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PAPER

Increasing stability for the inverse source problem in elastic waves with attenuation

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Abstract
This paper is concerned with the increasing stability of the inverse source problem for the elastic wave equation with attenuation in three dimensions. The stability estimate consists of the Lipschitz type data discrepancy and the high frequency tail of the source function, where the latter decreases as the upper bound of the frequency increases. The stability also shows exponential dependence on the attenuation coefficient. The ingredients of the analysis include Carleman estimates and time decay estimates for the elastic wave equation to obtain an exact observability bound, and the study of the resonance-free region and an upper bound of the resolvent in this region for the elliptic operator with respect to the complex frequency. The advantage of the method developed in this work is that it can be used to study the case of variable attenuation coefficient.

1. Introduction

Inverse source problems have been an active research topic in inverse scattering theory due to their significant applications in both science and engineering including antenna synthesis, seismic imaging and biomedical imaging [4, 6, 13, 15, 22]. Theoretically, it is known in general that there is no uniqueness for the inverse source problem at a fixed frequency due to the existence of non-radiating sources [1, 2, 16]. From the computational point of view, a more challenging issue is the lack of stability. A small variation of the data might lead to a huge error in the reconstruction. Recently, it has been realised that the use of multi-frequency data is an effective approach to overcome the difficulties of non-uniqueness and instability which are encountered at a single frequency [5]. The increasing stability estimates for the inverse source problems by multi-frequency measurements have been extensively studied in [6, 8, 11, 15, 19, 20, 23]. We also mention some recent works [3, 9, 10] on inverse scattering problems of elastic wave equations. This paper is concerned with the stability for the inverse source problem of the elastic wave equation with attenuation from multi-frequency boundary measurements.

We consider the three-dimensional elastic wave equation

$$-\Delta^*\mathbf{u} - \omega^2\mathbf{u} - \mathrm{i}\omega\sigma\mathbf{u} = \mathbf{f}(x), \quad x \in \mathbb{R}^3, \tag{1.1}$$

where the positive constant $\sigma > 0$ is a damping or attenuation coefficient, $\Delta^* = \lambda\Delta + (\lambda + \mu)\nabla\nabla\cdot$, $\omega > 0$ is the frequency. Here, λ and μ are the Lamé parameters satisfying $\mu > 0$ and $3\lambda + 2\mu > 0$, the vector $\mathbf{u}$ denotes the outgoing elastic field and the source function $\mathbf{f} \in L^\infty\left(\mathbb{R}^3\right)^3$ is assumed to have a compact support contained in the ball B_R. We are interested in the inverse problem of determining the source function $\mathbf{f}(x)$ by multi-frequency boundary measurements $\{\mathbf{u}(x,\omega)|_{\partial B_R}, \nabla\mathbf{u}(x,\omega)|_{\partial B_R}\}$ with ω given in a finite interval.

The main result in this paper is the derivation of the increasing stability estimate for the elastic wave modelled by the Navier equation with a damping coefficient. Motivated by [15], by the Fourier transform the inverse source problem is reduced to the identification of the initial data for the initial value problem of the time-domain damped elastic wave equation by lateral Cauchy data. Then, we obtain an exact observability bound for the source function using the Carleman estimates, which connects the scattering data and the unknown source function by taking the inverse Fourier transform. The Fourier transform is justified by proving an appropriate rate of time decay for the time-domain damped elastic wave equation. Using the resolvent estimates for the elastic wave equation, we obtain a sectorial resonance-free region and resolvent estimates for the data with respect to the complex frequency in this region, which lead to the bound of the analytic continuation of the data from the given data to the higher frequency data. By tracing the dependence of the bound for analytic continuation and of the exact observability bound for the elastic wave equation on the attenuation coefficient, we show the exponential dependence of increasing stability on the damping constant. An important ingredient of the analysis is the application of the Helmholtz decomposition to the elastic wave. The stability estimate consists of the Lipschitz type of data discrepancy and the high wavenumber tail for the source function. The latter decreases as the wavenumber of the data increases, which implies that the inverse problem is more stable when the higher wavenumber data is used. However, the stability deteriorates as the damping constant becomes larger. We point out that the method in this work can be used to deal with the case of variable attenuation coefficient.

This paper is organised as follows. Section 2 is devoted to the well-posedness of the direct scattering problem. In Section 3, we prove an exact observability bound for elastic wave equations. By Carleman estimates for wave equations, we trace the dependence of the exact observability bound on the attenuation coefficient. Section 4 is devoted to the proof of the stability estimate. We employ scattering theory to obtain resolvent estimates for the elliptic operator which gives explicit bounds for analytic continuation. The Fourier transform in time is justified using the decay estimates of the damped elastic wave equation, which is obtained from the decay estimates for the acoustic wave equation and the Helmholtz decomposition. Section 5 is devoted to the decay estimates of the damped acoustic wave.

2. Direct source problem

Using the Helmholtz decomposition, the source function $\mathbf{f} \in L^2(B_R)^3$ can be decomposed as

$$\mathbf{f} = \mathbf{f}_p + \mathbf{f}_s,$$

where $\mathbf{f}_p, \mathbf{f}_s \in L^2(B_R)^3$ with $\nabla\times\mathbf{f}_p = 0$ and $\nabla\cdot\mathbf{f}_s = 0$. Hence, the solution $\mathbf{u}$ to the equation (1.1) can be decomposed into the pressure wave $\mathbf{u}_p$ and shear wave $\mathbf{u}_s$

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s \tag{2.1}$$

where $\nabla\times\mathbf{u}_p = 0$, $\nabla\cdot\mathbf{u}_s = 0$ and

$$\Delta\mathbf{u}_p + k_p^2\mathbf{u}_p = \mathbf{f}_p, \quad \Delta\mathbf{u}_s + k_s^2\mathbf{u}_s = \mathbf{f}_s. \tag{2.2}$$

Here, $k_p = \sqrt{\dfrac{\omega^2 + i\omega\sigma}{\lambda + 2\mu}}$, $k_s = \sqrt{\dfrac{\omega^2 + i\omega\sigma}{\mu}}$ are the wavenumbers for the damped pressure and shear waves. Actually, one has

$$\mathbf{u}_p = -\frac{1}{k_p^2}\nabla\nabla\cdot\mathbf{u}, \quad \mathbf{u}_s = -\frac{1}{k_s^2}\nabla\times\nabla\times\mathbf{u} \quad \text{in } \mathbb{R}^3. \tag{2.3}$$

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Motivated by the Helmholtz decomposition (2.1)–(2.2), to investigate the direct scattering problem of the elastic wave we first study the damped Helmholtz equation. Consider the following Helmholtz equation:

$$-\Delta u(x,k) - k^2 u(x,k) - \mathrm{i}k\sigma u(x,k) = f(x), \quad x \in \mathbb{R}^3. \tag{2.4}$$

Here, we note that u and f are scalar-valued functions. The following theorem concerns its well-posedness.

Theorem 2.1. *Given $f \in L^2\left(\mathbb{R}^3\right)$ with a compact support, there exists a unique exponentially decaying outgoing solution $u \in H^2\left(\mathbb{R}^3\right)$ to (2.4) for every $k > 0$ with the following estimate:*

$$|u(x,k)| \le C(f)e^{-c(k,\sigma)|x|}$$

as $|x| \longrightarrow \infty$. Here, $C(f)$ and $c(k,\sigma)$ are positive constants depending on f and k, σ, respectively.

Proof. We define

$$u^*(x,k) = \int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi} \frac{1}{|\xi|^2 - k^2 - \mathrm{i}k\sigma} \hat{f}(\xi)\mathrm{d}\xi, \quad x \in \mathbb{R}^3,$$

where

$$\hat{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} f(x)e^{-\mathrm{i}x\cdot\xi}\mathrm{d}x.$$

This definition is motivated by taking the Fourier transform of $u(x,k)$ formally with respect to the spatial variable x. Then by the Plancherel's theorem, for each $k > 0$ one has that $u^*(\cdot, k) \in H^2\left(\mathbb{R}^3\right)$ and satisfies equation (2.4).

