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# The intersection theory of the moduli stack of vector bundles on $\mathbb{P}^1$

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*Abstract.* We determine the integral Chow and cohomology rings of the moduli stack $\mathcal{B}_{r,d}$ of rank $r$, degree $d$ vector bundles on $\mathbb{P}^1$-bundles. We work over a field $k$ of arbitrary characteristic. We first show that the rational Chow ring $A^*_{\mathbb{Q}}(\mathcal{B}_{r,d})$ is a free $\mathbb{Q}$-algebra on $2r+1$ generators. The isomorphism class of this ring happens to be independent of $d$. Then, we prove that the integral Chow ring $A^*(\mathcal{B}_{r,d})$ is torsion-free and provide multiplicative generators for $A^*(\mathcal{B}_{r,d})$ as a subring of $A^*_{\mathbb{Q}}(\mathcal{B}_{r,d})$. From this description, we see that $A^*(\mathcal{B}_{r,d})$ is not finitely generated as a $\mathbb{Z}$-algebra. Finally, when $k = \mathbb{C}$, the cohomology ring of $\mathcal{B}_{r,d}$ is isomorphic to its Chow ring.

## 1 Introduction

In this paper, we fully describe the intersection theory of the moduli stack $\mathcal{B}_{r,d}$ of vector bundles on $\mathbb{P}^1$-bundles. We work over an arbitrary field $k$ of any characteristic. Precisely, an object of $\mathcal{B}_{r,d}$ over a scheme $T$ is the data of a rank 2 vector bundle $W$ on $T$ and a rank $r$, relative degree $d$ vector bundle $E$ on $\mathbb{P}W$. To describe generators of the Chow or cohomology ring, let $\pi : \mathbb{P}\mathcal{W} \to \mathcal{B}_{r,d}$ be the universal $\mathbb{P}^1$-bundle and let $w_1 = c_1(\mathcal{W})$ and $w_2 = c_2(\mathcal{W})$ be Chern classes of the universal rank 2 bundle $\mathcal{W}$. Let $\mathcal{E}$ be the universal rank $r$ bundle on $\mathbb{P}\mathcal{W}$. If $z = c_1(\mathcal{O}_{\mathbb{P}\mathcal{W}}(1))$, then the Chern classes of $\mathcal{E}$ are uniquely expressible as $c_i(\mathcal{E}) = \pi^*(a_i) + \pi^*(a_i')z$ for $a_i \in A^i(\mathcal{B}_{r,d})$ and $a_i' \in A^{i-1}(\mathcal{B}_{r,d})$. We show that the rational Chow ring of $\mathcal{B}_{r,d}$ is freely generated by these classes. Then, we show that the integral Chow ring is torsion-free, and describe it as a subring of the rational Chow ring. This also determines the cohomology ring of $\mathcal{B}_{r,d}$, as it agrees with the Chow ring.

***Theorem 1.1*** *We have $A^*_{\mathbb{Q}}(\mathcal{B}_{r,d}) = \mathbb{Q}[w_1, w_2, a_1, \ldots, a_r, a_2', \ldots, a_r']$. The integral Chow ring $A^*(\mathcal{B}_{r,d}) \subset A^*_{\mathbb{Q}}(\mathcal{B}_{r,d})$ is the subring generated by $w_1, w_2$ and the Chern classes of $\pi_*\mathcal{E}(i)$ for $i = 0, 1, 2, \ldots$. If $k = \mathbb{C}$, then the cycle class map to Betti cohomology $A^*(\mathcal{B}_{r,d}) \to H^{2*}(\mathcal{B}_{r,d})$ is an isomorphism. Over an arbitrary field $k$, given a prime $\ell$ not equal to the characteristic, the cycle class map to $\ell$-adic étale cohomology $A^*(\mathcal{B}_{r,d}) \otimes \mathbb{Z}_\ell \to H^{2*}_{ét}(\mathcal{B}_{r,d}, \mathbb{Z}_\ell)$ is an isomorphism.*

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In the special case when the $\mathbb{P}^1$-bundle is trivial ($W$ is trivial), the integral Chow ring has a somewhat simpler description. This describes the Chow ring of the moduli stack $\mathcal{B}^\dagger_{r,d}$ of vector bundles on a fixed $\mathbb{P}^1$, often denoted $\mathrm{Bun}_{r,d}(\mathbb{P}^1)$. (Precisely, an object of $\mathcal{B}^\dagger_{r,d}$ over a scheme $T$ is a rank $r$, relative degree $d$ vector bundle $E$ on the trivial $\mathbb{P}^1$-bundle $\mathbb{P}^1 \times T$.) The map $\mathcal{B}^\dagger_{r,d} \to \mathcal{B}_{r,d}$ is the $\mathrm{GL}_2$-bundle associated with $\mathcal{W}$ on $\mathcal{B}_{r,d}$. By [17, Theorem 2] of Vistoli, the pullback $A^*(\mathcal{B}_{r,d}) \to A^*(\mathcal{B}^\dagger_{r,d})$ is surjective with kernel generated by $w_1, w_2$.

***Theorem 1.2*** *We have $A^*_{\mathbb{Q}}(\mathcal{B}^\dagger_{r,d}) = \mathbb{Q}[a_1, \ldots, a_r, a'_2, \ldots, a'_r]$. Integrally, $A^*(\mathcal{B}^\dagger_{r,d}) \subset A^*_{\mathbb{Q}}(\mathcal{B}^\dagger_{r,d})$ is the subring generated by $a_1, \ldots, a_r$ and the coefficients $e_i$ of the power series*

$$\sum_{i=0}^{\infty} e_i t^i = \exp\left( \int \frac{d \cdot (a_1 + a_2 t + \cdots + a_r t^{r-1}) - (a_2 + a'_3 t + \cdots + a'_r t^{r-2})}{1 + a_1 t + \cdots + a_r t^r} dt \right).$$

*If $k = \mathbb{C}$, then the cycle class map to Betti cohomology $A^*(\mathcal{B}^\dagger_{r,d}) \to H^{2*}(\mathcal{B}^\dagger_{r,d})$ is an isomorphism. Over an arbitrary field k, given a prime $\ell$ not equal to the characteristic, the cycle class map to $\ell$-adic étale cohomology $A^*(\mathcal{B}_{r,d}) \otimes \mathbb{Z}_\ell \to H^{2*}_{ét}(\mathcal{B}_{r,d}, \mathbb{Z}_\ell)$ is an isomorphism.*

***Remark 1.3*** (1) In the case $k = \mathbb{C}$, the cohomology of $\mathcal{B}^\dagger_{r,d}$ was first found by Atiyah and Bott [1, Proposition 2.20] using gauge theory. Our approach here is completely algebraic, first determining the integral Chow ring (over any field $k$) and finally showing that it must agree with cohomology (when $k = \mathbb{C}$). When $k = \mathbb{C}$, our classes $a_i, a'_i, e_i$ correspond to the cohomology classes $a_i, f_i, e_i$, respectively, in [1, Proposition 2.20]. In particular, the class $e_i$ is the $i$th Chern class of the derived push-forward $\pi_! \mathcal{E}(-1)$. The power series in our Theorem 1.2 thus makes explicit the "infinite sequence of integrality relations" mentioned in [1, p. 544] which are needed to determine which polynomials in $a_i, a'_i$ are integral.

(2) In [6], Bifet, Ghione, and Leticia take an algebreo-geometric approach to computing the $\ell$-adic cohomology of $\mathcal{B}^\dagger_{r,d}$, via studying a homotopic ind-variety $\mathbf{Div}^{r,d} = \bigcup_{D \geq 0} \mathrm{Div}^{r,d}(D)$. If $\pi : \mathbb{P}^1 \times \mathcal{B}^\dagger_{r,d} \to \mathcal{B}^\dagger_{r,d}$ denotes the projection map and $\mathcal{O}(D)$ the $D$th power of the relative $\mathcal{O}(1)$, then each $\mathrm{Div}^{r,d}(D)$ may be identified with an open subset of the total space of the vector bundle $\pi_* \mathcal{H}om(\mathcal{E}, \mathcal{O}(D)^{\oplus r})$ over $\mathcal{B}^\dagger_{r,d}$. (The compliment of this open subset is maps that are not generically injective, and its codimension goes to infinity as $D$ goes to infinity.) Bifet, Ghione, and Leticia find the cohomology of each $\mathrm{Div}^{r,d}(D)$—which in turn determines the cohomology of $\mathbf{Div}^{r,d}$ and so $\mathcal{B}^\dagger_{r,d}$—by producing a stratification into affine spaces. The fact that $A^*(\mathcal{B}^\dagger_{r,d}) \otimes \mathbb{Z}_\ell \to H^{2*}_{ét}(\mathcal{B}^\dagger_{r,d}, \mathbb{Z}_\ell)$ is an isomorphism is thus implied by their work.

We point out several interesting features of our results:

(1) Although $A^*_{\mathbb{Q}}(\mathcal{B}_{r,d})$ is obviously finitely generated as a $\mathbb{Q}$-algebra, $A^*(\mathcal{B}_{r,d})$ is *not* finitely generated as a $\mathbb{Z}$-algebra (see Corollary 5.7).

(2) The rational Chow ring $A^*_{\mathbb{Q}}(\mathcal{B}_{r,d})$ is independent of $d$. This may lead one to wonder if the isomorphism class of $\mathcal{B}_{r,d}$ could be independent of $d$. However,

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considering integral Chow rings, one can show that $\mathcal{B}_{2,1}$ and $\mathcal{B}_{2,0}$ (resp. $\mathcal{B}_{2,1}^\dagger$ and $\mathcal{B}_{2,0}^\dagger$) are *not* isomorphic (see Corollaries 5.8 and 5.9).

(3) To show $A^*(\mathcal{B}_{r,d})$ is torsion-free, we stratify by splitting loci, which in turn are modeled by spaces admitting affine stratifications. This stratification is also what allows us to see that the Chow and cohomology rings of $\mathcal{B}_{r,d}$ agree (see Lemma 5.3).

(4) Using the theory of higher Chow groups, we show that push-forward maps for including strata are all injective on Chow. This relies on a vanishing result for higher Chow groups of a point with torsion coefficients. Although this vanishing result only holds for an algebraically closed field of characteristic zero, we deduce from it a rank equality which allows us to establish our theorem in *all* characteristics.

