Nagoya Math. J., **254** (2024), 488–497
DOI 10.1017/nmj.2023.34

μ^*-ZARISKI PAIRS OF SURFACE SINGULARITIES

CHRISTOPHE EYRAL AND MUTSUO OKA

Abstract. Let f_0 and f_1 be two homogeneous polynomials of degree d in three complex variables z_1, z_2, z_3. We show that the Lê–Yomdin surface singularities defined by $g_0 := f_0 + z_i^{d+m}$ and $g_1 := f_1 + z_i^{d+m}$ have the same abstract topology, the same monodromy zeta-function, the same μ^*-invariant, but lie in distinct path-connected components of the μ^*-constant stratum if their projective tangent cones (defined by f_0 and f_1, respectively) make a Zariski pair of curves in $\mathbb{P}^2$, the singularities of which are Newton non-degenerate. In this case, we say that $V(g_0) := g_0^{-1}(0)$ and $V(g_1) := g_1^{-1}(0)$ make a μ^*-Zariski pair of surface singularities. Being such a pair is a necessary condition for the germs $V(g_0)$ and $V(g_1)$ to have distinct embedded topologies.

§1. Introduction and statement of the result

Let g_0 and g_1 be two polynomials in three complex variables z_1, z_2, z_3. We assume that they vanish at the origin $\mathbf{0} \in \mathbb{C}^3$ and that the corresponding germs of surfaces, $V(g_0) := g_0^{-1}(0)$ and $V(g_1) := g_1^{-1}(0)$, have an isolated singularity at $\mathbf{0}$. It is well known that if $V(g_0)$ and $V(g_1)$ have the same embedded topology (i.e., if the pairs $(\mathbb{C}^3, V(g_0))$ and $(\mathbb{C}^3, V(g_1))$ are homeomorphic in a neighborhood of the origin, or equivalently, by [28], if the pairs $(\mathbb{S}^5_\varepsilon, K_{g_0})$ and $(\mathbb{S}^5_\varepsilon, K_{g_1})$ are diffeomorphic for any ε small enough), then they have the same Milnor number (see [18], [23], [33]). Here, K_{g_l} denotes the *link* of g_l ($l \in \{0,1\}$), that is, $K_{g_l} := \mathbb{S}^5_\varepsilon \cap V(g_l)$ for ε small enough, where $\mathbb{S}^5_\varepsilon$ is the sphere with radius ε centered at $\mathbf{0} \in \mathbb{C}^3$. (Note that the diffeomorphism type of the embedded link $(\mathbb{S}^5_\varepsilon, K_{g_l})$ is independent of ε, provided that ε is small enough.) On the other hand, it is quite possible for two isolated surface singularities $V(g_0)$ and $V(g_1)$ to have the same Milnor number and non-diffeomorphic embedded links. In [3], [4], using Luengo's theory of superisolated singularities [20], Artal-Bartolo even showed that the embedded topology of the link of a superisolated surface singularity is not determined by the topology of the abstract link and the characteristic polynomial of the monodromy. However, in practice, given g_0 and g_1 with the same characteristic polynomial (or equivalently, the same monodromy zeta-function), the same abstract topology, and even with the same Teissier μ^*-invariant, it is extremely difficult to determine whether $(\mathbb{S}^5_\varepsilon, K_{g_0})$ and $(\mathbb{S}^5_\varepsilon, K_{g_1})$ are diffeomorphic or not. The goal of this paper is to investigate a special class of Lê–Yomdin surface singularities which are "likely to systematically produce" pairs of germs sharing all these invariants but having non-diffeomorphic embedded links. Such pairs are called *μ^*-Zariski pairs of surface singularities* and are defined as follows.

Consider a classical Zariski pair of (reduced) projective curves $C_0 = \{f_0 = 0\}$ and $C_1 = \{f_1 = 0\}$ of degree d in the complex projective plane $\mathbb{P}^2$, that is, there are regular neighborhoods N_0 and N_1 of C_0 and C_1, respectively, such that (N_0, C_0) and (N_1, C_1) are

Received January 24, 2023. Revised September 23, 2023. Accepted October 29, 2023.
2020 Mathematics subject classification: Primary 14M25, 14B05, 14J17, 32S55, 32S05.
Keywords: link of isolated surface singularities, Lê–Yomdin singularities, monodromy zeta-function, μ^*-constant stratum, abstract and embedded topology.

homeomorphic, while $(\mathbb{P}^2, C_0)$ and $(\mathbb{P}^2, C_1)$ are not. The first example of such a pair was found by Zariski [36] in the early 1930s, and their systematic study was initiated by Artal-Bartolo [5] in the mid-1990s (for a detailed survey on this topic, see [6], [25]). By a linear change of the coordinates z_1, z_2, z_3, we may assume that the singularities of the curves C_0 and C_1 are not located on the coordinate lines $z_i = 0$ $(1 \leq i \leq 3)$ and that their defining polynomials f_0 and f_1 are convenient[1] and Newton non-degenerate on any face Δ of their (common) Newton diagram if Δ is not top-dimensional. The fact that the singularities of the curves do not sit on the coordinate lines implies that for any integers $m \geq 1$ and $1 \leq i \leq 3$, the polynomials

$$g_0 := f_0 + z_i^{d+m} \quad \text{and} \quad g_1 := f_1 + z_i^{d+m}$$

define an *isolated* surface singularity at $\mathbf{0}$ (see [21, Th. 2]). Such singularities are called *m-Lê–Yomdin singularities* and were first investigated by Yomdin and Lê in [19], [13], respectively. The monodromy zeta-function (or the characteristic polynomial) of such a singularity was computed by Siersma [29], [30], Stevens [31], and Gusein-Zade, Luengo, and Melle-Hernández [11] (see also [26]). (The Milnor number was already known from [21].) In [7], Artal-Bartolo, Cogolludo-Agustín, and Martín-Morales gave a characterization for the abstract link of a Lê–Yomdin singularity to be a rational homology sphere.

