulators is reorganized to take the form of n interconnected subsystems. Each subsystem has a one-joint and one-link pair. The system parameters are deemed to be unknown. The adaptive distributed strategy controls one subsystem in each step, starting from the last one. The nth subsystem is controlled by assuming that the remaining subsystems are stable. Then, proceeding backward to the (n-1)th system, the same strategy is applied, and so on, until the first subsystem is reached. The gradient-based estimator is used to estimate the parameters of each subsystem. The control law of the ith subsystem uses its own estimated parameters and the estimated parameters of all upper level subsystems. The global stability of the error dynamics is proved using Lyapunov approach. This algorithm was implemented in real time on a two-flexible-link manipulator, and a comparison with the non-adaptive version shows the effectiveness of this approach.

KEYWORDS: Distributed control, Adaptive control, Flexible-link manipulators, Error dynamics, Stability

1. Introduction

Flexible link manipulators present some inherent advantages over conventional rigid robots, such as lower energy consumption, faster response, relatively smaller actuators, higher payload-to-weight ratio and lower overall cost. They can be found in a large diversity of applications, including nuclear maintenance, microsurgery, space robots, contouring control, collision control, pattern recognition and many others. Due to all these advantages and applications, the control of flexible link manipulators has received considerable attention in the literature.[1,2] They are multivariable systems, and their dynamics are strongly coupled and highly nonlinear. The dynamical model of flexible manipulators can be considered as one multi-input and multi-output (MIMO) system, and one controller is considered for all links and joints. Many control strategies are proposed in the literature using this configuration. When the system parameters are known, linear control,[3–5] feedback linearization control,[6–8] and sliding-mode control[9–12] have been proposed for flexible link manipulators control problems. A proportional, integral, derivative (PID) controller was used in ref. [5] to control the flexible manipulator. The controller is presented as a general second-order linear regulator and its parameters are systematically chosen by pole placement. A feedback control using a conventional

* Corresponding author. E-mail: rfareh@sharjah.ac.ae

motor with a gear actuator affected by nonlinear friction torque is presented in ref. [8] to solve the tracking problem of a flexible manipulator in joint space. A sliding mode controller is proposed in ref. [10] for a two-link flexible manipulator to control the end-point position. An inverse dynamics terminal sliding mode control strategy is proposed. Low-order adaptive controllers were applied to large scale examples in ref. [13] and had led to successful implementations of simple direct adaptive control. Decentralized simple adaptive control of nonlinear systems was studied in ref. [14]. The passivity results for linear time-invariant systems were extended to non-linear and non-stationary systems, thus guaranteeing stability of adaptive control of non-linear square systems. Adaptive control is used when the system parameters are unknown.[15–17] Non-adaptive and adaptive controllers are proposed in ref. [17] for a one-flexible link manipulator to track desired trajectories in the joint space. The asymptotic stability of the error dynamics is proved using Lyapunov theory. For a four-link flexible manipulator, a new robust control is proposed in ref. [18]. Indeed, an $H\infty$ controller for regional pole placement is designed for the uncertain linearized system. The controller is evaluated by simulation, and tests on an experimental manipulator show that the proposed controller performs better than an LQ controller. Intelligent control methods have been applied to flexible link manipulators. A two-stage direct fuzzy logic controller was developed in ref. [19]. One stage controls the rigid motion, while the second controls the flexible deformations and modifies the output of the first stage to reduce the induced vibrations. The latter approach was taken in order to reduce the number of rules needed in the fuzzy knowledge base controller. An intelligent proportional integral (iPI) controller was presented in refs. [20, 21] to control the extremity of a flexible-link manipulator. The authors conclude in ref. [20] that the PI and PID controllers produce better performance when the step and square-wave inputs are applied to a flexible manipulator, but the PI controller yielded better trajectory tracking performance.

When using the dynamical model as one MIMO system, all joints and links are controlled by a single controller. In general, the control algorithm is a single one, but not so the control actions, since the control gains may be different for each link. In this case, the control structure becomes more complex and the real-time implementation in industrial applications is not easy.[22] To overcome this problem, the dynamical model of a flexible manipulator can be viewed as an interconnection of multiple subsystems, where each subsystem is controlled using one controller. Using this configuration, decentralized controls were proposed in refs. [23-26] for flexible link manipulators. An indirect adaptive decentralized control for a class of two time-scale interconnected systems is proposed in ref. [23]. The dynamics of the slow subsystem are developed using the integral manifold method and the dynamics of the fast modes are presented by the fast subsystem. The adaptive controller uses the effects of unmodeled dynamics, identification errors and parameter variations. The virtual decomposition control (VDC) was used in ref. [26] for serial-chain manipulator. Every subsystem is controlled by its own VDC and the regressor-based adaptive control is used to estimate the unknown parameters. The distributed control strategy was also used for rigid manipulators[27] in order to track a desired trajectory in the workspace. The distributed control strategy consists in controlling the last subsystem while assuming that the remaining subsystems are stable. Then, going backward to the last-but-one subsystem, the same strategy is applied, and so on until the first one. This distributed control strategy is modified to take into account the flexibility of the links.[28,29]. Contrary to rigid manipulators, flexible manipulators are under-actuated systems, i.e., the deflection variables are not actuated. In this case, a subsystem has two parts: in addition to a joint as in a rigid manipulator, it must also include the corresponding flexible link. In ref. [29], the distributed control is applied to the two-flexible-link manipulator to track desired trajectories in the workspace. The redefinition output technique has been used to select a non-collocated output, ensuring the stability of internal dynamics. Only local stability was proved for this control strategy. An extension of this approach was proposed and tested on the two-flexible-link manipulator in order to ensure the global stability of the tracking error, the desired track trajectory in the joint space, and to minimize the links' vibration.[28] In these previous works, the distributed control strategy was applied assuming that the system parameters were perfectly known.

In this paper, the system parameters are assumed to be unknown, and a distributed adaptive control strategy is developed for flexible link manipulators. First, the dynamical model is reorganized to take the form of n interconnected subsystems, using a nonsingular transformation matrix. In this form, each subsystem includes one joint and its corresponding flexible link. Then, the distributed adaptive control strategy is used from the last subsystem proceeding backwards until the first one. Each subsystem is controlled while assuming that the remaining subsystems are stable and follow their desired trajectories. Since each subsystem contains its own parameters and the parameters of upper level subsystems, the unknown parameters of the last subsystem are first estimated and then used in the lower level (n-1)th subsystem. The adaptive control law of each subsystem is developed using its own estimated parameters and the parameters that were estimated in the previous steps. Lyapunov's theory is used to conclude the global stability. A real-time implementation on a two-flexible-link manipulator is given as an example of the proposed control strategy in order to show the effectiveness of this method.

The paper is organized as follows: Section 2 presents the modeling and problem formulation. The distributed adaptive control strategy of a multi-flexible-link manipulator is presented in Section 3. Section 4 presents the stability analysis of the errors dynamics. In Section 5, the proposed control strategy is applied to a two-flexible-link manipulator and experimental results are shown. Finally, conclusions are given in Section 6.

2. Modeling and Problem Formulation

The n-flexible-link manipulator is shown in Fig. 1. The links are presented in a cascade form and actuated with individual motors. An inertial payload is clamped to the end effectors. The motion of each link is assumed to be in the horizontal plane, and has a very small deflection. Using Lagrange equations, the dynamical model of an n DOF flexible manipulator is given by ref. [30]:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + Kq = L\tau, \tag{1}$$

where M is the inertia and mass matrix, $C(q,\dot{q})\dot{q}$ is the Coriolis and centrifugal forces vector, D is the friction matrix, K is the rigidity matrix, and L is the input matrix. q represents the vector of the generalized coordinates and τ is the vector of the applied torques. For n rigid coordinates and n flexible links, the deformation of the ith flexible link is given as follows:

$$v_i(x,t) = \sum_{j=1}^{z_i} \varphi_{ij}(x) q_{fij}(t) \quad i = 1,\ldots,n, \tag{2}$$

where q_{fij} is the jth generalized flexible coordinate, $\varphi_{ij}(x)$ is its jth shape function, and z_i is the number of the retained flexible modes of the ith flexible link. The total number of the flexible modes is $z = \sum_{i=1}^{n} z_i$ and the number of the rigid modes is n. Note that in terms of actual real-time implementation of the dynamics Eq. (1), it would be worth mentioning that O(n) methods exist for forward and inverse dynamics computations.[31] One approach to apply these methods to flexible-link manipulator dynamics equations is to discretize the flexible links into many small rigid-bodies connected by flexible elements. In such cases (as well as in cases of manipulators with many rigid links), this O(n) complexity is very important since it saves significantly on the O(n^3) or O(n^4) that would result from the direct implementation of the dynamics equations, as done in following sections.

Usually, the dynamical model (1) is written as an interconnection of rigid and flexible parts as follows:[30]

$$\begin{bmatrix} M_r & M_{rf} \\ M_{rf}^T & M_f \end{bmatrix}\begin{bmatrix} \ddot{q}_r \\ \ddot{q}_f \end{bmatrix} + \begin{bmatrix} C_r & C_{rf} \\ C_{rf}^T & C_f \end{bmatrix}\begin{bmatrix} \dot{q}_r \\ \dot{q}_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & D_f \end{bmatrix}\begin{bmatrix} \dot{q}_r \\ \dot{q}_f \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & K_f \end{bmatrix}\begin{bmatrix} q_r \\ q_f \end{bmatrix} = \begin{bmatrix} \tau \\ 0 \end{bmatrix}, \tag{3}$$

where M_r and M_f are the mass and inertia matrices for the rigid and the flexible parts, respectively. M_{rf} is a coupled element. The same decomposition is used for the Coriolis matrix $C(q,\dot{q})$. K_f is the stiffness diagonal matrix and D_f is the damping diagonal matrix of the flexible part. The subscripts r and f denote the rigid and flexible modes. In this work, we consider only the first flexible mode of each link, then: $z_i = 1$ and $z = n$.

The generalized coordinate ($q = [q_r^T \; q_f^T]^T$) includes the rigid coordinates in the first n elements and the remaining n elements are the flexible coordinates. In this paper, to develop the distributed control law, we need to reorganise the elements given in the generalized coordinates and the dynamical model to take the form of n interconnected subsystems. Each subsystem has a joint/link pair. Then,

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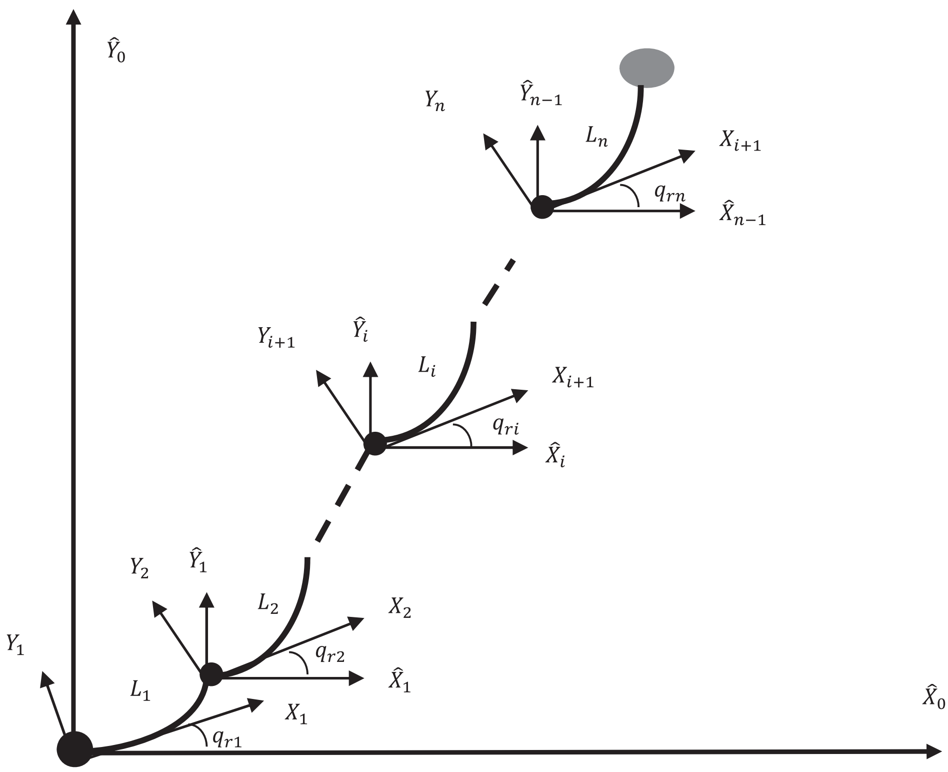

Fig. 1. Flexible-link manipulator.

there exists a non-singular matrix of transformation T_r such as

$$\bar{q} = T_r q, \tag{4}$$

where $q = [q_r^T \ q_f^T]^T = [q_{r1} \cdots q_{rn} \ \ q_{f1} \cdots q_{fn}]^T$ is the original generalized coordinate vector, $\bar{q} = [q_1 \ \cdots \ q_i \ \cdots \ q_n]^T = \Big[\underbrace{q_{r1}\, q_{f1}}_{q1} \cdots \underbrace{q_{ri}\, q_{fi}}_{qi} \cdots \underbrace{q_{rn}\, q_{fn}}_{q_n}\Big]^T$ is the new one, $q_i = [q_{ri} \ q_{fi}]^T$ is the generalized coordinate associated with the ith subsystem (ith joint and link), and the transformation matrix T_r is given by:

$$T_r = \begin{array}{c} \\ q_1 \\ \\ q_i \\ \\ q_n \end{array} \left[\begin{array}{cccccccccc c cccccccccc}
\overset{1}{} & \overset{2}{} & & & \overset{i}{} & & & & & \overset{n}{} & & \overset{n+1}{} & & & & \overset{n+i}{} & & & & & \overset{2n}{} \\
1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & \vdots & 1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
 & & & & & & & & & & \vdots & & & & & & & & & & \\
0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\
 & & & & & & & & & & \vdots & & & & & & & & & & \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1 & \vdots & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & 0 & 1
\end{array}\right], \tag{5}$$

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Using (4), the dynamical model (1) can be written as follows:

$$M\,(q)\,T_r^{-1}\ddot{\bar{q}} + C\,(q,\dot{q})\,T_r^{-1}\dot{\bar{q}} + DT_r^{-1}\dot{\bar{q}} + KT_r^{-1}\bar{q} = L\tau, \tag{6}$$

where $\ddot{\bar{q}} = T_r\ddot{q}$ and $\dot{\bar{q}} = T_r\dot{q}$.

