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Accurate evaluation of the conditions for generation of quantum effects in relativistic interactions between laser and electron beams

Alexandru Popa

National Institute for Laser, Plasma and Radiation Physics, Laser Department, P.O. Box MG-36, Bucharest, Romania 077125

Research Article

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Author for correspondence:
Alexandru Popa, National Institute for Laser, Plasma and Radiation Physics,
Laser Department, P.O. Box MG-36,
Bucharest, Romania 077125.
E-mail: ampopa@rdslink.ro

Abstract

The quantum behavior of the system composed of an electron in an electromagnetic field is described by the Dirac equation, whose solution is a wave function represented by a column matrix with four components. We prove, without using any approximation, that these components can be put in a form which reveals directly the values of the electron energy, laser beam intensity, or amplitude of the electric field intensity, for which the quantum electrodynamics effects are generated. Our results are in good agreement with the experimental data reported in the literature. We prove that the four components of the wave function verify the continuity equation of quantum electrodynamics. Our treatment is in good agreement with the Compton relation. We show that the interaction of electrons with laser beams could be modeled using classical approaches regardless of the laser beam intensity as long as the electrons are non-relativistic, in agreement with published experimental data.

Introduction

We consider a system composed of an electron which interacts with electromagnetic (EM) field. The analysis of the experiments from literature (Bula *et al.*, 1996; Burke *et al.*, 1997; Bamber *et al.*, 1999; Kirsebom *et al.*, 2001) shows that the quantum electrodynamics (QED) effects, including the strong interactions between electron spin magnetic momentum and EM field, occur for electron energies of the order of few tens of GeV or higher, for typical laser beam intensities comprised between 10^{18} and 10^{19} W/cm^2 (Bula *et al.*, 1996; Burke *et al.*, 1997; Bamber *et al.*, 1999), or for amplitude of the electric field intensity of the order of 10^{13} V/m (Kirsebom *et al.*, 2001). The solution of the Dirac equation, for this system, is a column matrix having four elements (Dirac, 1958). In this paper, we prove that the system of four partial differential equations, which is equivalent to the Dirac equation, leads to a simple solution, which reveals directly the conditions of the generation of the quantum effects, in good agreement with the above experimental results. Our solution is different of the solutions from literature, for similar interactions (Volkov, 1935; Brown and Kibble, 1964; Panek *et al.*, 2002; Boca and Florescu, 2009; Harvey *et al.*, 2009; Krajewska and Kaminski, 2012), the list not being exhaustive.

Our calculation is performed with the aid of a periodicity property of the system composed of an electron in EM field (Popa, 2011), which is validated by numerous published experimental data. We have used this periodicity property to simplify the modeling of many applications involving the interaction between electrons and laser beams. For instance, we proved that this property leads to new polarization effects of the radiation generated by the collision between laser beams at arbitrary angles and relativistic electron beams (REBs) (Popa, 2012). This property can be used to calculate the radiation damping parameters in the interaction between very intense laser beams and REBs (Popa, 2014a). We have presented a lot of other applications of this property in the books Popa (2014b and 2014c).

The paper is structured as follows. The section Initial data presents initial data necessary for our analysis, namely initial hypotheses related to the system composed of electron in an EM field and the Dirac equation, written for this system. In the section Solution of the system equivalent to the Dirac equation and its verification, we solve the system of equations equivalent to the Dirac equation in the laboratory frame of reference and give a verification of its solution. The section Solution of the system of equations in the rest frame of the relativistic electron presents the solution of the system of equations in the rest frame of reference of relativistic electrons, for interactions between laser beams and REBs. In the section Discussion of our solution in the light of experimental data from literature, we discuss our theoretical results in the light of experimental data from literature.

The equations are written in the International System (IS).

Initial data

Initial hypotheses

We analyze a system composed of a very intense EM field that interacts with an electron. We consider the following initial hypotheses:

(h1) We consider that the EM field is plane, elliptically polarized. In a Cartesian system of coordinates, the intensity of the electric field and of the magnetic induction vector, denoted, respectively, by $\overline{E}_L$ and $\overline{B}_L$, are polarized in the plane xy, while the wave vector, denoted by $\overline{k}_L$, is parallel to the axis oz. The expression of the electric field is

$$\overline{E}_L = E_{M1} \cos \eta \overline{i} + E_{M2} \sin \eta \overline{j} \tag{1}$$

with

$$\eta = \omega_L t - |\overline{k}_L| z + \eta_i \text{ and } |\overline{k}_L| c = \omega_L \tag{2}$$

where $\overline{i}$, $\overline{j}$, and $\overline{k}$ are versors of the ox, oy, and oz-axes, E_{M1}, E_{M2} are the amplitudes of the electric field oscillations in the ox and oy directions, ω_L is the angular frequency of the laser EM field, c is the light velocity, η_i is an arbitrary initial phase and t is the time in the xyz system in which the motion of the electron is studied.

From the properties of the EM field, it follows that the corresponding magnetic induction vector is

$$\overline{B}_L = -B_{M2} \sin \eta \overline{i} + B_{M1} \cos \eta \overline{j} \tag{3}$$

with

$$E_{M1} = cB_{M1},\ E_{M2} = cB_{M2} \text{ and } c\overline{B}_L = \overline{k} \times \overline{E}_L \tag{4}$$

where B_{M1} and B_{M2} are the amplitudes of the magnetic field oscillations in the oy and ox directions.

(h2) The electric potential of the field is constant. Since the magnetic potential vector of the field, denoted by $\overline{A}_L$, results from the relation $\overline{E}_L = -\partial \overline{A}_L / \partial t$, we obtain from (1)

$$\overline{A}_L = -A_{M1} \sin \eta \overline{i} + A_{M2} \cos \eta \overline{j} \tag{5}$$

with

$$A_{M1} = \frac{E_{M1}}{\omega_L} \text{ and } A_{M2} = \frac{E_{M2}}{\omega_L} \tag{6}$$

where A_{M1} and A_{M2} are the amplitudes of the magnetic potential vector oscillations in the ox and oy directions.

We note by a_1 and a_2 the relativistic parameters and have:

$$a_1 = \frac{eE_{M1}}{mc\omega_L} \text{ and } a_2 = \frac{eE_{M2}}{mc\omega_L} \tag{7}$$

Dirac equation, written in the laboratory frame system

We write the Dirac equation when the EM field is described by Eqs. (1)–(6). In virtue of hypothesis (h2), we write the Dirac equation [see Dirac (1958), page 257] using our notations, in IS, as follows:

$$[\widehat{p}_0 - \rho_1(\sigma, \widehat{p} + e\overline{A}_L) - \rho_3 mc]\psi = 0 \tag{8}$$

where

$$\widehat{p}_0 = \frac{i\hbar}{c}\frac{\partial}{\partial t},\ \widehat{p}_x = -i\hbar\frac{\partial}{\partial x},\ \widehat{p}_y = -i\hbar\frac{\partial}{\partial y} \text{ and } \widehat{p}_z = -i\hbar\frac{\partial}{\partial z} \tag{9}$$

e is the absolute value of the electron charge, ψ is the column matrix of the wave function, having four elements, and σ_1, σ_2, σ_3, ρ_1, ρ_2, and ρ_3 are the matrix operators which were been used by Dirac in his equation. These matrices are as follows:

$$\sigma_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$\rho_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

In agreement with the Dirac's notations, we have

$$(\sigma, \widehat{p} + e\overline{A}_L) = \sigma_1(\widehat{p}_x + eA_{Lx}) + \sigma_2(\widehat{p}_y + eA_{Ly}) + \sigma_3(\widehat{p}_z + eA_{Lz}) \tag{10}$$

Dirac has multiplied the relation (8) by the following factor

$$\widehat{p}_0 + \rho_1(\sigma, \widehat{p} + e\overline{A}_L) + \rho_3 mc \tag{11}$$

on the left, and obtained, after a mathematical processing, the following relation [see Dirac (1958), page 265]:

$$[\widehat{p}_0^2 - (\widehat{p} + e\overline{A}_L)^2 - m^2c^2]\psi + \left[-\hbar e(\sigma, \overline{B}_L) + i\rho_1 \frac{\hbar e}{c}(\sigma, \overline{E}_L)\right]\psi = 0 \tag{12}$$

Taking into account the relations (1), (3), and (9), Eq. (12)