In the special case where $m = 1$, a 1-Lê–Yomdin singularity is called a *superisolated singularity*. Superisolated singularities were introduced by Luengo [20] to answer important questions and conjectures. For example, in [20], Luengo gave examples of superisolated surface singularities for which the μ-constant stratum in the miniversal deformation is not smooth.

Now, let us make precise the notion of *Zariski pair of surface singularities*. Let $g_0 = f_0 + z_i^{d+m}$ and $g_1 = f_1 + z_i^{d+m}$ be two Lê–Yomdin surface singularities obtained from a Zariski pair of curves f_0 and f_1 as above.

- We say that $(V(g_0), V(g_1))$ is a *weak ζ-Zariski pair of surface singularities* if g_0 and g_1 have the same monodromy zeta-function (in particular, the same Milnor number).
- A weak ζ-Zariski pair for which the germs $V(g_0)$ and $V(g_1)$ (or equivalently, the links K_{g_0} and K_{g_1}) have the same abstract topology is called a ζ-Zariski pair (without the adjective "weak").
- A (weak) ζ-Zariski pair is said to be a *(weak) μ^*-Zariski pair* if g_0 and g_1 have the same μ^*-invariant while belonging to distinct path-connected components of the μ^*-constant stratum.
- A (weak) μ^*-Zariski pair is called a *(weak) μ-Zariski pair* if furthermore g_0 and g_1 lie in different path-connected components of the μ-constant stratum.
- Finally, a (weak) ζ-Zariski pair is called a *(weak) Zariski pair* if the germs $V(g_0)$ and $V(g_1)$ (or equivalently, K_{g_0} and K_{g_1}) have distinct embedded topologies.

Note that a (weak) Zariski pair of surface singularities $V(g_0)$ and $V(g_1)$ sharing the same μ^*-invariant is always a (weak) μ-Zariski pair, and hence a (weak) μ^*-Zariski pair. That is, being a (weak) μ^*-Zariski pair is a necessary condition for being a (weak) Zariski pair. Indeed, by [10, Th. 5.3], if g_0 and g_1 lie in the same path-connected component of the μ^*-constant stratum, then they can always be joined by a piecewise complex-analytic

[1] This means that the Newton diagram $\Gamma(f_l)$ of f_l $(l \in \{0,1\})$ meets each coordinate axis.

https://doi.org/10.1017/nmj.2023.34 Published online by Cambridge University Press

path (defined in the relevant natural way), and by a well-known theorem of Teissier [32, théorème 3.9], this in turn implies that the diffeomorphism type of the pairs $(\mathbb{S}^5_\varepsilon, K_{g_0})$ and $(\mathbb{S}^5_\varepsilon, K_{g_1})$ is identical.

In [20], Luengo proved that for superisolated singularities (i.e., for $m=1$), the abstract links K_{g_0} and K_{g_1} are homeomorphic. The second-named author showed a similar property for $m \geq 1$ if the singularities of the corresponding curves C_0 and C_1 are Newton non-degenerate (see [27, Th. 24 and Rem. 25]). In [3, théorème 4.4] and [4, théorème 1.6, §1.7, and corollaire 5.6.6], Artal-Bartolo proved that if $m=1$, then $V(g_0)$ and $V(g_1)$ also share the same characteristic polynomial of the monodromy, and if furthermore the Alexander polynomials of the curves C_0 and C_1 do not coincide, then $V(g_0)$ and $V(g_1)$ do not have the same embedded topology. In particular, combined with Luengo's result, this shows that, in this latter case, $(V(g_0), V(g_1))$ is a Zariski pair of surface singularities.

In this paper, we prove the following theorem.

Theorem 1.1. *If the singularities of the curves C_0 and C_1 are Newton non-degenerate in some suitable local coordinates,*[2] *then the pair made up of the m-Lê–Yomdin singularities $V(g_0)$ and $V(g_1)$ is a μ^*-Zariski pair of surface singularities.*

Again, we emphasize that being a μ^*-Zariski pair is a necessary condition for being a Zariski pair of surface singularities. We also highlight that in the above theorem, the Alexander polynomials of the curves C_0 and C_1 may coincide.

We expect that with the assumption of the theorem, $(V(g_0), V(g_1))$ is a μ-Zariski pair, and in fact, a Zariski pair of surface singularities. As mentioned above, in the special case of superisolated singularities (i.e., $m=1$), and provided that the curves have distinct Alexander polynomials (but not necessarily Newton non-degenerate singularities), this is already proved by combining Artal-Bartolo's [3], [4] and Luengo's [20] results.

§2. Proof of Theorem 1.1

First, we show that $(V(g_0), V(g_1))$ is a ζ-Zariski pair of surface singularities, and then we prove that it is in fact a μ^*-Zariski pair. To simplify, we assume that $i=1$, that is, $g_l = f_l + z_1^{d+m}$ $(l \in \{0,1\})$.