Since

$$G(x,k) = \int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi} \frac{1}{|\xi|^2 - k^2 - \mathrm{i}k\sigma} \mathrm{d}\xi = \frac{e^{\mathrm{i}\kappa|x|}}{4\pi|x|},$$

where $\kappa = (k^2 + \mathrm{i}k\sigma)^{\frac{1}{2}}$ with $\Im\kappa > 0$, one can rewrite $u^*(x,k)$ as

$$u^*(x,k) = (G * f)(x) = \int_{\mathbb{R}^3} \frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|} f(y)\mathrm{d}y. \tag{2.5}$$

Thus, since f has a compact support by (2.5) one has that the solution $u^*(x,k)$ satisfies the estimate

$$|u^*(x,k)| \le C(f)e^{-c(k,\sigma)|x|},$$

where $C(f)$ and $c(k,\sigma)$ are positive constants depending on f and k, σ, respectively. Using direct calculations, one may show that ∇u^* and Δu^* have similar exponential decay estimates.

Now we prove the uniqueness. Assume that $\tilde{u}^*(x,k)$ is another solution to (2.4). Then one has

$$(-\Delta - k^2 - \mathrm{i}k\sigma)(u^* - \tilde{u}^*) = 0$$

and applying Fourier transform to the above equation gives

$$\left(|\xi|^2 - k^2 - \mathrm{i}k\sigma\right)\left(\widehat{u^* - \tilde{u}^*}\right)(\xi) = 0.$$

Since for $k > 0$ one has that $|\xi|^2 - k^2 - ik\sigma \neq 0$ for all $\xi \in \mathbb{R}^3$, the inverse Fourier transform gives $u^* - \tilde{u}^* = 0$, which proves the uniqueness. □

The well-posedness of the direct scattering problem is a direct consequence of the Helmholtz decomposition (2.1)–(2.2) and Theorem 2.1 for the Helmholtz equation.

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Theorem 2.2. *Let* $\mathbf{f} \in L^2\left(\mathbb{R}^3\right)^3$ *with a compact support. Then there exists a unique outgoing solution* $\mathbf{u}$ *of Schwartz distribution to* (1.1) *for every* $\omega > 0$*. Moreover, the solution satisfies*

$$|\mathbf{u}(x, \omega)| \leq C(\mathbf{f}) e^{-c(\omega,\sigma,\lambda,\mu)|x|}$$

as $|x| \to \infty$*, where* $C(\mathbf{f})$ *and* $c(\omega, b, \lambda, \mu)$ *are positive constants depending on* $\mathbf{f}$*, and* ω, σ *and Lamé parameters, respectively.*

3. Exact observability bounds for elastic wave equations

In order to bound the unknown source $\mathbf{f}$ by boundary data of $\mathbf{u}$ (see Lemma 4.2), we will prove an observability bound for the initial data of the corresponding time-domain damped elastic wave equation by noting that the solution to the time-harmonic damped elastic wave equation can be connected with the time-domain damped elastic wave equation by Fourier transformation.

We will derive an exact observability bound for the initial data $\mathbf{f}$ of the following time-domain damped elastic wave equation

$$\begin{cases} \partial_t^2 \mathbf{U}(x,t) - \Delta^* \mathbf{U}(x,t) + \sigma \partial_t \mathbf{U}(x,t) = 0, & (x,t) \in \mathbb{R}^3 \times (0, +\infty), \\ \mathbf{U}(x,0) = 0, \quad \partial_t \mathbf{U}(x,0) = \mathbf{f}(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.1}$$

Using Helmholtz decomposition again to the vector-valued initial condition $\mathbf{f} \in L^2(B_R)^3$, one has

$$\mathbf{f} = \mathbf{f}_p + \mathbf{f}_s,$$

where $\mathbf{f}_p, \mathbf{f}_s \in L^2(B_R)^3$ with $\nabla \times \mathbf{f}_p = 0$ and $\nabla \cdot \mathbf{f}_s = 0$. As a consequence, we decompose the solution $\mathbf{U}$ to the equation (3.1) into a sum of pressure wave $\mathbf{U}_p$ and shear wave $\mathbf{U}_s$ (see, e.g., [4])

$$\mathbf{U} = \mathbf{U}_p + \mathbf{U}_s \tag{3.2}$$

where $\mathbf{U}_p$ and $\mathbf{U}_s$ satisfy $\nabla \times \mathbf{U}_p = 0$, $\nabla \cdot \mathbf{U}_s = 0$ and the following damped pressure and shear wave equations

$$\begin{cases} \partial_{tt} \mathbf{U}_p - c_p^2 \Delta \mathbf{U}_p + \sigma \partial_t \mathbf{U}_p = 0, & (x,t) \in \mathbb{R}^3 \times (0, \infty), \\ \mathbf{U}_p(x,0) = 0, \quad \partial_t \mathbf{U}_p(x,0) = \mathbf{f}_p(x), & x \in \mathbb{R}^3, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \partial_{tt} \mathbf{U}_s - c_s^2 \Delta \mathbf{U}_s + \sigma \partial_t \mathbf{U}_s = 0, & (x,t) \in \mathbb{R}^3 \times (0, \infty), \\ \mathbf{U}_s(x,0) = 0, \quad \partial_t \mathbf{U}_s(x,0) = \mathbf{f}_s(x), & x \in \mathbb{R}^3. \end{cases} \tag{3.4}$$

Here, $c_p = \sqrt{\lambda + 2\mu}$ and $c_s = \sqrt{\mu}$ are the wave speeds for pressure and shear waves, respectively.

The following Carleman estimate is useful in deriving the exact observability bound.

Lemma 3.1. *Let* $T > 2R + 1$, $\varphi(x,t) = |x - a|^2 - \theta^2 \left(t - \frac{T}{2}\right)^2$ *where* $a \notin \overline{B_R}$*, dist(a,* $\partial B_R) = 1$ *and* $\theta = \frac{1}{2}$*. Let* $\mathbf{U}$ *be a solution to* (3.1) *with* $\mathbf{f} \in H^1(B_R)^3$*, supp*$\mathbf{f} \subset B_R$*. Then we have the following Carleman estimate:*

$$\begin{aligned} &\sum_{|\alpha| \leq 1} s^{3-2|\alpha|} \int_{B_R \times (0,T)} |\partial_\alpha \mathbf{U}|^2 e^{2s\varphi} dxdt \\ &\lesssim \int_{\partial B_R \times (0,T)} \left(s^3 |\partial_{tt} \nabla \mathbf{U}|^2 + s^3 |\partial_t \mathbf{U}|^2 \right) e^{2s\varphi} d\Gamma(x) dt, \end{aligned}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^4$ *and* $\partial_\alpha = \partial_t^{\alpha_1} \partial_{x_1}^{\alpha_2} \partial_{x_2}^{\alpha_3} \partial_{x_3}^{\alpha_4}$.