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becomes

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi+\left[\begin{array}{l}-\hbar e(-\sigma_1 B_{\mathrm{M2}}\sin\eta+\sigma_2 B_{\mathrm{M1}}\cos\eta)\\ +i\rho_1\dfrac{\hbar e}{c}(\sigma_1 E_{\mathrm{M1}}\cos\eta+\sigma_2 E_{\mathrm{M2}}\sin\eta)\end{array}\right]\psi=0 \quad (13)$$

which can be written, with the aid of (4), as follows:

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi+\frac{\hbar e}{c}(-\sigma_2+i\rho_1\sigma_1)E_{\mathrm{M1}}\cos\eta\cdot\psi+\frac{\hbar e}{c}(\sigma_1+i\rho_1\sigma_2)E_{\mathrm{M2}}\sin\eta\cdot\psi=0 \quad (14)$$

We consider the above matrices σ_1, σ_2, and ρ_1. A simple calculation leads to the expressions of the matrices which enter in Eq. (14):

$$-\sigma_2+i\rho_1\sigma_1=i\begin{pmatrix}0&1&0&1\\-1&0&1&0\\0&1&0&1\\1&0&-1&0\end{pmatrix},\quad \sigma_1+i\rho_1\sigma_2=\begin{pmatrix}0&1&0&1\\1&0&-1&0\\0&1&0&1\\-1&0&1&0\end{pmatrix} \quad (15)$$

By introducing the expressions from Eq. (15) into Eq. (14), we obtain the following form of the Dirac equation

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}+\frac{\hbar e}{c}i\begin{pmatrix}0&1&0&1\\-1&0&1&0\\0&1&0&1\\1&0&-1&0\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}E_{\mathrm{M1}}\cos\eta+\frac{\hbar e}{c}\begin{pmatrix}0&1&0&1\\1&0&-1&0\\0&1&0&1\\-1&0&1&0\end{pmatrix}\begin{pmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{pmatrix}E_{\mathrm{M2}}\sin\eta=0$$

which can be written as follows:

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\begin{bmatrix}\psi_1\\\psi_2\\\psi_3\\\psi_4\end{bmatrix}+\frac{\hbar e}{c}\left(i\begin{bmatrix}\psi_2+\psi_4\\-\psi_1+\psi_3\\\psi_2+\psi_4\\\psi_1-\psi_3\end{bmatrix}E_{\mathrm{M1}}\cos\eta+\begin{bmatrix}\psi_2+\psi_4\\\psi_1-\psi_3\\\psi_2+\psi_4\\-\psi_1+\psi_3\end{bmatrix}E_{\mathrm{M2}}\sin\eta\right)=0 \quad (16)$$

Solution of the system equivalent to the Dirac equation and its verification

Solution of the system

Written in terms of the components ψ_1, …, ψ_4 of ψ, the Dirac equation, given by (16), is equivalent to the following system of four equations with four unknown:

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi_1+\frac{\hbar e}{c}(\psi_2+\psi_4)(iE_{\mathrm{M1}}\cos\eta+E_{\mathrm{M2}}\sin\eta)=0 \quad (17)$$

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi_2-\frac{\hbar e}{c}(\psi_1-\psi_3)(iE_{\mathrm{M1}}\cos\eta-E_{\mathrm{M2}}\sin\eta)=0 \quad (18)$$

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi_3+\frac{\hbar e}{c}(\psi_2+\psi_4)(iE_{\mathrm{M1}}\cos\eta+E_{\mathrm{M2}}\sin\eta)=0 \quad (19)$$

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi_4+\frac{\hbar e}{c}(\psi_1-\psi_3)(iE_{\mathrm{M1}}\cos\eta-E_{\mathrm{M2}}\sin\eta)=0 \quad (20)$$

We observe that the expressions from square brackets in Eqs. (17)–(20) are identical to the operator from the Klein–Gordon equation, divided by $-c^2$ (Messiah, 1962), and we have:

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t}\right)^2-\left(-i\hbar\nabla+e\overline{A}_{\mathrm{L}}\right)^2-m^2c^2\right]\psi_{\mathrm{KG}}=0 \quad (21)$$

where ψ_{KG} is the solution of the Klein–Gordon equation.

Supposing that $F_j(\eta)$, with $j = 1, 2, 3, 4$, are functions of η, and taking into account the relations (2), (5), and $|\overline{k}_{\mathrm{L}}|c=\omega_{\mathrm{L}}$, we can

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easily obtain $\nabla[e\overline{A}_{\mathrm{L}}F_j(\eta)] = 0$, $e\overline{A}_{\mathrm{L}}\nabla F_j(\eta) = 0$ and $[(1/c^2)(i\hbar\,\partial/\partial t)^2 - (-i\hbar\,\nabla)^2]F_j(\eta) = 0$. Taking into account these relations, together with (7) and (21), we obtain the following relation:

$$\left[\frac{1}{c^2}\left(i\hbar\,\frac{\partial}{\partial t}\right)^2 - (-i\hbar\,\nabla + e\overline{A}_{\mathrm{L}})^2 - m^2c^2\right]\psi_{\mathrm{KG}}F_j(\eta) = -\psi_{\mathrm{KG}}(e^2\overline{A}_{\mathrm{L}}^2 + m^2c^2)F_j(\eta) = - \psi_{\mathrm{KG}}\cdot m^2c^2(1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)F_j(\eta) \tag{22}$$

In virtue of Eq. (22) we can see easily that the solution of the system (17)–(20) is of the form:

$$\psi_1 = C\psi_{\mathrm{KG}}[1 + F_1(\eta)] \tag{23}$$

$$\psi_2 = C\psi_{\mathrm{KG}}[1 + F_2(\eta)] \tag{24}$$

$$\psi_3 = C\psi_{\mathrm{KG}}[-1 + F_1(\eta)] \tag{25}$$

$$\psi_4 = C\psi_{\mathrm{KG}}[1 - F_2(\eta)] \tag{26}$$

where c is a constant of normalization. This solution corresponds to $F_1 = F_3$ and $F_2 = F_4$.

We introduce the functions from Eqs. (23) to (26) in Eqs. (17)–(20) and, with the aid of Eq. (22), we have:

$$-\psi_{\mathrm{KG}}m^2c^2(1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)F_1(\eta) + \frac{\hbar\,e}{c}2\psi_{\mathrm{KG}}(iE_{\mathrm{M1}}\cos\,\eta + E_{\mathrm{M2}}\sin\,\eta) = 0 \tag{27}$$

$$-\psi_{\mathrm{KG}}m^2c^2(1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)F_2(\eta) - \frac{\hbar\,e}{c}2\psi_{\mathrm{KG}}(iE_{\mathrm{M1}}\cos\,\eta - E_{\mathrm{M2}}\sin\,\eta) = 0 \tag{28}$$

$$-\psi_{\mathrm{KG}}m^2c^2(1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)F_1(\eta) + \frac{\hbar\,e}{c}2\psi_{\mathrm{KG}}(iE_{\mathrm{M1}}\cos\,\eta + E_{\mathrm{M2}}\sin\,\eta) = 0 \tag{29}$$

$$\psi_{\mathrm{KG}}m^2c^2(1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)F_2(\eta) + \frac{\hbar\,e}{c}2\psi_{\mathrm{KG}}(iE_{\mathrm{M1}}\cos\,\eta - E_{\mathrm{M2}}\sin\,\eta) = 0 \tag{30}$$

We consider also the following relations

$$E_{1\mathrm{rms}} = \frac{1}{\sqrt{2}}E_{\mathrm{M1}},\ E_{2\mathrm{rms}} = \frac{1}{\sqrt{2}}E_{\mathrm{M2}} \tag{31}$$

$$E_{\mathrm{S}} = \frac{m^2c^3}{e\hbar},\ Y_{1\mathrm{e}} = \frac{E_{1\mathrm{rms}}}{E_{\mathrm{S}}}\ \text{and}\ Y_{2\mathrm{e}} = \frac{E_{2\mathrm{rms}}}{E_{\mathrm{S}}} \tag{32}$$

where $E_{1\mathrm{rms}}$ and $E_{2\mathrm{rms}}$ are the root mean squares of the components $E_{\mathrm{L}x}$ and $E_{\mathrm{L}y}$ of the electric field, E_{S} is the Schwinger electric field, while $Y_{1\mathrm{e}}$ and $Y_{2\mathrm{e}}$ are parameters entering in QED calculations (Bamber *et al.*, 1999).