To compute the monodromy zeta-function $\zeta_{g_l,\mathbf{0}}(t)$ of g_l, we use the classical formula of Siersma (see [29, Main theorem, p. 183] and [30, Th. 3.4 and Rem. 3.6]), Stevens (see [31, p. 140]), and Gusein-Zade, Luengo, and Melle-Hernández (see [11, p. 250]) (see also [26, Lem. 3.2 and Th. 3.7]). More precisely, the ordinary point blowing up at $\mathbf{0} \in \mathbb{C}^3$, denoted by $\pi\colon X \to \mathbb{C}^3$, being a biholomorphism over $\mathbb{C}^3 \setminus V(g_l)$, the tubular Milnor fibration of g_l at $\mathbf{0}$ can be lifted to X, so that the pullback $\pi^* g_l \equiv g_l \circ \pi$ is a locally trivial fibration which is isomorphic to it. Let $U_1 := \mathbb{P}^2 \setminus \{z_1 = 0\}$ be the standard affine chart of $\mathbb{P}^2$ with coordinates $(z_2/z_1, z_3/z_1)$. In the corresponding chart $X \cap (\mathbb{C}^3 \times U_1)$ of X, with coordinates $\mathbf{y} \equiv (y_1, y_2, y_3) := (z_1, z_2/z_1, z_3/z_1)$, the pullback $\pi^* g_l$ is written as

$$\pi^* g_l(\mathbf{y}) = y_1^d (f_l(1, y_2, y_3) + y_1^m).$$

The first factor, y_1^d, corresponds to the exceptional divisor $E \simeq \mathbb{P}^2$, while the second one represents the strict transform $\tilde{V}(g_l)$ of $V(g_l)$. Outside of the exceptional divisor, $\tilde{V}(g_l)$ has no singularities. On the exceptional divisor, it has a finite number of isolated singularities,

[2] For instance, this is always the case if the singularities are "simple" in the sense of Arnol'd [2].

which are given by the singular points $\mathbf{p}\in\Sigma(C_l)$ of the reduced curve C_l. Then the formula for the zeta-function mentioned above is written as

$$\zeta_{g_l,\mathbf{0}}(t)=\zeta_d(t)\times(1-t^d)^{\mu^{\text{tot}}(C_l)}\times\prod_{\mathbf{p}\in\Sigma(C_l)}\zeta_{\pi^*g_l,\mathbf{p}}(t), \tag{2.1}$$

where $\zeta_d(t)$ is the zeta-function of a Newton non-degenerate homogeneous polynomial of degree d (i.e., $\zeta_d(t)=(1-t^d)^{-d^2+3d-3}$), $\Sigma(C_l)$ is the set of singular points of C_l, and $\mu^{\text{tot}}(C_l)$ is the total Milnor number of C_l (i.e., the sum of the local Milnor numbers at the singular points of C_l).

By our assumption, there exist local coordinates $\mathbf{x}=(x_1,x_2,x_3)$ and $\mathbf{u}=(u_1,u_2,u_3)$ near $\mathbf{p}_0\in\Sigma(C_0)$ and $\mathbf{p}_1\in\Sigma(C_1)$, respectively, where $x_1=u_1=y_1$ and (x_2,x_3) and (u_2,u_3) are analytic coordinate changes of (y_2,y_3),[3] such that

$$\pi^*g_0(\mathbf{x})=x_1^d(h_0(x_2,x_3)+x_1^m)\quad\text{and}\quad\pi^*g_1(\mathbf{u})=u_1^d(h_1(u_2,u_3)+u_1^m),$$

where h_0 and h_1 are Newton non-degenerate. Moreover, if the singularities $(C_1,\mathbf{p}_1)$ and $(C_0,\mathbf{p}_0)$ are topologically equivalent, then we may assume that the Newton diagrams, $\Gamma(h_0)$ and $\Gamma(h_1)$, of h_0 and h_1 coincide. It follows that π^*g_0 and π^*g_1 are Newton non-degenerate with the same Newton diagram, and hence, by Varchenko's formula (see [34, Th. (4.1)]), we have

$$\zeta_{\pi^*g_0,\mathbf{p}_0}(t)=\zeta_{\pi^*g_1,\mathbf{p}_1}(t).$$

Since (C_0,C_1) is a Zariski pair of projective curves, the total Milnor numbers $\mu^{\text{tot}}(C_0)$ and $\mu^{\text{tot}}(C_1)$ coincide, and the equality $\zeta_{g_0,\mathbf{0}}(t)=\zeta_{g_1,\mathbf{0}}(t)$ follows immediately from (2.1).

To conclude that $(V(g_0),V(g_1))$ is a ζ-Zariski pair, it remains to observe that the links K_{g_0} and K_{g_1} have the same abstract topology; this is proved in [27, Th. 24 and Rem. 25].