***Remark 1.4*** Here, we are considering $\mathbb{P}^1$-bundles equipped with a relative degree 1 line bundle (i.e., our universal $\mathbb{P}^1$-bundle over $\mathcal{B}_{r,d} = [\mathcal{B}_{r,d}^\dagger/\mathrm{GL}_2]$ is pulled back from $\mathrm{BGL}_2$). However, not all families of genus 0 curves admit a relative degree 1 line bundle. The moduli stack of vector bundles on families of genus 0 curves would be $[\mathcal{B}_{r,d}^\dagger/\mathrm{PGL}_2]$ (whose family of genus 0 curves is pulled back from $\mathrm{BPGL}_2$, which is the moduli stack of genus 0 curves). Although the *integral* Chow ring of the $\mathrm{PGL}_2$-quotient is more subtle, our work does determine the *rational* Chow ring $A^*_{\mathbb{Q}}([\mathcal{B}_{r,d}^\dagger/\mathrm{PGL}_2])$, because it is equal to $A^*_{\mathbb{Q}}([\mathcal{B}_{r,d}^\dagger/\mathrm{SL}_2])$ (the $\mathrm{SL}_2$ quotient is a $\mu_2$-gerbe over the $\mathrm{PGL}_2$-quotient). To find the latter, note that $[\mathcal{B}_{r,d}^\dagger/\mathrm{SL}_2] \to \mathcal{B}_{r,d}$ is the $\mathbb{G}_m$-bundle associated with $\det \mathcal{W}$, so applying Vistoli's theorem [17, Theorem 2], we have

$$A^*_{\mathbb{Q}}([\mathcal{B}_{r,d}^\dagger/\mathrm{SL}_2]) = A^*_{\mathbb{Q}}(\mathcal{B}_{r,d})/\langle w_1\rangle = \mathbb{Q}[w_2, a_1, \dots, a_r, a_2', \dots, a_r'].$$

This result will be used to determine the intersection theory of low-degree Hurwitz spaces by Canning and the authors in [10, 11].

## 1.1 Relationship with the affine Grassmannian

Though not used in our proofs, we explain here a relationship of our results with a closely related moduli space. The *affine Grassmannian* for $\mathrm{GL}_r$ may be interpreted as the moduli space of rank $r$ vector bundles on $\mathbb{P}^1$ with a trivialization away from a point. The affine Grassmannian has components indexed by degree, each homotopic to the moduli stack $\mathcal{X}_{r,d}$ of rank $r$, degree $d$ vector bundles on $\mathbb{P}^1$ with a trivialization at a point. The natural map $\mathcal{X}_{r,d} \to \mathcal{B}_{r,d}^\dagger$ is the principal $\mathrm{GL}_r$-bundle associated with $\mathcal{E}|_{\mathcal{B}_{r,d}^\dagger \times 0}$. Thus, using Vistoli's theorem, we find $A^*(\mathcal{X}_{r,d}) = A^*(\mathcal{B}_{r,d}^\dagger)/\langle a_1, \dots, a_r\rangle$. In Remark 5.6, we explain why the Chow and cohomology rings of $\mathcal{X}_{r,d}$ agree. Thus, we recover the cohomology of the affine Grassmannian, which was first computed by Bott [9]. We note that the motive of the affine Grassmannian was determined by Bachmann in [2, Corollary 20], which also determines its Chow groups.

From this perspective, Theorem 1.2 should be interpreted as determining the $\mathrm{GL}_r$ *equivariant* Chow/cohomology of the components of the affine Grassmannian. (The fact that the rings $A^*(\mathcal{B}_{r,d}^\dagger) = A^*([\mathcal{X}_{r,d}/\mathrm{GL}_r])$ are not isomorphic for different degrees $d$ shows that the $\mathrm{GL}_r$-actions really are different on the components $\mathcal{X}_{r,d}$ for different degrees $d$!) Let us also mention related work of Lam–Shimozono. Let $T \subset \mathrm{GL}_r$ be a maximal torus. Then, the $T$-equivariant cohomology of the degree 0 component of the affine Grassmannian was found with rational coefficients in [14].

### 1.2 Overview

This paper is organized as follows. In Section 2, we briefly state necessary results concerning equivariant Chow rings and higher Chow groups. In Section 3, we describe a sequence of opens $\mathcal{U}_m$ that exhaust $\mathcal{B}_{r,d}$. Following a construction of Bolognesi–Vistoli, each $\mathcal{U}_m$ can be realized as a global quotient stack. In Section 4, we use this description to calculate the rational Chow ring of $\mathcal{B}_{r,d}$. Finally, in Section 5, we prove that the integral Chow ring is torsion-free and provide a description of the generators.

## 2 Preliminaries on equivariant Chow and higher Chow

Many of the stacks we encounter in this paper are quotients of open subsets $X$ of affine spaces by linear algebraic groups $G$. The Chow ring of a quotient stack $[X/G]$ is defined as the $G$-equivariant Chow ring of $X$, which in turn is defined in [12] using models based on Borel's mixing construction. More precisely, given a representation $V$ of $G$ and an open subset $U \subset V$ on which $G$ acts freely, if $\operatorname{codim} U^c > i$, we define $A^i([X/G]) = A^i(X \times_G U)$, where $X \times_G U$ is the quotient of $X \times U$ by the diagonal $G$ action. This is well defined because $A^*(V) \cong A^*(X)$ whenever $V$ is a vector bundle over $X$ ("homotopy") and $A^i(X - Z) = A^i(X)$ whenever $\operatorname{codim} Z > i$ ("excision").

For stacks that are locally of finite type, we define the Chow ring as in [3, Appendix A]. In the next section, we shall describe a *good filtration* of $\mathcal{B}_{r,d}$ by finite-type substacks $\mathcal{U}_1 \subset \mathcal{U}_2 \subset \mathcal{U}_3 \subset \cdots$, where $\mathcal{U}_m$ is a finite-type quotient stack. Thus, as explained in [3, Appendix A], for any fixed $i$, we shall have $A^i(\mathcal{B}_{r,d}) = A^i(\mathcal{U}_m)$ for $m$ suitably large.

***Example 2.1*** Let $V_N = \mathrm{Mat}_{r\times N}(k) = k^{\oplus rN}$ with $\mathrm{GL}_r$ acting on $V_N$ by left multiplication. The group $\mathrm{GL}_r$ acts freely on the open subset $U_N \subset V_N$ of full rank matrices. This determines a model $pt \times_{\mathrm{GL}_r} U_N = U_N/\mathrm{GL}_r = G(r,N)$. Since $\operatorname{codim} U_N^c = N - r + 1$, we have $A^i(\mathrm{BGL}_r) = A^i(G(r,N))$ for $i < N - r + 1$. It is a classical result that $A^*(G(r,N))$ is generated by the Chern classes $c_1, \ldots, c_r$ of the tautological rank $r$ bundle with no relations in degrees less than $N - r + 1$. Taking larger and larger $N$, it follows that $A^*(\mathrm{BGL}_r) = \mathbb{Z}[c_1, \ldots, c_r]$. ∎

Variants of the above construction allow one to approximate all quotient stacks in this paper with concrete models which are fiber bundles over Grassmannians.

The higher Chow groups of these quotient stacks will also be an important tool. In [7], Bloch defines the higher Chow groups of a quasi-projective variety $X$ as the homology of a complex $z^*(X,-)$ of free abelian groups, i.e., $\mathrm{CH}^*(X,n) = H_n(z^*(X,-))$. Higher Chow groups with coefficients in a ring $R = \mathbb{Z}/m$ or $\mathbb{Z}$ are

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defined similarly by $\mathrm{CH}^*(X,n,R) = H_n(z^*(X,-)\otimes R)$. Some properties of higher Chow groups are the following [7, pp. 268–269].

(1) Weight zero: we have $\mathrm{CH}^*(X,0,R) = A^*(X)\otimes R$.
(2) Functoriality: there are proper push forwards and flat pullbacks.
(3) Localization long exact sequence: if $Y \subset X$ is a closed subscheme of pure codimension $d$, then there is a long exact sequence

$$\begin{aligned} \ldots \to \mathrm{CH}^{*-d}(Y,1,R) &\to \mathrm{CH}^*(X,1,R) \to \mathrm{CH}^*(X-Y,1,R)\\ \to \mathrm{CH}^{*-d}(Y,0,R) &\to \mathrm{CH}^*(X,0,R) \to \mathrm{CH}^*(X-Y,0,R). \end{aligned}$$

(4) Homotopy: $\mathrm{CH}^*(X\times\mathbb{A}^m,n,R)\cong \mathrm{CH}^*(X,n,R)$. By (2) and (3), it follows that if $\widetilde{X}\to X$ is any affine bundle, then $\mathrm{CH}^*(X,n,R)\cong\mathrm{CH}^*(\widetilde{X},n,R)$.

Edidin and Graham [12, Section 2.7] extend the notion of higher Chow groups to quotients $[X/G]$ by defining them to be higher Chow groups of suitable models: $\mathrm{CH}^*([X/G],n,R) := \mathrm{CH}^*(X\times_G U,n,R)$, where $U$ is an open subset of a representation of $G$ whose complement has sufficiently high codimension and $G$ acts freely on $U$. This is well defined by the homotopy property, and Edidin and Graham obtain a localization long exact sequence for the corresponding quotients in (3) when $Y$ is $G$-equivariant [12, Proposition 5].

Over an algebraically closed field of characteristic zero, the higher Chow groups of a point with torsion coefficients are known:

$$\mathrm{CH}^i(pt,n,\mathbb{Z}/\ell) = \begin{cases}\mathbb{Z}/\ell, & \text{if } n = 2i,\\ 0, & \text{otherwise.}\end{cases} \tag{2.1}$$

This follows from [16, Corollary 4.3], which relates higher Chow groups to certain étale cohomology groups, although this special case was likely known earlier.[1] Using the long exact sequence and the homotopy property, it follows that over such a field, $\mathrm{CH}^*(X,1;\mathbb{Z}/\ell)=0$ for any $X$ admitting an affine stratification. In particular, since $\mathrm{BGL}_r$ is modeled by Grassmannians $G(r,N)$ (see Example 2.1), we have

$$\mathrm{CH}^*(\mathrm{BGL}_{r_1}\times\cdots\times\mathrm{BGL}_{r_s},1;\mathbb{Z}/\ell)=0 \tag{2.2}$$

over an algebraically closed field of characteristic zero.

## 3 Construction of the moduli stack

Given some rank $r\geq 0$ and degree $d\in\mathbb{Z}$, we define the moduli stack $\mathcal{B}_{r,d}$ of vector bundles of rank $r$ and degree $d$ on $\mathbb{P}^1$-bundles by

$$\mathcal{B}_{r,d}(T) = \left\{(W,E) : \begin{matrix} W \text{ a rank 2 vector bundle on } T\\ E \text{ a rank } r\text{, relative degree } d \text{ vector bundle on } \mathbb{P}W\end{matrix}\right\}.$$

An arrow $(W,E)\to(W',E')$ is the data of an isomorphism $\phi: W\to W'$—which induces an isomorphism $\mathbb{P}\phi:\mathbb{P}W\to\mathbb{P}W'$—and an isomorphism $\psi: E\to(\mathbb{P}\phi)^*E'$.

[1] Although we shall not need it, we mention that Kelly has extended Suslin's results to characteristic $p$ with $\mathbb{Z}[\frac{1}{p}]$ coefficients. In positive characteristic, our proof of the main theorem uses a calculation in characteristic zero, combined with a dimension counting trick.

