The limits of classical scattering. By P. M. S. BLACKETT, M.A., Fellow of King's College, Cambridge.

[*Received* 5 February, *read* 14 March 1927.]

The methods of geometrical optics provide an approximate solution of the equation of wave propagation, if the following condition, given by De Broglie*, is satisfied. If V is the phase velocity, λ the wave-length, l the direction of greatest increase of V, and θ the angle between l and V, then the condition is that

$$Z \equiv \frac{\cos\theta}{2\pi}\frac{1}{V}\frac{dV}{dl}\lambda << 1 \quad \text{.................(1)},$$

that is, the relative change of V over a distance of the order of λ must be small†.

If this condition is satisfied for the waves associated by De Broglie's theory with the motion of a material particle, the particle will move according to the laws of classical mechanics. For example, a charged particle will move classically in a hyperbolic orbit round another fixed charge only if $Z << 1$ at all points of the orbit. Wentzel‡ has recently shown that classical inverse square law scattering can in fact be obtained as an approximate solution of Schrödinger's differential equation. If this condition is not satisfied a rigorous solution of the wave equations is required to describe the phenomena. It is to be expected that this solution will correspond to some kind of diffraction effect.

Let a particle of mass m and charge e_1 and initial velocity v_0 move in a hyperbolic orbit in the repulsive field of a fixed charge e_2. If 2α is the angle between the asymptotes, the equation of the orbit in the tangential polar coordinates r, p is

$$\frac{l}{p^2} + \frac{2}{r} = \frac{1}{a} \quad \text{..........................(2)},$$

where $$l = a\tan^2\alpha \quad \text{..........................(3)},$$

and $$a = e_1 e_2/mv_0^2 \quad \text{..........................(4)}.$$

The orbit is determined in terms of the striking radius p_0 by the equation

$$p_0 = a\tan\alpha.$$

* De Broglie, *Journ. de Phys.* (1926), p. 322.

† In De Broglie's statement of the condition the direction l is left undefined. For a plane wave it can be proved that l must be taken as the direction of greatest increase of V. I am indebted to Mr L. H. Thomas for pointing out that if the wave is not plane, the condition (1) will contain other terms involving the curvature of the *wavefront*. It seems unlikely that these terms will be of importance in the cases discussed in this paper.

‡ Wentzel, *Zeit. für Phys.*, vol. XL (1926), p. 590.

The waves associated with the particle have a length

$$\lambda = h/mv, \quad (\lambda_0 = h/mv_0) \quad \text{..................(5)},$$

and are propagated with a velocity

$$V = c/v \quad \text{..............................(6)}.$$

Since $$v^2 = v_0^2 - 2e_1e_2/mr = v_0^2(1 - 2a/r) \quad \text{............(7)},$$

the radius vector r is also the direction of greatest increase of V, so that

$$dV/dl = dV/dr,$$

and $$\sin\theta = p/r \quad \text{..........................(8)}.$$

Thus from (1) and (6) we get

$$Z = \frac{\lambda}{4\pi}\frac{\cos\theta}{v^2}\frac{d(v^2)}{dr},$$

which with (5) gives

$$Z = \frac{\lambda_0}{4\pi}\frac{v_0}{v^3}\cos\theta\frac{d(v^2)}{dr}.$$

Using (2), (4), (7) and (8), we get

$$Z = \lambda_0 a(r^2 - 2ar - la)^{\frac{1}{2}}/2\pi r(r-2a)^2.$$

If we write $x = r/a$ and $\rho = l/a$, we get

$$Z = \lambda_0/kl,$$

where $$k = 2\pi x(x-2)^2/\rho(x^2 - 2x - \rho)^{\frac{1}{2}} \quad \text{..............(9)}.$$

The condition (1) will be satisfied at all points of the orbit if

$$\frac{\lambda_0}{k_m l} << 1 \quad \text{.........................(10)},$$

where k_m is the minimum value of k for the given orbit. We can find k_m numerically from (9). The values are given below for different values of the deflection $\phi = \pi - 2\alpha$,

$\phi°$	0	20	40	60	80	100	120	140	160	180
k_m	16·1	14·0	12·0	10·4	8·6	6·9	5·0	3·4	1·8	0.