The system (27)–(30), together with (31) and (32) lead to the following expressions of the functions F_1 and F_2:

$$F_1 = \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta + Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta} \tag{33}$$

$$F_2 = -\frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta - Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta} \tag{34}$$

Introducing the expressions from (33) and (34) into (23)–(26), we obtain:

$$\psi_1 = C\psi_{\mathrm{KG}}\left[1 + \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta + Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \tag{35}$$

$$\psi_2 = C\psi_{\mathrm{KG}}\left[1 - \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta - Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \tag{36}$$

$$\psi_3 = C\psi_{\mathrm{KG}}\left[-1 + \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta + Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \tag{37}$$

$$\psi_4 = C\psi_{\mathrm{KG}}\left[1 + \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta - Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \tag{38}$$

In the paper Popa (2011), at pages 023824–13, and in the book Popa (2014b), at page 26, we proved, without any approximation, that the Klein–Gordon equation is verified by the function

$$\psi_{\mathrm{KG}} = \exp\left(\frac{iS}{\hbar}\right) \tag{39}$$

where S is the action function, which is the solution of the relativistic Hamilton–Jacobi equation (Landau and Lifshitz, 1959; Jackson, 1999):

$$c^2(\nabla S + e\overline{A}_{\mathrm{L}})^2 - \left(\frac{\partial S}{\partial t}\right)^2 + (mc^2)^2 = 0. \tag{40}$$

In Appendix A we give a brief proof that the function ψ_{KG}, given by Eq. (39), verifies the Klein–Gordon equation and in Appendix B we calculate the expression of the S function.

With the aid of Eq. (39), the solution of the Dirac equation becomes:

$$\psi_1 = C\,\exp\left(\frac{iS}{\hbar}\right)\left[1 + \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta + Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \tag{41}$$

$$\psi_2 = C\,\exp\left(\frac{iS}{\hbar}\right)\left[1 - \frac{2\sqrt{2}(iY_{1\mathrm{e}}\cos\,\eta - Y_{2\mathrm{e}}\sin\,\eta)}{1 + a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \tag{42}$$

$$\psi_3 = C\,\exp\left(\frac{iS}{\hbar}\right)\left[-1+\frac{2\sqrt{2}(iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)}{1+a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \quad (43)$$

$$\psi_4 = C\,\exp\left(\frac{iS}{\hbar}\right)\left[1+\frac{2\sqrt{2}(iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)}{1+a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta}\right] \quad (44)$$

Verification of the solution

We prove now that the solution represented by Eqs. (41)–(44) verifies the continuity equation from QED. According to Eqs. (1.1.28) and (1.1.29) from Weinberg (1995), the continuity equation is given by the following relations:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\bar{J} = 0 \quad (45)$$

with

$$\rho = |\psi|^2,\ \bar{J} = c\psi^+\alpha\psi \quad (46)$$

where ρ is the probability density, $\bar{j}$ is the probability current density and the components of the matrix α are as follows:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix},\quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i\\ 0 & 0 & i & 0\\ 0 & -i & 0 & 0\\ i & 0 & 0 & 0\end{pmatrix},$$

$$\alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\end{pmatrix}$$

Using the above relations, we calculate the expressions of the probability density ρ, and of the components of the current probability, J_x, J_y, and J_z and obtain:

$$\rho = \begin{pmatrix}\psi_1^* & \psi_2^* & \psi_3^* & \psi_4^*\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4 \quad (47)$$

$$J_x = c\psi^+\alpha_1\psi = c\begin{pmatrix}\psi_1^*\psi_2^* & \psi_3^* & \psi_4^*\end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = c(\psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 + \psi_4^*\psi_1) \quad (48)$$

In a similar manner, we obtain

$$J_y = c\psi^+\alpha_2\psi = ci(-\psi_1^*\psi_4 + \psi_2^*\psi_3 - \psi_3^*\psi_2 + \psi_4^*\psi_1) \quad (49)$$

$$J_z = c\psi^+\alpha_3\psi = c(\psi_1^*\psi_3 - \psi_2^*\psi_4 + \psi_3^*\psi_1 - \psi_4^*\psi_2) \quad (50)$$

Noting

$$K = \frac{2\sqrt{2}}{1+a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta} \quad (51)$$

we introduce the components of the wave function, given by Eqs. (41)–(44) in Eq. (47) and obtain:

$$\begin{aligned}\rho &= \psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_3^*\psi_3 + \psi_4^*\psi_4\\ &= C^2[1+K(-iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)]\\ &\quad[1+K(iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)]\\ &\quad + C^2[1-K(-iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)]\\ &\quad[1-K(iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)]\\ &\quad + C^2[-1+K(-iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)]\\ &\quad[-1+K(iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)]\\ &\quad + C^2[1+K(-iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)]\\ &\quad[1+K(iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)]\\ &= C^2[1+K^2(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)]\\ &\quad + C^2\left[\begin{array}{l}K(-iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)\\ \quad + K(iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)\end{array}\right]\\ &\quad + C^2[1+K^2(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)]\\ &\quad + C^2\left[\begin{array}{l}-K(-iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)\\ \quad - K(iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)\end{array}\right]\\ &\quad + C^2[1+K^2(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)]\\ &\quad + C^2\left[\begin{array}{l}-K(-iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)\\ \quad - K(iY_{1e}\cos\,\eta + Y_{2e}\sin\,\eta)\end{array}\right]\\ &\quad + C^2[1+K^2(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)]\\ &\quad + C^2\left[\begin{array}{l}K(-iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)\\ \quad + K(iY_{1e}\cos\,\eta - Y_{2e}\sin\,\eta)\end{array}\right]\\ &= 4C^2[1+K^2(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)]\end{aligned} \quad (52)$$

which can be written as

$$\rho = 4C^2\left[1+\frac{8(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)}{(1+a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)^2}\right] \quad (53)$$

Similarly, we introduce the expressions of the components of the wave function given by Eqs. (41)–(44) in Eqs. (48)–(50) and, by simple calculations, obtain

$$J_x = 0,\quad J_y = 0 \quad (54)$$

$$J_z = 4cC^2\left[-1+\frac{8(Y_{1e}^2\cos^2\,\eta + Y_{2e}^2\sin^2\,\eta)}{(1+a_1^2\sin^2\,\eta + a_2^2\cos^2\,\eta)^2}\right] \quad (55)$$

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We observe that ρ and J_z are periodic functions of only one variable, which is η. From Eqs. (53)–(55) we have:

$$\begin{aligned}\frac{\partial\rho}{\partial t} &= \frac{d\rho}{d\eta}\frac{\partial\eta}{\partial t} = \frac{d\rho}{d\eta}\omega_{\mathrm{L}} \\ &= 32C^2\frac{d}{d\eta}\left[\frac{Y_{1e}^2\cos^2\eta + Y_{2e}^2\sin^2\eta}{(1+a_1^2\sin^2\eta + a_2^2\cos^2\eta)^2}\right]\omega_{\mathrm{L}}\end{aligned} \tag{56}$$

$$\begin{aligned}\nabla\bar{J} &= \frac{\partial J_z}{\partial z} = \frac{dJ_z}{d\eta}\frac{\partial\eta}{\partial z} = \frac{dJ_z}{d\eta}(-|\bar{k}_{\mathrm{L}}|) \\ &= 32cC^2\frac{d}{d\eta}\left[\frac{Y_{1e}^2\cos^2\eta + Y_{2e}^2\sin^2\eta}{(1+a_1^2\sin^2\eta + a_2^2\cos^2\eta)^2}\right](-|\bar{k}_{\mathrm{L}}|)\end{aligned} \tag{57}$$

We see that, in virtue of relations (56), (57), and $|\bar{k}_{\mathrm{L}}|c = \omega_{\mathrm{L}}$, the continuity Eq. (45) is verified.

Solution of the system of equations in the rest frame of the relativistic electron

The analysis of the experimental data from literature shows that the quantum effects are significant in interactions between very intense laser beams and REBs having energies of the order of few tens of GeV (Bula *et al.*, 1996; Burke *et al.*, 1997; Bamber *et al.*, 1999; Kirsebom *et al.*, 2001). For this regime, it is convenient to study the interaction between EM field and electron in the rest reference frame of the electron.

We consider an interaction between the laser beam and a REB which collide head-on with each other. The initial conditions, written in the laboratory reference system, denoted by $S(t, x, y, z)$, are as follows:

$$t=0,\quad x=y=z=0,\quad v_x=v_y=0,\quad v_z=-|V_0| \text{ and } \eta=\eta_i \tag{58}$$

where v_x, v_y, and v_z are the components of the electron velocity in the system S.