Now, let us show that $(V(g_0),V(g_1))$ is a μ^*-Zariski pair of surface singularities. For that, we must first show that g_0 and g_1 have the same μ^*-invariant at $\mathbf{0}$. We recall that the μ^*-invariant of g_l at $\mathbf{0}$, introduced by Teissier in [32], is the triple

$$\mu^*_{\mathbf{0}}(g_l):=(\mu_{\mathbf{0}}(g_l),\mu_{\mathbf{0}}(g_l|_H),\text{mult}_{\mathbf{0}}(g_l)-1),$$

where $\mu_{\mathbf{0}}(g_l)$ is the Milnor number of g_l at $\mathbf{0}$, $\mu_{\mathbf{0}}(g_l|_H)$ is the Milnor number at $\mathbf{0}$ of the restriction of g_l to a generic plane H of $\mathbb{C}^3$ through the origin (this number is usually denoted by $\mu^{(2)}_{\mathbf{0}}(g_l)$), and $\text{mult}_{\mathbf{0}}(g_l)$ is the multiplicity of g_l at $\mathbf{0}$.

By [21, Th. 2], for any $l\in\{0,1\}$, the Milnor number $\mu_{\mathbf{0}}(g_l)$ is given by

$$\mu_{\mathbf{0}}(g_l)=(d-1)^3+m\mu^{\text{tot}},$$

where μ^{tot} is the (common) total Milnor number of C_0 and C_1.

For a generic plane H of $\mathbb{C}^3$ through the origin, the restriction $f_l|_H$ is a homogeneous polynomial of degree d with an isolated singularity at $\mathbf{0}$, so that its Milnor number at $\mathbf{0}$ is $\mu_{\mathbf{0}}(f_l|_H)=(d-1)^2$. Since $f_l|_H$ is Newton non-degenerate and the term z_1^{d+m} is above the Newton diagram $\Gamma(g_l|_H)=\Gamma(f_l|_H)$, the restriction $g_l|_H$ is Newton non-degenerate too. Thus, its Milnor number at $\mathbf{0}$ is determined by $\Gamma(g_l|_H)$, and hence we have

$$\mu^{(2)}_{\mathbf{0}}(g_l):=\mu_{\mathbf{0}}(g_l|_H)=\mu_{\mathbf{0}}(f_l|_H)=(d-1)^2.$$

[3] Hereafter, such coordinates will be called *admissible coordinates.*

Lastly, since the multiplicities of g_0 and g_1 at $\mathbf{0}$ are equal to d, it follows that g_0 and g_1 have the same μ^*-invariant at $\mathbf{0}$, namely, for any $l \in \{0,1\}$, we have

$$\mu^*_{\mathbf{0}}(g_l) = ((d-1)^3 + m\mu^{\text{tot}}, (d-1)^2, d-1).$$

Finally, and this is the heart of the proof, we must now show that g_0 and g_1 lie in different path-connected components of the μ^*-constant stratum. To this end, we argue by contradiction. Suppose that g_0 and g_1 belong to the same component. Then, by [10, Th. 5.3], there exists a μ^*-constant piecewise complex-analytic family $\{g_s\}_{0\leq s\leq 1}$ connecting g_0 and g_1. In particular, the multiplicity $\text{mult}_{\mathbf{0}}(g_s)$ of g_s at $\mathbf{0}$ is independent of $s \in [0,1]$, and the initial polynomial $\text{in}(g_s)$ of g_s (i.e., the sum of the monomials of g_s of lowest degree) has degree d.

LEMMA 2.1. *For each $s \in [0,1]$, the homogeneous polynomial $\text{in}(g_s)$ is reduced, so that the projective curve $C_s \subseteq \mathbb{P}^2$ defined by $\text{in}(g_s)$ has only isolated singularities.*

Proof. We argue by contradiction. Suppose there exists $s_0 \in [0,1]$ such that $\text{in}(g_{s_0})$ is not reduced (i.e., C_{s_0} has non-isolated singularities). Then, for a generic linear plane H of $\mathbb{C}^3$, there are coordinates (x,y) for H and linear forms $\ell_1(x,y),\dots,\ell_q(x,y)$ such that

$$\text{in}(g_{s_0})|_H(x,y) = \ell_1(x,y)^{p_1}\cdots\ell_q(x,y)^{p_q}$$

with $p_1 \geq \cdots \geq p_q$ and $p_1 \geq 2$. By a linear change of coordinates, we may assume that $\ell_1(x,y) \equiv x$, so that

$$\text{in}(g_{s_0})|_H(x,y) = x^{p_1}h(x,y),$$

where h is a homogeneous polynomial of degree $d - p_1$ (in particular, $\text{in}(g_{s_0})|_H$ is not convenient with respect to the coordinates (x,y)). By adding monomials of the form x^α and y^β for α, β large enough, we may also assume that $g_{s_0}|_H$ is convenient. Now, since the integral point $(1, d-1)$ is not on the Newton diagram $\Gamma(\text{in}(g_{s_0})|_H)$ of $\text{in}(g_{s_0})|_H$ with respect to the coordinates (x,y), it follows[4] that

$$\nu(\Gamma_-(g_{s_0}|_H)) > \nu(\Gamma_-(g_0|_H))$$

(see Figure 1, where $\Gamma_+(\text{in}(g_{s_0})|_H)$ is the Newton polyhedron of $\text{in}(g_{s_0})|_H$ in the coordinates (x,y)). Here, $\nu(\cdot)$ denotes the Newton number (see [14] for the definition) and $\Gamma_-(g_{s_0}|_H)$ stands for the cone over $\Gamma(g_{s_0}|_H)$ with the origin as vertex. (Again, $\Gamma(g_{s_0}|_H)$ denotes the Newton diagram of $g_{s_0}|_H$ with respect to the coordinates (x,y).) The polyhedron $\Gamma_-(g_0|_H)$ is defined similarly. Since