Proof. Let $v = \nabla \cdot \mathbf{U}$ and $\mathbf{w} = \nabla \times \mathbf{U}$. Assume $\mathbf{U}$ is the solution to (3.1). By Carleman estimate in [14] or [7] one has that

$$
\begin{aligned}
&\sum_{|\alpha|\le 1} s^{3-2|\alpha|} \int_{B_R\times(0,T)} \Big(|\partial_\alpha \mathbf{U}|^2 + |\partial_\alpha v|^2 + |\partial_\alpha \mathbf{w}|^2\Big) e^{2s\varphi}\,\mathrm{d}x\mathrm{d}t \\
&\lesssim \int_{\partial B_R\times(0,T)} s\Big(|\partial_t v|^2 + |\nabla v|^2 + s^2 v^2\Big) e^{2s\varphi}\,\mathrm{d}\Gamma(x)\mathrm{d}t \\
&\quad + \int_{\partial B_R\times(0,T)} s\Big(|\partial_t \mathbf{w}|^2 + |\nabla \mathbf{w}|^2 + s^2 \mathbf{w}^2\Big) e^{2s\varphi}\,\mathrm{d}\Gamma(x)\mathrm{d}t.
\end{aligned} \tag{3.5}
$$

Since the Helmholtz decomposition (3.2) gives

$$
\Delta v = (c_p^2)^{-1}\partial_{tt} v + (c_p^2)^{-1}\sigma\,\partial_t v, \qquad \Delta \mathbf{w} = (c_s^2)^{-1}\partial_{tt}\mathbf{w} + (c_s^2)^{-1}\sigma\,\partial_t \mathbf{w},
$$

by the elliptic regularity theory one has

$$
\|v\|^2_{H^2(\partial B_R)} \lesssim \|\nabla \mathbf{U}\|^2_{L^2(\partial B_R)^{3\times 3}} + \|\partial_t \nabla \mathbf{U}\|^2_{L^2(\partial B_R)^{3\times 3}} + \|\partial_{tt} \nabla \mathbf{U}\|^2_{L^2(\partial B_R)^{3\times 3}}
$$

and

$$
\|\mathbf{w}\|^2_{H^2(\partial B_R)^3} \lesssim \|\nabla \mathbf{U}\|^2_{L^2(\partial B_R)^{3\times 3}} + \|\partial_t \nabla \mathbf{U}\|^2_{L^2(\partial B_R)^{3\times 3}} + \|\partial_{tt} \nabla \mathbf{U}\|^2_{L^2(\partial B_R)^{3\times 3}}.
$$

Hence, from (3.5) we have

$$
\begin{aligned}
&\sum_{|\alpha|\le 1} s^{3-2|\alpha|} \int_{B_R\times(0,T)} |\partial_\alpha \mathbf{U}|^2 \mathrm{d}x\mathrm{d}t \\
&\lesssim \int_{\partial B_R\times(0,T)} \Big(s|\partial_{tt}\nabla \mathbf{U}|^2 + s|\partial_t \nabla \mathbf{U}|^2 + s|\partial_t \mathbf{U}|^2 + s^3|\nabla \mathbf{U}|^2 + s^3|\mathbf{U}|^2\Big) e^{2s\varphi}\,\mathrm{d}\Gamma(x)\mathrm{d}t.
\end{aligned}
$$

Since $\mathbf{f}$ has compact support contained in B_R, one has

$$
\|\mathbf{U}\|_{L^2(\partial B_R\times(0,T))^3} \lesssim \|\partial_t \mathbf{U}\|_{L^2(\partial B_R\times(0,T))^3}
$$

and

$$
\|\nabla \mathbf{U}\|_{L^2(\partial B_R\times(0,T))^{3\times 3}},\ \|\partial_t \nabla \mathbf{U}\|_{L^2(\partial B_R\times(0,T))^{3\times 3}} \lesssim \|\partial_{tt} \nabla \mathbf{U}\|_{L^2(\partial B_R\times(0,T))^{3\times 3}},
$$

which completes the proof. □

Using Lemma 3.1 and following the arguments in the proof of [[15], Theorem 3.1], we obtain the exact observability bound.

Theorem 3.2. *Assume that the observation time T satisfies $4(2R+1) < T < 5(2R+1)$. There exists a constant C depending on the domain B_R such that*

$$
\|\mathbf{f}\|^2_{L^2(B_R)^3} \le C e^{C\sigma^2}\Big(\|\partial_t \mathbf{U}\|^2_{L^2(\partial B_R\times(0,T))^3} + \|\partial_{tt}\nabla \mathbf{U}\|^2_{L^2(\partial B_R\times(0,T))^{3\times 3}}\Big) \tag{3.6}
$$

for all $\mathbf{U}$ solving (3.1) with $\mathbf{f} \in H^1(B_R)^3$, supp$\mathbf{f} \subset B_R$.

4. Inverse source problem

In this section, we study the inverse source problem for elastic waves. Denote the resolvent of the elliptic operator $-\Delta^*$ by

$$
\mathbf{R}_0(\zeta) = (-\Delta^* - \zeta^2)^{-1}, \quad \zeta \in \mathbb{C}. \tag{4.1}
$$

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The following results on analyticity and resolvent estimates of $\mathbf{R}_0(\zeta)$ with respect to complex frequency $\zeta \in \mathbb{C}$ are useful in the subsequent analysis of the stability estimate. The proof utilises the method for the classical Schrödinger operator in [11].

Proposition 4.1. *Fix a smooth cut-off function $\rho \in C_0^\infty(\mathbb{R}^3)$. The free resolvent defined by (4.4) is analytic in $\mathbb{C}$ with respect to ζ as a family of operators*

$$\rho \mathbf{R}_0(\zeta)\rho : L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3$$

with the following resolvent estimate:

$$\|\rho \mathbf{R}_0(\zeta)\rho\|_{L^2(\mathbb{R}^3)^3 \to H^j(\mathbb{R}^3)^3} = \mathcal{O}\left(\langle\zeta\rangle^{j-1} e^{L(\Im \widetilde{k}_s)_-}\right), \quad j = 0, 1, 2, \tag{4.2}$$

Here, $\langle\zeta\rangle = \sqrt{1 + |\zeta|^2}$, $\widetilde{k}_s = \zeta\sqrt{\frac{1}{\mu}}$, $(\Im \widetilde{k}_s)_- = \max\{0, -\Im \widetilde{k}_s\}$ and L satisfies $c_s L >$ diamsuppρ with diamsupp $\rho = sup\{|x-y| : x, y \in supp \rho\}$.

Proof. Recall the wavenumbers for compressional and shear waves given by

$$\widetilde{k}_p = \zeta\sqrt{\frac{1}{\lambda + 2\mu}}, \quad \widetilde{k}_s = \zeta\sqrt{\frac{1}{\mu}}, \tag{4.3}$$

respectively. Denote the resolvents of the pressure and shear waves in the frequency domain respectively by

$$\mathbf{R}_p(\zeta) = \left(-\Delta - \widetilde{k}_p^2\right)^{-1}, \quad \mathbf{R}_s(\zeta) = \left(-\Delta - \widetilde{k}_s^2\right)^{-1}.$$

Fix a smooth cut-off function $\rho \in C_0^\infty(\mathbb{R}^3)$ and denote diamsupp $\rho = \sup\{|x-y| : x, y \in \operatorname{supp}\rho\}$. As in [11], for $\zeta \in \mathbb{C}$ the resolvents of the wave speed for pressure and shear waves can be represented by

$$\rho \mathbf{R}_p(\zeta)\rho = \int_0^L e^{i\widetilde{k}_p t}\rho \mathbf{W}(t)\rho \,\mathrm{d}t, \quad \rho \mathbf{R}_s(\zeta)\rho = \int_0^L e^{i\widetilde{k}_s t}\rho \mathbf{W}(t)\rho \,\mathrm{d}t,$$

where $\mathbf{W}(t) = \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}$ and L satisfies $c_s L >$ diamsuppρ. Consequently, by (3.2), (3.3) and (3.4) we obtain that

$$\rho \mathbf{R}_0(\zeta)\rho\, \mathbf{f} = \rho \mathbf{R}_p(\zeta)\rho\, \mathbf{f}_p + \rho \mathbf{R}_p(\zeta)\rho\, \mathbf{f}_s. \tag{4.4}$$