The relations (1)–(6) remain valid in the S system, and they are used to calculate the components of the EM field in the inertial system of reference denoted by $S'(t', x', y', z')$, in which the initial velocity of the electron is zero. The Cartesian axes in the systems $S(t, x, y, z)$ and $S'(t', x', y', z')$ are parallel. In our case, the S' system moves with velocity $-|V_0|$ along the oz-axis.

Since the phase of the EM field is a relativistic invariant (Jackson, 1999), the initial conditions in the inertial system S' are as follows:

$$t'=0,\quad x'=y'=z'=0,\quad v'_{x'}=v'_{y'}=v'_{z'}=0 \quad\text{and}\quad \eta=\eta'=\eta_i \tag{59}$$

where $v'_{x'}$, $v'_{y'}$ and $v'_{z'}$ are the components of the electron velocity in S'.

We calculate now the parameters of the laser field, denoted by $\bar{E}'_{\mathrm{L}}$, $\bar{B}'_{\mathrm{L}}$, $\bar{k}'_{\mathrm{L}}$, and ω'_{L}, in the S' system. We denote the four-dimensional wave vectors by $(\omega_{\mathrm{L}}/c, k_{\mathrm{L}x}, k_{\mathrm{L}y}, k_{\mathrm{L}z})$ and $(\omega'_{\mathrm{L}}/c, k'_{\mathrm{L}x'}, k'_{\mathrm{L}y'}, k'_{\mathrm{L}z'})$ in the systems S and S', respectively.

In virtue of the Lorentz transformation, given by relations (11.22) of Jackson (1999), we have

$$\frac{\omega'_{\mathrm{L}}}{c} = \frac{\omega_{\mathrm{L}}}{c}\gamma_0(1+|\bar{\beta}_0|) \tag{60}$$

$$k'_{\mathrm{L}z'} = |\bar{k}'_{\mathrm{L}}| = |\bar{k}_{\mathrm{L}}|\gamma_0(1+|\bar{\beta}_0|) \tag{61}$$

$$k'_{\mathrm{L}x'} = k_{\mathrm{L}x} = k'_{\mathrm{L}y'} = k_{\mathrm{L}y} = 0 \tag{62}$$

where

$$\bar{\beta}_0 = -\frac{|\bar{V}_0|}{c}\bar{k} \text{ and } \gamma_0 = (1-\bar{\beta}_0^2)^{-\frac{1}{2}} \tag{63}$$

The phase of the EM wave is

$$\eta = \omega_{\mathrm{L}}t - \bar{k}_{\mathrm{L}}\cdot\bar{r} + \eta_i = \omega'_{\mathrm{L}}t' - \bar{k}'_L\cdot\bar{r}' + \eta_i = \eta' \tag{64}$$

where $\bar{r}$ and $\bar{r}'$ are the positions vectors of the electron in the two systems.

We write equations (11.149) from Jackson (1999) in IS and with the aid of Eqs. (4) and (63), we obtain the following expressions for the components of the EM field in the S' system:

$$\bar{E}'_{\mathrm{L}} = \gamma_0(\bar{E}_{\mathrm{L}} + \bar{\beta}_0\times c\bar{B}_{\mathrm{L}}) = \gamma_0(1+|\bar{\beta}_0|)\bar{E}_{\mathrm{L}} \tag{65}$$

$$\bar{B}'_{\mathrm{L}} = \gamma_0(\bar{B}_{\mathrm{L}} - \bar{\beta}_0\times\bar{E}_{\mathrm{L}}/c) = \gamma_0(1+|\bar{\beta}_0|)\bar{B}_{\mathrm{L}} \tag{66}$$

where the relations between the amplitudes of the components of the electric field, in the systems S and S', are as follows

$$E'_{\mathrm{M}1} = \gamma_0(1+|\bar{\beta}_0|)E_{\mathrm{M}1} \text{ and } E'_{\mathrm{M}2} = \gamma_0(1+|\bar{\beta}_0|)E_{\mathrm{M}2} \tag{67}$$

In virtue of the relations (60) and (67), it follows that the relativistic parameters are relativistic invariants because we have

$$a'_1 = \frac{eE'_{\mathrm{M}1}}{mc\omega'_{\mathrm{L}}} = \frac{eE_{\mathrm{M}1}}{mc\omega_{\mathrm{L}}} = a_1 \text{ and } a'_2 = \frac{eE'_{\mathrm{M}2}}{mc\omega'_{\mathrm{L}}} = \frac{eE_{\mathrm{M}2}}{mc\omega_{\mathrm{L}}} = a_2 \tag{68}$$

According to the theory presented in Section 68 from Dirac (1958), the Dirac equation has the same form in the inertial systems S and S', and the Eqs. (8) and (12) from pages 257 and 265 of Dirac (1958), can be written in the system S', as follows:

$$[\widehat{p}'_{0'} - \rho_1(\sigma, \widehat{\bar{p}}' + e\bar{A}'_{\mathrm{L}}) - \rho_3 mc]\psi' = 0 \tag{69}$$

and

$$\begin{aligned}&[(\widehat{p}'_{0'})^2 - (\sigma, \widehat{\bar{p}}' + e\bar{A}'_{\mathrm{L}})^2 - m^2c^2]\psi' \\ &+ \left[-\hbar e(\sigma, \bar{B}'_{\mathrm{L}}) + i\rho_1\frac{\hbar e}{c}(\sigma, \bar{E}'_{\mathrm{L}})\right]\psi' = 0\end{aligned} \tag{70}$$

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where

$$(\sigma, \widehat{p}' + e\overline{A}'_{\mathrm{L}}) = \sigma_1(\widehat{p}'_{x'} + eA'_{\mathrm{L}x'}) + \sigma_2(\widehat{p}'_{y'} + eA'_{\mathrm{L}y'}) + \sigma_3(\widehat{p}'_{z'} + eA'_{\mathrm{L}z'}) \tag{71}$$

$$\widehat{p}'_{0'} = \frac{i\hbar}{c}\cdot\frac{\partial}{\partial t'}, \quad \widehat{p}'_{x'} = -i\hbar\frac{\partial}{\partial x'}, \quad \widehat{p}'_{y'} = -i\hbar\frac{\partial}{\partial y'} \text{ and } \widehat{p}'_{z'} = -i\hbar\frac{\partial}{\partial z'} \tag{72}$$

and the matrices ρ and σ have the same form in the systems S and S'.

An identical treatment, as that presented in the subsection Solution of the system, leads to the following system, which is equivalent to the Dirac equation:

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t'}\right)^2 - (-i\hbar\nabla' + e\overline{A}'_{\mathrm{L}})^2 - m^2c^2\right]\psi'_1 + \frac{\hbar e}{c}(\psi'_2 + \psi'_4)(iE'_{\mathrm{M1}}\cos\eta' + E'_{\mathrm{M2}}\sin\eta') = 0 \tag{73}$$

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t'}\right)^2 - (-i\hbar\nabla' + e\overline{A}'_{\mathrm{L}})^2 - m^2c^2\right]\psi'_2 - \frac{\hbar e}{c}(\psi'_1 - \psi'_3)(iE'_{\mathrm{M1}}\cos\eta' - E'_{\mathrm{M2}}\sin\eta') = 0 \tag{74}$$

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t'}\right)^2 - (-i\hbar\nabla' + e\overline{A}'_{\mathrm{L}})^2 - m^2c^2\right]\psi'_3 + \frac{\hbar e}{c}(\psi'_2 + \psi'_4)(iE'_{\mathrm{M1}}\cos\eta' + E'_{\mathrm{M2}}\sin\eta') = 0 \tag{75}$$

$$\left[\frac{1}{c^2}\left(i\hbar\frac{\partial}{\partial t'}\right)^2 - (-i\hbar\nabla' + e\overline{A}'_{\mathrm{L}})^2 - m^2c^2\right]\psi'_4 + \frac{\hbar e}{c}(\psi'_1 - \psi'_3)(iE'_{\mathrm{M1}}\cos\eta - E'_{\mathrm{M2}}\sin\eta) = 0 \tag{76}$$

where $\psi'_1, \ldots, \psi'_4$ of ψ' are the four unknown and the expressions from square brackets in Eqs. (73)–(76) are identical to the operator from the Klein–Gordon equation, divided by $-c^2$ (Messiah, 1962).