The condition for the classical description of an orbit is then that the wave-length λ_0 corresponding to the initial velocity of the particle should be small compared with k_m times the semi-latus rectum l of the orbit.

In view of (3), (4) and (5) the condition can be written

$$\frac{\lambda_0}{a}\frac{\cot^2\alpha}{k_m} = \frac{hv_0}{e_1e_2}\frac{\cot^2\alpha}{k_m} << 1 \quad \text{..............(11)}.$$

It is interesting to compare this condition with the qualitative one mentioned by Schrödinger* and given also by De Broglie as the quantitative equivalent of condition (1), namely, that the wave-length should be small compared with the radius of curvature of the orbit. Now for a hyperbolic orbit minimum radius of curvature occurs at the apse and is numerically equal to the semi-latus rectum l. The condition is then that

$$\lambda/l << 1,$$

or, giving λ its value at the apse,

$$\lambda_0/k'l << 1,$$

where $$k' = \{(\sec\alpha - 1)/(\sec\alpha + 1)\}^{\frac{1}{2}}.$$

This constant k' is found to be very nearly $k_m/16{\cdot}1$, for all values of ϕ. In this particular case Schrödinger's rough condition differs from De Broglie's only by a constant numerical factor†.

It can be shown by using a theorem due to Darwin that condition (11) is also valid in the general case of two particles of comparable mass, if the angle α is replaced by Θ, the angle of projection of the particle initially at rest.

The condition for classical scattering is always satisfied, as it must be, if h is put equal to zero. It is also satisfied when e_1e_2 is large and when v is small (except when α is very nearly zero), that is for the scattering of macroscopic particles. But diffraction effects are to be expected for angles of deflection ϕ sufficiently near 180° even if hv_0/e_1e_2 is very small.

As an example of the application of (11) to the scattering of alpha particles the following cases will be discussed. For an alpha particle of 5 cms. range scattered through 30° by an aluminium nucleus we find $Z \doteqdot 1/100$. Approximate normal scattering is to be expected and is in fact found‡. If the angle of scattering is 90° we find $Z \doteqdot 1/4$ and large deviations from normal scattering are both expected and are found. If the observed deviations in the latter case are in fact due to a diffraction effect we should expect similar deviations in other cases when Z is approximately 1/4. For the collision of an 8·6 cm. alpha particle with a proton, deviations are to be expected at angles of projection about 75°, that is for deflections of the alpha particle of 6°. Such deviations

* Schrödinger, *Ann. d. Phys.* vol. LXXIX (1926), p. 496.

† It should be noticed that though the curvature has its greatest value at the apse, the quantity Z has its maximum some distance away from the apse. Since Z contains the factor $\cos\theta$ it is zero at the apse. There can therefore be no general relation between Schrödinger's and De Broglie's conditions. For instance, for a particle in a circular orbit, $Z=0$, whatever the size of the orbit.

‡ Bieler, *Proc. Roy. Soc.* A, vol. CV (1924), p. 434.

certainly occur*. For the collision of 5 cm. alpha particles with gold nuclei, deviations are to be expected at angles of about 125°. The recent experimental results of Rutherford and Chadwick† are in complete disagreement with this result. Of the many possible causes of this discrepancy two may be mentioned. The deviations observed with aluminium for $Z = 1/4$ may be a *structure* effect; the diffraction effect may not become noticeable till Z is, say, roughly unity, or even greater. The diffraction effect may be present but may not be of a type to be revealed by the measurements; for instance the fairly wide limits of scattering angle in the experiments of Rutherford and Chadwick may have left unrevealed any rapid variations of scattering with angle.

It seems therefore not impossible that the observed deviations from normal scattering, which have hitherto been explained with the aid of classical mechanics by postulating arbitrary deviations from the inverse square law, may now receive an explanation with the aid of the wave mechanics and the inverse square law alone. If other forces connected with the structure of the nucleus, such as polarisation or magnetic forces, have to be taken into account, their effect may complicate, but cannot in general be expected to mask, the diffraction scattering due to the inverse square law alone.