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Given a vector bundle $E$ on a $\mathbb{P}^1$-bundle $\mathbb{P}W \to T$, we write $E(m) := E \otimes \mathscr{O}_{\mathbb{P}W}(1)^{\otimes m}$. There are equivalences $\mathcal{B}_{r,d} \cong \mathcal{B}_{r,d+mr}$ sending $(W, E) \mapsto (W, E(m))$. Thus, it would suffice to study $\mathcal{B}_{r,\ell}$ for $0 \leq \ell < r$. Throughout, $d = \ell + mr$ will be an arbitrary degree in the residue class of $\ell$ modulo $r$.

The stack $\mathcal{B}_{r,\ell}$ is a union of open substacks

$$\mathcal{B}_{r,\ell} = \bigcup_{m=0}^{\infty} \mathcal{U}_m \qquad \text{with} \qquad \mathcal{U}_0 \subset \mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots,$$

where $\mathcal{U}_m := \mathcal{U}_{m,r,\ell}$ is defined by

$$\begin{aligned}\mathcal{U}_{m,r,\ell}(T) &= \{(W, E) \in \mathcal{B}_{r,\ell}(T) : E(m) \text{ is globally generated on each fiber over } T\} \\ &= \{(W, E) \in \mathcal{B}_{r,\ell}(T) : R^1\pi_* E(m-1) = 0 \text{ for } \pi : \mathbb{P}W \to T \text{ projection}\}.\end{aligned}$$

The stack $\mathcal{U}_{0,r,\ell}$ is equal to the stack of globally generated rank $r$, degree $\ell$ vector bundles on $\mathbb{P}^1$-bundles, which we shall denote $\mathcal{V}_{r,\ell}$. Via the twist by $\mathscr{O}_{\mathbb{P}W}(m)$, there are isomorphisms $\mathcal{U}_{m,r,\ell} \cong \mathcal{V}_{r,\ell+mr}$ for each $m$.

## 3.1 Splitting loci

Every vector bundle on $\mathbb{P}^1$ splits as a direct sum of line bundles, say $E = \mathscr{O}(e_1) \oplus \cdots \oplus \mathscr{O}(e_r)$ for $e_1 \leq \cdots \leq e_r$. We call $\vec{e} = (e_1, \ldots, e_r)$ the *splitting type* of $E$, and abbreviate the corresponding sum of line bundles by

$$\mathscr{O}(\vec{e}) := \mathscr{O}(e_1) \oplus \cdots \oplus \mathscr{O}(e_r).$$

Given a vector bundle $E$ on a $\mathbb{P}^1$-bundle $\pi : \mathbb{P}W \to T$, we define *splitting loci*

$$\Sigma_{\vec{e}}(E) := \{b \in B : E|_{\pi^{-1}(b)} \cong \mathscr{O}(\vec{e})\}.$$

Each splitting locus is a locally closed subvariety of $T$ (see, e.g., [13, Theorem 14.7]). In turn, the moduli stack $\mathcal{B}_{r,\ell}$ admits a stratification by the splitting loci of the universal vector bundle. Let us denote the universal splitting locus for splitting type $\vec{e}$ by $\Sigma_{\vec{e}} \subset \mathcal{B}_{r,\ell}$. Equivalently, $\Sigma_{\vec{e}}$ is the moduli stack of vector bundles that have splitting type $\vec{e}$ on each fiber of a $\mathbb{P}^1$-bundle. There is a partial order on splitting types defined by $\vec{e}' \leq \vec{e}$ if $e_1' + \cdots + e_j' \leq e_1 + \cdots + e_j$ for all $j$ and equality holds when $j = r$. The splitting locus $\Sigma_{\vec{e}'}$ is contained in the closure of $\Sigma_{\vec{e}}$ if and only if $\vec{e}' \leq \vec{e}$.

Suppose that $\mathscr{O}(\vec{e}) = \bigoplus_{i=1}^{s} \mathscr{O}(d_i)^{\oplus r_i}$ with $d_1 < \cdots < d_s$ (so the $d_i$ are the *distinct* degrees in $\vec{e}$ and the $r_i$ their multiplicities). Let us identify $\operatorname{Aut}(\mathscr{O}(\vec{e}))$ with block upper triangular matrices whose entries in the $i, j$ block are homogeneous polynomials on $\mathbb{P}^1$ of degree $d_j - d_i$. The block diagonal matrices correspond to the subgroup $\prod_{i=1}^{s} \mathrm{GL}_{r_i} \hookrightarrow \operatorname{Aut}(\mathscr{O}(\vec{e}))$. The group $\mathrm{GL}_2$ acts on the off-diagonal blocks via change of coordinates on $\mathbb{P}^1$. The data of a vector bundle $E$ on a $\mathbb{P}^1$-bundle $\mathbb{P}W \to T$ whose restriction to each fiber has splitting type $\vec{e}$ are the same as a principal bundle for the semidirect product $H_{\vec{e}} := \operatorname{Aut}(\mathscr{O}(\vec{e})) \ltimes \mathrm{GL}_2$. (To see this, consider transition data between local trivializations of such a family.) In other words, $\Sigma_{\vec{e}}$ is equivalent to the classifying stack $BH_{\vec{e}}$. From this, we see that

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$$\begin{aligned}\operatorname{codim}\Sigma_{\vec{e}} &= \dim \mathcal{B}_{r,d} - \dim \Sigma_{\vec{e}} = -r^2 + \dim H_{\vec{e}} = -r^2 + h^0(\mathbb{P}^1, \operatorname{End}(\mathcal{O}(\vec{e})))\\ &= h^1(\mathbb{P}^1, \operatorname{End}(\mathcal{O}(\vec{e})) = \sum_{i,j} \max\{0, e_i - e_j - 1\} =: u(\vec{e}).\end{aligned} \tag{3.1}$$

***Remark 3.1*** In [5, Section 7], Behrend and Dhillon use the splitting-type stratification on $\mathcal{B}^{\dagger}_{r,d} = \operatorname{Bun}_G(\mathbb{P}^1)$ to compute its class in the Grothendieck ring of stacks.

The complement of $\mathcal{U}_{m,r,\ell} \subset \mathcal{B}_{r,\ell}$ is the union of splitting loci $\Sigma_{\vec{e}}$ with $e_1 < -m$. In particular, using (3.1), one sees that this union of splitting loci has codimension $\ell + mr + 1$ (see also [15, Section 5]). In particular, as $m$ increases, the codimension of the complement of $\mathcal{U}_m$ goes to infinity. The key to understanding the intersection theory of $\mathcal{B}_{r,\ell}$ is therefore to understand each $\mathcal{U}_{m,r,\ell}$, which is equivalent to the stack of globally generated vector bundles $\mathcal{V}_{r,d}$ for $d = \ell + mr$.

## 3.2 Globally generated vector bundles

The stack $\mathcal{V}_{r,d}$ of rank $r$, degree $d$ globally generated vector bundles on $\mathbb{P}^1$ was constructed by Bolognesi and Vistoli in [8]. We briefly motivate and review their construction. If $E$ is a globally generated vector bundle on $\mathbb{P}^1$, then there is a surjection $H^0(E) \otimes \mathcal{O}_{\mathbb{P}^1} \to E$. Since $h^0(E) = r + d$, the kernel of this map is rank $d$, degree $-d$. Furthermore, the kernel has no global sections, so it must be $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d}$. That is, given a globally generated $E$, it sits naturally in a sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d} \xrightarrow{\psi} \mathcal{O}_{\mathbb{P}^1}^{\oplus(r+d)} \longrightarrow E \longrightarrow 0. \tag{3.2}$$

Let $M_{r,d} := \operatorname{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus d}, \mathcal{O}_{\mathbb{P}^1}^{\oplus(r+d)})$ be the space of $d \times (r+d)$ matrices of linear forms on $\mathbb{P}^1$. The sequence (3.2) determines an element $\psi \in M_{r,d}$ which is well defined up to the choice of framings of the source and target. Moreover, $\psi$ lies in the open subvariety $\Omega_{r,d} \subset M_{r,d}$ of matrices of linear forms having full rank $d$ at each point on $\mathbb{P}^1$. Let $\mathrm{GL}_d$ act on $M_{r,d}$ by left multiplication and $\mathrm{GL}_{r+d}$ act by right multiplication. In addition, let $\mathrm{GL}_2$ act on $M_{r,d}$ by change of coordinates on the entries, which live in the two-dimensional vector space $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1))$. These three actions commute, so we obtain an action of $\mathrm{GL}_d \times \mathrm{GL}_{d+r} \times \mathrm{GL}_2$ on $M_{r,d}$. The locus $\Omega_{r,d} \subset M_{r,d}$ is preserved by this action and thus inherits an action of $\mathrm{GL}_d \times \mathrm{GL}_{d+r} \times \mathrm{GL}_2$.

***Theorem 3.2*** (Bolognesi and Vistoli [8, Theorem 4.4]) *There is an isomorphism of fibered categories $\mathcal{V}_{r,d} \cong [\Omega_{r,d}/\mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_2]$.*

In other words, Theorem 3.2 says $\mathcal{V}_{r,d}$ is an open substack of a vector bundle over $\mathrm{BGL}_d \times \mathrm{BGL}_{r+d} \times \mathrm{BGL}_2$. In particular, by the homotopy and excision properties, we have a surjection

$$A^*(\mathrm{BGL}_d \times \mathrm{BGL}_{r+d} \times \mathrm{BGL}_2) \twoheadrightarrow A^*(\mathcal{V}_{r,d}). \tag{3.3}$$

Let $\mathcal{W}$ denote the universal rank 2 vector bundle on $\mathcal{V}_{r,d}$ (pulled back from the $\mathrm{BGL}_2$ factor), and let $\mathcal{E}^{gg} := \mathcal{E}^{gg}_{r,d}$ be the universal globally generated rank $r$, degree $d$ vector

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bundle on $\pi : \mathbb{P}\mathcal{W} \to \mathcal{V}_{r,d}$. Let $T_d$ and $T_{r+d}$ denote the universal vector bundles on $\mathrm{BGL}_d$ and $\mathrm{BGL}_{r+d}$. We wish to identify their pullbacks to $\mathcal{V}_{r,d}$ in terms of $\mathcal{E}^{gg}$.