An identical procedure, as that presented in the subsection Solution of the system, leads to the following solution of this system:

$$\psi'_1 = C'\psi'_{\mathrm{KG}}\left[1 + \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta + Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{77}$$

$$\psi'_2 = C'\psi'_{\mathrm{KG}}\left[1 - \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta - Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{78}$$

$$\psi'_3 = C'\psi'_{\mathrm{KG}}\left[-1 + \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta + Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{79}$$

$$\psi'_4 = C'\psi'_{\mathrm{KG}}\left[1 + \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta - Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{80}$$

where ψ'_{KG} is the solution of the Klein–Gordon equation, written in the S' system, C' is the constant of normalization, and $Y'_{1\mathrm{e}}$ and $Y'_{2\mathrm{e}}$ are given by the following relations:

$$Y'_{1\mathrm{e}} = \frac{E'_{1\mathrm{rms}}}{E_{\mathrm{S}}} = \frac{1}{\sqrt{2}}\cdot\frac{E'_{\mathrm{M1}}}{E_{\mathrm{S}}} = \frac{1}{\sqrt{2}}\cdot\frac{\gamma_0(1+|\overline{\beta}_0|)E_{\mathrm{M1}}}{E_{\mathrm{S}}} \tag{81}$$

$$Y'_{2\mathrm{e}} = \frac{E'_{2\mathrm{rms}}}{E_{\mathrm{S}}} = \frac{1}{\sqrt{2}}\cdot\frac{E'_{\mathrm{M2}}}{E_{\mathrm{S}}} = \frac{1}{\sqrt{2}}\cdot\frac{\gamma_0(1+|\overline{\beta}_0|)E_{\mathrm{M2}}}{E_{\mathrm{S}}} \tag{82}$$

In Appendix A we prove that

$$\psi'_{\mathrm{KG}} = \exp\left(\frac{iS'}{\hbar}\right) \tag{83}$$

where S' is the action function, written in the inertial system S'. Introducing this expression in Eqs. (77)–(80), we obtain the following solution of the system of equations in the inertial system S'.

$$\psi'_1 = C'\exp\left(\frac{iS'}{\hbar}\right)\left[1 + \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta + Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{84}$$

$$\psi'_2 = C'\exp\left(\frac{iS'}{\hbar}\right)\left[1 - \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta - Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{85}$$

$$\psi'_3 = C'\exp\left(\frac{iS'}{\hbar}\right)\left[-1 + \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta + Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{86}$$

$$\psi'_4 = C'\exp\left(\frac{iS'}{\hbar}\right)\left[1 + \frac{2\sqrt{2}(iY'_{1\mathrm{e}}\cos\eta - Y'_{2\mathrm{e}}\sin\eta)}{1 + a_1^2\sin^2\eta + a_2^2\cos^2\eta}\right] \tag{87}$$

We observe that both, the system of equations which is equivalent to Dirac equation, and its solutions, have the same form if they wrote in the S and S' systems.

Discussion of our solution in the light of experimental data from the literature

In order to simplify the analysis, without loss of generality, we assume in this section that the incident laser field is linearly polarized. That is, its components are

$$\overline{E}_{\mathrm{L}} = E_{\mathrm{M}}\cos\eta\bar{i}, \quad \overline{B}_{\mathrm{L}} = B_{\mathrm{M}}\cos\eta\bar{j} \text{ and } \overline{A}_{\mathrm{L}} = -A_{\mathrm{M}}\sin\eta\bar{i} \tag{88}$$

where

$$E_M = cB_M \quad \text{and} \quad A_M = \frac{E_M}{\omega_L} \tag{89}$$

In this case, the components of the wave function in the system S' become

$$\psi'_1 = C' \exp\left(\frac{iS'}{\hbar}\right)\left(1 + 2\sqrt{2}\cdot iY'_e \cdot \frac{\cos\eta'}{1 + a^2 \sin^2\eta'}\right) \tag{90}$$

$$\psi'_2 = C' \exp\left(\frac{iS'}{\hbar}\right)\left(1 - 2\sqrt{2}\cdot iY'_e \cdot \frac{\cos\eta'}{1 + a^2 \sin^2\eta'}\right) \tag{91}$$

$$\psi'_3 = C' \exp\left(\frac{iS'}{\hbar}\right)\left(-1 + 2\sqrt{2}\cdot iY'_e \cdot \frac{\cos\eta'}{1 + a^2 \sin^2\eta'}\right) \tag{92}$$

$$\psi'_4 = C' \exp\left(\frac{iS'}{\hbar}\right)\left(1 + 2\sqrt{2}\cdot iY'_e \cdot \frac{\cos\eta'}{1 + a^2 \sin^2\eta'}\right) \tag{93}$$

where

$$a = \frac{eE_M}{mc\omega_L} \tag{94}$$

and

$$Y'_e = \frac{E'_{rms}}{E_S} = \frac{1}{\sqrt{2}}\frac{E'_M}{E_S} = \frac{1}{\sqrt{2}}\frac{\gamma_0(1 + |\overline{\beta}_0|)E_M}{E_S} \tag{95}$$

To our knowledge, the only experiments showing QED effects in the literature involve interactions between very intense EM beams and REB. In this respect, we confine ourselves to discussing the experimental data from Bula *et al.* (1996); Burke *et al.* (1997); Bamber *et al.* (1999); and Kirsebom *et al.* (2001). In all these cases the calculations are made in the rest frame of the electron.

Bula *et al.* (1996) and Bamber *et al.* (1999) present experimental data for nonlinear Compton scattering of Nd:glass laser beams at 1.054 and 0.527 μm wavelengths on relativistic electron bunches. The peak laser intensities and electron energies are $I_p = 10^{18}$ W/cm^2 and $E_e = 46.6$ GeV in Bula *et al.* (1996) and $I_p = 0.5 \times 10^{18}$ W/cm^2 and $E_e = 46.6$ and 49.1 GeV in Bamber *et al.* (1999). The values of the QED parameter Y'_e for which the nonlinear Compton scattering effect takes place, are, respectively, 0.17 for $\lambda_L = 1.054$ μm and 0.27 for $\lambda_L = 0.527$ μm (Bamber *et al.*, 1999). In other words, the quantum effects are evidenced experimentally when the parameter Y'_e, given by Eq. (95), has values of the order of few tenths. This happens when the root mean square of the electric field in the system S' is of the order of few tenths of Schwinger field.

Burke *et al.* (1997) present an experiment in which electron–positron pairs are produced at the interaction between a Nd:glass laser beam having $I_p = 10^{19}$ W/cm^2 at a wavelength of 0.527 μm, and a 46.6 GeV electron energy. The conditions of generation of quantum effects are roughly similar to those from Bula *et al.* (1996); Bamber *et al.* (1999).

In Kirsebom *et al.* (2001) the influence of the electron spin on the energy loss of ultra-relativistic electrons in huge EM fields is shown experimentally. According to Fig. 2 from that paper, the minimum value of the electron energy at which this effect takes place is $E_e = 35$ GeV, when the amplitude of the intensity of the electric field is $E_M = 10^{13}$ V/m. With the aid of the relations $\gamma_0 = E_e/(mc^2)$, $|\overline{\beta}_0| = (1 - 1/\gamma_0^2)^{1/2} \cong 1$ and $Y'_e = (2\gamma_0 E_M)/(\sqrt{2}E_S)$, we obtain $\gamma_0 = 6.85 \times 10^4$, and $Y'_e = 0.734$.

On the other hand, the term Y'_e can be written as follows:

$$Y'_e = \frac{E'_{rms}}{E_S} = \frac{e\hbar\, cB'_{rms}}{m^2c^3} = 2\left(\frac{e\hbar}{2m}B'_{rms}\right)\frac{1}{mc^2} \tag{96}$$

where the expression in parenthesis is the energy of interaction between electron spin and a magnetic field having an intensity equal to B'_{rms}. In virtue of the experimental data presented in Kirsebom *et al.* (2001), the interactions between the electron spin and magnetic field are significant when $E_e > 35$ GeV and Y'_e has values of the order of a few tenths of unity. This happens when the value of the energy of interaction between the electron spin and magnetic field is of the order of a few tenths of the value of the rest energy of the electron.

In conclusion, according to the experiments from Bula *et al.* (1996); Burke *et al.* (1997); Bamber *et al.* (1999); Kirsebom *et al.* (2001), the QED effects are evidenced when the electron energy, laser beam intensity and the amplitude of the intensity of the electric field have, respectively, values of the order $E_e = 40$ GeV, $I_p = 10^{18}\ldots10^{19}$ W/cm^2 and $E_M = 10^{13}$ V/m, while the parameter Y'_e has values of a few tenths of unity.