Moreover, since by [[11], Theorem 3.1]

$$\|\rho \mathbf{R}_p(\zeta)\rho\|_{L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3} = \mathcal{O}\left((1 + |\zeta|)^{-1} e^{L(\Im \widetilde{k}_p)_-}\right)$$

and

$$\|\rho \mathbf{R}_s(\zeta)\rho\|_{L^2(\mathbb{R}^3)^3 \to L^2(\mathbb{R}^3)^3} = \mathcal{O}\left((1 + |\zeta|)^{-1} e^{L(\Im \widetilde{k}_s)_-}\right)$$

where $(t)_- := \max\{0, -t\}$, we obtain from (4.4) and (4.3) that

$$\begin{aligned}\|\rho \mathbf{R}_0(\zeta)\rho\, \mathbf{f}\|_{L^2(\mathbb{R}^3)^3} &\lesssim (1 + |\zeta|)^{-1} e^{L(\Im \widetilde{k}_s)_-}\left(\|\mathbf{f}_p\|_{L^2(\mathbb{R}^3)^3} + \|\mathbf{f}_s\|_{L^2(\mathbb{R}^3)^3}\right) \\ &\lesssim (1 + |\zeta|)^{-1} e^{L(\Im \widetilde{k}_s)_-}\|\mathbf{f}\|_{L^2(\mathbb{R}^3)^3}.\end{aligned} \tag{4.5}$$

Now we can derive the analyticity of the free resolvent $\rho \mathbf{R}_0(\zeta)\rho$ from those of $\rho \mathbf{R}_p(\zeta)\rho$ and $\rho \mathbf{R}_s(\zeta)\rho$. In what follows, we prove the resolvent estimates (4.2) by standard regularity theory of elliptic equations. The case for $j = 0$ is a consequence of (4.5). For $j = 2$, taking $\tilde{\rho} \in C_0^\infty(\mathbb{R}^3)$ such that $\tilde{\rho} = 1$ near the support of ρ, we have from [[21], (7.13)] that

$$\|\rho \mathbf{u}\|_{H^2(\mathbb{R}^3)^3} \le C\left(\|\tilde{\rho}\mathbf{u}\|_{L^2(\mathbb{R}^3)^3} + \|\tilde{\rho}(-\Delta^*)\mathbf{u}\|_{L^2(\mathbb{R}^3)^3}\right),$$

where $C > 0$ is a constant. Let $\mathbf{u} = \mathbf{R}_0(\zeta)(\rho\mathbf{f})$ with $\mathbf{f} \in L^2(\mathbb{R}^3)^3$. One has

$$\|\rho \mathbf{R}_0(\zeta)(\rho\mathbf{f})\|_{H^2(\mathbb{R}^3)^3} \le C\left(\|\tilde{\rho}\mathbf{R}_0(\zeta)(\rho\mathbf{f})\|_{L^2(\mathbb{R}^3)^3} + \|\tilde{\rho}(-\Delta^*)(\mathbf{R}_0(\zeta)(\rho\mathbf{f}))\|_{L^2(\mathbb{R}^3)^3}\right).$$

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Since

$$\|\tilde{\rho}(-\Delta^*)(\mathbf{R}_0(\zeta)(\rho\mathbf{f}))\|_{L^2(\mathbb{R}^3)^3} = \|\rho\mathbf{f} + \tilde{\rho}|\zeta|^2\mathbf{R}_0(\zeta)(\rho\mathbf{f})\|_{L^2(\mathbb{R}^3)^3}$$
$$\lesssim \langle\zeta\rangle^2 e^{L(\Im\widetilde{k}_s)_-}\|\mathbf{f}\|_{L^2(\mathbb{R}^3)^3},$$

we get

$$\|\rho\mathbf{R}_0(\zeta)\rho\|_{L^2(\mathbb{R}^3)^3\to H^2(\mathbb{R}^3)^3} \lesssim \langle\zeta\rangle^2 e^{L(\Im\widetilde{k}_s)_-}.$$

Finally, the case of $j=1$ can be obtained from the interpolation between $j=0$ and $j=2$, which completes the proof. □

Define a vector-valued real function space

$$\mathcal{C}_Q = \left\{\mathbf{f}\in H^n(B_R)^3 : n\geq 5, \|\mathbf{f}\|_{H^n(B_R)^3}\leq Q, \mathrm{supp}\mathbf{f}\subset B_R, \mathbf{f}:B_R\to\mathbb{R}^3\right\}.$$

Due to the exact observability bound (3.6), we introduce the boundary measurement of the elastic wave

$$\|\mathbf{u}(x,\omega)\|^2_{\partial B_R} := \int_{\partial B_R}\left(\omega^2|\mathbf{u}(x,\omega)|^2 + \omega^4|\nabla\mathbf{u}(x,\omega)|^2\right)\mathrm{d}\Gamma(x). \tag{4.6}$$

In the following lemma, we bound the unknown source by the boundary measurements (4.6).

Lemma 4.2. *There holds*

$$\|\mathbf{f}\|^2_{L^2(B_R)^3} \lesssim e^{C\sigma^2}\int_0^{+\infty}\|\mathbf{u}(x,\omega)\|^2_{\partial B_R}d\omega,$$

where $\mathbf{u}(x,\omega)$ *is the solution to the direct scattering problem (1.1) with* $\mathbf{f}\in\mathcal{C}_Q$.

Remark 4.3. The proof of Lemma 4.2 depends on the exact observability inequality (3.6) and Fourier transform. Intuitively, by letting $\mathbf{U}(x,t)=0$, $t<0$, we can see $\mathbf{u}(x,\omega)=\int_{-\infty}^{+\infty}\mathbf{U}(x,t)e^{\mathrm{i}\omega t}dt = \int_0^{+\infty}\mathbf{U}(x,t)e^{\mathrm{i}\omega t}dt$. By Theorem 3.6 and Plancherel's theorem, we have

$$\begin{aligned}\|\mathbf{f}\|^2_{L^2(B_R)^3} &\leq Ce^{C\sigma^2}\left(\|\partial_t\mathbf{U}\|^2_{L^2(\partial B_R\times(0,T))^3} + \|\partial_{tt}\nabla\mathbf{U}\|^2_{L^2(\partial B_R\times(0,T))^{3\times 3}}\right)\\ &\leq Ce^{C\sigma^2}\left(\|\partial_t\mathbf{U}\|^2_{L^2(\partial B_R\times(0,+\infty))^3} + \|\partial_{tt}\nabla\mathbf{U}\|^2_{L^2(\partial B_R\times(0,+\infty))^{3\times 3}}\right)\\ &= \frac{C}{2\pi}e^{C\sigma^2}\int_0^{+\infty}\|\mathbf{u}(x,\omega)\|^2_{\partial B_R}\mathrm{d}\omega.\end{aligned} \tag{4.7}$$

Thus, we formally proved Lemma 4.2. To justify the Fourier transform rigorously, one can obtain the decay estimates of the solution $\mathbf{U}$ to the time-domain initial-valued elastic wave (3.1) by combining the Helmholtz decomposition (3.2) and the decay estimates for acoustic wave equations proved in Theorem 5.1 assuming that the source $\mathbf{f}$ has sufficient regularity ($\mathbf{f}\in H^n, n\geq 5$). As the proof follows the arguments in [[17], Lemma 3.1] in a straightforward way, we omit it for brevity.

Define

$$I_0(s) = \int_0^s \omega^2\int_{\partial B_R}\mathbf{u}(x,\omega)\cdot\mathbf{u}(x,-\omega)\mathrm{d}\Gamma(x)\mathrm{d}\omega,$$
$$I_1(s) = \int_0^s \omega^4\int_{\partial B_R}\nabla\mathbf{u}(x,\omega):\nabla\mathbf{u}(x,-\omega)\mathrm{d}\Gamma(x)\mathrm{d}\omega,$$

where : denotes the double dot product which is defined by $A:B=\sum_{i,j=1}^3 a_{ij}b_{ij}$ for $A=(a_{ij})^3_{i,j=1}$, $B=(b_{ij})^3_{i,j=1}$. Since the integrands are entire analytic functions of ω, the integrals in $I_0(s)$ and $I_1(s)$ with respect to ω can be taken over any path joining points 0 and s of the complex plane. Consequently, $I_0(s)$ and $I_1(s)$ are entire analytic functions of $s=s_1+\mathrm{i}s_2$.