***Lemma 3.3*** *Let $\gamma : \mathcal{V}_{r,d} \to \mathrm{BGL}_d \times \mathrm{BGL}_{r+d}$ be the natural map. We have*

$$\gamma^* T_d = (\pi_* \mathcal{E}^{gg}(-1)) \otimes \det \mathcal{W}^\vee \qquad \text{and} \qquad \gamma^* T_{r+d} = \pi_* \mathcal{E}^{gg}. \tag{3.4}$$

*In particular, $A^*(\mathcal{V}_{r,d})$ is generated by the Chern classes of the three vector bundles $\mathcal{W}$, $\pi_* \mathcal{E}^{gg}(-1)$, and $\pi_* \mathcal{E}^{gg}$.*

**Proof** By the construction of $\mathcal{V}_{r,d}$ as a quotient of $\Omega_{r,d} \subset M_{r,d}$, the universal $\mathbb{P}^1$-bundle $\pi : \mathbb{P}\mathcal{W} \to \mathcal{V}_{r,d}$ is equipped with an exact sequence of vector bundles globalizing (3.2)

$$0 \to (\pi^* \gamma^* T_d)(-1) \to \pi^* \gamma^* T_{r+d} \to \mathcal{E}^{gg} \to 0. \tag{3.5}$$

By the theorem on cohomology and base change and the fact that $h^i(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, we have $R^i \pi_* (\pi^* \gamma^* T_d)(-1) = 0$ for $i = 0, 1$. Therefore, pushing forward (3.5) by $\pi$ induces an isomorphism

$$\gamma^* T_{r+d} \cong \pi_* \pi^* \gamma^* T_{r+d} \xrightarrow{\sim} \pi_* \mathcal{E}^{gg}.$$

On the other hand, tensoring (3.5) with $\mathcal{O}_{\mathbb{P}\mathcal{W}}(-1)$ and pushing forward by $\pi$ induces an isomorphism

$$\pi_* \mathcal{E}^{gg}(-1) \xrightarrow{\sim} R^1 \pi_* ((\pi^* \gamma^* T_d)(-2)) \cong \gamma^* T_d \otimes R^1 \pi_* \mathcal{O}_{\mathbb{P}\mathcal{W}}(-2). \tag{3.6}$$

Noting that the relative dualizing sheaf of $\pi$ is $\omega_\pi = \mathcal{O}_{\mathbb{P}\mathcal{W}}(-2) \otimes \det \mathcal{W}^\vee$, Serre duality provides an isomorphism of the right-hand term in (3.6) with $\gamma^* T_d \otimes \det \mathcal{W}$. Having identified the tautological vector bundles, (3.3) now establishes the claim about generators of $A^*(\mathcal{V}_{r,d})$. ∎

***Remark 3.4*** Lemma 3.3 provides a quick proof of the existence half of [15, Theorem 1.2]: Pulling back the classes of closures of the universal splitting loci $\overline{\Sigma}_{\vec{e}}$ on $\mathcal{B}_{r,d}$, it follows that when the splitting loci of a vector bundle $E$ on a $\mathbb{P}^1$-bundle $\mathbb{P}W \to B$ have the expected codimension, their classes in the Chow ring of $B$ are given by a universal formula in terms of the Chern classes of the rank 2 bundle $\pi_* \mathcal{O}_{\mathbb{P}W}(1)$ and the bundles $\pi_* E(i)$ for suitable $i$. This observation does not, however, give an indication of how to find these formulas, as done in [15, Section 6].

Now, let $\mathcal{E} := \mathcal{E}_{r,\ell}$ denote the universal rank $r$, degree $\ell$ vector bundle on $\mathbb{P}\mathcal{W} \to \mathcal{B}_{r,\ell}$. The restriction of $\mathcal{E}$ to $\mathcal{U}_{0,r,\ell} \subset \mathcal{B}_{r,\ell}$ is just $\mathcal{E}|_{\mathcal{U}_{0,r,\ell}} = \mathcal{E}^{gg}_{r,\ell}$. More generally, we have each $\mathcal{U}_{m,r,\ell} \cong \mathcal{V}_{r,\ell+mr}$, and via this identification, $\mathcal{E}|_{\mathcal{U}_{m,r,\ell}} = \mathcal{E}^{gg}_{r,\ell+mr}(-m)$, equivalently $\mathcal{E}(m)|_{\mathcal{U}_{m,r,\ell}} = \mathcal{E}^{gg}_{r,\ell+mr}$. This establishes the following.

***Lemma 3.5*** *The Chow ring $A^*(\mathcal{U}_m)$ is generated over $\mathbb{Z}$ by the Chern classes of $\mathcal{W}$ and the vector bundles $\pi_* \mathcal{E}(m-1)$ and $\pi_* \mathcal{E}(m)$ on $\mathcal{U}_m$. Thus, $A^*(\mathcal{B}_{r,\ell})$ is generated by the Chern classes of $\mathcal{W}$ and $\pi_* \mathcal{E}(i)$ for $i = 0, 1, 2, \ldots$.*

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We shall later describe the Chow ring as a subring of a finitely generated $\mathbb{Q}$-algebra, which gives rise to an implicit description of the relations among these generators.

## 4 The rational Chow ring

The rational Chow ring of $\mathcal{B}_{r,\ell}$ can be described with fewer generators than the integral generators of Lemma 3.5. Let $w_1 = c_1(\mathcal{W})$, $w_2 = c_2(\mathcal{W})$, and $z = c_1(\mathscr{O}_{\mathbb{P}\mathcal{W}}(1))$. The Chern classes of $\mathcal{E}$ can be written as $c_i(\mathcal{E}) = \pi^*(a_i) + \pi^*(a_i')z$ for unique $a_i \in A^i(\mathcal{B}_{r,\ell})$ and $a_i' \in A^{i-1}(\mathcal{B}_{r,\ell})$. (Note that $a_1' = \ell$.) By Grothendieck–Riemann–Roch, the Chern classes of the vector bundle $\pi_*\mathcal{E}(m)$ on $\mathcal{U}_m$ are expressible in terms of the $a_i$ and $a_i'$ for any $m$, so these classes are generators for the rational Chow ring of each $\mathcal{U}_m$ and therefore of $\mathcal{B}_{r,\ell}$. The main result of this section will be that there are no relations among these generators on $\mathcal{B}_{r,\ell}$. We first consider relations on an open $\mathcal{U}_m \cong \mathcal{V}_{r,d}$, where $d = mr + \ell$.

***Theorem 4.1*** *The ring $A^*_{\mathbb{Q}}(\mathcal{V}_{r,d})$ is a quotient of $\mathbb{Q}[w_1, w_2, a_1, \dots, a_r, a_2', \dots, a_r']$ with all relations in degrees $d+1$ and higher.*

**Proof** Set $G := \mathrm{GL}_d \times \mathrm{GL}_{r+d} \times \mathrm{GL}_2$. We let $T_d$, $T_{r+d}$ and $\mathcal{W}$ denote the corresponding tautological bundles on $B := BG$. Let $t_i = c_i(T_d)$, $u_i = c_i(T_{r+d})$, and $w_i = c_i(\mathcal{W})$ be their Chern classes, which freely generate $A^*(B)$. Next, define $M := [M_{r,d}/G]$, which is the total space of the vector bundle $\mathcal{H}om(T_d, T_{r+d}) \otimes \mathcal{W}^\vee$ over $B$. Theorem 3.2 says that $\mathcal{V}_{r,d}$ is an open substack of $M$. Let us write $X := [\Omega^c_{r,d}/G]$ for its closed complement. Here, $\Omega^c_{r,d} \subset M_{r,d}$ is the space of matrices of linear forms that drop rank along some point on $\mathbb{P}^1$. By excision, we have a right exact sequence

$$A^{*-\operatorname{codim} X}(X) \to A^*(M) \to A^*(\mathcal{V}_{r,d}) \to 0. \tag{4.1}$$

We shall see soon that $X$ is irreducible of codimension $r$, which immediately implies that there are no relations among generators of $A^*(M)$ restricted to $\mathcal{V}_{r,d}$ in degrees less than $r$. In what follows, we describe relations among the restrictions of these Chern classes in degrees $r$ up to $d$.

First, we construct a space $\widetilde{X}$ whose total space maps properly to $M$ with image $X$. Let $\mathbb{P}(T_d)$ be the projectivization of the tautological rank $d$ bundle, and let $\sigma : \mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d) \to B$ be the map to the base. On $\mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d)$, we have a surjection of vector bundles

$$\sigma^* M = \sigma^*(\mathcal{H}om(T_d, T_{r+d}) \otimes \mathcal{W}^\vee) \to \mathscr{O}_{\mathbb{P}(T_d)}(1) \otimes \sigma^* T_{r+d} \otimes \mathscr{O}_{\mathbb{P}\mathcal{W}}(1), \tag{4.2}$$

corresponding to the evaluation of the map along a one-dimensional subspace of the fiber of $T_d$. Let $\widetilde{X} \subset \sigma^* M$ denote the total space of the kernel vector bundle. Informally,

(4.3)
$$\widetilde{X} = \{(p, \Lambda, \psi) : p \in \mathbb{P}\mathcal{W}, \Lambda \subset (T_d)_{\pi(p)}, \psi \in M_{\pi(p)}, \Lambda \subset \ker \psi(p) \subset (T_d)_{\pi(p)}\}.$$

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We have a commutative diagram:

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\iota"] \arrow[dr, "\rho''"'] & \sigma^* M \arrow[r, "\sigma'"] \arrow[d, "\rho'"] & M \arrow[d, "\rho"] \\
 & \mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d) \arrow[r, "\sigma"] & B,
\end{tikzcd}$$

where $\rho$, $\rho'$, and $\rho''$ are all vector bundle maps. Since $X$ consists of maps that drop rank along some point on $\mathbb{P}^1$, we have $\sigma'(\iota(\widetilde{X})) = X$ by construction. Considering (4.3), we see that the fiber of $\widetilde{X} \to X$ over a point corresponding to $\psi$ is $\{(p, \Lambda) : p \in \mathbb{P}^1, \Lambda \subset \ker \psi(p)\}$ (in words, the projectivization of kernels over points on $\mathbb{P}^1$ where $\psi$ drops rank). Alternatively, we can think of $\psi$ as defining a degree 1 map $\mathbb{P}^1 \to \mathbb{P}\mathrm{Mat}_{d\times d+r}$ (so long as $\psi(p)$ is not the zero matrix at some $p$, which is a high codimension condition). Let $\widetilde{Z} \to Z \subset \mathbb{P}\mathrm{Mat}_{d\times d+r}$ be the usual resolution of the locus of matrices that are not full rank by marking a one-dimensional subspace of the kernel. Then, the fiber of $\widetilde{X} \to X$ over $\psi$ is the fiber product

$$\begin{tikzcd}
(\sigma' \circ \iota)^{-1}(\psi) \arrow[r] \arrow[d] & \widetilde{Z} \arrow[d] \\
\mathbb{P}^1 \arrow[r] & \mathbb{P}\mathrm{Mat}_{d\times d+r}.
\end{tikzcd}$$

The loci of matrices in $\mathbb{P}\mathrm{Mat}_{d\times d+r}$ that are not full rank have codimension $r+1 \geq 2$, so a general line $\mathbb{P}^1 \to \mathbb{P}\mathrm{Mat}_{d\times d+r}$ that meets $Z$ meets it in a single general point of $Z$. In particular, $\widetilde{X} \to X$ is generically one-to-one and has projective fibers. This establishes that

$$\dim X = \dim \widetilde{X} = \dim M + \dim(\mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d)) - \operatorname{rank} T_{k+r} = \dim M - r.$$

Since $\sigma' \circ \iota : \widetilde{X} \to X$ is surjective and projective, the push forward of rational Chow groups $A^*_{\mathbb{Q}}(\widetilde{X}) \to A^*_{\mathbb{Q}}(X)$ is surjective. (Take the preimage of any cycle. Since the map is surjective, the fibers are nonempty. If the fibers are generically finite, the push forward is some nonzero multiple of the original cycle. Otherwise, slice with enough copies of the hyperplane class to get a cycle mapping with generically finite, nonempty fibers to the same image.) It follows that the image of $A^{*-r}_{\mathbb{Q}}(X) \to A^*_{\mathbb{Q}}(M)$ via push forward of cycles is the same as the image of $(\sigma' \circ \iota)_* : A^{*-r}_{\mathbb{Q}}(\widetilde{X}) \to A^*_{\mathbb{Q}}(M)$.