We call "quantum terms" the terms appearing in the expressions of the components of the wave function, which contain the parameter Y'_e. We observe that when the quantum terms are negligible, the components of the wave function in Eqs. (90)–(93) reduce to the wave function associated with classical motion, namely $C' \exp iS'/\hbar$.

The factor $2\sqrt{2}\cos\eta'/(1 + a^2\sin^2\eta')$ from the equations of the components of wave functions, namely Eqs. (90)–(93), takes values between $-2\sqrt{2}$ and $2\sqrt{2}$. A simple calculation shows that the quantum terms from these equations have significant values, of the order of unity, for the values of Y'_e at which quantum effects are evidenced experimentally. These values are shown in Table 1. It follows that the quantum terms are significant when they correspond to parameters at which quantum effects are evidenced experimentally.

It is easy to show that when the electron energy, E_e, is strongly diminished, the quantum terms are much smaller than unity, and the system can be treated classically. It follows that the interaction of electrons with laser beams could be modeled using classical approaches, regardless of the laser beam intensity, as long as the relation $E_e \ll 40$ GeV is valid. This property is also valid in the case of the interactions between non-relativistic electrons and laser beams. In the books Popa (2014b and 2014c), we presented a synthesis of our papers referring to classical approaches of these interactions, when $E_e \ll 40$ GeV. All these approaches lead to results which are in good agreement with the experimental data from the literature.

We show now that our results are in agreement with the Compton relation. The coefficient which enters in the Compton relation, and reflects the quantum behavior of the system, written in the inertial system S', is $Y'_C = \hbar\omega'_L/(mc^2)$. When the coefficient Y'_C has values of the order of unity, the quantum effects

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Table 1. Typical values of parameters a and Y'_e, in the rest frame of the electron, in interactions between very intense laser beams and relativistic electron beams, when quantum effects are significant. Calculations are made for the electron energy $E_e = 40$ GeV, when $\lambda_L = 1.054$ and 0.527 μm, for two peak values of laser intensity, $I_p = 10^{18}$ W/cm^2 and $I_p = 10^{19}$ W/cm^2

	$I_p = 10^{18}$ W/cm^2		$I_p = 10^{19}$ W/cm^2	
λ_L (μm)	1.054	0.527	1.054	0.527
a	0.6372	0.3186	2.0149	1.0075
Y'_e	0.1628	0.1628	0.5148	0.5148

are significant, while when $Y_C \ll 1$, the system is at the classical limit. We consider a typical value of the electron energy, $E_e = 40$ GeV, at which, according to our calculations, the quantum effects become significant, and two values of the wavelength of the EM field, namely $\lambda_L = 1.054 \times 10^{-6}$ m and $\lambda_L = 0.527 \times 10^{-6}$ m. We use the relations $\gamma_0 = E_e/(mc^2)$, $|\overline{\beta}_0| = (1 - 1/\gamma_0^2)^{1/2} \cong 1$, $\omega_L = 2\pi c/\lambda_L$, and $\omega'_L = 2\gamma_0\omega_L$ and obtain, respectively, the values 0.7208 and 0.3604 for Y'_C. In virtue of the Compton relation, these values correspond to significant quantum effects. It follows that there is a concordance between our calculations and the Compton relation. Our treatment is self-consistent, because all our classical approaches presented in Popa (2014b and 2014c) correspond to the conditions $\hbar\,\omega_L/mc^2 \ll 1$ or $\hbar\,\omega'_L/mc^2 \ll 1$, which are valid when the Compton relation is at the classical limit.

The property that the quantum effects are evidenced when the initial electron energies are of the order of few tents of GeV is reflected by the fact that the references "Bamber, Bula and Burke" refer to similar experiments which need very energetic electron accelerators and laser beams having intensities of the order $I_p = 10^{18} \ldots 10^{19}$ W/cm^2.

We note that the experimental data from Bula *et al.* (1996); Burke *et al.* (1997); Bamber *et al.* (1999); Kirsebom *et al.* (2001), for nonlinear Compton scattering, electron–positron pairs production, or for the interaction between electron spin and huge EM fields, correspond to values of the parameter Y'_e of the order of few tenths of unity. For these values, the quantum terms in our relations begin to be significant.

We conclude that our relations are in agreement with the experimental data from the literature, as the quantum terms from the wave functions have significant values for electron energies and laser beam intensities at which quantum effects are evidenced experimentally.

Conclusions

We presented a solution of the system of equations which is equivalent to the Dirac equation, written for the system composed of an electron in EM field, which depends on the phase of the field and on the QED parameter Y_e. This solution predicts significant quantum effects in the case of interactions between very intense laser fields and relativistic electrons, for electron energy and laser beam intensity, having values, respectively, of the orders $E_e = 40$ GeV and $I_p = 10^{18} \ldots 10^{19}$ V/cm^2, in agreement with experiments reported in the literature. The classical models are valid for interactions between very intense laser fields and nonrelativistic electrons, regardless of the laser beam intensity, in agreement with numerous classical approaches from literature.

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Appendix A Solution of the Klein–Gordon equation

We prove briefly that the function $\exp(iS/\hbar)$ verifies the Klein–Gordon equation [see pages 023824–13 from Popa (2011) or page 26 from the book Popa (2014b)]. We need to present this demonstration to show that it is valid also in the inertial system S'. We solve first the relativistic system of the equations of the motion of the electron: Taking into account (1) and (3), the equations of

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motion of the electron are

$$m\frac{d}{dt}(\gamma v_x) = -eE_{M1}\cos\eta + ev_zB_{M1}\cos\eta \tag{A1}$$

$$m\frac{d}{dt}(\gamma v_y) = -eE_{M2}\sin\eta + ev_zB_{M2}\sin\eta \tag{A2}$$

$$m\frac{d}{dt}(\gamma v_z) = -ev_xB_{M1}\cos\eta - ev_yB_{M2}\sin\eta \tag{A3}$$

where v_x, v_y, and v_z are the components of the electron velocities and

$$\gamma = (1-\beta_x^2-\beta_y^2-\beta_z^2)^{-\frac{1}{2}} \tag{A4}$$

with $\beta_x = v_x/c$, $\beta_y = v_y/c$ and $\beta_z = v_z/c$. In our relations e is the absolute value of the electron charge and m is the electron mass.

We consider the following initial conditions

$$\begin{gathered} t=0,\quad x=y=z=0,\quad v_x=v_{xi},\\ v_y=v_{yi},\quad v_z=v_{zi}\ \text{ and }\ \eta=\eta_i \end{gathered} \tag{A5}$$

Using (4), the equations of motion become

$$\frac{d}{dt}(\gamma\beta_x) = -a_1\omega_L(1-\beta_z)\cos\eta \tag{A6}$$

$$\frac{d}{dt}(\gamma\beta_y) = -a_2\omega_L(1-\beta_z)\sin\eta \tag{A7}$$

$$\frac{d}{dt}(\gamma\beta_z) = -\omega_L(a_1\beta_x\cos\eta + a_2\beta_y\sin\eta) \tag{A8}$$

where a_1 and a_2 are the relativistic parameters, which are given by Eq. (7).

We presented an exact solution of the system (A6)–(A8) in the paper Popa (2011). We will present now briefly this solution, as follows.

We multiply (A6), (A7), and (A8), respectively, by β_x, β_y, and β_z. Since $\beta_x^2+\beta_y^2+\beta_z^2 = 1-1/\gamma^2$ their sum leads to

$$\frac{d\gamma}{dt} = -\omega_L(a_1\beta_x\cos\eta + a_2\beta_y\sin\eta) \tag{A9}$$

Taking into account (A8) and (A9) we obtain $d(\gamma\beta_z)/dt = d\gamma/dt$. We integrate this relation with respect to time between 0 and t, using the initial conditions (5), and obtain $\gamma-\gamma_i = \gamma\beta_z - \gamma_i\beta_{zi}$. From (2) we have $d\eta/dt = \omega_L(1-\beta_z)$. From the last two relations, we obtain:

$$1-\beta_z = \frac{1}{\omega_L}\frac{d\eta}{dt} = \frac{f_0}{\gamma}\quad \text{where } f_0 = \gamma_i(1-\beta_{zi}) \tag{A10}$$

with $\beta_{xi} = v_{xi}/c$, $\beta_{yi} = v_{yi}/c$, $\beta_{zi} = v_{zi}/c$ and $\gamma_i = 1/\sqrt{1-\beta_{xi}^2-\beta_{yi}^2-\beta_{zi}^2}$.