$$\mu_{\mathbf{0}}(g_{s_0}|_H) \geq \nu(\Gamma_-(g_{s_0}|_H))$$

[4] Let us briefly show it, for instance, in the special case where the Newton boundaries are as in Figure 1, the general case being completely similar. Clearly, in this case,

$$\nu(\Gamma_-(g_{s_0}|_H)) = 2S' - (d+c) - (d+e) + 1,$$

where $S' = S + cq/2 + ep/2$ with $p \geq p_1 \geq 2$ and S is the area of the triangle $(0,d,d)$. Similarly, $\nu(\Gamma_-(g_0|_H)) = 2S - 2d + 1$. Since $p \geq 2$, it follows that

$$\nu(\Gamma_-(g_{s_0}|_H)) - \nu(\Gamma_-(g_0|_H)) = c(q-1) + e(p-1) > 0$$

(note that if $q = 0$, then $c = 0$, and the above inequality still holds true).

https://doi.org/10.1017/nmj.2023.34 Published online by Cambridge University Press

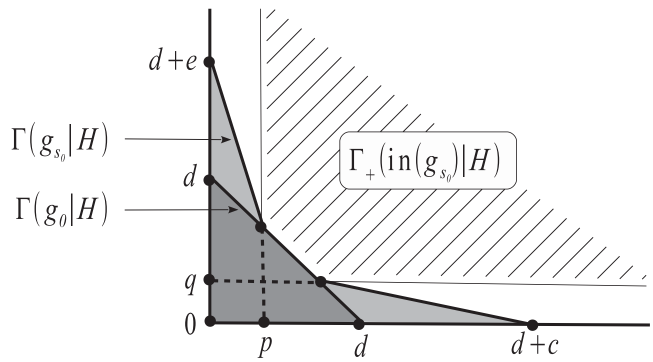

Figure 1.
Newton diagrams.

(see [14, théorème 1.10]), altogether we have

$$\mu_{\mathbf{0}}^{(2)}(g_{s_0}) = \mu_{\mathbf{0}}(g_{s_0}|_H) \geq \nu(\Gamma_-(g_{s_0}|_H)) > \nu(\Gamma_-(g_0|_H)) = (d-1)^2 = \mu_{\mathbf{0}}^{(2)}(g_0),$$

which is a contradiction to the μ^*-constancy. ∎

LEMMA 2.2. *The zeta-function $\zeta_{g_s,\mathbf{0}}(t)$ is independent of $s \in [0,1]$.*

Proof. It is well known that in a μ^*-constant piecewise complex-analytic family $\{g_s\}$, the diffeomorphism type of the embedded link $(\mathbb{S}_\varepsilon^5, K_{g_s})$ is independent of s (see [32, théorème 3.9 and remarque 3.12]). Alternatively, we may use [27, Lem. 12], which asserts that in a μ-constant (*a fortiori* in a μ^*-constant) piecewise complex-analytic family $\{g_s\}$, the zeta-function $\zeta_{g_s,\mathbf{0}}(t)$ is independent of s. ∎

Now, by the A'Campo formula (see [1, théorème 3]), we know that the zeta-function $\zeta_{g_s,\mathbf{0}}(t)$ is uniquely written as

$$\zeta_{g_s,\mathbf{0}}(t) = \prod_{i=1}^{\ell} (1-t^{d_i})^{\nu_i}, \tag{2.2}$$

where $d_1,\ldots,d_\ell$ are mutually disjoint and $\nu_1,\ldots,\nu_\ell$ are nonzero integers. The smallest integer d_{i_0} among $d_1,\ldots,d_\ell$ is called the *zeta-multiplicity of* g_s and is denoted by $m_\zeta(g_s)$. We define the *zeta-multiplicity factor of* $\zeta_{g_s,\mathbf{0}}(t)$ as the factor $(1-t^{d_{i_0}})^{\nu_{i_0}}$ of (2.2) corresponding to the zeta-multiplicity $d_{i_0} \equiv m_\zeta(g_s)$. Note that, by Lemma 2.2, the zeta-multiplicity of g_s and the zeta-multiplicity factor of $\zeta_{g_s,\mathbf{0}}(t)$ are independent of s. Moreover, by [27, Prop. 11], we know that $m_\zeta(g_s) \geq \mathrm{mult}_{\mathbf{0}}(g_s) = d$, and the formula (2.1) shows that for $s=0$ we have $m_\zeta(g_0) \leq d$. So, altogether, $m_\zeta(g_s) = d$ for any $s \in [0,1]$.

LEMMA 2.3. *For any $s \in [0,1]$, the zeta-multiplicity factor of $\zeta_{g_s,\mathbf{0}}(t)$ is given by*

$$(1-t^d)^{-d^2+3d-3+\mu^{\mathrm{tot}}(C_s)},$$

and since the latter is independent of s, so is the total Milnor number $\mu^{\mathrm{tot}}(C_s)$.