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The following lemma gives estimates of $I_0(s)$ and $I_1(s)$. The proof employs the resolvent estimates in Proposition 4.1.

Lemma 4.4. *Denote* $S=\{z=x+\mathrm{i}y\in\mathbb{C}:-\frac{\pi}{4}<\arg z<\frac{\pi}{4}\}$. *Let* $s=s_1+is_2\in S$. *The following estimates hold:*

$$|I_0(s)|\lesssim(1+|s|)^3e^{C(4s_1+\sigma)}\|\mathbf{f}\|^2_{L^2(B_R)^3},$$
$$|I_1(s)|\lesssim(1+|s|)^5e^{C(4s_1+\sigma)}\|\mathbf{f}\|^2_{L^2(B_R)^3}.$$

Proof. We first show that $\kappa(\omega)=\sqrt{\omega^2+\mathrm{i}\omega\sigma}$ is analytic for $\omega\in S$. When $\omega=\omega_1+\mathrm{i}\omega_2\in S$ the image of $\omega^2+\mathrm{i}\sigma\omega$ satisfies

$$\{\omega^2+\mathrm{i}\sigma\omega|\omega\in S\}\cap\mathrm{i}(-\infty,0]=\phi.$$

In fact, assume that $\omega\in S$. Since

$$\omega^2+\mathrm{i}\sigma\omega=(\omega_1^2-\omega_2^2-\sigma\omega_2)+\mathrm{i}(2\omega_1\omega_2+\sigma\omega_1),$$

if $\omega^2+\mathrm{i}\sigma\omega\in\mathrm{i}(-\infty,0]$, then we have

$$\omega_1^2-\omega_2^2-\sigma\omega_2=0,\quad 2\omega_1\omega_2+\sigma\omega_1\le 0,$$

which gives $\omega_2\le-\frac{\sigma}{2}$. Then, we have

$$\omega_1^2=\omega_2^2+\sigma\omega_2\le\omega_2^2-\frac{\sigma^2}{2}<\omega_2^2,$$

which is in contradiction with the assumption that $\omega\in S$ where $\omega_1\ge|\omega_2|$. Therefore, by choosing the branch cut of $\sqrt{z}$ to be $z\in\mathbb{C}\backslash\mathrm{i}(-\infty,0]$ we obtain that $\kappa(\omega)=\sqrt{\omega^2+\mathrm{i}\omega b}$ is analytic for $\omega\in S$.

Let $\kappa(\omega)=\kappa_1(\omega)+\mathrm{i}\kappa_2(\omega)$. A direct calculation gives $|\kappa_2(\omega)|^2\le(1+\sqrt{2})\Big(\omega_1+\frac{\sigma}{4}\Big)^2$ for $\omega\in S$. Consequently, by the resolvent estimates in Proposition 4.1 we have that for $j=0,1,2$

$$\begin{aligned}\|\mathbf{u}(x,\omega)\|_{H^j(B_R)^3}&=\|\mathbf{R}_0(\kappa(\omega))\mathbf{f}\|_{H^j(B_R)^3}\\&\lesssim e^{C(\Im\omega)_-}(1+|\omega|)^{j-1}\|\mathbf{f}\|_{L^2(B_R)^3}\\&\lesssim(1+|\omega|)^{j-1}e^{C(\omega_1+\frac{\sigma}{4})}\|\mathbf{f}\|_{L^2(B_R)^3}\\&\lesssim(1+|\omega|)^{j-1}e^{C(4\omega_1+\sigma)}\|\mathbf{f}\|_{L^2(B_R)^3}.\end{aligned}$$

Then letting $\omega=st$, $t\in[0,1]$, we obtain that

$$\begin{aligned}|I_0(s)|&\lesssim|s|^3\int_0^1t^2\left(\int_{\partial B_R}|\mathbf{u}(x,st)|^2\mathrm{d}\Gamma(x)\right)^{1/2}\left(\int_{\partial B_R}|\mathbf{u}(x,-st)|^2\mathrm{d}\Gamma(x)\right)^{1/2}\mathrm{d}t\\&\lesssim(1+|s|)^3\int_0^1t^2\|\mathbf{u}(x,st)\|_{H^1(B_R)^3}\|\mathbf{u}(x,-st)\|_{H^1(B_R)^3}\mathrm{d}t\\&\lesssim(1+|s|)^3e^{C(4s_1+\sigma)}\|\mathbf{f}\|^2_{L^2(B_R)^3}.\end{aligned}$$

For $I_1(s)$ repeating the above arguments for $I_0(s)$, we have that

$$\begin{aligned}|I_1(s)|&\lesssim|s|^5\int_0^1t^4\left(\int_{\partial B_R}|\nabla\mathbf{u}(x,st)|^2\mathrm{d}\Gamma(x)\right)^{1/2}\left(\int_{\partial B_R}|\nabla\mathbf{u}(x,-st)|^2\mathrm{d}\Gamma(x)\right)^{1/2}\mathrm{d}t\\&\lesssim|s|^5\int_0^1t^4\|\mathbf{u}(x,st)\|_{H^2(B_R)^3}\|\mathbf{u}(x,-st)\|_{H^2(B_R)^3}\mathrm{d}t\\&\lesssim(1+|s|)^5e^{C(4s_1+\sigma)}\|\mathbf{f}\|^2_{L^2(B_R)^3}.\end{aligned}$$

The proof is completed. □

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The following lemma proved in [6] provides an estimate of the high frequency tail of $\|\mathbf{u}(x,\omega)\|^2_{\partial B_R}$.

Lemma 4.5. *Let* $\mathbf{f} \in \mathcal{C}_Q$. *Then the following estimate holds:*

$$\int_s^\infty \|\mathbf{u}(x,\omega)\|^2_{\partial B_R} d\omega \lesssim \frac{1}{s^{2n-3}} \|\mathbf{f}\|^2_{H^n(B_R)^3}.$$

The following lemma on analytic continuation is proved in [8] which will be useful in the subsequent analysis.

Lemma 4.6. *Let* $J(z)$ *be analytic in* $S = \{z = x + \mathrm{i}y \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$ *and continuous in* $\bar{S}$ *satisfying*

$$\begin{cases} |J(z)| \le \varepsilon, & z \in (0, K], \\ |J(z)| \le M, & z \in S, \\ |J(0)| = 0. \end{cases}$$

Then there exits a function $\beta(z)$ *satisfying*

$$\begin{cases} \beta(z) \ge \frac{1}{2}, & z \in \left(K, 2^{\frac{1}{4}}K\right), \\ \beta(z) \ge \frac{1}{\pi}\left(\left(\frac{z}{K}\right)^4 - 1\right)^{-\frac{1}{2}}, & z \in \left(2^{\frac{1}{4}}K, \infty\right) \end{cases}$$

such that

$$|J(z)| \le M\varepsilon^{\beta(z)} \quad \forall\, z \in (K, \infty).$$

The following lemma is a direct consequence of Lemmas 4.4 and 4.6.

