# INTEGRALS INVOLVING $E$-FUNCTIONS

*by* T. M. MACROBERT

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**1. Introductory.** The following two integrals will be established in § 2.

If $m$ is a positive integer, if $p \geqq q+1$ and if $R(\alpha_r+k_t)>0 \quad (r=1, 2, \ldots, p, t=1, 2, \ldots, m)$,

$$\int_0^\infty e^{-\lambda_1}\lambda_1^{k_1-1}\,d\lambda_1\int_0^\infty \cdots \int_0^\infty e^{-\lambda_m}\lambda_m^{k_m-1}E\begin{pmatrix}p;\ \alpha_r:\ \lambda_1\lambda_2\cdots\lambda_m z\\ q;\ \rho_s\end{pmatrix}d\lambda_m$$

$$=\frac{\pi^m}{\sin k_1\pi\ldots\sin k_m\pi}E\begin{pmatrix}p;\ \alpha_r & :\ \omega z\\ \rho_1,\ldots,\rho_q,\ 1-k_1,\ldots,1-k_m\end{pmatrix}$$

$$-\sum_{t=1}^{m}\frac{\pi^m z^{-k_t}}{\sin k_t\pi\prod_{u=1}^{m}{}'\sin(k_u-k_t)\pi}E\begin{pmatrix}p;\ \alpha_r+k_t & :\ \omega z\\ 1+k_t,\ \rho_1+k_t,\ldots,\rho_q+k_t,\ k_t-k_1+1,\ldots * \ldots,k_t-k_m+1\end{pmatrix}, \quad (1)$$

where $\omega$ is 1 or $e^{\pm i\pi}$ according as $m$ is even or odd, the dash denotes that the factor $\sin(k_t-k_t)\pi$ does not appear and the asterisk that the parameter $k_t-k_t+1$ is omitted. If $p \leqq q$ the result holds if the integral is convergent.

If $m$ is a positive integer, if $p \geqq q+1$, and if $R(k+m\alpha_r)>0 \quad (r=1, 2, \ldots, m)$,

$$\int_0^\infty e^{-\lambda}\lambda^{k-1}E(p;\ \alpha_r:q;\ \rho_s:z\lambda^m)\,d\lambda$$

$$=2^{\frac{1}{2}m-\frac{1}{2}}m^{k-\frac{1}{2}}\pi^{\frac{1}{2}m+\frac{1}{2}}$$

$$\times\left[\frac{1}{\sin k\pi}E\begin{Bmatrix}p;\ \alpha_r & ;\ \omega m^m z\\ \rho_1,\rho_2,\ldots,\rho_q,\ \Delta(m;\ 1-k)\end{Bmatrix} -\sum_{t=1}^{m}\frac{(-1)^{m+t}z^{(t-k)/m-1}}{m\sin\left(\frac{t-k}{m}\pi\right)}E\begin{Bmatrix}p;\ \alpha_r-\frac{t-k}{m}+1:\ \omega m^m z\\ \hline 2-\frac{t-k}{m},\ \rho_1-\frac{t-k}{m}+1,\ldots,\rho_q-\frac{t-k}{m}+1,\\ 1+\frac{1-t}{m},\ 1+\frac{2-t}{m},\ldots * \ldots,1+\frac{m-t}{m}\end{Bmatrix}\right], \quad (2)$$

where $\omega$ is 1 or $e^{\pm i\pi}$ according as $m$ is even or odd, $\Delta(m;\ \alpha)$ denotes the set of parameters

$$\frac{\alpha}{m},\ \frac{\alpha+1}{m},\ \ldots,\ \frac{\alpha+m-1}{m},$$

and the asterisk indicates that the parameter $1+\frac{t-t}{m}$ is omitted. If $p \leqq q$ the result holds if the integral is convergent.

When $m = 1$ each of these integrals reduces to an integral previously given by Ragab [**2**, p. 408, **3**, p. 192].

The proof depends on the expression in terms of $E$-functions of the generalised $E$-function

$$E\begin{pmatrix} p;\ \alpha_r \mid m;\ \rho_{q+s} : z \\ q;\ \rho_s \mid l;\ \alpha_{p+r} \end{pmatrix} \equiv \frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)\prod_{s=1}^{m}\Gamma(\zeta-\rho_{q+s}+1)}{\prod_{s=1}^{q}\Gamma(\rho_s-\zeta)\prod_{r=1}^{l}\Gamma(\zeta-\alpha_{p+r}+1)}\, z^{\zeta}\, d\zeta, \tag{3}$$

where $l$ and $m$ are positive integers; and the contour passes up the $\eta$-axis from $-\infty$ to $+\infty$, with loops, if necessary, to ensure that the poles of the integrand at the origin and at $\rho_{q+1}-1, \ldots, \rho_{q+m}-1$ lie to the left and the poles at $\alpha_1, \ldots, \alpha_p$ to the right of the contour; when necessary the contour is bent to the left or the right at both ends till it is parallel to the $\xi$-axis.