Proof. Here, to compute $\zeta_{g_s,\mathbf{0}}(t)$, we apply a method developed by the second-named author in [24]. This method, inspired by an approach of Clemens [8], was used in [24, Chap. I, Proof of Th. 5.2] to generalize the classical zeta-function formula of A'Campo [1]. Roughly, the idea is to decompose the lifted Milnor fibration $\pi^* g_s$ (which is isomorphic to the original Milnor fibration of g_s at $\mathbf{0}$) into its restrictions along "controlled" tubular neighborhoods of the strata in a canonical regular stratification of $\pi^{-1}(V(g_s))$. Then, by the multiplicativeness of the zeta-function, it suffices to compute the zeta-functions of the induced restricted fibrations. More precisely, let $\mathbf{p}_1,\dots,\mathbf{p}_{k_0}$ be the points of the singular set $\Sigma(C_s)$ of C_s, and for each $\mathbf{p}_k$, let $B_\varepsilon(\mathbf{p}_k)$ be a small ball centered at $\mathbf{p}_k$. Put

$$B := \bigcup_{k=1}^{k_0} B_\varepsilon(\mathbf{p}_k),$$

and consider tubular neighborhoods $N(C_s)$ and $N(E)$ of $C_s \setminus B$ and $E \setminus (N(C_s) \cup B)$, respectively. As in [24, Chap. I, p. 56], we assume that the triple

$$\{B, N(C_s), N(E)\}, \tag{2.3}$$

together with its natural associated projections and distance functions, makes a family of "control data" in the sense of Mather [22, §7]. Consider the restrictions of $\hat{g}_s := \pi^* g_s$ to $N(E)$, $N(C_s)$ and the balls $B_\varepsilon(\mathbf{p}_k)$, respectively. The relations (5.2.4) and (5.2.5), together with Lemmas (5.3) and (5.4), of [24, Chap. I] say that

$$\zeta_{g_s,\mathbf{0}}(t) \equiv \zeta_{\hat{g}_s}(t) = \zeta_{\hat{g}_s|_{N(E)}}(t) \cdot \zeta_{\hat{g}_s|_{N(C_s)}}(t) \cdot \prod_{k=1}^{k_0} \zeta_{\hat{g}_s|_{B_\varepsilon(\mathbf{p}_k)}}(t). \tag{2.4}$$

Thus, it suffices to compute each piece $\zeta_{\hat{g}_s|_{N(E)}}(t)$, $\zeta_{\hat{g}_s|_{N(C_s)}}(t)$, and $\zeta_{\hat{g}_s|_{B_\varepsilon(\mathbf{p}_k)}}(t)$ separately.

We start with the calculation of the zeta-function $\zeta_{\hat{g}_s|_{N(E)}}(t)$ of the fibration $\hat{g}_s|_{N(E)}$. For admissible coordinates $\mathbf{x} = (x_1, x_2, x_3)$ in a neighborhood $U_\mathbf{p}$ of a point $\mathbf{p} \in E' := E \setminus (N(C_s) \cup B)$, we may assume that the projection

$$p \colon U_\mathbf{p} \cap N(E) \to E'$$

associated with the family of control data (2.3) is given by $\mathbf{x} \mapsto (0, x_2, x_3)$, so that E' is defined by $x_1 = 0$ and the restriction of $\hat{g}_s$ to $p^{-1}(\mathbf{p})$ is given by x_1^d. Then, by the relation (5.2.5) of [24, Chap. I], the *normal zeta-function* $\zeta^\perp_{E'}(t)$ of $\hat{g}_s$ along E' (see [24, Chap. I, p. 59] for the definition) is given by

$$\zeta^\perp_{E'}(t) = (1 - t^d)^{-1}.$$

Thus, by [24, Chap. I, Lems. (5.3) and (5.4)], we get

$$\begin{aligned} \zeta_{\hat{g}_s|_{N(E)}}(t) &= (\zeta^\perp_{E'}(t))^{\chi(E \setminus \tilde{V}(g_s))} = (\zeta^\perp_{E'}(t))^{\chi(\mathbb{P}^2 \setminus C_s)} = (\zeta^\perp_{E'}(t))^{\chi(\mathbb{P}^2) - \chi(C_s)} \\ &= (1 - t^d)^{-\chi(\mathbb{P}^2) + \chi(C_s)} = (1 - t^d)^{-3 + \chi(C_s)} = (1 - t^d)^{-3 + 3d - d^2 + \mu^{\mathrm{tot}}(C_s)}. \end{aligned}$$

Here, $\chi(\cdot)$ denotes the Euler–Poincaré characteristic, and we recall that for a reduced curve C_s of degree d, we have $\chi(C_s) = 3d - d^2 + \mu^{\mathrm{tot}}(C_s)$ (see, e.g., [35, Cor. 7.1.4]).

Next, we look at the zeta-function $\zeta_{\hat{g}_s|_{N(C_s)}}(t)$. This time, for admissible coordinates $\mathbf{x} = (x_1, x_2, x_3)$ in a neighborhood $U_\mathbf{p}$ of a point $\mathbf{p} \in C'_s := C_s \setminus B$, we may assume that the projection

$$p' \colon U_\mathbf{p} \cap N(C_s) \to C'_s$$

associated with the family of control data (2.3) is given by $\mathbf{x} \mapsto (0, x_2, 0)$, so that C'_s is defined by $x_1 = x_3 = 0$ and the restriction of $\hat{g}_s$ to $p'^{-1}(\mathbf{p})$ is given by $x_1^d x_3$. Then, by

https://doi.org/10.1017/nmj.2023.34 Published online by Cambridge University Press