Lemma 4.7. *Let* $f \in \mathcal{C}_Q$. *Then there exists a function* $\beta(s)$ *satisfying*

$$\begin{cases} \beta(s) \ge \frac{1}{2}, & s \in \left(K, 2^{\frac{1}{4}}K\right), \\ \beta(s) \ge \frac{1}{\pi}\left(\left(\frac{s}{K}\right)^4 - 1\right)^{-\frac{1}{2}}, & s \in \left(2^{\frac{1}{4}}K, \infty\right), \end{cases} \tag{4.8}$$

such that

$$|I_0(s) + I_1(s)| \lesssim Q^2 e^{(C+1)s} \epsilon^{2\beta(s)}, \quad \forall s \in (K, \infty),$$

where

$$\epsilon^2 = \int_0^K \|\mathbf{u}(x,\omega)\|^2_{\partial B_R} d\omega.$$

Proof. It follows from Lemma 4.4 that there exists $C > 0$ such that

$$|(I_0(s) + I_1(s))\, e^{-(C+1)|s|}| \lesssim Q^2, \quad \forall s \in S.$$

Moreover, we have

$$|(I_0(s) + I_1(s))\, e^{-(C+1)|s|}| \le \epsilon^2, \quad s \in [0, K].$$

A direct application of Lemma 4.6 shows that there exists a function $\beta(s)$ satisfying (4.8) such that

$$|(I_0(s) + I_1(s))\, e^{-(C+1)s}| \lesssim Q^2 \epsilon^{2\beta}, \quad \forall s \in (K, \infty),$$

which completes the proof. □

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In the following theorem, we present an increasing stability estimate for the inverse source problem following the arguments in [19].

Theorem 4.8. *Let* $\mathbf{u}(x, \omega)$ *be the outgoing solution of the scattering problem* (1.1) *corresponding to the source* $\mathbf{f} \in \mathcal{C}_Q$. *Then for* ϵ *sufficiently small the following estimate holds:*

$$\|\mathbf{f}\|^2_{L^2(B_R)^3} \lesssim e^{C\sigma^2}\left(\epsilon^2 + \frac{Q^2}{\left(K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}\right)^{2n-3}}\right) \tag{4.9}$$

where

$$\epsilon^2 = \int_0^K \|\mathbf{u}(x,\omega)\|^2_{\partial B_R} d\omega.$$

Proof. We can assume that $\epsilon < e^{-1}$, otherwise the estimate is obvious. Let

$$s = \begin{cases} \dfrac{1}{((C+3)\pi)^{\frac{1}{3}}} K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}, & 2^{\frac{1}{4}}((C+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln\epsilon|^{\frac{1}{4}}, \\ K, & |\ln\epsilon|^{\frac{1}{4}} \le 2^{\frac{1}{4}}((C+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}}. \end{cases}$$

If $2^{\frac{1}{4}}((C+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln\epsilon|^{\frac{1}{4}}$, then we have from Lemma 4.7 that

$$\begin{aligned} |I_0(s) + I_1(s)| &\lesssim Q^2 e^{(C+3)s} e^{-\frac{2|\ln\epsilon|}{\pi}\left(\left(\frac{s}{K}\right)^4 - 1\right)^{-\frac{1}{2}}} \\ &\lesssim Q^2 e^{\frac{(C+3)}{((C+3)\pi)^{\frac{1}{3}}}K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}} - \frac{2|\ln\epsilon|}{\pi}\left(\frac{K}{s}\right)^2} \\ &= M^2 e^{-2\left(\frac{(C+3)^2}{\pi}\right)^{\frac{1}{3}} K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}}\left(1 - \frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}}\right)}. \end{aligned}$$

Noting

$$\frac{1}{2}|\ln\epsilon|^{-\frac{1}{4}} < \frac{1}{2}, \quad \left(\frac{(C+3)^2}{\pi}\right)^{\frac{1}{3}} > 1,$$

we have

$$|I_0(s) + I_1(s)| \lesssim Q^2 e^{-K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-x} \le \frac{(6n-9)!}{x^{3(2n-3)}}, \quad x > 0, \tag{4.10}$$

we get

$$|I_0(s) + I_1(s)| \lesssim \frac{Q^2}{\left(\dfrac{K^2|\ln\epsilon|^{\frac{3}{2}}}{(6n-9)^3}\right)^{2n-3}}. \tag{4.11}$$

If $|\ln\epsilon|^{\frac{1}{4}} \le 2^{\frac{1}{4}}((C+3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}}$, then $s = K$. We have that

$$|I_0(s) + I_1(s)| \le \epsilon^2.$$

Here, we have used the fact that

$$I_0(s) + I_1(s) = \int_0^s \|\mathbf{u}(x,\omega)\|^2_{\partial B_R} \mathrm{d}\omega, \quad s > 0.$$

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Hence, we obtain from Lemma 4.5 and (4.11) that

$$
\begin{aligned}
&\int_0^{\infty} \|\mathbf{u}(x,\omega)\|^2_{\partial B_R} \mathrm{d}\omega \\
&\leq I_0(s) + I_1(s) + \int_s^{\infty} \|\mathbf{u}(x,\omega)\|^2_{\partial B_R} \mathrm{d}\omega \\
&\lesssim \epsilon^2 + \frac{Q^2}{\left(\left(\frac{K^2|\ln\epsilon|}{(6n-9)^3}\right)^{\frac{3}{2}}\right)^{2n-3}} + \frac{Q^2}{\left(2^{-\frac{1}{4}}((C+3)\pi)^{-\frac{1}{3}}K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}\right)^{2n-3}}.
\end{aligned}
$$

By Lemma 4.2, we have

$$
\|\mathbf{f}\|^2_{L^2(B_R)} \lesssim e^{C\sigma^2}\left(\epsilon^2 + \frac{Q^2}{\left(K^2|\ln\epsilon|^{\frac{3}{2}}\right)^{2n-3}} + \frac{Q^2}{\left(K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}}\right)^{2n-3}}\right).
$$

Since $K^{\frac{2}{3}}|\ln\epsilon|^{\frac{1}{4}} \leq K^2|\ln\epsilon|^{\frac{3}{2}}$ when $K > 1$ and $|\ln\epsilon| > 1$, we finish the proof and obtain the stability estimate (4.9). □

The stability estimate (4.9) consists of two parts: the first part is the Lipschitz type of data discrepancy and the second part is the high frequency tail of the source function. As the upper bound K of the frequency increases, the stability estimate (4.9) tends to a Lipschitz-type stability which suggests that the inverse source problem becomes more stable when data of higher frequency are used. It also shows that the stability deteriorates if the attenuation σ becomes larger.

5. Useful decay estimates for acoustic waves

To justify the Fourier transform in Remark 4.3, we need to prove some decay estimates for the solution $\mathbf{U}$ to the time-domain initial-valued elastic wave (3.1). As mentioned in Remark 4.3, in order to obtain the decay estimates of the solution to the time-domain initial-valued elastic wave (3.1), we just need to prove decay estimates for the dumped acoustic wave equation since we have the Helmholtz decomposition (3.2). In this section, we will prove some decay estimates which can guarantee that the Fourier transform for the solution of the dumped acoustic wave equation is well defined (see Remarks 5.2 and 5.3).

We consider the following initial-valued damped acoustic wave equation in $\mathbb{R}^3$

$$
\begin{cases}
\partial_t^2 U(x,t) - \Delta U(x,t) + \sigma\partial_t U(x,t) = 0, & (x,t) \in \mathbb{R}^3 \times (0,+\infty), \\
U(x,0) = 0, \quad \partial_t U(x,0) = f(x), & x \in \mathbb{R}^3,
\end{cases} \tag{5.1}
$$

where $f(x) \in L^1(\mathbb{R}^3) \cap H^s(\mathbb{R}^3)$. The regularity assumption $H^s(\mathbb{R}^3)$ for $f(x)$ will be specified later. Since the equation in (5.1) has constant coefficients, the decay estimates of the solution $U(x,t)$ can be derived by the standard Fourier transform.