The pullback maps $(\rho'')^*$, $(\rho')^*$, and $\rho^*$ all induce isomorphisms on Chow rings. Recall that $\widetilde{X}$ is the kernel of (4.2), so it is defined inside the total space $\sigma^* M$ by the vanishing of a section of a vector bundle $(\rho')^* \mathscr{O}_{\mathbb{P}(T_d)}(1) \otimes \sigma^* T_{r+d} \otimes \mathscr{O}_{\mathbb{P}\mathcal{W}}(1)$. Therefore, the fundamental class of $\widetilde{X}$ in the Chow ring of $\sigma^* M$ is equal to $(\rho')^* \beta$ where

$$\beta := c_{r+d}(\mathscr{O}_{\mathbb{P}(T_d)}(1) \otimes \sigma^* T_{r+d} \otimes \mathscr{O}_{\mathbb{P}\mathcal{W}}(1)).$$

We can write every class in $A^{*-r}(\widetilde{X})$ as $(\rho'')^*\alpha$ for a unique class $\alpha \in A^{*-r}(\mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d))$. Then, the effect of $(\sigma' \circ \iota)_*$ is

(4.4)
$$\sigma'_* \iota_* (\rho'')^* \alpha = \sigma'_* \iota_* \iota^* (\rho')^* \alpha = \sigma'_*([\widetilde{X}] \cdot (\rho')^* \alpha) = \sigma'_*(\rho')^*(\beta \cdot \alpha) = \rho^* \sigma_*(\beta \cdot \alpha).$$

Now, let $z = c_1(\mathcal{O}_{\mathbb{P}\mathcal{W}}(1))$ and $\zeta = c_1(\mathcal{O}_{\mathbb{P}(T_d)}(1))$. By the projective bundle theorem,

(4.5)
$$A^*(\mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d)) = A^*(B)[z, \zeta]/(z^2 + w_1 z + w_2, \zeta^d + \zeta^{d-1} t_1 + \cdots + \zeta t_{d-1} + t_d).$$

Thus, each class $\alpha \in A^*(\mathbb{P}\mathcal{W} \times_B \mathbb{P}(T_d))$ is uniquely expressible as

$$\alpha = \sum_{i=0}^{d-1} (\sigma^* \gamma_i) \zeta^i + \sum_{i=1}^{d} (\sigma^* \gamma'_i) z \zeta^{i-1}, \tag{4.6}$$

where $\gamma_i \in A^*(B)$ with $\deg \gamma_i = \deg \gamma'_i = \deg \alpha - i$.

By (4.4) and (4.6), the image of $(\sigma' \circ \iota)_* : A^{*-r}_{\mathbb{Q}}(\widetilde{X}) \to A^*_{\mathbb{Q}}(X) \cong A^*_{\mathbb{Q}}(B)$ is the ideal generated by the classes

$$f_{i,j} := \sigma_*(\beta \cdot z^i \zeta^j) \qquad \text{for } 0 \le i \le 1,\ 0 \le j \le d-1.$$

As $\rho^*$ is an isomorphism on Chow, we omit it above and in what follows for ease of notation. By the excision sequence (4.1), we have

$$A^*_{\mathbb{Q}}(\mathcal{V}_{r,d}) = \frac{A^*_{\mathbb{Q}}(B)}{\langle f_{i,j} : 0 \le i \le 1, 0 \le j \le d-1 \rangle} \cong \frac{\mathbb{Q}[w_1, w_2, t_1, \ldots, t_d, u_1, \ldots, u_{r+d}]}{\langle f_{i,j} : 0 \le i \le 1, 0 \le j \le d-1 \rangle}.$$

Since $\sigma$ has relative dimension $d$, the codimension of $f_{i,j}$ is $(r+d) + i + j - d = r + i + j$. Recall that there are no relations among the generators of $A^*(M) \cong A^*(B)$, so $f_{i,j}$ is a unique polynomial of codimension $i + j + r$ in the $t$'s, $u$'s, and $w$'s. Next, we shall determine the coefficients of $t_{i+j+r}$ and $u_{i+j+r}$ in this expression for $f_{i,j}$. We shall then use this to obtain expressions for each of $t_{j+r}$ and $u_{j+r}$ in terms of $f_{0,j}$, $f_{1,j-1}$, and the $t_n, u_m$ with $m, n < j + r$. This will inductively allow us to write each $t_{j+r}$ and $u_{j+r}$ in terms of the $t_n, u_m$ with $n < r$ and $m \le r$.

Using the splitting principle as in [13, Proposition 5.17], we have

$$\beta = c_{r+d}(\mathcal{O}_{\mathbb{P}(T_d)}(1) \otimes \sigma^* T_{r+d} \otimes \mathcal{O}_{\mathbb{P}\mathcal{W}}(1)) = \sum_{i=0}^{r+d} (\zeta + z)^{r+d-i} \sigma^* u_i,$$

and applying (4.5), this becomes

$$\begin{aligned} &= (\zeta^{r+d} + (r+d) z \zeta^{r+d-1}) + (\zeta^{r+d-1} + (r+d-1) z \zeta^{r+d-2}) \sigma^* u_1 \\ &\quad + \cdots + (\zeta^d + d z \zeta^{d-1}) \sigma^* u_r + \cdots + \sigma^* u_{r+d} + \langle \sigma^* w_1, \sigma^* w_2 \rangle. \end{aligned}$$

The push forward of any term involving $\sigma^* w_1$ or $\sigma^* w_2$ cannot contribute to the coefficient of $t_{i+j+r}$ or $u_{i+j+r}$. Since $z^2 \in \langle \sigma^* w_1, \sigma^* w_2 \rangle$, after we multiply $z^i \zeta^j$ with $\beta$, we only care about the resulting terms where the power of $z$ is 1 (if the power of $z$ is zero, then the push forward vanishes).

To compute the push forward of such terms, first note that $\sigma_*(z\zeta^n) = 0$ for $n < d-1$ and $\sigma_*(z\zeta^{d-1}) = 1$ (see, e.g., [13, Lemma 9.7]). Therefore, $\sigma_*(z\zeta^{d-1+i})$ is equal to

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the coefficient of $\zeta^{d-1}$ in the expression for $\zeta^{d-1+i}$ in terms of $\zeta^n$ with $n \leq d-1$. We find such an expression by using (4.5) repeatedly. Each contribution to the coefficient of $\zeta^{d-1}$ in $\zeta^{d-1+i}$ comes from an ordered partition $i = \lambda_1 + \lambda_2 + \cdots + \lambda_s$ with $\lambda_n \leq d$; the term for such a partition contributes the degree $i$ class $\prod_{n=1}^{s}(-t_{\lambda_n})$ to the coefficient of $\zeta^{d-1}$. We therefore conclude

$$\sigma_*(z\zeta^{d-1+i}) = \begin{cases} 0, & \text{if } i < 0, \\ 1, & \text{if } i = 0, \\ \sum_{m_1\cdot 1+\cdots+m_d\cdot d=i}(-1)^{m_1+\cdots+m_d}\cdot\frac{(m_1+\cdots+m_d)!}{m_1!\cdots m_d!}\cdot t_1^{m_1}\cdots t_d^{m_d}, & \text{if } i \geq 1. \end{cases}$$

Above, the coefficient in front of a monomial for $(m_1,\ldots,m_d)$ is the number of ordered partitions of $i$ so that $j$ appears with multiplicity $m_j$. In particular, we compute

$$f_{1,j-1} = \sigma_*(\beta z\zeta^{j-1}) = -t_{j+r} + u_{j+r} + (\text{terms with subscript } < j+r), \tag{4.7}$$

$$f_{0,j} = \sigma_*(\beta\zeta^{j}) = -(r+d)t_{j+r} + (d-j)u_{j+r} + (\text{terms with subscript } < j+r). \tag{4.8}$$

When $j = 0$, we can write

$$t_r = \frac{d}{r+d}u_r + (\text{terms with subscript } < r). \tag{4.9}$$

When $0 < j \leq d-r$, taking appropriate linear combinations of (4.7) and (4.8), we have

$$u_{j+r} = \frac{1}{j+r}((r+d)f_{1,j-1} - f_{0,j}) + (\text{terms with subscript } < j+r), \tag{4.10}$$

$$t_{j+r} = \frac{1}{r+j}((d-j)f_{1,j-1} - f_{0,j}) + (\text{terms with subscript } < j+r). \tag{4.11}$$

In $A^*_{\mathbb{Q}}(\mathcal{V}_{r,d})$, the classes $f_{i,j}$ are zero. Thus, using (4.9)–(4.11) inductively, the classes $t_{j+r}$ for $0 \leq j \leq d-r$ and $u_{j+r}$ for $0 < j \leq d-r$ are expressible as polynomials in $w_1, w_2, t_1, \ldots, t_{r-1}, u_1, \ldots, u_r$. The remaining $f_{0,j}$ and $f_{1,j-1}$ with $j > d-r$ give rise to relations in degrees greater than $d$. Hence, the map

$$\mathbb{Q}[w_1, w_2, t_1, t_2, \ldots, t_{r-1}, u_1, \ldots, u_r] \to A^*_{\mathbb{Q}}(\mathcal{V}_{r,d}) \tag{4.12}$$

is an isomorphism in degrees $* \leq d$. For dimension reasons, there can be no relations among the generators $w_1, w_2, a_1, \ldots, a_r, a'_2, \ldots, a'_r$ in degrees less than $d$ because they are a list of the same number of generators in the same degrees as (4.12). ∎

**Corollary 4.2** *We have* $A^*_{\mathbb{Q}}(\mathcal{B}_{r,\ell}) = \mathbb{Q}[w_1, w_2, a_1, \ldots, a_r, a'_2, \ldots, a'_r]$.

**Proof** Recall that the codimension of the complement of $\mathcal{U}_{m,r,\ell}$ is $\ell + mr + 1$. Thus, choosing $m$ such that $\ell + mr + 1 > i$, we see

$$\begin{aligned}\dim A^i_{\mathbb{Q}}(\mathcal{B}_{r,\ell}) &= \dim A^i_{\mathbb{Q}}(\mathcal{U}_{m,r,\ell}) = \dim A^i_{\mathbb{Q}}(\mathcal{V}_{r,mr+\ell})\\ &= \dim \mathbb{Q}[w_1, w_2, a_1, \dots, a_r, a'_2, \dots, a'_r]_i.\end{aligned}$$

We already know $\mathbb{Q}[w_1, w_2, a_1, \dots, a_r, a'_2, \dots, a'_r] \twoheadrightarrow A^*(\mathcal{B}_{r,\ell})$, so we conclude that it has no kernel for dimension reasons. ∎

## 5 The integral Chow ring

Our plan is to describe the integral Chow ring of $\mathcal{B}_{r,\ell}$ by building it up as a union of the splitting loci $\Sigma_{\vec{e}}$ using excision. The reader may be interested to compare the ideas and use of higher Chow groups here with the work of Bae and Schmitt in their study of the moduli stack of pointed genus 0 curves [4].