We integrate (A6) with respect to time between 0 and t, and to phase between η_i and η, taking into account the initial conditions (A5), and, changing the variables t and η in agreement with (A10), we obtain

$$\begin{aligned} \gamma\beta_x - \gamma_i\beta_{xi} &= -a_1\omega_L\int_0^t (1-\beta_z)\cos\eta\, dt\\ &= -a_1\int_{\eta_i}^{\eta}\cos\eta\, d\eta = -a_1(\sin\eta - \sin\eta_i) \end{aligned} \tag{A11}$$

or

$$\beta_x = \frac{f_1}{\gamma}\quad \text{where } f_1 = -a_1(\sin\eta - \sin\eta_i) + \gamma_i\beta_{xi} \tag{A12}$$

Similarly, integrating (A7) and taking into account (A5) and (A10), we obtain

$$\beta_y = \frac{f_2}{\gamma}\quad \text{where } f_2 = -a_2(\cos\eta_i - \cos\eta) + \gamma_i\beta_{yi} \tag{A13}$$

From (A10) we obtain

$$\beta_z = \frac{f_3}{\gamma}\quad \text{where } f_3 = \gamma - f_0 \tag{A14}$$

We substitute the expressions of β_x, β_y, and β_z, respectively, from (A12), (A13), and (A14) into (A4) and obtain:

$$\gamma = \frac{1}{2f_0}(1+f_0^2+f_1^2+f_2^2) \tag{A15}$$

Using the same procedure, in Appendix B we calculate the expressions of the electron coordinates and of the action.

The analysis of the relations (A12)–(A15) shows that the functions β_x, β_y, and β_z are periodic functions of only one variable, that is η.

We consider now the following relations (Landau and Lifshitz, 1959; Jackson, 1999):

$$\bar{p} = \gamma m\bar{v},\quad H = \gamma mc^2 \tag{A16}$$

$$\nabla S = \bar{p} - e\bar{A}_L \text{ and } H = -\frac{\partial S}{\partial t} \tag{A17}$$

With the aid of these relations, together with (A12)–(A15), we need to calculate the following expression:

$$\begin{aligned} &\left[\nabla(\nabla S + e\bar{A}_L) - \frac{\partial^2 S}{c^2\partial t^2}\right] = \nabla\bar{p} + \frac{1}{c^2}\frac{\partial H}{\partial t}\\ &= \frac{\partial}{\partial x}(mc\gamma\beta_x) + \frac{\partial}{\partial y}(mc\gamma\beta_y)\\ &\quad + \frac{\partial}{\partial z}(mc\gamma\beta_z) + \frac{1}{c^2}\frac{\partial}{\partial t}(mc^2\gamma)\\ &= mc\frac{df_1}{d\eta}\frac{\partial\eta}{\partial x} + mc\frac{df_2}{d\eta}\frac{\partial\eta}{\partial y} + mc\frac{df_3}{d\eta}\frac{\partial\eta}{\partial z}\\ &\quad + m\frac{d\gamma}{d\eta}\frac{\partial\eta}{\partial t} = -mc\frac{df_3}{d\eta}|\bar{k}_L| + m\frac{d\gamma}{d\eta}\omega_L \end{aligned} \tag{A18}$$

From (2) and (A14) we have, respectively, $c|k_L| = \omega_L$ and $df_3/d\eta = d\gamma/d\eta$, so (A18) becomes:

$$\left[\nabla(\nabla S + e\bar{A}_L) - \frac{\partial^2 S}{c^2\partial t^2}\right] = 0 \tag{A19}$$

We observe that the expression from the first member of the above relation is the divergence of the energy-momentum four vector.

The relativistic Hamilton–Jacobi and Klein–Gordon equations are given by the following relations (Landau and Lifshitz, 1959; Messiah, 1962; Jackson, 1999):

$$c^2(\nabla S + e\bar{A}_L)^2 - \left(\frac{\partial S}{\partial t}\right)^2 + (mc^2)^2 = 0. \tag{A20}$$

$$\left[c^2(-i\hbar\nabla + e\bar{A}_L)^2 - \left(i\hbar\frac{\partial}{\partial t}\right)^2 + (mc^2)^2\right]\psi = 0 \tag{A21}$$

where ψ is the wave function. Recall that e is the absolute value of the electron charge.

In the paper Popa (2011), at pages 023824–13, we proved that the Klein–Gordon equation is verified by the function $\exp(iS/\hbar)$. In order to check this, we substitute $\psi = \exp(iS/\hbar)$ in Eq. (A21), and obtain

$$\begin{aligned}&\left[c^2(-i\hbar\nabla + e\bar{A}_{\mathrm{L}})^2 - \left(i\hbar\frac{\partial}{\partial t}\right)^2 + (mc^2)^2\right]\exp\left(\frac{iS}{\hbar}\right)\\ &= \left[c^2(\nabla S + e\bar{A}_{\mathrm{L}})^2 - \left(\frac{\partial S}{\partial t}\right)^2 + (mc^2)^2\right]\exp\left(\frac{iS}{\hbar}\right)\\ &\quad - i\hbar c^2\left[\nabla(\nabla S + e\bar{A}_{\mathrm{L}}) - \frac{\partial^2 S}{c^2\partial t^2}\right]\exp\left(\frac{iS}{\hbar}\right)\end{aligned} \quad (A22)$$

Taking into account Eqs. (A19) and (A20), the relation (A22) becomes

$$\left[c^2(-i\hbar\nabla + e\bar{A}_{\mathrm{L}})^2 - \left(i\hbar\frac{\partial}{\partial t}\right)^2 + (mc^2)^2\right]\exp\left(\frac{iS}{\hbar}\right) = 0 \quad (A23)$$

and the solution of the Klein–Gordon equation is verified.

We write now the above equations in the inertial system S', which is described in the section Solution of the system of equations in the rest frame of the relativistic electron. It is easy to prove that the relativistic system of equations of motion of the electron, (A6)–(A8), has the same form in the inertial systems S and S' [see (Popa, 2011), pages 023824–9], and we can write

$$\frac{d}{dt'}(\gamma'\beta'_{x'}) = -a'_1\omega'_{\mathrm{L}}(1-\beta'_{z'})\cos\eta' \quad (A24)$$

$$\frac{d}{dt'}(\gamma'\beta'_{y'}) = -a'_2\omega'_{\mathrm{L}}(1-\beta'_{z'})\sin\eta' \quad (A25)$$

$$\frac{d}{dt'}(\gamma'\beta'_{z'}) = -\omega'_{\mathrm{L}}(a'_1\beta'_{x'}\cos\eta' + a'_2\beta'_{y'}\sin\eta') \quad (A26)$$

where $a'_1 = a_1$ and $a'_2 = a_2$ are given by Eq. (68) and the initial conditions in the systems S and S' are given, respectively, by Eqs. (58) and (59).

An identical solution, as that for Eqs. (A6)–(A8), leads to

$$\beta'_{x'} = \frac{f'_1}{\gamma'} \quad \text{where } f'_1 = -a_1(\sin\eta' - \sin\eta_i) \quad (A27)$$

$$\beta'_{y'} = \frac{f'_2}{\gamma'} \quad \text{where } f'_2 = -a_2(\cos\eta_i - \cos\eta') \quad (A28)$$

$$\beta'_{z'} = \frac{f'_3}{\gamma'} \quad \text{where } f'_3 = \gamma' - 1 \quad (A29)$$

where

$$\gamma' = \frac{1}{2}(2 + f'^2_1 + f'^2_2) \quad (A30)$$

Since the relativistic relations (A16) and (A17) have the same form in the inertial systems S and S', we have $\bar{p}' = \gamma' m\bar{v}'$, $H' = \gamma' mc^2$, $\nabla' S' = \bar{p}' - e\bar{A}'_{\mathrm{L}}$ and $H' = -\partial S'/\partial t'$. According to these relations, together with (A27)–(A30), an identical calculation, as that from Eq. (A18), leads to the following relation:

$$\left[\nabla'(\nabla' S' + e\bar{A}'_{\mathrm{L}}) - \frac{\partial^2 S'}{c^2\partial t'^2}\right] = 0 \quad (A31)$$

It follows that the divergence of the energy-momentum four vector is also equal to zero in the system S'.