This expansion is [**2**, p. 419]

$$E\begin{pmatrix} p;\ \alpha_r \mid m;\ \rho_{q+s}: z \\ q;\ \rho_s \mid l;\ \alpha_{p+r} \end{pmatrix} = \pi^{m-l}\prod_{r=1}^{l}\sin(\alpha_{p+r}\pi)\prod_{s=1}^{m}\operatorname{cosec}(\rho_{q+s}\pi)\, E(p+l;\ \alpha_r : q+m;\ \rho_s :\ \omega z)$$

$$-\sum_{s=1}^{m}\frac{\pi^{m-l}\prod_{r=1}^{l}\sin(\rho_{q+s}-\alpha_{p+r})\pi z^{\rho_{q+s}-1}}{\sin(\rho_{q+s}\pi)\prod_{t=1}^{m}{}'\sin(\rho_{q+s}-\rho_{q+t})\pi}E\begin{pmatrix} p+l;\ \alpha_r-\rho_{q+s}+1 & :\ \omega z \\ 2-\rho_{q+s}, \rho_1-\rho_{q+s}+1, \ldots, * \ldots, \rho_{q+m}-\rho_{q+s}+1 & \end{pmatrix}, \tag{4}$$

where the dash and the asterisk denote that the factor $\sin(\rho_{q+s}-\rho_{q+s})\pi$ and the parameter $\rho_{q+s}-\rho_{q+s}+1$ are omitted; and $\omega$ is equal to 1 or $e^{\pm i\pi}$ according as $l+m$ is even or odd.

If in (3) $m$ is replaced by $m-1$ and then $\zeta$, $\alpha_r$ and $\rho_s$ by $\zeta-\rho_{q+m}+1$, $\alpha_r-\rho_{q+m}+1$ and $\rho_s-\rho_{q+m}+1$ respectively, the function, on being multiplied by $z^{\rho_{q+m}-1}$, becomes Meijer's function [**1**, pp. 206–222]

$$G\begin{pmatrix} p;\ \alpha_r \mid m;\ \rho_{q+s} : z \\ q;\ \rho_s \mid l;\ \alpha_{p+r} \end{pmatrix} \equiv \frac{1}{2\pi i}\int \frac{\prod_{r=1}^{p}\Gamma(\alpha_r-\zeta)\prod_{s=1}^{m}\Gamma(\zeta-\rho_{q+s}+1)}{\prod_{r=1}^{q}\Gamma(\rho_s-\zeta)\prod_{s=1}^{l}\Gamma(\zeta-\alpha_{p+r}+1)}\, z^{\zeta}\, d\zeta. \tag{5}$$

From (4) it follows that

$$G\begin{pmatrix} p;\ \alpha_r \mid m;\ \rho_{q+s} : z \\ q;\ \rho_s \mid l;\ \alpha_{p+r} \end{pmatrix}$$

$$= \pi^{m-l-1}\sum_{s=1}^{m}\frac{\prod_{r=1}^{m}{}'\sin(\rho_{q+s}-\rho_{q+r})\pi}{\prod_{t=1}^{l}{}'\sin(\rho_{q+s}-\alpha_{p+t})\pi}\, z^{\rho_{q+s}-1}E\begin{pmatrix} p+l;\ \alpha_r-\rho_{q+s}+1 & :\ \omega z \\ \rho_1-\rho_{q+s}+1, \ldots * \ldots, \rho_{q+m}-\rho_{q+s}+1 & \end{pmatrix}, \tag{6}$$

where $\omega$ is equal to $e^{\pm i\pi}$ or 1 according as $l+m$ is even or odd.

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The following formulae will also be required.

If $m$ is a positive integer,

$$\Gamma(mz) = (2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{mz-\frac{1}{2}} \prod_{s=0}^{m-1} \Gamma\left(z+\frac{s}{m}\right). \tag{7}$$

If $m$ is a positive integer,

$$\prod_{s=0}^{m-1} \sin\left(\frac{k+s}{m}\pi\right) = 2^{1-m} \sin k\pi. \tag{8}$$

If $s = 1, 2, \ldots, m-1$,

$$\sin\frac{s\pi}{m}\sin\frac{(s-1)\pi}{m}\ldots\sin\frac{\pi}{m}\sin\frac{\pi}{m}\sin\frac{2\pi}{m}\ldots\sin\frac{(m-s-1)\pi}{m} = 2^{1-m}m. \tag{9}$$

**2. Proofs.** On the left of (1) replace the $E$-function by

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod\Gamma(\alpha_r-\zeta)}{\prod\Gamma(\rho_s-\zeta)}(\lambda_1\lambda_2\ldots\lambda_m z)^{\zeta}\,d\zeta.$$

Then, on changing the order of integration and evaluating the integrals, the multiple integral becomes

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod\Gamma(\alpha_r-\zeta)\prod\Gamma(\zeta+k_t)}{\prod\Gamma(\rho_s-\zeta)}z^{\zeta}\,d\zeta;$$

and, on applying (4) with $l = 0$ and $\rho_{q+t} = 1-k_t$ $(t = 1, 2, \ldots, m)$, the expression on the right of (1) is obtained.

Again, on substituting for the $E$-function on the left of (2) and changing the order of integration, the integral is found to be equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod\Gamma(\alpha_r-\zeta)\Gamma(m\zeta+k)}{\prod\Gamma(\rho_s-\zeta)}z^{\zeta}\,d\zeta.$$

Here apply (7) to $\Gamma(m\zeta+k)$, and the integral becomes

$$(2\pi)^{\frac{1}{2}-\frac{1}{2}m} m^{k-\frac{1}{2}}\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\prod\Gamma(\alpha_r-\zeta)\prod_{s=1}^{m}\Gamma\left(\zeta-\frac{s-k}{m}+1\right)}{\prod\Gamma(\rho_s-\zeta)}(m^m z)^{\zeta}\,d\zeta.$$

Hence, on applying (4), (8) and (9), formula (2) is obtained.

## REFERENCES


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