Equation (6) is equivalent to the following expression:

$$\bar{M}\,(q)\,\ddot{\bar{q}} + \bar{C}\,(q,\dot{q})\,\dot{\bar{q}} + \bar{D}\dot{\bar{q}} + \bar{K}\bar{q} = \bar{L}\tau, \tag{7}$$

where $\bar{M}(q) = T_r M(q) T_r^{-1}$; $\bar{C}(q,\dot{q}) = T_r C(q,\dot{q}) T_r^{-1}$; $\bar{D} = T_r D T_r^{-1}$; $\bar{K} = T_r K T_r^{-1}$, $\bar{L} = T_r L$, and $\bar{q}$ is given previously.

The dynamical model (7) takes the form of n interconnected subsystems given as follows:

$$\begin{bmatrix} \bar{M}_1^T\,(q) \\ \vdots \\ \bar{M}_n^T\,(q) \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} \bar{C}_1^T\,(q,\dot{q}) \\ \vdots \\ \bar{C}_n^T\,(q,\dot{q}) \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} \bar{D}_1^T \\ \vdots \\ \bar{D}_n^T \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{bmatrix} + \begin{bmatrix} \bar{K}_1^T \\ \vdots \\ \bar{K}_n^T \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} \bar{L}_1 \\ \vdots \\ \bar{L}_n \end{bmatrix}, \tag{8}$$

where $q_i = [q_{ri}\ q_{fi}]^T$; $\bar{M}_i^T = [\bar{M}_{i1}\ \ldots\ \bar{M}_{ii}\ \ldots\ \bar{M}_{in}]_{2\times 2n}$; $\bar{C}_i^T = [\bar{C}_{i1}\ \ldots\ \bar{C}_{ii}\ \ldots\ \bar{C}_{in}]_{2\times 2n}$ $\bar{D}_i^T = [0\ \ldots\ 0\ \bar{D}_{ii}\ 0\ \ldots\ 0]_{2\times 2n}$; $\bar{K}_i^T = [0\ \ldots\ 0\ \bar{K}_{ii}\ 0\ \ldots\ 0\,]_{2\times 2n}$, $\bar{M}_{ij} = \begin{bmatrix} M_{r\bar{M}_{ij}} & M_{rf\bar{M}_{ij}} \\ M_{rf\bar{M}_{ij}} & M_{f\bar{M}_{ij}} \end{bmatrix}_{2\times 2}$; $\bar{C}_{ij} = \begin{bmatrix} C_{r\bar{C}_{ij}} & C_{rf\bar{C}_{ij}} \\ C_{rf\bar{C}_{ij}} & C_{f\bar{C}_{ij}} \end{bmatrix}$; $\bar{D}_{ii} = \begin{bmatrix} 0 & 0 \\ 0 & D_{fi} \end{bmatrix}$; $\bar{K}_{ii} = \begin{bmatrix} 0 & 0 \\ 0 & K_{fi} \end{bmatrix}$; $\bar{L}_i = \begin{bmatrix} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}_{2\times n}$.

Let us define the sliding surface as follows:

$$s = \begin{bmatrix} s_r \\ s_f \end{bmatrix} = \begin{bmatrix} \dot{\tilde{q}}_r + \lambda_r\tilde{q}_r \\ \dot{\tilde{q}}_f + \lambda_f\tilde{q}_f \end{bmatrix} = \begin{bmatrix} \dot{q}_{rd} - \dot{q}_r + \lambda_r\tilde{q}_r \\ \dot{q}_{fd} - \dot{q}_f + \lambda_f\tilde{q}_f \end{bmatrix} = \begin{bmatrix} \dot{u}_r - \dot{q}_r \\ \dot{u}_f - \dot{q}_f \end{bmatrix} = \dot{u} - \dot{q}, \tag{9}$$

where $s = [s_r^T\ s_f^T]^T = [s_{r1}\cdots s_{rn}\ \ s_{f1}\cdots s_{fn}]^T$. λ_r and λ_f are two positive gains; $\tilde{q}_r$, $\tilde{q}_f$ are errors of rigid and flexible parts, respectively.

Using the transformation matrix, a new form of the sliding surface can be written as follows:

$$\bar{s} = T_r s = \begin{bmatrix} s_1 & \cdots & s_i & \cdots & s_n \end{bmatrix}^T = \begin{bmatrix} s_{r1} & s_{f1} & \cdots & s_{ri} & s_{fi} & \ldots & s_{rn} & s_{fn} \end{bmatrix}^T. \tag{10}$$

The same idea is used for u, then $\dot{u} = [\dot{u}_r\ \dot{u}_f]^T = [\dot{u}_{r1}\cdots\dot{u}_{rn}\ \ \dot{u}_{f1}\cdots\dot{u}_{fn}]^T$ and

$$\dot{\bar{u}} = T_r\dot{u} = \begin{bmatrix} \dot{u}_1 & \cdots & \dot{u}_i & \cdots & \dot{u}_n \end{bmatrix}^T = \begin{bmatrix} \dot{u}_{r1} & \dot{u}_{f1} & \cdots & \dot{u}_{ri} & \dot{u}_{fi} & \ldots & \dot{u}_{rn} & \dot{u}_{fn} \end{bmatrix}^T. \tag{11}$$

The properties that will be used in the control law development can be deduced from the dynamical model as follows:

P1: M, M_{rr}, M_{ff}, D_{ff} and K_{ff} are positive definite matrices.

P2: The inertia-mass matrix $M(q)$ and the Coriolis matrix $C(q,\dot{q})$ satisfy the following skew-symmetric property:[30]

$$X^T\left(\dot{M}(q,\dot{q}) - 2C(q,\dot{q})\right)X = 0 \quad \forall X \in \mathcal{R}^{(n+z)}. \tag{12}$$

Let the desired trajectory associated with the rigid part of the ith subsystem, its first- and second-order derivatives, be $q_{rdi}(t)$, $\dot{q}_{rdi}(t)$ and $\ddot{q}_{rdi}(t)$, respectively, and $q_{fdi}(t)$, $\dot{q}_{fdi}(t)$ and $\ddot{q}_{fdi}(t)$ are those associated with the flexible part of this subsystem. The objective is to track the desired trajectories in the joint space and to reduce the vibrations of the links. The desired positions of the flexible modes are then set to zero.

3. Adaptive Distributed Control Strategy

This section presents the development of a distributed adaptive control strategy for an n-flexible-link manipulator to track-desired trajectories in the joint space while reducing links vibrations. The distributed control strategy consists in controlling the flexible manipulator, starting from the last subsystem and then working backward until the first one. Each subsystem is controlled by assuming that the remaining ones are stable. For the nth subsystem, the equation of motion is derived from (8) as follows:

$$\bar{M}_n^T(q)\ddot{\bar{q}} + \bar{C}_n^T(q,\dot{q})\dot{\bar{q}} + \bar{D}_n^T\dot{\bar{q}} + \bar{K}_n^T\bar{q} = \bar{L}_n\tau, \tag{13}$$

where $\bar{M}_n^T = [\bar{M}_{n1}\ \ldots\ \bar{M}_{ni}\ \ldots\ \bar{M}_{nn}]_{2\times 2n}$; $\bar{C}_n^T = [\bar{C}_{n1}\ \ldots\ \bar{C}_{ni}\ \ldots\ \bar{C}_{nn}]_{2\times 2n}$; $\bar{D}_n^T = [0_{2\times 2}\ \ldots\ 0_{2\times 2}\ \bar{D}_{nn}]_{2\times 2n}$; $\bar{K}_n^T = [0_{2\times 2}\ \ldots\ 0_{2\times 2}\ \bar{K}_{nn}]_{2\times 2n}$ and $\bar{L}_n = [\begin{smallmatrix} 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{smallmatrix}]_{2\times n}$.

The generalized coordinate associated with the nth subsystem is given by:

$$\bar{Q}_n = \begin{bmatrix} q_{rd1} & q_{fd1} & \cdots & q_{rd(n-1)} & q_{fd(n-1)} & q_{rn} & q_{fn} \end{bmatrix}^T = \begin{bmatrix} q_{d1} & \cdots & q_{d(n-1)} & q_n \end{bmatrix}^T. \tag{14}$$

Note that the rigid and flexible coordinates of the nth subsystem are the controlled ones, while the rigid and flexible coordinates of the remaining subsystems are the desired ones. Using the new coordinate, the corresponding equation of motion becomes:

$$\bar{M}_n^T(\bar{Q}_n)\ddot{\bar{Q}}_n + \bar{C}_n^T\left(\bar{Q}_n,\dot{\bar{Q}}_n\right)\dot{\bar{Q}}_n + \bar{D}_n^T\dot{\bar{Q}}_n + \bar{K}_n^T\bar{Q}_n = \bar{L}_n\tau, \tag{15}$$

where the velocity $\dot{\bar{Q}}_n$ is the time derivative of $\bar{Q}_n$ and the acceleration $\ddot{\bar{Q}}_n$ is the time derivative of $\dot{\bar{Q}}_n$.

There exists a vector $p_n = [\begin{smallmatrix} p_{nr} \\ p_{nf} \end{smallmatrix}] \in \Re^{b_{nr}+b_{nf}}$ with components depending on the manipulators' parameters (masses, moments in inertia, etc.), such as:

$$W_n\left(\bar{Q}_n,\dot{\bar{Q}}_n,t\right)p_n + RM_n = \bar{M}_n^T(\bar{Q}_n)\ddot{u} + \bar{C}_n^T\left(\bar{Q}_n,\dot{\bar{Q}}_n\right)\dot{u} + \bar{D}_n^T\dot{u}, \tag{16}$$

where $\dot{u}$ is given in (11), and $W_n = \left[\begin{smallmatrix} W_{nr} & 0 \\ 0 & W_{nf} \end{smallmatrix}\right]$ is the regressor matrix which contains all known functions, $W_{nr}^T \in \Re^{b_{nr}}$ and $W_{nf}^T \in \Re^{b_{nf}}$, b_{nr} and b_{nf} are the number of unknown parameters of rigid and flexible parts. RM_n is the remaining term, which is independent of the parameters p_n.