The Chow rings of our strata $\Sigma_{\vec{e}}$ are particularly nice. Fix some $\vec{e}$ and write $\mathcal{O}(\vec{e}) = \bigoplus_{i=1}^s \mathcal{O}(d_i)^{\oplus r_i}$ as in Section 3.1. There is an inclusion of groups $\prod \mathrm{GL}_{r_i} \times \mathrm{GL}_2 \hookrightarrow H_{\vec{e}}$, which induces a map on classifying stacks

$$\prod \mathrm{BGL}_{r_i} \times \mathrm{BGL}_2 \to BH_{\vec{e}}. \tag{5.1}$$

The following lemma is a special case of a standard result about Chow rings of classifying spaces of parabolic groups, using higher homotopy invariance.

***Lemma 5.1*** *The map (5.1) factors as a sequence of affine bundles. In particular $A^*(\Sigma_{\vec{e}}) = A^*(\prod \mathrm{BGL}_{r_i} \times \mathrm{BGL}_2)$, which is a free $\mathbb{Z}$-algebra. Furthermore, over $\mathbb{C}$, the first higher Chow groups satisfy*

$$\mathrm{CH}^*(\Sigma_{\vec{e}}, 1, \mathbb{Z}/p\mathbb{Z}) = \mathrm{CH}^*\left(\prod \mathrm{BGL}_{r_i} \times \mathrm{BGL}_2, 1, \mathbb{Z}/p\mathbb{Z}\right) = 0$$

*for all primes $p$.* ∎

**Proof** We induct on $s$. The $s = 1$ case is immediate. Let $G' = \mathrm{Aut}(\oplus_{i=1}^{s-1} \mathcal{O}(d_i)^{\oplus r_i})$ and $H = G' \times \mathrm{GL}_{r_s} \subset \mathrm{Aut}(\mathcal{O}(\vec{e}))$. There is a quotient map $\mathrm{Aut}(\mathcal{O}(\vec{e})) \to H$ defined by forgetting blocks not in $H$, which expresses $\mathrm{Aut}(\mathcal{O}(\vec{e}))$ as a semidirect product $N \rtimes H$ where $N \cong H^0(\mathcal{O}_{\mathbb{P}^1}(d_s - d_1))^{\oplus(r_1 r_s)} \oplus \cdots \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(d_s - d_{s-1}))^{\oplus(r_{s-1} r_s)}$ is affine. Moreover,

$$H_{\vec{e}} = (N \rtimes H) \rtimes \mathrm{GL}_2 = N \rtimes ((G' \rtimes \mathrm{GL}_2) \times \mathrm{GL}_{r_s}).$$

Now, let $H_{\vec{e}}$ act on $N$ where elements of $N$ act by left multiplication and elements of $(G' \rtimes \mathrm{GL}_2) \times \mathrm{GL}_{r_s}$ act by conjugation. This action is affine linear, so the quotient, which is $B((G' \rtimes \mathrm{GL}_2) \times \mathrm{GL}_{r_s})$, is an affine bundle over $BH_{\vec{e}}$. By the homotopy property, the Chow ring and higher Chow groups of $BH_{\vec{e}}$ agree with that of $B(G' \rtimes \mathrm{GL}_2) \times \mathrm{BGL}_{r_s}$, which, by induction, are isomorphic to those of $(\mathrm{BGL}_{r_1} \times \cdots \times \mathrm{BGL}_{r_{s-1}} \times \mathrm{BGL}_2) \times \mathrm{BGL}_{r_s}$. The vanishing of the first higher Chow group over $\mathbb{C}$ is equation (2.2). ∎

Using the above lemma, we find that inclusion of cycle classes from strata is injective and deduce that the Chow ring of $\mathcal{B}_{r,\ell}$ is torsion-free.

***Lemma 5.2*** *The Chow group $A^i(\mathcal{B}_{r,\ell})$ is a finitely generated free $\mathbb{Z}$-module for all $i$.*

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**Proof** Suppose that $U$ is a finite union of strata and $\Sigma_{\vec{e}}$ is a disjoint stratum which is closed in $X = U \cup \Sigma_{\vec{e}}$. By Lemma 5.1, we know $A^i(\Sigma_{\vec{e}})$ is a finitely generated free $\mathbb{Z}$-module for all $i$. By induction, we may also assume that $A^i(U)$ is a finitely generated free $\mathbb{Z}$-module. It suffices to show that $A^i(X)$ is also a finitely generated free $\mathbb{Z}$-module.

Let us first deduce the result over $\mathbb{C}$. We have a localization long exact sequence

$$\begin{aligned}\cdots &\to \mathrm{CH}^{*-u(\vec{e})}(\Sigma_{\vec{e}},1,\mathbb{Z}/p) \to \mathrm{CH}^*(X,1,\mathbb{Z}/p) \to \mathrm{CH}^*(U,1,\mathbb{Z}/p)\\ &\to A^{*-u(\vec{e})}(\Sigma_{\vec{e}}) \otimes \mathbb{Z}/p\mathbb{Z} \to A^*(X) \otimes \mathbb{Z}/p\mathbb{Z} \to A^*(U) \otimes \mathbb{Z}/p\mathbb{Z} \to 0.\end{aligned}$$

By Lemma 5.1, we have $\mathrm{CH}^{*-u(\vec{e})}(\Sigma_{\vec{e}},1,\mathbb{Z}/p) = 0$, so if $\mathrm{CH}^*(U,1,\mathbb{Z}/p\mathbb{Z}) = 0$, then we have $\mathrm{CH}^*(X,1,\mathbb{Z}/p) = 0$ too. It follows by induction that $\mathrm{CH}^*(U,1,\mathbb{Z}/p) = 0$ for any finite union of strata. Hence, we have an exact sequence

$$(5.2)\quad 0 \to A^{*-u(\vec{e})}(\Sigma_{\vec{e}}) \otimes \mathbb{Z}/p\mathbb{Z} \to A^*(X) \otimes \mathbb{Z}/p\mathbb{Z} \to A^*(U) \otimes \mathbb{Z}/p\mathbb{Z} \to 0$$

for each prime $p$. Because the $\mathbb{Z}$-modules involved are finitely generated, the exactness of (5.2) for all $p$ implies that

$$0 \to A^{i-u(\vec{e})}(\Sigma_{\vec{e}}) \to A^i(X) \to A^i(U) \to 0 \tag{5.3}$$

is exact. Since $A^{i-u(\vec{e})}(\Sigma_{\vec{e}})$ and $A^i(U)$ are finitely generated free $\mathbb{Z}$-modules, so is $A^i(X)$. This concludes the proof over $\mathbb{C}$.

The exactness of (5.3) over $\mathbb{C}$ also tells us that

$$\operatorname{rank} A^i(\mathcal{B}_{r,\ell}) = \sum_{\vec{e}} \operatorname{rank} A^{i-u(\vec{e})}(\Sigma_{\vec{e}}). \tag{5.4}$$

We claim that (5.4) in fact holds over any ground field. Indeed, Corollary 4.2 holds over any field, so the left-hand side is independent of the ground field. Similarly, using Lemma 5.1, we have $A^*(\Sigma_{\vec{e}}) = A^*(\prod \mathrm{BGL}_{r_i} \times \mathrm{BGL}_2)$ and the latter is independent of the ground field, so the right-hand side of (5.4) is independent of the ground field.

Now, working over any ground field, we claim that the map $A^{i-u(\vec{e})}(\Sigma_{\vec{e}}) \to A^i(X)$ for attaching each stratum is injective. If not, then the image of $A^{i-u(\vec{e})}(\Sigma_{\vec{e}}) \to A^i(X)$ would have rank strictly less than $\operatorname{rank} A^{i-u(\vec{e})}(\Sigma_{\vec{e}})$. Then, $\operatorname{rank} A^i(\mathcal{B}_{r,\ell})$ would be less than the sum $\sum_{\vec{e}} \operatorname{rank} A^{i-u(\vec{e})}(\Sigma_{\vec{e}})$, violating (5.4). Hence, we must have an exact sequence as in (5.3) for each $\vec{e}$. Arguing as before, we know $A^{i-u(\vec{e})}(\Sigma_{\vec{e}})$ and $A^i(U)$ are finitely generated free $\mathbb{Z}$-modules, so because (5.3) is exact, $A^i(X)$ is also a finitely generated free $\mathbb{Z}$-module. ∎

An analogous argument in cohomology can be used to show that Chow and cohomology rings of $\mathcal{B}_{r,d}$ agree.

***Lemma 5.3*** *If $k = \mathbb{C}$, then the cycle class map $A^*(\mathcal{B}_{r,d}) \to H^{2*}(\mathcal{B}_{r,d})$ is an isomorphism. If $k$ is arbitrary and $\ell$ is a prime not equal to the characteristic of $k$, then the cycle class map $A^*(\mathcal{B}_{r,d}) \otimes \mathbb{Z}_\ell \to H^{2*}_{\acute{e}t}(\mathcal{B}_{r,d}, \mathbb{Z}_\ell)$ is an isomorphism.*

**Proof** First, consider the case $k = \mathbb{C}$. As before, suppose that $U$ is a finite union of strata and $\Sigma_{\vec{e}}$ is a disjoint stratum which is closed in $X = U \cup \Sigma_{\vec{e}}$. Because $\Sigma_{\vec{e}}$ and

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$X$ are smooth, the cohomology of the pair $H^*(X, U)$ is the reduced cohomology of the Thom space of the normal bundle of $\Sigma_{\vec{e}} \subset X$. By the Thom isomorphism, the cohomology of this pair is then $H^*(X, U) \cong H^{*-2u(\vec{e})}(\Sigma_{\vec{e}})$. By Lemma 5.1, we have $H^*(\Sigma_{\vec{e}}) = H^*(\prod \mathrm{BGL}_{r_i} \times \mathrm{BGL}_2)$, which vanishes in odd degrees. By induction, we may assume that the odd cohomology of $U$ vanishes. Thus, the long exact sequence for the pair $(X, U)$ gives an exact sequence

$$0 \to H^{2(i-u(\vec{e}))}(\Sigma_{\vec{e}}) \to H^{2i}(X) \to H^{2i}(U) \to 0 \tag{5.5}$$

for each $i$. The cycle class map sends the short exact sequence (5.3) to (5.5). Because $A^*(\mathrm{BGL}_{r_i}) \to H^{2i}(\mathrm{BGL}_{r_i})$ is an isomorphism, using Lemma 5.1, we see that $A^{i-u(\vec{e})}(\Sigma_{\vec{e}}) \to H^{2(i-u(\vec{e}))}(\Sigma_{\vec{e}})$ is an isomorphism. By induction, we may assume that $A^i(U) \to H^{2i}(U)$ is an isomorphism. By the five lemma, we see that $A^i(X) \to H^{2i}(X)$ is also an isomorphism.