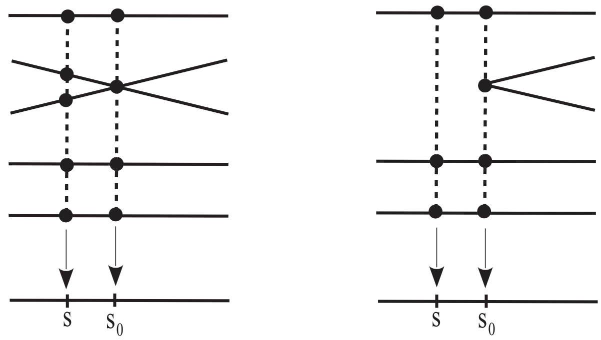

Figure 2.
Bifurcation of singularities.

the relation (5.2.5) of [24, Chap. I], the normal zeta-function of $\hat{g}_s$ along C'_s is given by $\zeta^{\perp}_{C'_s}(t) = 1$, and hence, by [24, Chap. I, Lems. (5.3) and (5.4)] again, we get

$$\zeta_{\hat{g}_s|_{N(C_s)}}(t) = 1.$$

As for the zeta-function $\zeta_{\hat{g}_s|_{B_\varepsilon(\mathbf{p}_k)}}(t)$, since the zeta-multiplicity of g_s is d and the (usual) multiplicity of $\hat{g}_s$ at $\mathbf{p}_k$ is greater than or equal to $d+1$, it follows from [27, Prop. 11] that $\zeta_{\hat{g}_s|_{B_\varepsilon(\mathbf{p}_k)}}(t)$ does not contribute to the zeta-multiplicity factor of $\zeta_{\hat{g}_s}(t)$.

So, altogether, the unique contribution to the zeta-multiplicity factor of $\zeta_{\hat{g}_s}(t)$ comes from the zeta-function $\zeta_{\hat{g}_s|_{N(E)}}(t)$ and is given by $(1-t^d)^{-3+3d-d^2+\mu^{\text{tot}}(C_s)}$. ∎

We can now easily complete the proof of Theorem 1.1 thanks to two theorems of Lê. Indeed, we first observe that if there exists $s_0 \in [0,1]$ such that the family $\{\text{in}(g_s)\}$ has a *bifurcation of the singularities* in a small ball B centered at a singular point $\mathbf{p}_0$ of C_{s_0},[5] then, by [17, théorème B] (see also [12], [15]), for $s \neq s_0$ near s_0, we have

$$\sum_{\mathbf{p}\in B\cap\Sigma(C_s)} \mu_{\mathbf{p}}(\text{in}(g_s)) < \mu_{\mathbf{p}_0}(\text{in}(g_{s_0})),$$

and hence $\mu^{\text{tot}}(C_s) < \mu^{\text{tot}}(C_{s_0})$, which contradicts Lemma 2.3. Therefore, there is no such an s_0. But in this case it follows from [16] and the discussion in [9, pp. 17–18, 121] that the topological type of the pair $(\mathbb{P}^2, C_s)$ is independent of s, so that (C_0, C_1) is not a Zariski pair—again a contradiction.

References

[1] N. A'Campo, *La fonction zêta d'une monodromie*, Comment. Math. Helv. **50** (1975), 233–248.

[2] V. I. Arnol'd, *Normal forms of functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 3–25.

[3] E. Artal-Bartolo, *Sur la monodromie des singularités superisolées*, C. R. Acad. Sci. **312** (1991), no. 8, 601–604.

[4] E. Artal-Bartolo, *Forme de Jordan de la monodromie des singularités superisolées de surfaces*, Memoirs of the American Mathematical Society, Vol. 109 (525), American Mathematical Society, Providence, RI, 1994.

[5] That is, $\mathbf{p}_0$ is the only singular point of C_{s_0} in B and it is either a "newly born" singularity or a singularity obtained as a "merging" of several singularities of C_s for $s \neq s_0$ near s_0. In other words, s_0 is a point where the natural projection $\{(\mathbf{p}, s) \in \mathbb{P}^2 \times [0,1]; \mathbf{p} \in \Sigma(C_s)\} \to [0,1]$ fails to be a covering map (see Figure 2).