The control law of the last subsystem can be proposed as follows:

$$\tau_n = K_{dnr}s_{nr} + T_n + W_{nr}\hat{p}_{nr} + RM_{nr} - \delta\tau_{nr}, \tag{17}$$

where

$$T_n = \begin{cases} \dfrac{s_{nf}\left(K_{dnf}s_{nf} + K_{fn}\tilde{q}_{fn} + W_{nf}\hat{p}_{nf} + RM_{nf} + \delta\tau_{nf}\right)}{s_{nr}} \\ 0 \end{cases}, \tag{18}$$

$$W_n\hat{p}_n = \begin{bmatrix} W_{nr}\hat{p}_{nr} \\ W_{nf}\hat{p}_{nf} \end{bmatrix} = \hat{\bar{M}}_n^T(\bar{Q}_n)\ddot{u} + \hat{\bar{C}}_n^T\left(\bar{Q}_n,\dot{\bar{Q}}_n\right)\dot{u} + \hat{\bar{D}}_n^T\dot{u}, \tag{19}$$

$$\delta\tau_n = \begin{bmatrix} \delta\tau_{nr} \\ \delta\tau_{nf} \end{bmatrix} = \left(\delta\bar{M}_n^T\right)\ddot{u} + \left(\delta\bar{C}_n^T\right)\dot{u}, \tag{20}$$

and $K_{dn} = [\begin{smallmatrix} K_{dnr} & 0 \\ 0 & K_{dnf} \end{smallmatrix}]$; $\delta M_{nj} = \sum_{j=1}^{n-1} \frac{\partial M_{nj}(q)}{\partial q_j}|_{q_{jd}}\tilde{q}_j + \mathcal{O}_{M_{nj}}(\tilde{q}_j)\,\delta C_{nj} = \sum_{j=1}^{n-1} \frac{\partial C_{nj}(q,\dot{q})}{\partial q_j}|_{q_{jd}}\tilde{q}_j + \sum_{j=1}^{n-1} \frac{\partial C_{nj}(q,\dot{q})}{\partial \dot{q}_j}|_{q_{jd}}\dot{\tilde{q}}_j + \mathcal{O}_{C_{nj}}(\tilde{q}_j,\dot{\tilde{q}}_j)\,\mathcal{O}_{M_{ij}}$ and $\mathcal{O}_{C_{ij}}$ are the high-order terms of the Taylor series for $M_{ij}(q)$ and $C_{ij}(q)$, respectively.

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The adaptive laws for the rigid and flexible parts are given as follows:

$$\begin{aligned} \dot{\hat{p}}_{nr} &= K_{vnr} W_{nr}^T s_{nr}, \\ \dot{\hat{p}}_{nf} &= K_{vnf} W_{nf}^T s_{nf}, \end{aligned} \tag{21}$$

where K_{vnr} and K_{vnr} are some positive definite gains.

The same strategy is used backward for the remaining subsystems. Taking for example the ith subsystem, the equation of motion using the new associated coordinate is given as follows:

$$\bar{M}_i^T \left(\bar{Q}_i\right) \ddot{\bar{Q}}_i + \bar{C}_i^T \left(\bar{Q}_i, \dot{\bar{Q}}_i \dot{\bar{Q}}_i\right) \dot{\bar{Q}}_i + \bar{D}_i^T \dot{\bar{Q}}_i + \bar{K}_i^T \bar{Q}_i = \bar{L}_i \tau, \tag{22}$$

where $\bar{M}_i^T = [\, \bar{M}_{i1} \ \cdots \ \bar{M}_{ii} \ \ldots \ \bar{M}_{in} \,]^T$; $\bar{C}_i^T = [\, \bar{C}_{i1} \ \cdots \ \bar{C}_{ii} \ \ldots \ \bar{C}_{in} \,]^T$; $\bar{L}_i = [\begin{smallmatrix} 0 \ \ldots 010 \cdots 0 \\ 0 \ \ldots 000 \cdots 0 \end{smallmatrix}]_{2\times n}$; $\bar{D}_i^T = [\, \bar{D}_{i1} \ \cdots \ \bar{D}_{ii} \ \ldots \ \bar{D}_{in} \,]^T$; $\bar{K}_i^T = [\, \bar{K}_{i1} \ \cdots \ \bar{K}_{ii} \ \ldots \ \bar{K}_{in} \,]^T$; $\bar{M}_{ii} = [\begin{smallmatrix} M_{r_{\bar{M}_{ii}}} & M_{rf_{\bar{M}_{ii}}} \\ M_{rf_{\bar{M}_{ii}}} & M_{f_{\bar{M}_{ii}}} \end{smallmatrix}]_{2\times 2}$; $\bar{C}_{ii} = [\begin{smallmatrix} C_{r_{\bar{C}_{ii}}} & C_{rf_{\bar{C}_{ii}}} \\ C_{rf_{\bar{C}_{ii}}} & C_{f_{\bar{C}_{ii}}} \end{smallmatrix}]_{2\times 2}$; $\bar{D}_{ii} = [\begin{smallmatrix} 0 & 0 \\ 0 & D_{fi} \end{smallmatrix}]_{2\times 2}$ and $\bar{K}_{ii} = [\begin{smallmatrix} 0 & 0 \\ 0 & K_{fi} \end{smallmatrix}]_{2\times 2}$ and the corresponding coordinate is:

$$\bar{Q}_i = \left[\, q_{d1} \ \cdots \ q_{d(i-1)} \ q_i \ q_{d(i+1)} \ \ q_{dn} \,\right]^T, \tag{23}$$

$q_{dj} = [\, q_{rdj} \ q_{fdj} \,]^T$, $j = 1, \ \ldots, \ n$ and $j \neq i$, and $q_i = [\, q_{ri} \ q_{fi} \,]^T$.

There exists a vector $p_i = [\, p_{ir} \ \ p_{if} \,]^T \in \Re^{b_{ir}+b_{if}}$ with components depending on the manipulators' parameters and the regressor matrix $W_i = [\begin{smallmatrix} W_{ir} & 0 \\ 0 & W_{if} \end{smallmatrix}]$, which contains all known functions such as:

$$W_i \left(\bar{Q}_i, \dot{\bar{Q}}_i, t\right) p_i + RM_i = \bar{M}_i^T \left(\bar{Q}_i\right) \ddot{u} + \bar{C}_i^T \left(\bar{Q}_i, \dot{\bar{Q}}_i\right) \dot{u} + \bar{D}_i^T \dot{u}, \tag{24}$$

where RM_i are the terms independent of p_i.

The proposed control law is given as follows:

$$\tau_i = K_{dir} s_{ir} + T_i + W_{ir} \hat{p}_{ir} + RM_{ir} - \delta\tau_{ir}, \tag{25}$$

where

$$T_i = \begin{cases} \dfrac{s_{if}\left(K_{dif} s_{if} + K_{fi} \tilde{q}_{fi} + W_{if} \hat{p}_{if} + RM_{if} + \delta\tau_{if}\right)}{s_{ir}} & ; \ s_{ir} \neq 0 \\ 0 & ; s_{ir} = 0 \end{cases} \tag{26}$$

$$W_i \hat{p}_i = \begin{bmatrix} W_{ir} \hat{p}_{ir} \\ W_{if} \hat{p}_{if} \end{bmatrix} = \hat{\bar{M}}_i^T \left(\bar{Q}_i\right) \ddot{u} + \hat{\bar{C}}_i^T \left(\bar{Q}_i, \dot{\bar{Q}}_i\right) \dot{u} + \hat{\bar{D}}_i^T \dot{u}, \tag{27}$$

$$\delta\tau_i = \begin{bmatrix} \delta\tau_{ir} \\ \delta\tau_{if} \end{bmatrix} = \delta\bar{M}_i^T \ddot{u} + \delta\bar{C}_i^T \dot{u}, \tag{28}$$

and $\delta\bar{M}_{ij} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \bar{M}_{ij}(q)}{\partial q_j}|_{q_{jd}} \tilde{q}_j + \mathcal{O}_{\bar{M}_{ij}}\left(\tilde{q}_j\right)$; $K_{di} = [\begin{smallmatrix} K_{dir} & 0 \\ 0 & K_{dif} \end{smallmatrix}]$ $\delta\bar{C}_{ij} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \bar{C}_{ij}(q,\dot{q})}{\partial q_j}|_{q_{jd}} \tilde{q}_j + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \bar{C}_{ij}(q,\dot{q})}{\partial \dot{q}_j}|_{q_{jd}} \dot{\tilde{q}}_j + \mathcal{O}_{\bar{C}_{ij}}\left(\tilde{q}_j, \dot{\tilde{q}}_j\right)$ $\mathcal{O}_{M_{ij}}$ and $\mathcal{O}_{C_{ij}}$ are the high-order terms of the Taylor series for $M_{ij}(q)$ and $C_{ij}(q)$, respectively.

The corresponding adaptive laws are given as follows:

$$\begin{aligned} \dot{\hat{p}}_{ir} &= K_{vir} W_{ir}^T s_{ir}, \\ \dot{\hat{p}}_{if} &= K_{vif} W_{if}^T s_{if}. \end{aligned} \tag{29}$$

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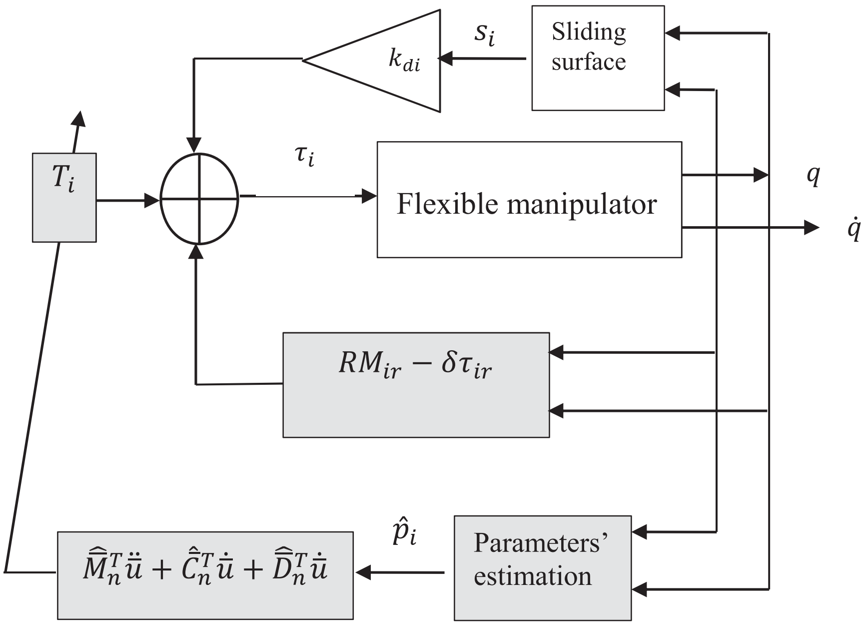

Fig. 2. The ith control law.

The ith control law is presented in Fig. 2.

The n control laws can be written as follows:

$$\tau = K_{dr}s_r + T + W_r\hat{p}_r + RM - \delta\tau_r, \tag{30}$$

where $W_r\hat{p}_r = [W_{1r}\hat{p}_{1r} \ldots W_{nr}\hat{p}_{nr}]^T$, $T = [T_1 \ldots T_n]^T$, $\delta\tau_r = [\delta\tau_{r1} \ldots \delta\tau_{rn}]^T$, $s_r = [s_{r1} \ldots s_{rn}]^T$ and $K_{dr} = \text{diag}(K_{dri})$

The adaptive laws are given as follows:

$$\begin{aligned} \dot{\hat{p}}_r &= K_{vr}W_r^T s_r, \\ \dot{\hat{p}}_f &= K_{vf}\ W_f^T s_f, \end{aligned} \tag{31}$$

where $K_{vr} = diag(K_{vri})$ and $K_{vf} = diag(K_{vfi})$ are some positive definite gain matrices.

Figure 3 shows the distributed adaptive control strategy. Starting with the last subsystem, and using its own estimated parameters, the nth control law is developed. Then, going backward to the (n-1)th subsystem, the control law is computed using its own estimated parameters and the estimated parameters of the nth (upper level) subsystem. For an ith subsystem, the controller depends on its own estimated parameters and the estimated parameters of all upper level subsystems ((i+1)th, ..., nth). The same strategy is used for each subsequent subsystem until the first one.