The relativistic Hamilton–Jacobi and Klein–Gordon equations can be written in the system S', as follows:

$$c^2(\nabla' S' + e\bar{A}'_{\mathrm{L}})^2 - \left(\frac{\partial S'}{\partial t'}\right)^2 + (mc^2)^2 = 0. \quad (A32)$$

$$\left[c^2(-i\hbar\nabla' + e\bar{A}'_{\mathrm{L}})^2 - \left(i\hbar\frac{\partial}{\partial t'}\right)^2 + (mc^2)^2\right]\psi' = 0 \quad (A33)$$

We substitute $\psi' = \exp(iS'/\hbar)$ in Eq. (A33), and obtain

$$\begin{aligned}&\left[c^2(-i\hbar\nabla' + e\bar{A}'_{\mathrm{L}})^2 - \left(i\hbar\frac{\partial}{\partial t'}\right)^2 + (mc^2)^2\right]\exp\left(\frac{iS'}{\hbar}\right)\\ &= \left[c^2(\nabla' S' + e\bar{A}'_{\mathrm{L}})^2 - \left(\frac{\partial S'}{\partial t'}\right)^2 + (mc^2)^2\right]\exp\left(\frac{iS'}{\hbar}\right)\\ &\quad - i\hbar c^2\left[\nabla'(\nabla' S' + e\bar{A}'_{\mathrm{L}}) - \frac{\partial^2 S'}{c^2\partial t'^2}\right]\exp\left(\frac{iS'}{\hbar}\right)\end{aligned} \quad (A34)$$

Taking into account the Eqs. (A31) and (A32), the relation (A34) becomes

$$\left[c^2(-i\hbar\nabla' + e\bar{A}'_{\mathrm{L}})^2 - \left(i\hbar\frac{\partial}{\partial t'}\right)^2 + (mc^2)^2\right]\exp\left(\frac{iS'}{\hbar}\right) = 0 \quad (A35)$$

and the solution of the Klein–Gordon equation, in the S' system, is verified.

Appendix B Calculations of electron coordinates and action

We obtain the expressions of the electron coordinates using the procedure presented in Appendix A. Thus, to calculate x, we integrate the expression of β_x, given by Eq. (A12), between 0 and t, taking into account (A10) and the initial conditions (A5). From (A10) we have $dt = [\gamma/(\omega_{\mathrm{L}}f_0)]d\eta$ and obtain

$$\begin{aligned}x &= c\int_0^t \beta_x dt = \frac{c}{\omega_{\mathrm{L}}f_0}\int_{\eta_i}^{\eta}[-a_1(\sin\eta - \sin\eta_i) + \gamma_i\beta_{xi}]d\eta\\ &= \frac{c}{\omega_{\mathrm{L}}f_0}[a_1(\cos\eta - \cos\eta_i) + c_1(\eta - \eta_i)]\end{aligned} \quad (B1)$$

Similarly, integrating the expressions of β_y and β_z, given by Eqs. (A13) and (A14), we obtain

$$y = \frac{c}{\omega_{\mathrm{L}}f_0}[a_2(\sin\eta - \sin\eta_i) + c_2(\eta - \eta_i)] \quad (B2)$$

$$\begin{aligned}z &= c\int_0^t \beta_z dt = \frac{c}{\omega_{\mathrm{L}}f_0}\int_{\eta_i}^{\eta}(\gamma - f_0)d\eta\\ &= \frac{c}{\omega_{\mathrm{L}}2f_0^2}\int_{\eta_i}^{\eta}[1 - f_0^2 + (-a_1\sin\eta + c_1)^2\\ &\quad + (a_2\cos\eta + c_2)^2]d\eta\\ &= -\frac{c}{8\omega_{\mathrm{L}}f_0^2}(a_1^2 - a_2^2)(\sin 2\eta - \sin 2\eta_i)\\ &\quad + \frac{c}{\omega_{\mathrm{L}}f_0^2}[a_1c_1(\cos\eta - \cos\eta_i)\\ &\quad + a_2c_2(\sin\eta - \sin\eta_i)] + \frac{cc_3}{2\omega_{\mathrm{L}}f_0^2}(\eta - \eta_i)\end{aligned} \quad (B3)$$

The constants c_1, c_2, and c_3 from Eqs. (B1)–(B3) are given by the following relations:

$$c_1 = a_1\sin\eta_i + \gamma_i\beta_{xi} \quad (B4)$$

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$$c_2 = -a_2 \cos \eta_i + \gamma_i \beta_{yi} \tag{B5}$$

$$c_3 = 1 - f_0^2 + \frac{1}{2}(a_1^2 + a_2^2) + c_1^2 + c_2^2 \tag{B6}$$

The analysis of the relations (B1)–(B3) shows that the coordinates x, y, and z are functions of only one variable, which is η.

We calculate now the S action. In virtue of the relations (5), (A16), and (A17), the variation of the action can be written

$$\begin{aligned} dS &= \frac{\partial S}{\partial x}dx + \frac{\partial S}{\partial y}dy + \frac{\partial S}{\partial z}dz + \frac{\partial S}{\partial t}dt \\ &= (p_x - eA_{Lx})dx + (p_y - eA_{Ly})dy + p_z dz - \gamma mc^2 dt \\ &= (\overline{p} \cdot \overline{v} - eA_{Lx}v_x - eA_{Ly}v_y - \gamma mc^2)dt \end{aligned} \tag{B7}$$

where $A_{Lx} = -A_{M1}\sin \eta$ and $A_{Ly} = A_{M2}\cos \eta$.

We calculate the terms of this relation, as follows. From (A4) and (A16) we have:

$$\overline{p} \cdot \overline{v} - \gamma mc^2 = -mc^2\gamma(1 - \overline{\beta}^2) = -\frac{mc^2}{\gamma} \tag{B8}$$

From (6), (7), (A12), (A13), (B4), and (B5) we obtain

$$\begin{aligned} &-eA_{Lx}v_x - eA_{Ly}v_y = ceA_{M1} \sin \eta \frac{f_1}{\gamma} - ceA_{M2} \cos \eta \frac{f_2}{\gamma} \\ &= \frac{mc^2 a_1}{\gamma}(-a_1 \sin \eta + c_1) \sin \eta - \frac{mc^2 a_2}{\gamma}(a_2 \cos \eta + c_2) \cos \eta \end{aligned} \tag{B9}$$

From Eq. (A10) we have $dt = [\gamma/(\omega_L f_0)]d\eta$. We use this relation together with Eqs. (B7)–(B9) and obtain:

$$\begin{aligned} dS = &-\frac{mc^2 a_1^2}{\omega_L f_0}\sin^2 \eta \cdot d\eta - \frac{mc^2 a_2^2}{\omega_L f_0}\cos^2 \eta \cdot d\eta \\ &+ \frac{mc^2 a_1 c_1}{\omega_L f_0}\sin \eta \cdot d\eta - \frac{mc^2 a_2 c_2}{\omega_L f_0}\cos \eta \cdot d\eta - \frac{mc^2}{\omega_L f_0}d\eta \end{aligned} \tag{B10}$$

We integrate (B10) with respect to phase between η_i and η and have

$$\begin{aligned} S = &\frac{mc^2}{4\omega_L f_0}(a_1^2 - a_2^2)\int_{2\eta_i}^{2\eta} \cos 2\eta \cdot d(2\eta) \\ &+ \frac{mc^2 a_1 c_1}{\omega_L f_0}\int_{\eta_i}^{\eta} \sin \eta \cdot d\eta - \frac{mc^2 a_2 c_2}{\omega_L f_0} \\ &\times \int_{\eta_i}^{\eta} \cos \eta \cdot d\eta - \frac{mc^2}{\omega_L f_0}\left[1 + \frac{1}{2}(a_1^2 + a_2^2)\right](\eta - \eta_i) \end{aligned} \tag{B11}$$

or

$$\begin{aligned} S = &\frac{mc^2}{4\omega_L f_0}(a_1^2 - a_2^2)(\sin 2\eta - \sin 2\eta_i) \\ &- \frac{mc^2}{\omega_L f_0}[a_1 c_1(\cos \eta - \cos \eta_i) + a_2 c_2(\sin \eta - \sin \eta_i)] \\ &- \frac{mc^2}{\omega_L f_0}\left[1 + \frac{1}{2}(a_1^2 + a_2^2)\right](\eta - \eta_i) \end{aligned} \tag{B12}$$

It follows that S is written in the form of a function of only one variable, which is η.