In étale cohomology, there is a Gysin long exact sequence for $\Sigma_{\vec{e}} \subset X$ and the open compliment $U = X \smallsetminus \Sigma_{\vec{e}}$. Moreover, we have $H^*_{\text{ét}}(\Sigma_{\vec{e}}, \mathbb{Z}_\ell) = H^*_{\text{ét}}(\prod \mathrm{BGL}_{r_i} \times \mathrm{BGL}_2, \mathbb{Z}_\ell)$, which vanishes in odd degrees. Thus, upon tensoring Chow with $\mathbb{Z}_\ell$ the same proof establishes that $A^*(\mathcal{B}_{r,d}) \otimes \mathbb{Z}_\ell \to H^{2*}_{\text{ét}}(\mathcal{B}_{r,d}, \mathbb{Z}_\ell)$ is an isomorphism. ■

**Proof** Corollary 4.2 determines $A^*_{\mathbb{Q}}(\mathcal{B}_{r,\ell})$. Lemma 5.2 shows that $A^*(\mathcal{B}_{r,\ell})$ is a subring of $A^*_{\mathbb{Q}}(\mathcal{B}_{r,\ell})$, and Lemma 3.5 identifies it as the subring generated by the Chern classes of the sheaves $\pi_*\mathcal{E}(m)$. Lemma 5.3 shows that the cycle class map is an isomorphism. ■

To make this subring more explicit, we provide formulas for the Chern classes of $\pi_*\mathcal{E}(m)$ in terms of the rational generators (working modulo $\langle w_1, w_2\rangle$). Recall that $a_1' = \ell$ is the relative degree of $\mathcal{E}$.

***Lemma 5.4*** *Let*

$$F(t) = \sum_{i=0}^{\infty} e_i t^i = \exp\left(\int \frac{a_1'(a_1 + a_2 t + \cdots + a_r t^{r-1}) - (a_2' + a_3' t + \cdots + a_r' t^{r-2})}{1 + a_1 t + \cdots + a_r t^r} dt\right). \tag{5.6}$$

*Then,*

$$\sum_{i=0}^{\infty} c_i(\pi_*\mathcal{E}(m))t^i = F(t)(1 + a_1 t + \cdots + a_r t^r)^{m+1} \mod \langle w_1, w_2, t^{mr+\ell+1}\rangle. \tag{5.7}$$

**Proof** We will use Grothendieck–Riemann–Roch to compute the Chern characters and make use of formal manipulations that turn power sums (Chern characters) into elementary symmetric functions (Chern classes). It is convenient to package this information in generating functions.

Given some $\alpha_1, \ldots, \alpha_r$, let $\sigma_j := \sigma_j(\alpha_1, \ldots, a_r) = \sum_{i_1 < i_2 < \cdots < i_j} \alpha_{i_1}\alpha_{i_2}\cdots a_{i_j}$ denote the $j$th elementary symmetric function in the $\alpha_i$. For each $j \geq 0$, there is a polynomial

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$p_j(x_1, \dots, x_r)$ such that $p_j(\sigma_1, \dots, \sigma_r) = \alpha_1^j + \cdots + \alpha_r^j$. These polynomials satisfy

$$\log(1 + x_1 t + \cdots + x_r t^r) = \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} p_j(x_1, \dots, x_r) t^j. \tag{5.8}$$

For any vector bundle $E$, the $j$th Chern character is related to the Chern classes by

$$\mathrm{ch}_j(E) = \frac{1}{j!} p_j(c_1(E), \dots, c_r(E)).$$

In what follows, we work modulo the ideal $\langle w_1, w_2 \rangle$, so that $z^2 = 0$. Recall that, for any polynomial $f(x_1, \dots, x_r)$, we have

$$f(x_1, \dots, x_i + \varepsilon, \dots, x_r) = f(x_1, \dots, x_r) + \varepsilon \frac{\partial f}{\partial x_i}(x_1, \dots, x_r) + \langle \varepsilon^2 \rangle.$$

Applying this repeatedly, one sees more generally that

$$f(x_1 + h_1\varepsilon, x_2 + h_2\varepsilon, \dots, x_r + h_r\varepsilon) = f(x_1, \dots, x_r) + \varepsilon \cdot \sum_{i=1}^{r} h_i \frac{\partial f}{\partial x_i}(x_1, \dots, x_r) + \langle \varepsilon^2 \rangle.$$

In particular,

$$\begin{aligned}
\mathrm{ch}_j(\mathcal{E}) &= \frac{1}{j!} p_j(\pi^* a_1 + \pi^* a_1' z, \dots, \pi^* a_r + \pi^* a_r' z) \\
&= \frac{1}{j!} p_j(\pi^* a_1, \dots, \pi^* a_r) + z \cdot \frac{1}{j!} \left( \sum_{i=1}^{r} \pi^* a_i' \frac{\partial p_j}{\partial x_i}(\pi^* a_1, \dots, \pi^* a_r) \right).
\end{aligned}$$

Now, let

$$c_j = \frac{1}{j!} p_j(a_1, \dots, a_r) \qquad \text{and} \qquad c_j' = \frac{1}{j!} \sum_{i=1}^{r} a_i' \frac{\partial p_j}{\partial x_i}(a_1, \dots, a_r), \tag{5.9}$$

so that $\mathrm{ch}_j(\mathcal{E}) = \pi^*(c_j) + \pi^*(c_j')z$, with $c_j \in A^j_{\mathbb{Q}}(\mathcal{B}_{r,\ell})$ and $c_j' \in A^{j-1}_{\mathbb{Q}}(\mathcal{B}_{r,\ell})$. In addition, let us write $c = c_0 + c_1 + c_2 + \cdots$ and $c' = c_1' + c_2' + \cdots$, so that $\mathrm{ch}(\mathcal{E}) = \pi^*(c) + \pi^*(c')z$.

The relative tangent bundle of $\pi : \mathbb{P}\mathcal{W} \to \mathcal{B}_{r,\ell}$ has Todd class $\mathrm{td}(T_\pi) = 1 + \frac{1}{2}c_1(T_\pi) = 1 + z$. Moreover, $\mathrm{ch}(\mathcal{E}(m)) = \mathrm{ch}(\mathcal{E})\mathrm{ch}(\mathcal{O}_{\mathbb{P}\mathcal{W}}(m)) = \mathrm{ch}(\mathcal{E})(1 + mz)$. On $\mathcal{U}_m$, we have $R^1\pi_*\mathcal{E}(m) = 0$, so Grothendieck–Riemann–Roch tells us

$$\begin{aligned}
\mathrm{ch}(\pi_*\mathcal{E}(m)) &= \pi_*(\mathrm{ch}(\mathcal{E}(m))\mathrm{td}(T_\pi)) \\
&= \pi_*((\pi^*(c) + \pi^*(c')z)(1 + (m+1)z)) = c' + (m+1)c.
\end{aligned}$$

To recover the Chern classes, by (5.8), we have to evaluate

$$\begin{aligned}
&\exp\left(\sum_{j=1}^{\infty}(j-1)!(-1)^{j+1}\mathrm{ch}_j(\pi_*\mathcal{E}(m))t^j\right)\\
&\quad = \exp\left(\sum_{j=1}^{\infty}(j-1)!(-1)^{j+1}(c'_{j+1}+(m+1)c_j)t^j\right)\\
&\quad = \exp\left(\sum_{j=1}^{\infty}\left((j-1)!(-1)^{j+1}\cdot\frac{1}{j!}\sum_{i=1}^{r}a'_i\frac{\partial p_j}{\partial x_i}(a_1,\ldots,a_r)\right)t^j\right)\\
&\qquad \times \exp\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j!}p_j(a_1,\ldots,a_r)t^j\right)^{m+1}\\
&\quad = \exp\left(\sum_{i=1}^{r}a'_i\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(j+1)}\frac{\partial p_{j+1}}{\partial x_i}(a_1,\ldots,a_r)t^j\right)(1+a_1t+\cdots+a_rt^r)^{m+1}.
\end{aligned} \tag{5.10}$$

To evaluate the infinite sums inside (5.10), we consider

$$\frac{d}{dt}\left(\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(j+1)}\frac{\partial p_{j+1}}{\partial x_i}(a_1,\ldots,a_r)t^j\right)=\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j+1}\frac{\partial p_{j+1}}{\partial x_i}(a_1,\ldots,a_r)t^{j-1} \tag{5.11}$$

$$=\sum_{j=2}^{\infty}\frac{(-1)^{j}}{j}\frac{\partial p_{j}}{\partial x_i}(a_1,\ldots,a_r)t^{j-2}. \tag{5.12}$$

Note that $\partial p_j/\partial x_i = 0$ if $j < i$. Therefore taking the partial derivative of (5.8) with respect to $x_i$, we see that (5.12) is equal to

$$\begin{aligned}
&= \frac{-1}{t^2}\left(\frac{\partial}{\partial x_i}\log(1+a_1t+\cdots+a_rt^r)-\delta_{i1}t\right)\\
&= \begin{cases}(a_1+a_2t+\cdots+a_rt^{r-1})/(1+a_1t+\cdots+a_rt^r), & \text{if } i=1,\\ -t^{i-2}/(1+a_1t+\cdots+a_rt^r), & \text{if } i\geq 2.\end{cases}
\end{aligned} \tag{5.13}$$

Taking the formal integral of the left-hand side of (5.11), which equals (5.13), we have

$$\begin{aligned}
\sum_{i=1}^{r}a'_i\sum_{j=1}^{\infty}\frac{(-1)^{j+1}}{j(j+1)}\frac{\partial p_{j+1}}{\partial x_i}(a_1,\ldots,a_r)t^j = \int a'_1\cdot\frac{a_1+a_2t+\cdots+a_rt^{r-1}}{1+a_1t+\cdots+a_rt^r}\\
-\frac{a'_2+a'_3t+\cdots+a'_rt^{r-2}}{1+a_1t+\cdots+a_rt^r}.
\end{aligned}$$

Exponentiating this quantity and multiplying by $(1+a_1t+\cdots+a_rt^r)^{m+1}$ shows that (5.10) is equal to the claimed formula (5.7). ∎

***Remark 5.5*** The Chern classes of $\pi_*\mathcal{E}(m)$ (not just modulo $\langle w_1, w_2\rangle$) are also expressible in terms of exponentials of formal integrals of rational functions in the

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$a_i$ and $a'_i$ and the Chern classes of $\mathcal{W}$. Equation (5.9) is made up of terms with higher partial derivatives of the polynomials $p_j$ multiplied by the Chern classes of $\mathcal{W}$. The sums of these higher partial derivatives that appear in expanding (5.10) can again be collected into rational functions and formal integrals of rational functions in the $a_i$. However, we no longer have the sequence $0 \to \pi_*\mathcal{E}(m-1) \to \pi_*\mathcal{E}(m) \to \mathcal{E}|_{B\times\{0\}} \to 0$ that holds for vector bundles on trivial families $B \times \mathbb{P}^1$, so the relationship between twists is not as nice.