4. Stability Analysis

The global stability is studied by inserting the control law (30) in the initial dynamical model (1). From (9), the velocity $\dot{q}$ and the acceleration can be written as follows:

$$\begin{aligned} \dot{q} &= \dot{u} - \text{s}, \\ \ddot{q} &= \ddot{u} - \dot{\text{s}}. \end{aligned} \tag{32}$$

Using (32) and (1), the error dynamics can be deduced as follows:

$$M(q)\dot{s} + C(q,\dot{q})s + Ds + K_d s = M(q)\ddot{u} + C(q,\dot{q})\dot{u} + D\dot{u} + K_d s + Kq - L\tau. \tag{33}$$

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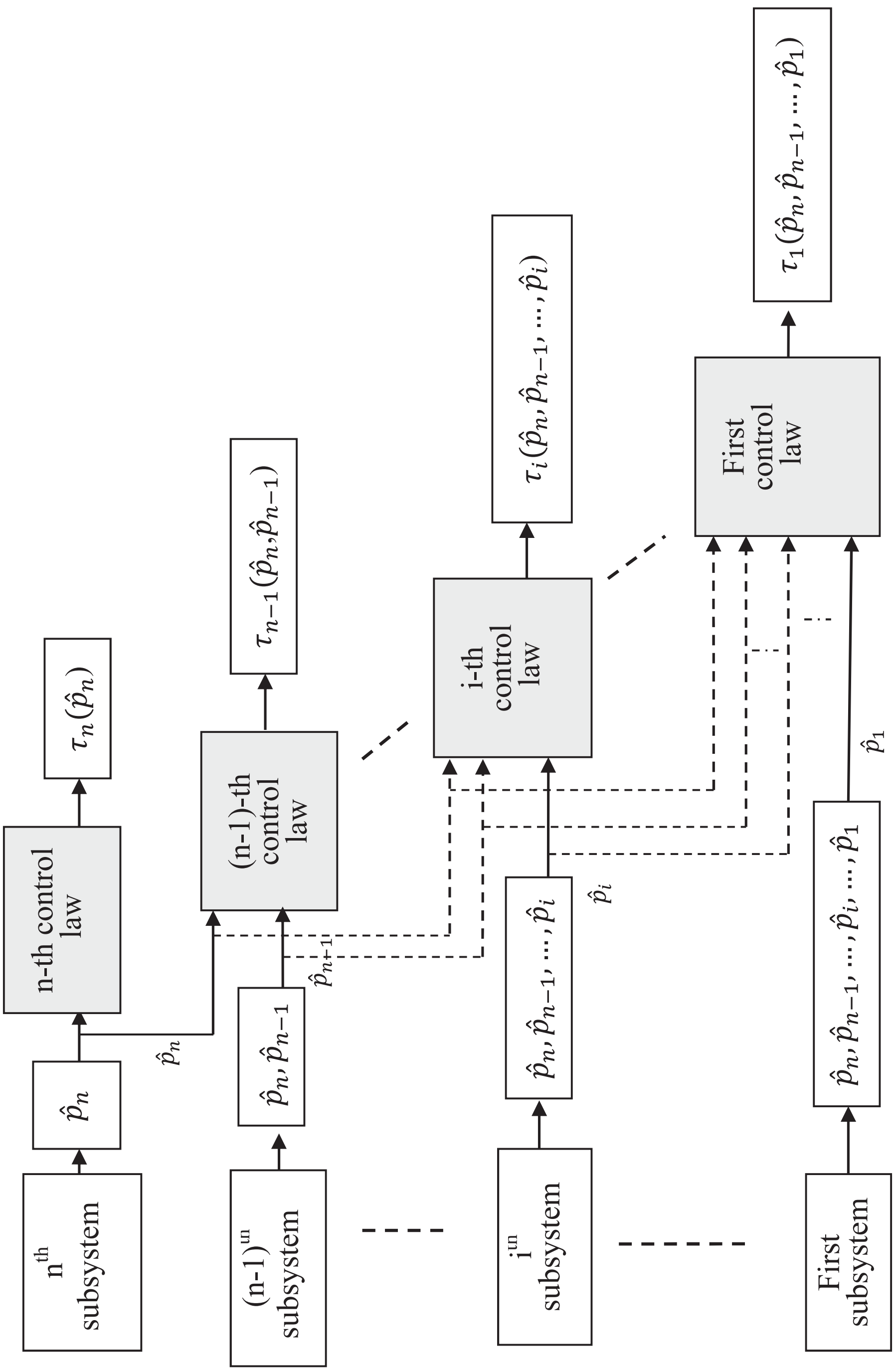

Fig. 3. Distributed adaptive control strategy.

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Proposition 1. The error dynamics (33) is equivalent to the following expression:

$$M(q)\dot{s} + C(q,\dot{q})s + Ds + K_d s = \begin{bmatrix} -T + W_r\tilde{p}_r \\ W_f p_f + RM_f + \delta\tau_f + K_{df}s_f + K_{ff}\tilde{q}_f \end{bmatrix}. \tag{34}$$

Proof: See Appendix.

For the stability analysis, let us define the following positive Lyapunov function:

$$V = \frac{1}{2}s^T M(q)s + \frac{1}{2}\tilde{p}^T K_v^{-1}\tilde{p}, \tag{35}$$

where $\tilde{p} = [\begin{smallmatrix}\tilde{p}_r \\ \tilde{p}_f\end{smallmatrix}]$; $K_v = \text{diag}(K_{vr}, K_{vf})$, $\tilde{p}^T K_v^{-1}\tilde{p} = \tilde{p}_r^T K_{vr}^{-1}\tilde{p}_r + \tilde{p}_f^T K_{vf}^{-1}\tilde{p}_f$

The time derivative of $V(t)$ gives:

$$\begin{aligned}
\dot{V}(t) &= \tfrac{1}{2}s^T M(q)\dot{s} + \tfrac{1}{2}s^T\dot{M}(q)s + \dot{\tilde{p}}^T K_v^{-1}\tilde{p}, \\
&= s^T\left(-C(q,\dot{q})s - Ds - K_d s + \begin{bmatrix} -T + W_r\tilde{p}_r \\ W_f p_f + RM_f + \delta\tau_f + K_{df}s_f + K_{ff}e_f \end{bmatrix}\right) \\
&\quad + \tfrac{1}{2}s^T\dot{M}(q)s + \dot{\tilde{p}}^T K_v^{-1}\tilde{p}, \\
&= -s^T(D + K_d)s + s^T\left(\begin{bmatrix} -T + W_r\tilde{p}_r \\ W_f p_f + RM_f + \delta\tau_f + K_{df}s_f + K_{ff}e_f \end{bmatrix}\right) + \dot{\tilde{p}}^T K_v^{-1}\tilde{p}, \\
&= -s^T(D + K_d)s + R_s,
\end{aligned}$$

where

$$R_s = s^T\left(\begin{bmatrix} -T + W_r\tilde{p}_r \\ W_f p_f + RM_f + \delta\tau_f + K_{df}s_f + K_{ff}e_f \end{bmatrix}\right) + \dot{\tilde{p}}^T K_v^{-1}\tilde{p} = s^T\tau_s + \dot{\tilde{p}}^T K_v^{-1}\tilde{p}. \tag{36}$$

Proposition 2. Using the dynamical model (1) and the control law (30), the error dynamics is globally asymptotically stable and the time derivative of $V(t)$ is equivalent to the following expression:

$$\dot{V}(t) = -s^T(D + K_d)s, \tag{37}$$

Proof: See Appendix

5. Experimental Results

The system considered in this work is the two-flexible-link manipulator manufactured by Quanser, shown in Fig. 4.

The system consists of two motors, two flexible links, and a payload. It moves in the horizontal plane and is connected by rigid revolute joints. Two motors actuate the system and generate the torques. Each flexible link is assumed uniform, and has a mass m_i and length l_i. $(\hat{X}_0, \hat{Y}_0)$ is the fixed reference frame. (X_1, Y_1) moves with the first link, while (X_2, Y_2) moves with the second link. The flexible links are modeled as Euler–Bernoulli beams, and the deformations of the links are assumed to be small. The tips' deflection is measured using two strain gauges clamped at the base of each flexible beam. Table I shows the system parameters.

The model of the two-flexible-link system given in ref. [32] is modified in this work by only considering the first flexible mode of each link. Then, we have $n = 2$, $z = 2$ and $z_1 = z_2 = 1$. The

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Table I. System parameters.

Parameter	Link 1	Link 2
Link length (l_i)	0.202 m	0.2 m
Link moment of inertia	0.17 kg.m^2	0.0064 10–6 kg.m^2
Elasticity	2.068 10^{11} N/m^2	2.068 10^{11} N/m^2
Gear ratio	100	50
Drive torque constant	0.119 Nm/A	0.0234 N.m/A
Drive moment of inertia	7.63 10^{-4} kg.m^2	44.55 10^{-6} kg.m^2
Rotor moment of inertia	6.28 10^{-6} kg.m^2	1.03 10^{-6} kg.m^2
Maximum rotation	+/− 90 deg	+/− 90 deg

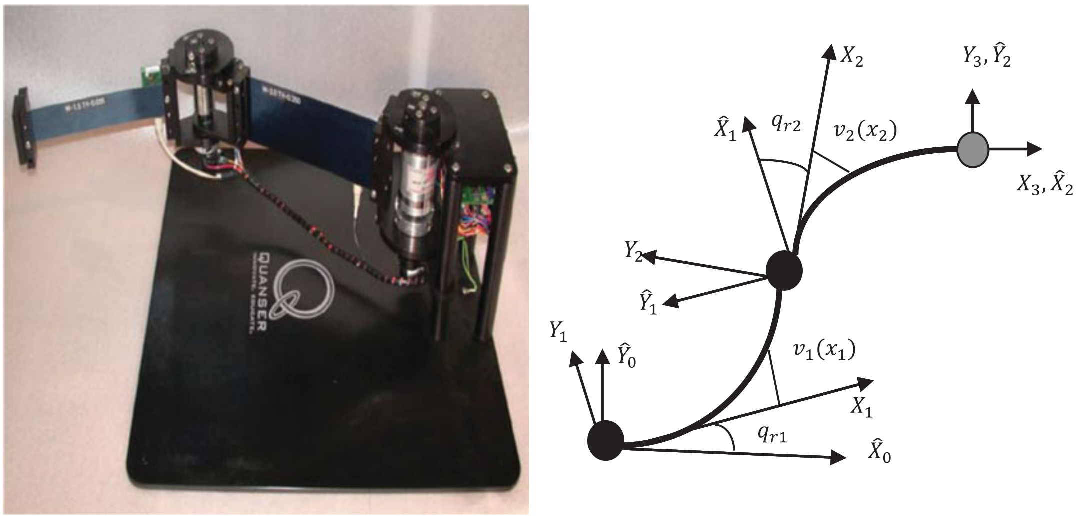

Fig. 4. Two-flexible-link manipulator.

dynamical model of the two-flexible-link manipulator is given in (3), where

$$M_{rr} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}; M_{rf} = \begin{bmatrix} M_{13} & M_{14} \\ M_{23} & M_{24} \end{bmatrix}; M_{ff} = \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}; C_{rr} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix};$$

$$C_{rf} = \begin{bmatrix} C_{13} & C_{14} \\ C_{23} & C_{24} \end{bmatrix}; C_{ff} = \begin{bmatrix} C_{33} & C_{34} \\ C_{43} & C_{44} \end{bmatrix}; D_{ff} = \begin{bmatrix} D_{f1} & 0 \\ 0 & D_{f2} \end{bmatrix} \text{ and } K_{ff} = \begin{bmatrix} K_{f1} & 0 \\ 0 & K_{f2} \end{bmatrix}.$$

The control strategies presented in the previous sections are applied and implemented on this system.

The experimental setup of the serial two-flexible-link robot manipulator is shown in Fig. 5. It consists of a Q8 terminal board, a DAQ system, sensors such as strain gauges, an encoder and limit switches. The proposed controllers are tested in real time using Workshop (RTW) of Mathworks® (Fig. 6).

The desired trajectories of the joints are represented by polynomial functions given by the following expressions[33] (Fig. 7):

$$q_{rd1}(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3 + a_{14}t^4 + a_{15}t^5, \quad (38)$$

$$q_{rd2}(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3 + a_{24}t^4 + a_{25}t^5, \quad (39)$$

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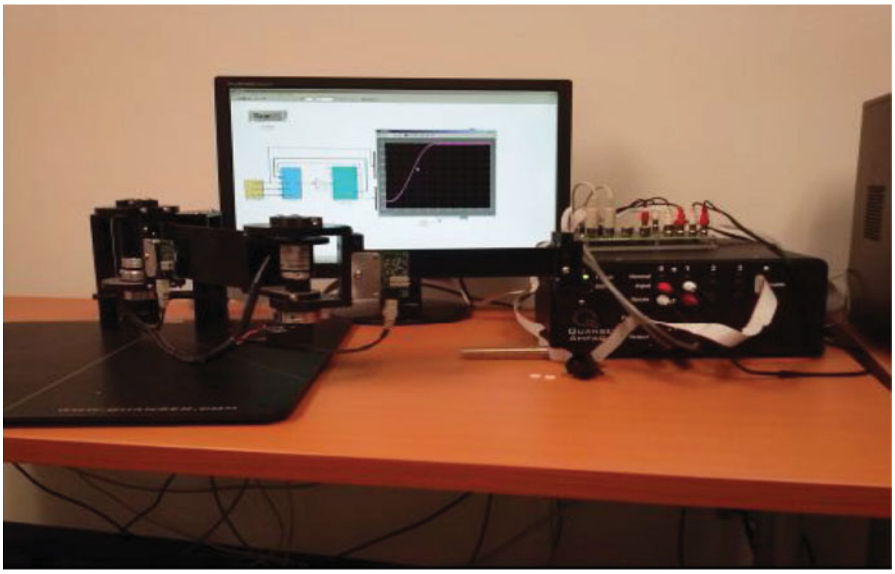

Fig. 5. Quanser two-link flexible robot.