Since the product $e_1 e_2$ but not the mass m enters into (11), the limits of classical scattering of electrons by electrons will be given by this same condition. When the field is attractive, as in the scattering of electrons by nuclei, the condition for classical scattering will still be of the form (11) but k_m will have a different value.

The dimensional part $\lambda/a \propto hv/e_1 e_2$ of (11) can be obtained from considerations of dimensions alone. This method also applies when the law of force is of the general type μr^{-n}. From the analogy with optics the required condition is that the ratio λ/L should be small, where L is a length playing the part of the *size* of the diffracting centre. The only dimensional quantities on which L can depend are μ, m and v. It must therefore be of the form

$$(\mu/mv^2)^{1/n-1}.$$

We have therefore
$$\lambda/L \propto hv^{\frac{3-n}{n-1}} m^{\frac{2-n}{n-1}} \mu^{\frac{1}{1-n}} \quad \text{.................(12)}.$$

For $n = 2$, $\mu = e_1 e_2$ and λ/L is proportional to $hv/e_1 e_2$, in agreement with (11). When the forces vary inversely as the cube of the distance, $n = 3$ and λ/L is proportional to

$$h/(m\mu)^{\frac{1}{2}}.$$

Here the velocity does not enter into the expression for λ/L and so the limits of normal scattering should be independent of the velocity.

* Chadwick and Bieler, *Phil. Mag.*, vol. XLII (1921), p. 923.
† Rutherford and Chadwick, *Phil. Mag.*, vol. L (1925), p. 889.

https://doi.org/10.1017/S0305004100011208 Published online by Cambridge University Press

When the forces vary inversely as a high power of the distance we approximate to the case of the scattering by a rigid body. Then we have $L = \text{const.}$ and $\lambda/L \propto h/mv$. The deviations from normal scattering now occur when v is small, instead of when v is large, as for the inverse square law. This is due to the fact that in the latter case the effective *size* of the scattering centre of force is proportional to a, and therefore to $1/v^2$, while in the former case the effective *size* is constant.

Elsasser* has suggested that the long free path of very slow electrons observed by Ramsauer can be associated with the decrease of scattering by a rigid obstacle as the wave-length of the scattered wave is increased. Why this effect should only appear with atoms of high symmetry appears clearly from the dimensional argument given above. Elsasser's explanation assumes a diffracting disc whose size is independent of the velocity. This is only obtained when n in (12) is large, that is when the law of force outside the atom varies very rapidly with the distance. Large values of n can only be expected with the most symmetrical atoms.

In optics not only is the condition for geometrical optics that the ratio of the wave-length to a linear dimension of the object should be small, but the intensity in any direction when diffraction occurs is a function of the same ratio. In the dynamical case the numbers of particles scattered in a given direction should therefore be a function of λ/L, that is of the R.H.S. of (12). For the law of the inverse square the *type* of scattering observed should be the same for all cases when $v/e_1 e_2$ has the same value. For the inverse cube law the *type* of diffraction pattern should be independent of the velocity. This may be the origin of the fixed peak in the distribution of scattered electrons from helium, found by Dymond†.

Note added in Proof. In a recent paper (*Göttinger Nachrichten*, 1927) Born has derived the classical scattering formula as an approximate solution of Schrödinger's wave equations, for the scattering of an alpha particle by a neutral hydrogen atom. In both Wentzel's and Born's results deviations from the inverse square law scattering occur for low velocity particles. This appears to be connected with the screening effect of the electron in the hydrogen atom. That normal scattering by a naked nucleus must occur for slow particles, seems certain in view of the occurrence of the product hv in (11) above and in the quantity z_0 in equation (32) of Born's paper. Since we must obtain classical scattering when $h = 0$, we must also obtain classical scattering as v approaches zero.

* Elsasser, *Naturwissenschaften*, vol. XIII (1925), p. 711.

† Dymond, *Nature*, vol. CXVIII (1926), p. 336.