Applying the Fourier transform to the solution $U(x,t)$ to (5.1) with respect to the spatial variable x, we obtain that

$$
U(x,t) = \mathcal{F}^{-1}(m_\sigma(t,\xi)\hat{f}(\xi))(x),
$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform, the multiplies $m_\sigma(t,\xi)$ takes the form

$$
m_\sigma(t,\xi) = \frac{e^{-\frac{\sigma}{2}t}}{\sqrt{\sigma^2 - 4|\xi|^2}}\left(e^{\frac{1}{2}t\sqrt{\sigma^2-4|\xi|^2}} - e^{-\frac{1}{2}t\sqrt{\sigma^2-4|\xi|^2}}\right), \tag{5.2}
$$

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and $\hat{f}(\xi)$ is the Fourier transform of f defined as follows:

$$\hat{f}(\xi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{-\mathrm{i}x\cdot\xi} f(x)\mathrm{d}x.$$

Assume that $\sqrt{\sigma^2 - 4|\xi|^2} = \mathrm{i}\sqrt{4|\xi|^2 - \sigma^2}$ when $|\xi|^2 > \frac{\sigma^2}{4}$. Then (5.2) becomes

$$m_\sigma(t,\xi) = \begin{cases} 2e^{-\frac{\sigma}{2}t} \dfrac{\sinh\left(\frac{t}{2}\sqrt{\sigma^2-4|\xi|^2}\right)}{\sqrt{\sigma^2-4|\xi|^2}}, & |\xi|^2 < \frac{\sigma^2}{4}, \\ e^{-\frac{\sigma}{2}t} \dfrac{\sin\left(\frac{t}{2}\sqrt{4|\xi|^2-\sigma^2}\right)}{\sqrt{4|\xi|^2-\sigma^2}}, & |\xi|^2 > \frac{\sigma^2}{4}. \end{cases}$$

Notice from the representation of the multiplies $m_\sigma(t,\xi)$ above that the solution $U(x,t)$ behaves as a "parabolic type" of $e^{-t\Delta}f$ in the low frequency, while for the high frequency part it behaves like a "dispersive type" of $e^{it\Delta}f$.

Theorem 5.1. *Let $U(x,t)$ be the solution of (5.1). Then $U(x,t)$ satisfies the decay estimate*

$$sup_{x\in\mathbb{R}^3}|\partial_x^\alpha \partial_t^j U(x,t)| \lesssim (1+t)^{-\frac{3+|\alpha|}{2}} \|f\|_{L^1(\mathbb{R}^3)} + e^{-ct}\|f\|_{H^s(\mathbb{R}^3)}, \tag{5.3}$$

where $j \in \mathbb{N}$, α is a multi-index vector in $\mathbb{N}^3$ such that $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $s > j + |\alpha| + \frac{1}{2}$ and $c > 0$ is some positive constant. In particular, for $|\alpha| = j = 0$, the following estimate holds:

$$sup_{x\in\mathbb{R}^3}|U(x,t)| \lesssim (1+t)^{-\frac{3}{2}} \left(\|f\|_{L^1(\mathbb{R}^3)} + \|f\|_{H^1(\mathbb{R}^3)}\right). \tag{5.4}$$

Remark 5.2. The estimate (5.4) provides a time decay of the order $O((1+t)^{-\frac{3}{2}})$ for $U(x,t)$ uniformly for all $x \in \mathbb{R}^3$, which gives

$$\sup_{x\in\mathbb{R}^3} \int_0^\infty |U(x,t)|^2 \mathrm{d}t \lesssim \int_0^\infty (1+t)^{-3}\mathrm{d}t < +\infty.$$

Hence, letting $U(x,t) = 0$ when $t < 0$ then $U(x,t)$ has a Fourier transform $\hat{U}(x,k) \in L^2(\mathbb{R})$ for each $x \in \mathbb{R}^3$. Moreover, the following Plancherel equality holds:

$$\int_0^{+\infty} |U(x,t)|^2 \mathrm{d}t = \int_{-\infty}^{+\infty} |\hat{U}(x,k)|^2 \mathrm{d}k.$$

Remark 5.3. To study the inverse source problem, it suffices to assume that $f \in H^s\left(\mathbb{R}^3\right)$, $s \geq 5$. In this case, it follows from the above theorem that both $\partial_t U(x,t)$ and $\partial_{tt}\nabla U(x,t)$ are continuous functions. Moreover, we have from (5.3) that the following estimate holds:

$$\sup_{x\in\mathbb{R}^3}|\partial_t U(x,t)| \lesssim (1+t)^{-\frac{3}{2}} \|f\|_{L^1(\mathbb{R}^3)} + e^{-ct}\|f\|_{H^s(\mathbb{R}^3)},$$
$$\sup_{x\in\mathbb{R}^3}|\partial_{tt}\nabla U(x,t)| \lesssim (1+t)^{-2} \|f\|_{L^1(\mathbb{R}^3)} + e^{-ct}\|f\|_{H^s(\mathbb{R}^3)}.$$

Proof. Without loss of generality, we may assume that $\sigma = 1$, and then

$$m(t,\xi) \triangleq m_\sigma(t,\xi) = \frac{e^{-\frac{1}{2}t}}{\sqrt{1-4|\xi|^2}}\left(e^{\frac{1}{2}t\sqrt{1-4|\xi|^2}} - e^{-\frac{1}{2}t\sqrt{1-4|\xi|^2}}\right).$$

First we prove (5.3) for $j = 0$. Choose $\chi \in C_0^\infty\left(\mathbb{R}^3\right)$ such that supp$\chi \subset B(0,\frac{1}{4})$ and $\chi(\xi) = 1$ for $|\xi| \leq \frac{1}{16}$. Let

$$\begin{aligned} U(x,t) &= \mathcal{F}^{-1}(m(t,\xi)\chi(\xi)\hat{f}) + \mathcal{F}^{-1}(m(t,\xi)(1-\chi(\xi))\hat{f}) \\ &:= U_1(x,t) + U_2(x,t). \end{aligned}$$

For $U_1(x,t)$, since $\sqrt{1-4|\xi|^2} \leq 1 - 2|\xi|^2$ when $0 \leq |\xi| \leq \frac{1}{4}$, we have for $|\xi| \leq \frac{1}{4}$ that

$$\frac{1}{\sqrt{1-4|\xi|^2}} e^{-\frac{t}{2}\left(1\pm\sqrt{1-4|\xi|^2}\right)} \leq 2e^{-t|\xi|^2}, \quad t \geq 0.$$

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For each $x \in \mathbb{R}^3$, we have

$$\partial_x^\alpha U_1(x,t) = \int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi} (\mathrm{i}\xi)^\alpha m(t,\xi)\chi(\xi)\hat{f}(\xi)\mathrm{d}\xi,$$

which gives

$$\sup_{x\in\mathbb{R}^3} |\partial_x^\alpha U_1(x,t)| \le \int_{|\xi|\le\frac{1}{4}} |\xi|^\alpha e^{-t|\xi|^2} |\hat{f}(\xi)|\mathrm{d}\xi \lesssim \|\hat{f}\|_{L^\infty(\mathbb{R}^3)} \int_{|\xi|\le\frac{1}{4}} |\xi|^\alpha e^{-t|\xi|^2}\mathrm{d}\xi.$$

Since

$$\int_{|\xi|\le\frac{1}{4}} |\xi|^\alpha e^{-t|\xi|^2}\mathrm{d}\xi \lesssim \begin{cases} C, & 0\le t\le 1, \\ t^{-\frac{3+|\alpha|}{2}}, & t\ge 1, \end{cases}$$

and $\|\hat{f}\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f\|_{L^1(\mathbb{R}^3)}$, we obtain

$$\sup_{x\in\mathbb{R}^3} |\partial_x^\alpha U_1(x,t)| \lesssim (1+t)^{-\frac{3+|\alpha|}{2}} \|f\|_{L^1(\mathbb{R}^3)} \quad \forall \alpha\in\mathbb{N}^3. \tag{5.5}$$