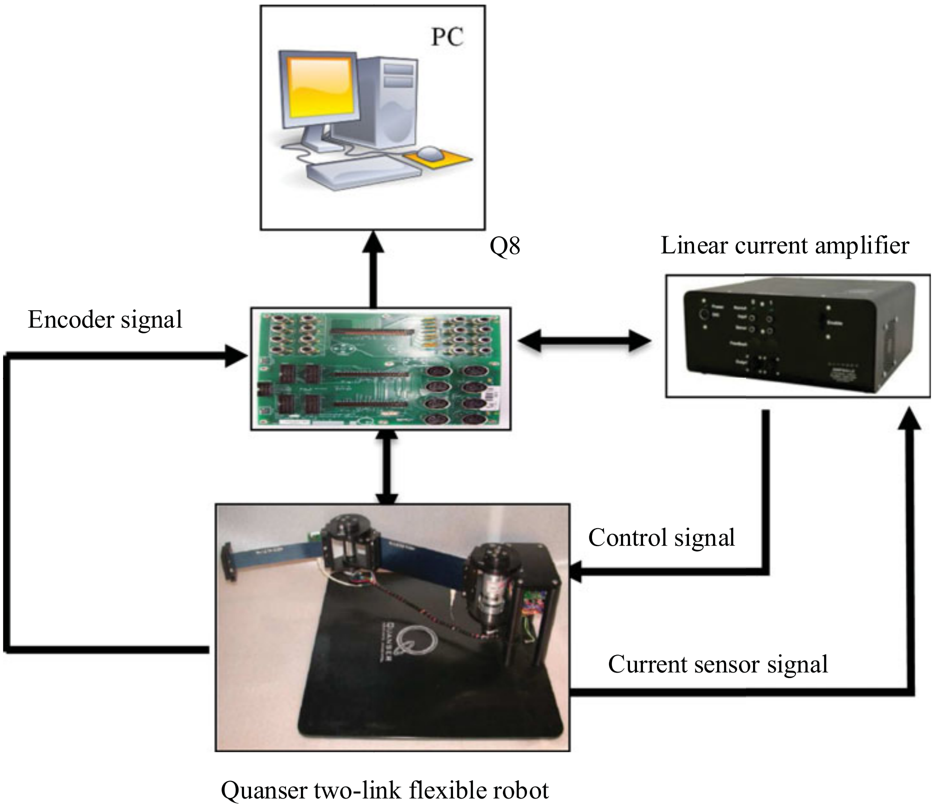

Fig. 6. Real time setup.

for $0 \leq t \leq T_f = 5s$. For $t \geq T_f$, $q_{rd1}(t) = \frac{\pi}{4}$ and $q_{rd2}(t) = \frac{\pi}{8}$. For the flexible mode:

$$q_{fd1}(t) = q_{fd2}(t) = 0. \tag{40}$$

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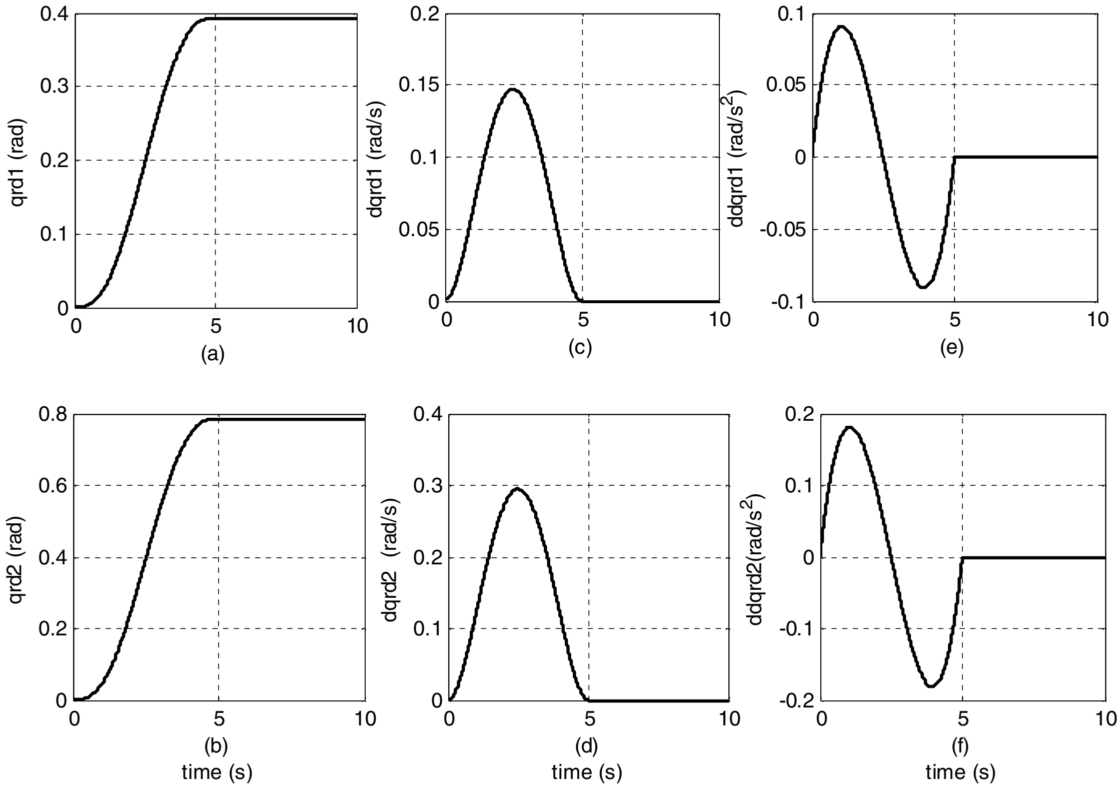

Fig. 7. Desired trajectories of rigid part: (a)–(b) position, (c)–(d) velocity, (e)–(f) acceleration.

Using a trial and error method, the controller's gains are chosen as follows: for the distributed adaptive control strategy, the selected gains are $K_{d1r} = 15$; $K_{d1f} = 10$, $K_{d2r} = 25$ and $K_{d2f} = 10$.

For an ith subsystem, the previous theoretical development of the control law is given in the general case. For the sake of simplicity and the real-time implementation, a finite-order Taylor series is fixed. The Taylor series is limited to the first order. Then, $\mathcal{O}_{M_{ij}} = \mathcal{O}_{C_{ij}} = 0$.

The uncertain parameters are chosen as follows:

- For the second subsystem, $p_2 = [\,p_{2r}\ p_{2f}\,]^T$: $p_{2r} = [\,b_{121}\ b_{123}\ b_{251}\,]^T$ and $p_{2f} = [\,b_{153}\ b_{551}\,]^T$. Then, τ_2 is developed using $\hat{p}_2$.
- For the first subsystem, $p_1 = [\,p_{1r}\ p_{1f}\,]^T$: $p_{1r} = [\,p_2\ J_{eq1}\,]^T$ and $p_{1f} = [\,p_{1r}\ b_{331}\,]^T$; τ_1 is developed using $\hat{p}_2$, $\hat{J}_{eq1}$ and $\hat{b}_{331}$

where b_{ijk} are given in ref. [30] and $J_{eq1} = J_{h1} + J_{o1} + m_{h2}l_1^2 + m_2 l_1^2 + m_p l_1^2$.

The experimental results of the adaptive distributed control are shown in Figs. 8–10.

To show the contribution of the developed adaptive distributed control method, the results were compared with a non-adaptive control version and a classical controller like PD. The experimental results of the non-adaptive controller[28] are given in Figs. 11–13, and PD control results are shown in Figs. 14–16.

According to the experimental results, a good tracking is obtained in the joint space for the distributed adaptive control strategy. The tracking of the desired trajectory of joint 1 is shown in Fig. 8a and that of joint 2 is given in Fig. 8b. The tracking errors of joints 1 and 2 are given in Fig. 8c and d, respectively. The good quality of the tracking obtained is confirmed by the tracking errors, which do not exceed 0.0025 rad for joint 1 and 0.001 rad for joint 2. For the flexible part, the desired trajectories are set to zero to reduce vibrations in the links. The errors, shown in Fig. 9, are less than 0.0005. The control input for joints 1 and 2 are given in Fig. 10. The maximum torque reaches up to 5N. The non-adaptive version and the PD control were tested on the two-flexible-link manipulator using the same desired trajectories. Figure 11 shows the tracking of the two joints, and

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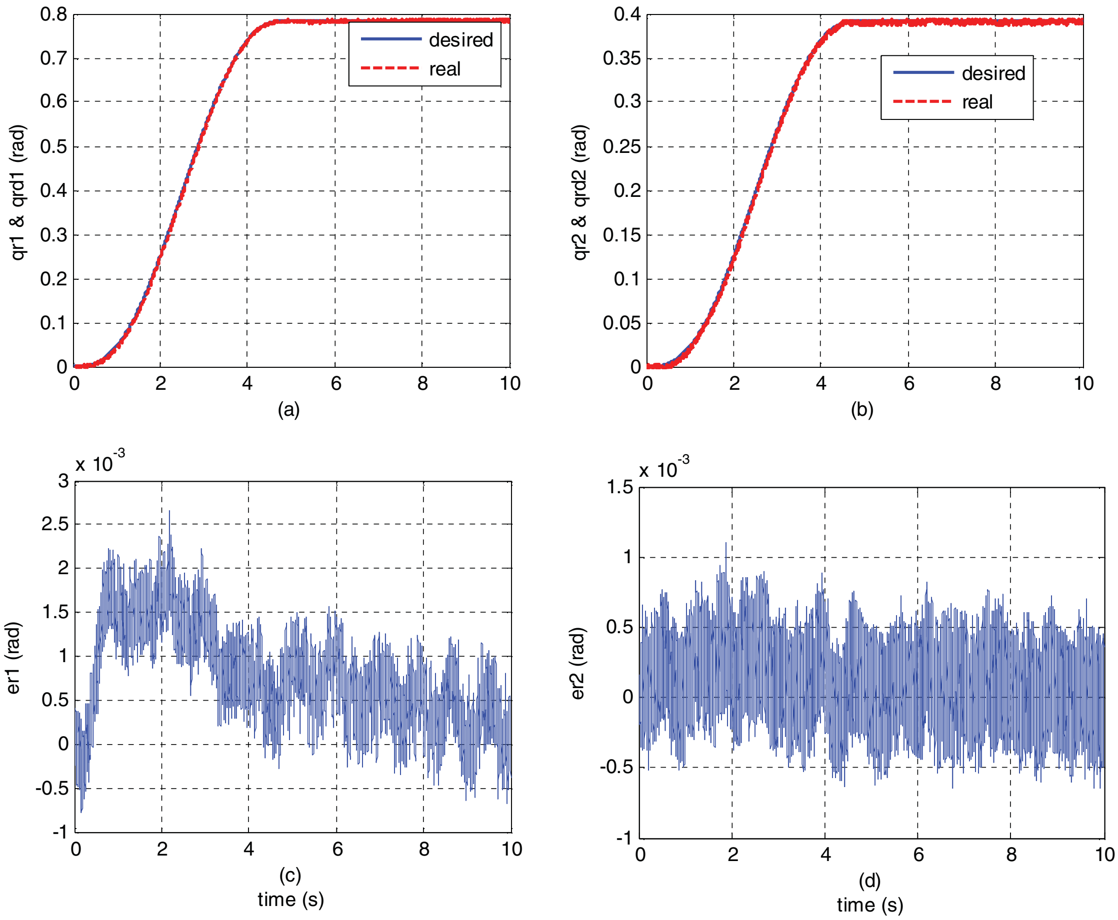

Fig. 8. Adaptive Distributed Control: (a)–(b) joints tracking trajectories; (c)–(d) joints tracking errors.