[5] E. Artal-Bartolo, *Sur les couples de Zariski*, J. Algebraic Geom. **3** (1994), 223–247.
[6] E. Artal-Bartolo, J. I. Cogolludo and H. Tokunaga, "A survey on Zariski pairs" in *Algebraic geometry in East Asia–Hanoi 2005*, Advanced Studies in Pure Mathematics, Vol. 50, Mathematical Society of Japan, Tokyo, 2008, 1–100.
[7] E. Artal-Bartolo, J. I. Cogolludo-Agustín and J. Martín-Morales, "Cremona transformations of weighted projective planes, Zariski pairs, and rational cuspidal curves" in *Singularities and their interaction with geometry and low dimensional topology—In honor of András Némethi*, Trends in Mathematics, Birkhäuser/Springer, Cham, 2021, 117–157.
[8] C. H. Clemens, Jr., *Picard–Lefschetz theorem for families of nonsingular algebraic varieties acquiring ordinary singularities*, Trans. Amer. Math. Soc. **136** (1969), 93–108.
[9] A. Dimca, *Singularities and topology of hypersurfaces*, Universitext, Springer, New York, 1992.
[10] C. Eyral and M. Oka, *On paths in the μ-constant and μ^*-constant strata*, to appear in Hiroshima Math. J. **54**, no. 2.
[11] S. M. Gusein-Zade, I. Luengo and A. Melle-Hernández, *Partial resolutions and the zeta-function of a singularity*, Comment. Math. Helv. **72** (1997), no. 2, 244–256.
[12] C. Haş Bey, *Sur l'irréductibilité de la monodromie locale; application à l'équisingularité*, C. R. Acad. Sci. Paris Sér. A–B. **275** (1972), A105–A107.
[13] I. N. Iomdin, *Complex surfaces with a one-dimensional set of singularities*, Sibirsk. Mat. Ž. **15** (1974), 1061–1082, 1181 (in Russian). English translation: Siberian Math. J. **15** (1974), no. 5, 748–762 (1975).
[14] A. G. Kouchnirenko, *Polyèdres de Newton et nombres de Milnor*, Invent. Math. **32** (1976), 1–31.
[15] F. Lazzeri, "A theorem on the monodromy of isolated singularities" in *Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci. de Cargèse, 1972)*, Astérisque, Vols. 7–8, Société Mathématique de France, Paris, 1973, 269–275.
[16] D. T. Lê, *Sur un critère d'équisingularité,* C. R. Acad. Sci. Paris Sér. A–B. **272** (1971), A138–A140.
[17] D. T. Lê, *Une application d'un théorème d'A'Campo à l'équisingularité,* Nederl. Akad. Wetensch. Proc. Ser. A. **76** (= Indag. Math. **35**) (1973), 403–409.
[18] D. T. Lê, "Topologie des singularités des hypersurfaces complexes" in *Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972),* Astérisque, Vols. 7–8, Société Mathématique de France, Paris, 1973, 171–182.
[19] D. T. Lê, "Ensembles analytiques complexes avec lieu singulier de dimension un (d'après I. N. Iomdine)" in *Seminar on singularities (Paris, 1976/1977),* Publications Mathématiques de l'Université Paris VII, Vol. 7, Université Paris VII, Paris, 1980, 87–95.
[20] I. Luengo, *The μ-constant stratum is not smooth*, Invent. Math. **90** (1987), no. 1, 139–152.
[21] I. Luengo and A. Melle, *A formula for the Milnor number*, C. R. Acad. Sci. **321** (1995), no. 11, 1473–1478.
[22] J. Mather, *Notes on topological stability*, Bull. Amer. Math. Soc. (N.S.). **49** (2012), no. 4, 475–506.
[23] J. Milnor, *Singular points of complex hypersurfaces*, Annals of Mathematics Studies, Vol. 61, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1968.
[24] M. Oka, *Non-degenerate complete intersection singularity*, Hermann, Paris, 1997.
[25] M. Oka, "A survey on Alexander polynomials of plane curves" in *Singularités Franco-Japonaises*, Séminaire et Congrès, Vol. 10, Société Mathématique de France, Paris, 2005, 209–232.
[26] M. Oka, "Almost non-degenerate functions and a Zariski pair of links" in *Essays in geometry, dedicated to N. A'Campo*, edited by A. Papadopoulos, IRMA Lectures in Mathematics and Theoretical Physics, Vol. 34, European Mathematical Society, Zürich, 2023, 601–628.
[27] M. Oka, *On μ -Zariski pairs of links*, J. Math. Soc. Japan. **75** (2023), no. 4, 1227–1259.
[28] O. Saeki, *Topological types of complex isolated hypersurface singularities*, Kodai Math. J. **12** (1989), no. 1, 23–29.
[29] D. Siersma, *The monodromy of a series of hypersurface singularities*, Comment. Math. Helv. **65** (1990), no. 2, 181–197.
[30] D. Siersma, "The vanishing topology of non-isolated singularities" in *New developments in singularity theory (Cambridge, 2000)*, NATO Science Series II: Mathematics, Physics and Chemistry, Vol. 21, Kluwer Academic Publishers, Dordrecht, 2001, 447–472.
[31] J. Stevens, *On the μ-constant stratum and the V-filtration: An example*, Math. Z. **201** (1989), no. 1, 139–144.
[32] B. Teissier, "Cycles évanescents, sections planes et conditions de Whitney" in *Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972)*, Astérisque, Vols. 7–8, Société Mathématique de France, Paris, 1973, 285–362.

[33] B. Teissier, "Déformations à type topologique constant" in *Quelques problèmes de modules (Sém. de Géométrie Analytique, École Norm. Sup., Paris, 1971–1972)*, Astérisque, Vol. 16, Société Mathématique de France, Paris, 1974, 215–249.
[34] A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), no. 3, 253–262.
[35] C. T. C. Wall, *Singular points of plane curves*, London Mathematical Society Student Texts, Vol. 63, Cambridge University Press, Cambridge, 2004.
[36] O. Zariski, *On the problem of existence of algebraic functions of two variables possessing a given branched curve*, Amer. J. Math. **51** (1929), no. 2, 305–328.

Christophe Eyral
Institute of Mathematics
Polish Academy of Sciences
ul. Śniadeckich 8
00-656 Warsaw
Poland
cheyral@impan.pl

Mutsuo Oka
Professor Emeritus of Tokyo
Institute of Technology
3-19-8 Nakaochiai
Shinjuku-ku
Tokyo 161-0032
Japan
okamutsuo@gmail.com

https://doi.org/10.1017/nmj.2023.34 Published online by Cambridge University Press