To estimate $U_2(x,t)$, noting

$$(1-\Delta)^{\frac{p}{2}} U_2(x,t) = \int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi}(1+|\xi|^2)^{\frac{p}{2}} m(t,\xi)(1-\chi(\xi))\hat{f}(\xi)\mathrm{d}\xi,$$

we have from Plancherel's theorem that

$$\int_{\mathbb{R}^3} |(1-\Delta)^{\frac{p}{2}} U_2(x,t)|^2\mathrm{d}x = \int_{\mathbb{R}^3} (1+|\xi|^2)^p |m(t,\xi)(1-\chi(\xi))\hat{f}(\xi)|^2\mathrm{d}\xi. \tag{5.6}$$

There exists a positive constant c such that

$$|m(t,\xi)| \le \begin{cases} te^{-\frac{t}{2}(1-\sqrt{1-4|\xi|^2})}\left|\frac{1-e^{-t\sqrt{1-4|\xi|^2}}}{t\sqrt{1-4|\xi|^2}}\right| \lesssim e^{-ct}, & \frac{1}{16}<|\xi|\le\frac{1}{2}, \\ \frac{1}{2}te^{-\frac{t}{2}}\frac{\sin\frac{t}{2}\sqrt{4|\xi|^2-1}}{\frac{t}{2}\sqrt{4|\xi|^2-1}} \lesssim e^{-ct}, & \frac{1}{2}<|\xi|\le 1, \\ \frac{e^{-\frac{t}{2}}}{\sqrt{4|\xi|^2-1}}|\sin\frac{t}{2}\sqrt{4|\xi|^2-1}| \le \frac{e^{-ct}}{\sqrt{4|\xi|^2-1}}, & |\xi|>1. \end{cases}$$

Hence, when $|\xi|\ge\frac{1}{16}$ we have

$$|(1+|\xi|^2)^{1/2} m(t,\xi)| \lesssim e^{-ct}.$$

It follows from (5.6) that

$$\begin{aligned}\|U_2(x,t)\|^2_{H^p(\mathbb{R}^3)} &\le \int_{|\xi|\ge\frac{1}{16}} |(1+|\xi|^2)^{\frac{p}{2}} m(t,\xi)\hat{f}(\xi)|^2\mathrm{d}\xi \\ &\le e^{-2ct}\int_{\mathbb{R}^3} |(1+|\xi|^2)^{-\frac{1}{2}+\frac{p}{2}}\hat{f}(\xi)|^2\mathrm{d}\xi = e^{-2ct}\|f\|^2_{H^{p-1}(\mathbb{R}^3)}.\end{aligned}$$

On the other hand, by Sobolev's theorem, we have for $p>\frac{3}{2}$ that

$$\sup_{x\in\mathbb{R}^3} |U_2(x,t)| \le \|U_2(\cdot,t)\|_{H^p(\mathbb{R}^3)} \lesssim e^{-ct}\|f\|_{H^{p-1}(\mathbb{R}^3)}.$$

More generally, for any $\alpha\in\mathbb{N}^3$ it holds that

$$(1-\Delta)^{\frac{p}{2}}\partial_x^\alpha U_2(x,t) = \mathcal{F}^{-1}((1+|\xi|^2)^{\frac{p}{2}} m(t,\xi)(1-\chi(\xi))\widehat{\partial_x^\alpha f}),$$

which leads to

$$\sup_{x\in\mathbb{R}^3} |\partial_x^\alpha U_2(x,t)| \lesssim e^{-ct}\|\partial_x^\alpha f\|_{H^{p-1}(\mathbb{R}^3)} \lesssim e^{-ct}\|f\|_{H^s(\mathbb{R}^3)}. \tag{5.7}$$

Here $s=p-1+|\alpha|>|\alpha|+\frac{1}{2}$ by choosing $p>\frac{3}{2}$. Combining the estimate (5.5) with (5.7) yields (5.3) for $j=0$.

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Next we consider the general case with $j \neq 0$. Noting

$$\partial_t^j U(x,t) = \int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi} \partial_t^j m(t,\xi)\hat{f}(\xi)\mathrm{d}\xi,$$

we obtain from direct calculations that

$$\begin{aligned}\partial_t^j m(t,\xi) &= \partial_t^j \left(\frac{e^{-\frac{1}{2}t}}{\sqrt{1-4|\xi|^2}} \left(e^{\frac{1}{2}t\sqrt{1-4|\xi|^2}} - e^{-\frac{1}{2}t\sqrt{1-4|\xi|^2}} \right) \right) \\ &= \sum_{l=0}^{j} \begin{pmatrix} l \\ j \end{pmatrix} (-1)^{j-l} 2^{-j} \left(\sqrt{1-4|\xi|^2}\right)^{l-1} e^{-\frac{t}{2}} \left(e^{\frac{1}{2}t\sqrt{1-4|\xi|^2}} + (-1)^{l+1} e^{-\frac{1}{2}t\sqrt{1-4|\xi|^2}} \right) \\ &:= \sum_{l=0}^{j} m_l(t,\xi),\end{aligned}$$

where $\begin{pmatrix} l \\ j \end{pmatrix} = \frac{j!}{(j-l)!l!}$. Hence, we can write $\partial_t^j U(x,t)$ as

$$\partial_t^j U(x,t) = \sum_{l=0}^{j} \int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi} m_l(t,\xi)\hat{f}(\xi)\mathrm{d}\xi := \sum_{l=0}^{j} W_l(x,t). \tag{5.8}$$

For each $0 \le l \le j$, $j \neq 0$, using similar arguments for the case $j = 0$ we obtain

$$\sup_{x\in\mathbb{R}^3} |\partial_x^\alpha W_l(x,t)| \le (1+t)^{-\frac{3+|\alpha|}{2}} \|f\|_{L^1(\mathbb{R}^3)} + e^{-ct}\|f\|_{H^s(\mathbb{R}^3)} \tag{5.9}$$

for $s > l + |\alpha| - \frac{1}{2}$. Combining (5.8) and (5.9), we obtain the general estimate (5.3). ☐

6. Conclusion

We have presented an increasing stability result for the inverse source problem of the elastic wave equation with attenuation. A key ingredient in the proof is the use of the scattering theory to analyse the resolvent of the elliptic operator. The advantage of this method is that it can be used to study the case of a variable attenuation coefficient. For instance, consider the following elastic wave equation:

$$-\Delta^*\mathbf{u} - \omega^2\mathbf{u} - \mathrm{i}\omega\sigma(x)\mathbf{u} + V(x)\mathbf{u} = \mathbf{f}(x), \qquad x \in \mathbb{R}^3.$$

where V is a bounded potential function. Formally, one has from the classical resolvent identity that

$$\mathbf{u} = \mathbf{R}_0(\omega)(\mathbf{I} + (\mathrm{i}\omega\sigma(x) + V(x))\mathbf{R}_0(\omega))^{-1}\mathbf{f},$$

where $\mathbf{I}$ is the identity operator. The operator $\mathbf{I} + (\mathrm{i}\omega\sigma(x) + V)\mathbf{R}_0(\omega)$ is invertible if the attenuation coefficient $\sigma(x)$ is assumed to be small. Indeed, for small $\sigma(x)$ using the free resolvent estimate (4.2) one has for $\Im\omega > 0$ sufficiently large that

$$\|(\mathrm{i}\omega\sigma(x) + V(x))\mathbf{R}_0(\omega)\mathbf{u}\|_{L^2(\mathbb{R}^3)} \le \frac{1}{2}\|\mathbf{u}\|_{L^2(\mathbb{R}^3)},$$

which gives by the Neumann series argument that the operator $\mathbf{I} + (\mathrm{i}\omega\sigma(x) + V)\mathbf{R}_0(\omega)$ is invertible. As a consequence, using the standard perturbation argument in scattering theory one may prove a similar resolvent estimates as Proposition 4.1 for the resolvent $(-\Delta^* - \mathrm{i}\omega\sigma(x) - \omega^2)^{-1}$ with variable attenuation. An exact observability may also be derived using Carleman estimates for wave equation. A more challenging problem is to remove the smallness assumption. We hope to report the progress on this problem elsewhere.

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