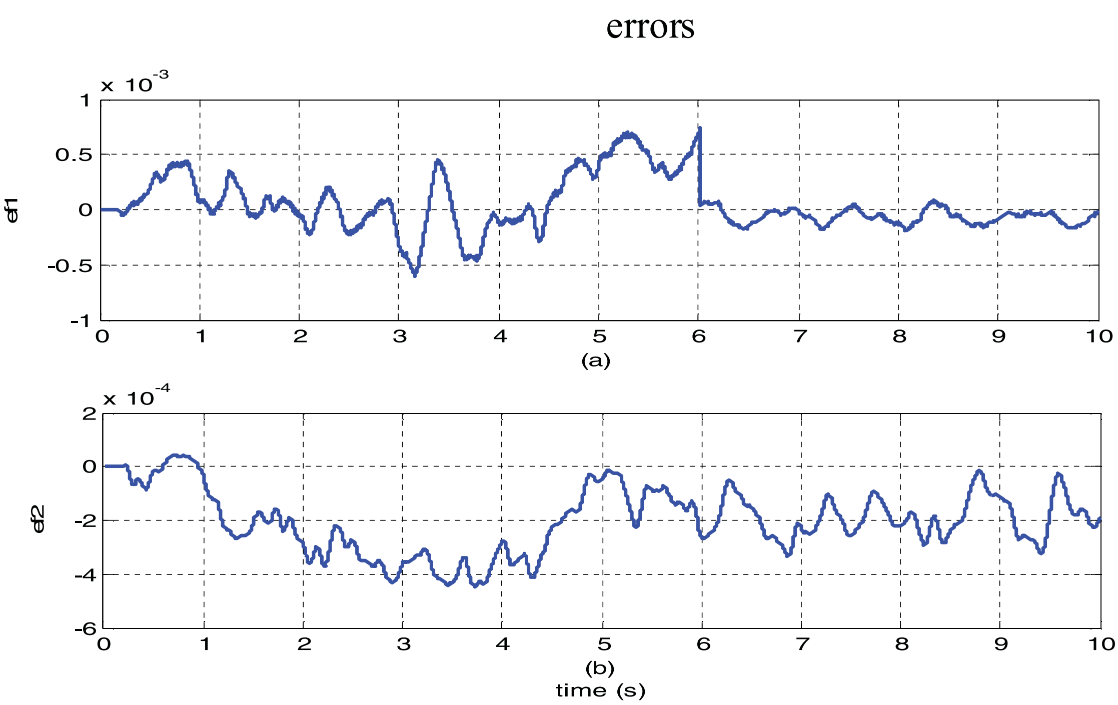

Fig. 9. Adaptive control: errors of flexible part.

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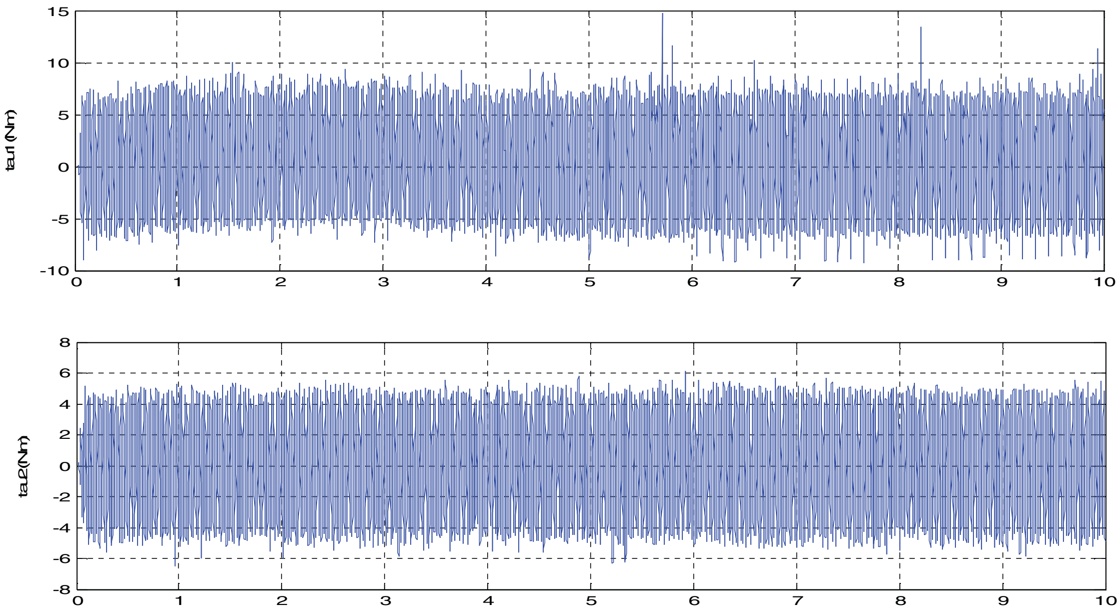

Fig. 10. Adaptive control: control input.

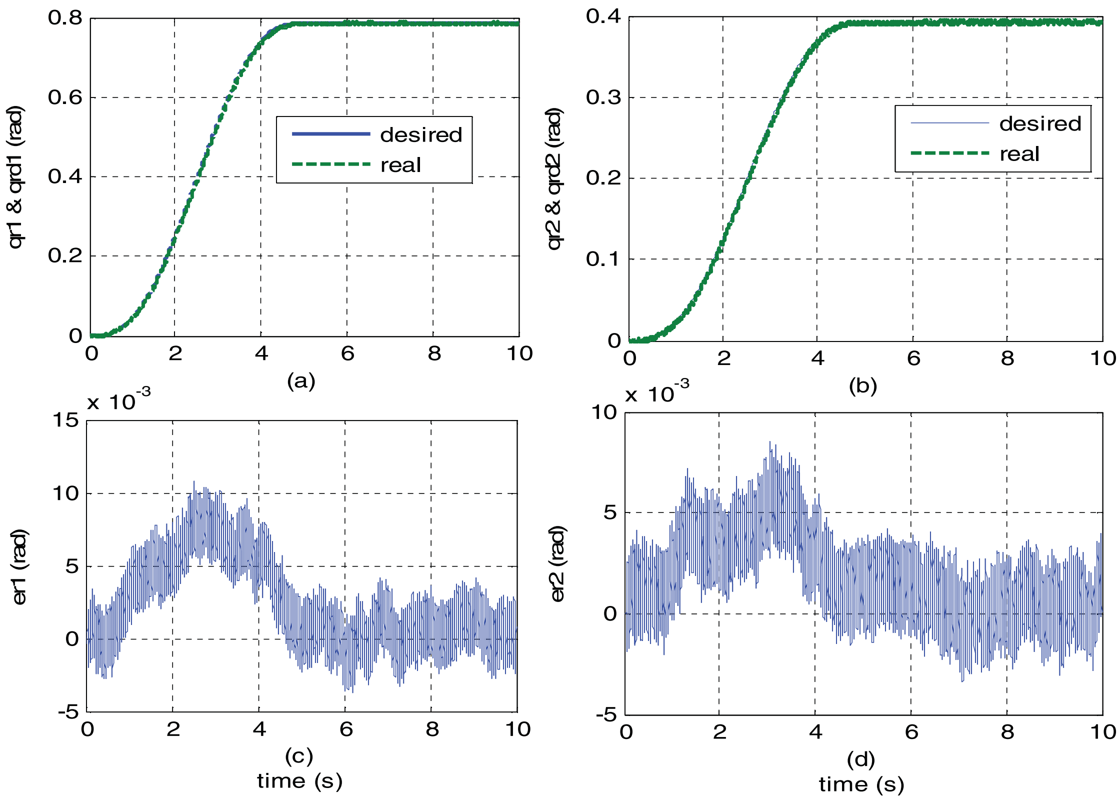

Fig. 11. Non-adaptive Distributed Control: (a)–(b) joints tracking trajectories; (c)–(d) joints tracking errors.

the corresponding tracking errors. The tracking errors of the flexible part are given in Fig. 12, while the input torque signals are given in Fig. 13. The results of the PD control are given in Figs. 14–16. Indeed, Fig. 14 shows the tracking of the first and second joints, as well as the corresponding tracking errors. Figure 15 presents the tracking errors of the flexible part, and the input torques are shown in Fig. 16. For the adaptive distributed control method, the tracking errors of the joints and the links'

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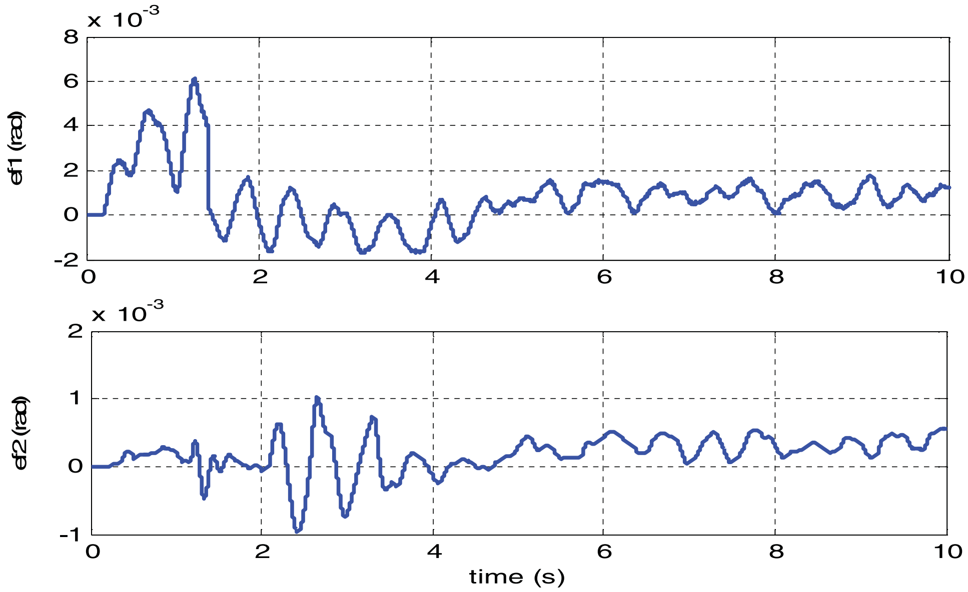

Fig. 12. Distributed control: errors of the flexible parts.

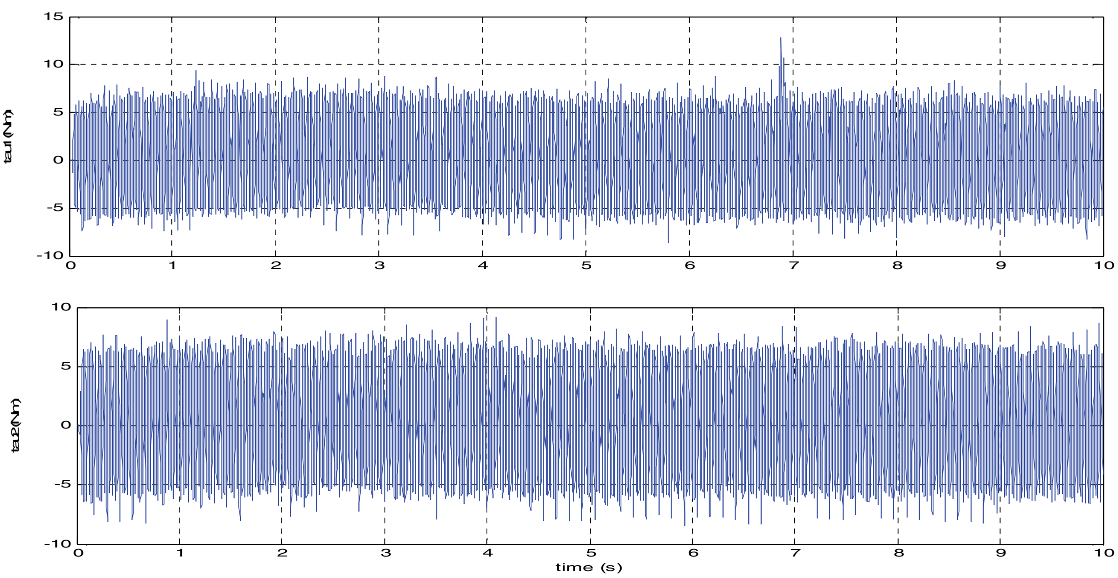

Fig. 13. Distributed control: control input.

vibrations are smaller than those resulting from non-adaptive control and PD control. Therefore, we can conclude that the adaptive controller reduces the links' vibrations and provides a good tracking of the desired trajectories when compared to the non-adaptive and/or PD controllers.

6. Conclusion

This paper presents a distributed adaptive control strategy for flexible link manipulators. This control strategy takes advantage of the configuration of n serial-link manipulators by using the dynamical model as a set of interconnected subsystems. Each subsystem has a joint/link pair. Next, the distributed control strategy consists in controlling the last subsystem while assuming that the

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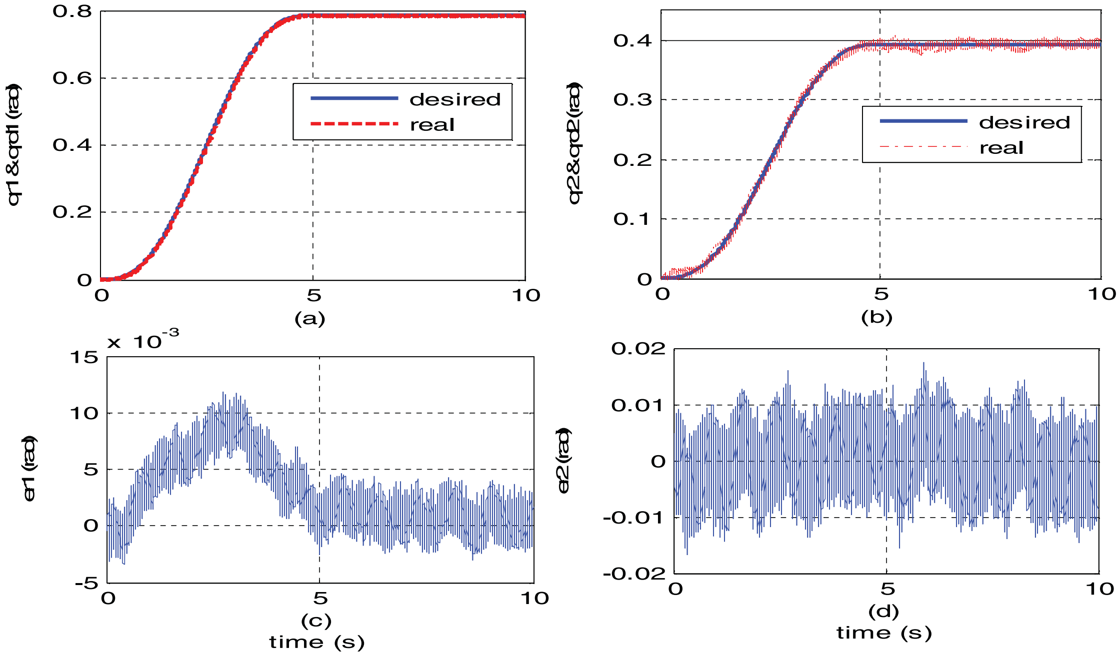

Fig. 14. PD control: (a)–(b) joint tracking trajectories; (c)–(d) joint tracking errors.

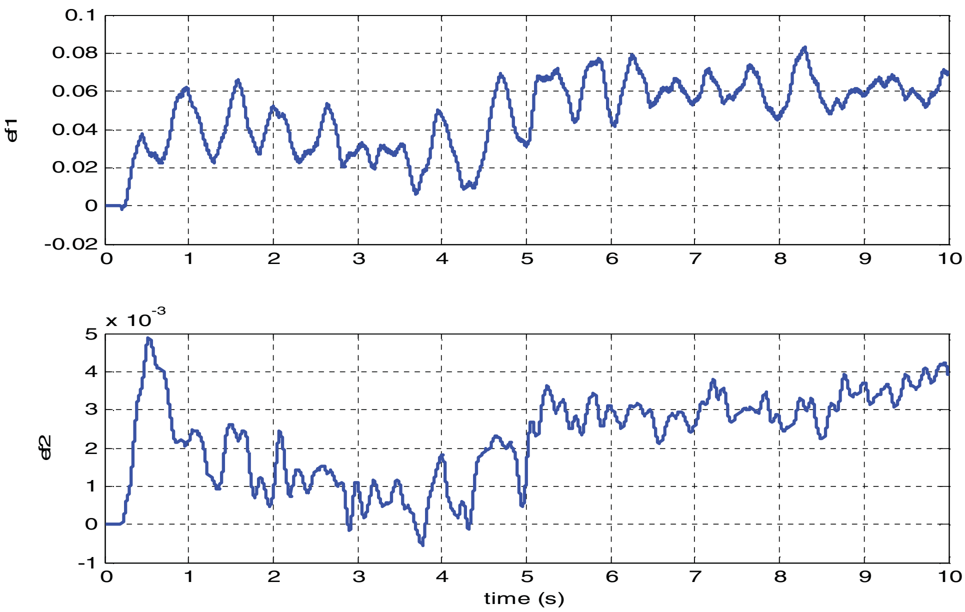

Fig. 15. PD control: errors of flexible parts.

remaining subsystems are stable. Then, going backward to the last-but-one subsystem, the same strategy is applied, and so on, until the first one. The control law of a subsystem uses its own estimated parameters and the parameters already estimated in the upper level subsystems. The global stability is proved using Lyapunov theory. Experimental results compared with a non-adaptive controller and PD controller show that the distributed adaptive control strategy provides good tracking and reduces

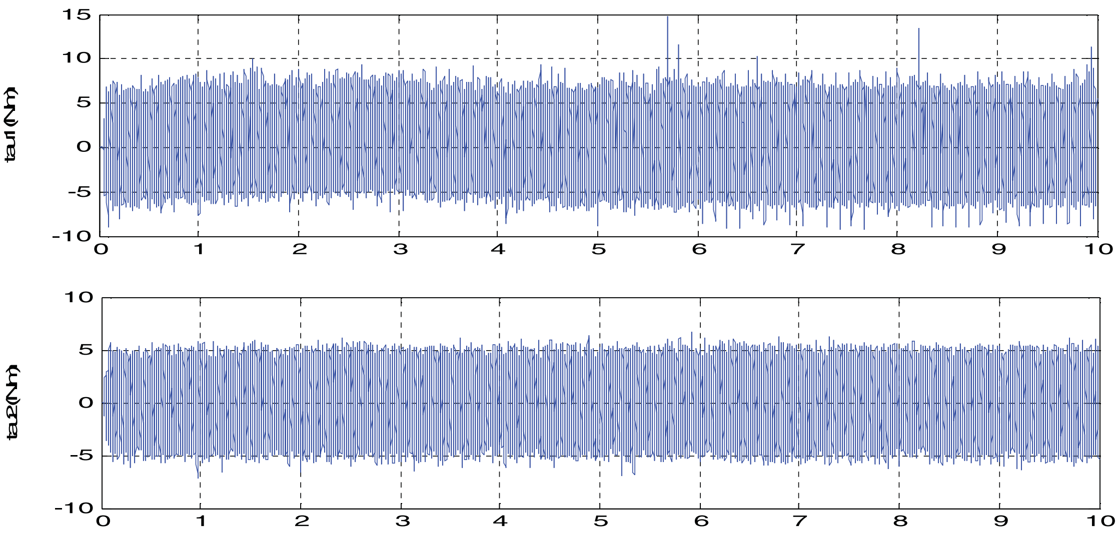

Fig. 16. PD control: input torques.

the links' vibrations. The workspace tracking control is a good challenge control problem for flexible links manipulators. This approach could be extended in an eventual future work for controlling the manipulator's tip position, but additional constraints must be taken into consideration such as the stability of the internal dynamics.

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Appendix

Proof of Proposition 1

From the error dynamics (33), let us define the second term as follows:

$$\tau_s = M(q)\ddot{u} + C(q,\dot{q})\dot{u} + D\dot{u} + K_d s + Kq - L\tau. \tag{A1}$$

Using the transformation matrix T_r and Eqs. (4), (9) and (11), we can write $\ddot{u} = T_r^{-1}\ddot{\bar{u}}$; $\dot{u} = T_r^{-1}\dot{\bar{u}}$; $s = T_r^{-1}\bar{s}$ and $q = T_r^{-1}\bar{q}$. Equation (A1) becomes:

$$\tau_s = M(q)T_r^{-1}\ddot{\bar{u}} + C(q,\dot{q})T_r^{-1}\dot{\bar{u}} + DT_r^{-1}\dot{\bar{u}} + K_d T_r^{-1}\bar{s} + KT_r^{-1}\bar{q} - L\tau. \tag{A2}$$

By multiplying (A2) by the transformation matrix T_r, we obtain:

$$\bar{\tau}_s = \bar{M}(q)\ddot{\bar{u}} + \bar{C}(q,\dot{q})\dot{\bar{u}} + \bar{D}\dot{\bar{u}} + \bar{K}_d\bar{s} + \bar{K}\bar{q} - \bar{L}\tau, \tag{A3}$$

where $\bar{\tau}_s = T_r\tau_s$; $\bar{M}(q) = T_r M(q) T_r^{-1}$; $\bar{C}(q,\dot{q}) = T_r C(q,\dot{q}) T_r^{-1}$; $\bar{D} = T_r D T_r^{-1}$; $\bar{K} = T_r K T_r^{-1}$; $\bar{L} = T_r L$.

For the ith subsystem, $\bar{\tau}_{si}$ is the ith element of $\bar{\tau}_s$ and is given as follows:

$$\bar{\tau}_{si} = \bar{M}_i^T(q)\ddot{u} + \bar{C}_i^T(q,\dot{q})\dot{u} + \bar{D}_i^T\dot{u} + \bar{K}_i^T\bar{q} + \bar{K}_{di}^T\bar{s} - \bar{L}_i\tau, \tag{A4}$$

where $\bar{M}_i^T = [\bar{M}_{i1} \ \ldots \ \bar{M}_{ii} \ \ldots \ \bar{M}_{in}]_{2\times 2n}$; $\bar{C}_i^T = [\bar{C}_{i1} \ \ldots \ \bar{C}_{ii} \ \ldots \ \bar{C}_{in}]_{2\times 2n}$; $\bar{L}_i\tau = [\,\tau_i \ 0\,]^T$.

$$\bar{D}_i^T = \left[0_{2\times 2} \ \ldots \bar{D}_{ii} \ \ldots \ 0_{2\times 2}\right]_{2\times 2n}; \ \bar{K}_i^T = \left[0_{2\times 2} \ \ldots \bar{K}_{ii} \ldots \ 0_{2\times 2}\right]_{2\times 2n},$$
$$\bar{K}_{di}^T = \left[0_{2\times 2} \ \ldots \bar{K}_{dii} \ldots \ 0_{2\times 2}\right]^T.$$

Using the Taylor series, we can write

$$\bar{M}_{ij}(q) = \bar{M}_{ij}(Q_i) + D\bar{M}_{ij}(Q_i)^T(q - Q_i) + \frac{1}{2!}(q - Q_i)^T\left\{D^2\bar{M}_{ij}(Q_i)\right\}(q - Q_i) + \ldots$$

where $D\bar{M}_{ij}(Q_i)$ is the gradient of $\bar{M}_{ij}$ evaluated at $q = Q_i$ and $D^2\bar{M}_{ij}(Q_i)$ is a Hessian matrix. Then, we can write:

$$\bar{M}_{ij}(q) = \bar{M}_{ij}(Q_i) + \frac{\partial \bar{M}_{ij}(q)}{\partial\left[q_1 \cdots q_{i-1} q_{i+1} \ldots q_n\right]}\Bigg|\begin{bmatrix} q_{1d} \\ \vdots \\ q_{(i-1)d} \\ q_{(i+1)d} \\ \vdots \\ q_{nd}\end{bmatrix}\begin{bmatrix} q_1 - q_{1d} \\ \vdots \\ q_{i-1} - q_{(i-1)d} \\ q_{i+1} - q_{(i+1)d} \\ \vdots \\ q_n - q_{nd}\end{bmatrix} + \mathcal{O}_{M_{ij}}\left(\tilde{q}_j\right),$$
$$= \bar{M}_{ij}(Q_i) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \bar{M}_{ij}(q)}{\partial q_j}\Bigg|_{q_{jd}} \tilde{q}_j + \mathcal{O}_{M_{ij}}\left(\tilde{q}_j\right),$$

$$\bar{M}_{ij}(q) = \bar{M}_{ij}(Q_i) - \delta\bar{M}_{ij}, \tag{A5}$$

where $\delta\bar{M}_{ij} = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \bar{M}_{ij}(q)}{\partial q_j}|_{q_{jd}}\tilde{q}_j - \mathcal{O}_{M_{ij}}(\tilde{q}_j)$ and $\mathcal{O}_{M_{ij}}$ are the remaining terms given as follows: $\mathcal{O}_{M_{ij}}(\tilde{q}_j) = \bar{M}_{ij}(q) - \bar{M}_{ij}(Q_i) + \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\partial \bar{M}_{ij}(q)}{\partial q_j}|_{q_{jd}}\tilde{q}_j$. The same idea is used for the Coriolis matrix C

$$\bar{C}_{ij}(q,\dot{q}) = \bar{C}_{ij}\left(Q_i, \dot{Q}_i\right) - \delta\bar{C}_{ij}. \tag{A6}$$

Using (A5) and (A6), we can write:

$$\begin{aligned} \bar{M}_i^T(q) &= \bar{M}_i^T(Q_i) - \delta\bar{M}_i^T, \\ \bar{C}_i^T(q,\dot{q}) &= \bar{C}_i^T\left(Q_i, \dot{Q}_i\right) - \delta\bar{C}_i^T, \end{aligned} \tag{A7}$$

and (A4) becomes

$$\bar{\tau}_{si} = \bar{M}_i^T(Q_i)\ddot{u} + \bar{C}_i^T\left(Q_i, \dot{Q}_i\right)\dot{u} - \delta\bar{M}_i^T\ddot{u} - \delta\bar{C}_i^T\dot{u} + \bar{D}_i^T\dot{u} + \bar{K}_i^T\bar{q} + \bar{K}_{di}^T\bar{s} - \bar{L}_i\tau, \tag{A8}$$

where $\delta\tau_i = [\delta\tau_{ir} \quad \delta\tau_{if}]^T = \delta\bar{M}_i^T\ddot{u} + \delta\bar{C}_i^T\dot{u}$; $\bar{K}_i^T\bar{q} = [\,0 \ K_{ffi}q_{fi}\,]^T$; $\bar{K}_{di}^T\bar{s} = [\,K_{dir}s_{ir} \ K_{dif}s_{if}\,]^T$.

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Using (24), the last Eq. (A8) can be written as follows:

$$\bar{\tau}_{si} = \begin{bmatrix} W_{ir}p_{ir} + RM_{ir} + K_{dir}s_{ir} - \delta\tau_{ir} - \tau_i \\ W_{if}p_{if} + RM_{if} - \delta\tau_{if} + K_{dif}s_{if} + K_{ffi}q_{fi} \end{bmatrix}. \tag{A9}$$

Using (25), we can deduce

$$\bar{\tau}_{si} = \begin{bmatrix} -T_i + W_{ir}\tilde{p}_{ir} \\ W_{if}p_{if} + RM_{if} - \delta\tau_{if} + K_{dif}s_{if} + K_{ffi}\tilde{q}_{if} \end{bmatrix}, \tag{A10}$$

where $\tilde{p}_{ir} = p_{ir} - \hat{p}_{ir}$ and $\dot{\tilde{p}}_{ir} = -\dot{\hat{p}}_{ir}$ (the system parameters p_{ir} are assumed constant, then $\dot{p}_{ir} = 0$); $\tilde{q}_{if} = q_{fi} - q_{fdi} = q_{fi}$.

The n elements of $\bar{\tau}_s$ can be written as follows:

$$\bar{\tau}_s = \begin{bmatrix} \bar{\tau}_{s1} & \ldots & \bar{\tau}_{sn} \end{bmatrix}^T. \tag{A11}$$

As for q, $\dot{u}$ and $\dot{s}$, τ_s can be reorganized as rigid and flexible parts using the matrix T_r

$$\tau_s = T_r^{-1}\bar{\tau}_s = \begin{bmatrix} \tau_{sr1} & \cdots & \tau_{srn} & \tau_{sf1} & \cdots & \tau_{sfn} \end{bmatrix}^T, \tag{A12}$$

where $\tau_{sir} = -T_i + W_{ir}\tilde{p}_{ir}$ and $\tau_{sif} = W_{if}p_{if} + RM_{if} - \delta\tau_{if} + K_{dif}s_{if} + K_{ffi}e_{fi}$.

Then, (A3) can be written as

$$\tau_s = \begin{bmatrix} -T + W_r\tilde{p}_r \\ W_f p_f + RM_f - \delta\tau_f + K_{df}s_f + K_{ff}\tilde{q}_f \end{bmatrix}, \tag{A13}$$

where $T = [\,T_1 \ldots T_n\,]^T$; $W_r\tilde{p}_r = [\,W_{1r}\tilde{p}_{1r} \ldots W_{nr}\tilde{p}_{nr}\,]^T$; $W_f p_f = [\,W_{1f}p_{1f} \ldots W_{nf}p_{nf}\,]^T$

$$\delta\tau_f = \begin{bmatrix} \delta\tau_{f1} & \ldots & \delta\tau_{fn} \end{bmatrix}^T; \;\; K_{df} = diag\left(K_{dfi}\right); \;\; K_{ff} = diag\left(K_{ffi}\right); \; i = 1 \ldots n.$$

The error dynamics (33) is equivalent to:

$$M(q)\dot{s} + C(q,\dot{q})s + Ds + K_d s = \begin{bmatrix} -T + W_r\tilde{p}_r \\ W_f p_f + RM_f - \delta\tau_f + K_{df}s_f + K_{ff}\tilde{q}_f \end{bmatrix}. \tag{A14}$$

Proof of Proposition 2

As given in Section 2, $\bar{s}$, $\bar{\tau}_s$, $\bar{K}_v^{-1}\bar{\tilde{p}}$ and $\dot{\bar{\tilde{p}}}$ can be obtained by changing the order of elements of the vectors: s, τ_s, $K_v^{-1}\tilde{p}$ and $\dot{\tilde{p}}$. Then, we can write

$$s^T\tau_s = \bar{s}^T\bar{\tau}_s = \sum_{i=1}^{n} \bar{s}_i^T\bar{\tau}_{si} = \sum_{i=1}^{n} s_i^T\tau_{si}, \tag{A15}$$

$$\dot{\tilde{p}}^T K_v^{-1}\tilde{p} = \dot{\bar{\tilde{p}}}^T \bar{K}_v^{-1}\bar{\tilde{p}} = \sum_{i=1}^{n} \dot{\bar{\tilde{p}}}_i^T \bar{K}_{vi}^{-1}\bar{\tilde{p}}_i = \sum_{i=1}^{n} \dot{\tilde{p}}_i^T K_{vi}^{-1}\tilde{p}_i, \tag{A16}$$

and

$$R_s = \sum_{i=1}^{n} \bar{s}_i^T\bar{\tau}_{si} + \dot{\bar{\tilde{p}}}_i^T \bar{K}_{vi}^{-1}\bar{\tilde{p}}_i = \sum_{i=1}^{n} s_i^T\tau_{si} + \dot{\tilde{p}}_i^T K_{vi}^{-1}\tilde{p}_i = \sum_{i=1}^{n} R_{si}, \tag{A17}$$

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where $R_{si} = s_i^T \tau_{si} + \dot{\tilde{p}}_i^T K_{vi}^{-1} \tilde{p}_i$; $\bar{\tau}_{si} = \left[\begin{smallmatrix} -T_i + W_{ir}\tilde{p}_{ir} \\ W_{if}p_{if} + RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if} \end{smallmatrix} \right]$; $\tilde{p}_i = [\, \tilde{p}_{ir} \;\; \tilde{p}_{if} \,]^T$.

We can write

$$\dot{\tilde{p}}_i^T K_{vi}^{-1} \tilde{p}_i = \left[\, \dot{\tilde{p}}_{ir}^T \quad \dot{\tilde{p}}_{if}^T \,\right] \begin{bmatrix} K_{vir}^{-1} & 0 \\ 0 & K_{vif}^{-1} \end{bmatrix} \begin{bmatrix} \tilde{p}_{ri} \\ \tilde{p}_{fi} \end{bmatrix} = \dot{\tilde{p}}_{ir}^T K_{vir}^{-1} \tilde{p}_{ir} + \dot{\tilde{p}}_{if}^T K_{vif}^{-1} \tilde{p}_{if}. \tag{A18}$$

Then,

$$\begin{aligned} R_{si} &= \left[\, s_{ir} \quad s_{if} \,\right] \begin{bmatrix} -T_i + W_{ir}\tilde{p}_{ir} \\ W_{if}p_{if} + RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if} \end{bmatrix} + \dot{\tilde{p}}_{ir}^T K_{vir}^{-1} \tilde{p}_{ir} + \dot{\tilde{p}}_{if}^T K_{vif}^{-1} \tilde{p}_{if}, \\ &= -s_{ir}T_i + s_{ir}W_{ir}\tilde{p}_{ir} + s_{if}\left(W_{if}p_{if} + RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if}\right) \\ &\quad + \dot{\tilde{p}}_{ir}^T K_{vir}^{-1} \tilde{p}_{ir} + \dot{\tilde{p}}_{if}^T K_{vif}^{-1} \tilde{p}_{if}, \\ &= -s_{ir}T_i + \left(s_{ir}W_{ir} + \dot{\tilde{p}}_{ir}^T K_{vir}^{-1}\right)\tilde{p}_{ir} + s_{if}\left(W_{if}p_{if} + RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if}\right) \\ &\quad + \dot{\tilde{p}}_{if}^T K_{vif}^{-1}\left(p_{if} - \hat{p}_{if}\right), \\ &= \left(s_{ir}W_{ir} + \dot{\tilde{p}}_{ir}^T K_{vir}^{-1}\right)\tilde{p}_{ir} + \left(s_{if}W_{if} + \dot{\tilde{p}}_{if}^T K_{vif}^{-1}\right)p_{if} - s_{ir}T_i \\ &\quad + s_{if}\left(RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if}\right) - \dot{\tilde{p}}_{if}^T K_{vif}^{-1}\hat{p}_{if}. \end{aligned}$$

The system parameters p_{ir} and p_{if} are assumed constant, then: $\tilde{p}_{ir} = p_{ir} - \hat{p}_{ir}$; $\tilde{p}_{if} = p_{if} - \hat{p}_{if}$; $\dot{\tilde{p}}_{ir} = -\dot{\hat{p}}_{ir}$ and $\dot{\tilde{p}}_{if} = -\dot{\hat{p}}_{if}$. Then: $R_{si} = \left(s_{ir}W_{ir} - \dot{\hat{p}}_{ir}^T K_{vir}^{-1}\right)\tilde{p}_{ir} + \left(s_{if}W_{if} - \dot{\hat{p}}_{if}^T K_{vif}^{-1}\right)p_{if} - s_{ir}T_i + s_{if}\left(RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if}\right) + \dot{\hat{p}}_{if}^T K_{vif}^{-1}\hat{p}_{if}$

Using the adaptive laws of the ith subsystem:

$$\dot{\hat{p}}_{ir} = K_{vir}W_{ir}^T s_{ir}, \tag{A19}$$

$$\dot{\hat{p}}_{if} = K_{vif}W_{if}^T s_{if}, \tag{A20}$$

then,

$$R_{si} = -s_{ir}T_i + s_{if}\left(RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if}\right) + \dot{\hat{p}}_{if}^T K_{vif}^{-1}\hat{p}_{if}. \tag{A21}$$

Using (26), the Eq. (A21) becomes:

$$\begin{aligned} R_{si} &= -s_{ir}\left[\frac{s_{if}\left(K_{dif}s_{if} + K_{fi}\tilde{q}_{fi} + W_{if}\hat{p}_{if} + RM_{if} + \delta\tau_{if}\right)}{s_{ir}}\right] + \dot{\hat{p}}_{if}^T K_{vif}^{-1}\hat{p}_{if}, \\ &\quad + s_{if}\left(RM_{if} + \delta\tau_{fi} + K_{dfi}s_{fi} + K_{fi}\tilde{q}_{if}\right) \\ &= -s_{if}\left(K_{dif}s_{if} + K_{fi}\tilde{q}_{fi} + W_{if}\hat{p}_{if} + RM_{if} + \delta\tau_{if}\right) \\ &\quad + s_{if}\left(K_{dif}s_{if} + K_{fi}\tilde{q}_{fi} + RM_{if} + \delta\tau_{if}\right) + \dot{\hat{p}}_{if}^T K_{vif}^{-1}\hat{p}_{if}, \\ &= -s_{if}\left(K_{dif}s_{if} + K_{fi}\tilde{q}_{fi} + RM_{if} + \delta\tau_{if}\right) + s_{if}\left(K_{dif}s_{if} + K_{fi}\tilde{q}_{fi} + RM_{if} + \delta\tau_{if}\right) \\ &\quad + \left(\dot{\hat{p}}_{if}^T K_{vif}^{-1} - s_{if}W_{if}\right)\hat{p}_{if}. \end{aligned}$$

Using the adaptive law (A20), we can conclude that $R_{si} = 0$. Then $R_s = 0$ and the time derivative of V becomes:

$$\dot{V}(t) = -s^T(D + K_d)s \tag{A22}$$

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Since D and K_d are positive matrices, $\dot{V}(t) \leq 0$. $V(t)$ is a continuous function of $\tilde{p}$. Using (A22), $V(t)$ is non-increasing in t. We can conclude then that $s, \ \tilde{p} \in L_\infty$. From the definition of s given in (9), $\tilde{q} \ \epsilon L_\infty$ and from the definition of T, $T \epsilon L_\infty$. Using the error dynamics (34), $\dot{s} \epsilon L_\infty$. On the other hand, we have $-\int_0^\infty \dot{V} dt = V(0) - V(\infty) < \infty$ or equivalently, $\int_0^\infty \| s \|^2 dt < \infty$, i.e., $s \epsilon L_2$. Assuming that the Lyapunov derivative is uniformly continuous and using Barbalat Lemma,[34] we can conclude that $s \rightarrow 0$ as $t \rightarrow \infty$, i.e., $\tilde{q}, \dot{\tilde{q}} \rightarrow 0$ asymptotically as $t \rightarrow \infty$. Note that recent work[35,36] mitigates LaSalle's Invariance Principle stability conditions[37] reaching stability conclusions for non-autonomous systems under milder conditions, in particular without requiring the hard uniform continuity condition.