**Proof** By Vistoli's theorem, we have $A^*(\mathcal{B}^\dagger_{r,\ell}) = A^*(\mathcal{B}_{r,\ell})/\langle w_1, w_2\rangle$. By Lemma 5.4, after setting $w_1 = w_2 = 0$, the Chern classes of $\pi_*\mathcal{E}(m)$ for all $m$ are expressible in terms of the $e_i$ in (5.6) and $a_1, \dots, a_r$. To see that the cycle class map is an isomorphism, one may argue as in Lemmas 5.1 and 5.3, but using the splitting loci $\Sigma^\dagger_{\vec{e}}$ on $\mathcal{B}^\dagger_{r,\ell}$. One has $\Sigma^\dagger_{\vec{e}} \cong B\,\mathrm{Aut}(\mathcal{O}(\vec{e}))$, so the proof is essentially the same after dropping the $B\mathrm{GL}_2$ factor from Lemma 5.1. ∎

***Remark 5.6*** Let $\mathcal{X}_{r,d}$ be the moduli stack of vector bundles on $\mathbb{P}^1$ with a trivialization at a point, so $\mathcal{X}_{r,d} \to \mathcal{B}^\dagger_{r,d}$ is a principal $\mathrm{GL}_r$-bundle. We can stratify $\mathcal{X}_{r,d}$ by splitting loci $\widetilde{\Sigma}_{\vec{e}}$. Similar to Lemma 5.1, there is a natural map $\mathrm{GL}_r/\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_s} \to \widetilde{\Sigma}_{\vec{e}}$ which is an affine bundle. Meanwhile, $\mathrm{GL}_r/\mathrm{GL}_{r_1} \times \cdots \times \mathrm{GL}_{r_s}$ is an affine bundle over the partial flag variety $\mathrm{Fl} := \mathrm{Fl}(r_1, r_1 + r_2, \dots, r_1 + \cdots + r_{s-1}; r)$. Since Fl admits a stratification into affines, its higher Chow groups vanish and it has no odd cohomology. Thus, arguing as in Lemma 5.3, we conclude that the Chow and cohomology rings of $\mathcal{X}_{r,d}$ also agree.

***Corollary 5.7*** *The integral Chow rings $A^*(\mathcal{B}_{r,\ell})$ and $A^*(\mathcal{B}^\dagger_{r,\ell})$ are not finitely generated as $\mathbb{Z}$-algebras.*

**Proof** Since $A^*(\mathcal{B}_{r,\ell})$ surjects onto $A^*(\mathcal{B}^\dagger_{r,\ell})$, it suffices to prove the claim for $\mathcal{B}^\dagger_{r,\ell}$. Suppose to the contrary that $A^*(\mathcal{B}^\dagger_{r,\ell})$ were finitely generated as a $\mathbb{Z}$-algebra. Then, there would exist a finite set of primes $\{p_1, \dots, p_n\}$ such that for all $\eta \in A^*(\mathcal{B}^\dagger_{r,\ell})$, there exist $m_1, \dots, m_n \in \mathbb{Z}$ such that $p_1^{m_1} \cdots p_n^{m_n} \cdot \eta \in \mathbb{Z}[a_1, \dots, a_r, a'_2, \dots, a'_r]$. Let $q$ be a prime not in $\{p_1, \dots, p_n\}$. Let us consider the coefficient $f_q$ in (5.6). Performing the formal integral, we see that $F(t) = \exp(-a'_2 t + \cdots)$, so $(a'_2)^q$ appears in $f_q$ with coefficient $\frac{1}{q!}$. Hence, $p_1^{m_1} \cdots p_n^{m_n} \cdot f_q \notin \mathbb{Z}[a_1, \dots, a_r, a'_2, \dots, a'_r]$ for any $m_1, \dots, m_n$. ∎

As an example of the utility of the formulas in Lemma 5.6, we show that the integral Chow ring $A^*(\mathcal{B}^\dagger_{r,\ell})$ depends on $\ell$.

***Corollary 5.8*** *The integral Chow rings $A^*(\mathcal{B}^\dagger_{r,0})$ and $A^*(\mathcal{B}^\dagger_{r,\ell})$ are not isomorphic when $\ell \neq 0$ mod r. Hence, $\mathcal{B}^\dagger_{r,0}$ is not isomorphic to $\mathcal{B}^\dagger_{r,\ell}$.*

**Proof** Let $\sum e_i t^i = F(t)$ as in (5.6). Then, $A^*(\mathcal{B}^\dagger_{r,\ell}) \subset \mathbb{Q}[a_1, \dots, a_r, a'_2, \dots, a'_r]$ is the subring generated by the classes $e_i$ together with $a_1, \dots, a_r, a'_2, \dots, a'_r$. Let

$$S = A^1(\mathcal{B}^\dagger_{r,\ell}) \cdot A^{r-1}(\mathcal{B}^\dagger_{r,\ell}) + A^2(\mathcal{B}^\dagger_{r,\ell}) \cdot A^{r-1}(\mathcal{B}^\dagger_{r,\ell}) + \cdots + A^{\lfloor r/2 \rfloor}(\mathcal{B}^\dagger_{r,\ell}) \cdot A^{\lceil r/2 \rceil}(\mathcal{B}^\dagger_{r,\ell}) \subset A^r(\mathcal{B}^\dagger_{r,\ell}),$$

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so $S$ is the subspace of $A^r(\mathcal{B}^\dagger_{r,\ell})$ generated by products of integral elements from lower codimension. Taking $\frac{d}{dt}$ of (5.6), we have

$$(1 + a_1 t + \cdots + a_r t^r) \sum_{i=1}^{\infty} i e_i t^{i-1} = (\ell(a_1 + \cdots + a_r t^{r-1}) - (a_2' + \cdots + a_r' t^{r-2})) \sum_{j=0}^{\infty} e_j t^j.$$

Equating the coefficient of $t^{i-1}$ on both sides for $i \leq r$, we find

$$\begin{aligned} i e_i + (i-1)e_{i-1}a_1 + \cdots + e_1 a_{i-1} &= \ell(e_{i-1}a_1 + \cdots + e_1 a_{i-1} + a_i) \\ &\quad - (e_{i-1}a_2' + \cdots + e_1 a_i' + a_{i+1}'). \end{aligned} \tag{5.14}$$

(Above, $a_{i+1}' = 0$ if $i = r$.) In particular, setting $i = r$, we find

$$r e_r - \ell a_r \in S.$$

This establishes that $A^r(\mathcal{B}^\dagger_{r,\ell})/S$ is a quotient of $\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ where $m = \gcd(r, \ell)$.

If $\ell \neq 0 \bmod r$, then $m < r$, so $A^r(\mathcal{B}^\dagger_{r,\ell})/S$ has no element of order $r$. On the other hand, if $\ell = 0$, then $m = r$, and we claim that $e_r$ is an element of order $r$ in $A^r(\mathcal{B}^\dagger_{r,0})/S$.

If $\ell = 0$, we have $e_1 = -a_2'$, and, more generally, no copy of $a_{r-1}$ appears in $e_{r-1}$. In particular, using (5.14), we find that $e_r = -\frac{1}{r} a_2' a_{r-1} + \cdots$. Meanwhile, since $a_{r-1}$ does not appear in $e_{r-1}$, elements of $A^{r-1}(\mathcal{B}^\dagger_{r,0})$ necessarily have integer coefficient of $a_{r-1}$. Hence, elements of $S$ always have integer coefficient of $a_2' a_{r-1}$. It follows that $n e_r \notin S$ for any $n < r$. ∎

To get results like Corollary 5.8 for $A^*(\mathcal{B}_{r,\ell})$ is more challenging because we cannot ignore the $w_1$ and $w_2$ terms and work with the nice generating function (5.6). However, in specific cases, one can write down integral generators using Macaulay2 and, using ad hoc arguments, prove results such as the following.

***Corollary 5.9*** *The integral Chow rings $A^*(\mathcal{B}_{2,1})$ and $A^*(\mathcal{B}_{2,0})$ are not isomorphic. Hence, $\mathcal{B}_{2,0}$ is not isomorphic to $\mathcal{B}_{2,1}$.*

**Proof** The codimension of the complement of $\mathcal{U}_{3,2,\ell} \subset \mathcal{B}_{2,\ell}$ is $6 + \ell$. In particular, for $\ell = 0, 1$, we have $A^4(\mathcal{B}_{2,\ell}) = A^4(\mathcal{U}_{3,2,\ell})$. By Lemma 3.3, we have that $A^4(\mathcal{B}_{2,\ell})/(A^1(\mathcal{B}_{2,\ell}) \cdot A^3(\mathcal{B}_{2,\ell}) + A^2(\mathcal{B}_{2,\ell})^2)$ is generated by $t_4 := c_4(\pi_* \mathcal{E}(2))$ and $u_4 := c_4(\pi_* \mathcal{E}(3))$. These classes are determined by the splitting principle and Grothendieck–Riemann–Roch, and one can calculate them quickly using the Schubert2 package in Macaulay2. When $\ell = 0$, we have

$$\begin{aligned} t_4 = &-\frac{1}{4} a_1^3 a_2' + \frac{11}{24} a_1^2 a_2'^2 - \frac{1}{4} a_1 a_2'^3 + \frac{1}{24} a_2'^4 + 3a_1^3 w_1 - 4a_1^2 a_2' w_1 + a_1 a_2'^2 w_1 + \frac{1}{12} a_1'^3 w_1 + 15 a_1^2 w_1^2 \\ &- \frac{25}{4} a_1 a_2' w_1^2 + \frac{13}{24} a_2'^2 w_1^2 + 18 a_1 w_1^3 - \frac{3}{2} a_2' w_1^3 + 4w_1^4 + 3a_1^2 a_2 - 4a_1 a_2 a_2' + \frac{7}{6} a_2 a_2'^2 + 18 a_1 a_2 w_1 \\ &- 5a_2 a_2' w_1 + 18 a_2 w_1^2 + 12 a_1^2 w_2 + a_1 a_2' w_2 - \frac{19}{6} a_2'^2 w_2 + 48 a_1 w_1 w_2 + 2a_2' w_1 w_2 + 40 w_1^2 w_2 + 3a_2^2 + 16 w_2^2, \\ u_4 = &\, a_1^4 - \frac{25}{12} a_1^3 a_2' + \frac{35}{24} a_1^2 a_2'^2 - \frac{5}{12} a_1 a_2'^3 + \frac{1}{24} a_2'^4 + 30 a_1^3 a w_1 - \frac{53}{2} a_1^2 a_2' w_1 + \frac{20}{3} a_1 a_2'^2 w_1 - \frac{5}{12} a_2'^3 w_1 + 185 a_2^2 w_1^2 \\ &- \frac{845}{12} a_2 a_2' w_1^2 + \frac{133}{24} a_2'^2 w_1^2 + 360 a_1 w_1^3 - 45 a_2' w_1^3 + 193 w_1^4 + 12 a_1^2 a_2 - \frac{55}{6} a_1 a_2 a_2' + \frac{5}{3} a_2 a_2'^2 + 90 a_1 a_2 w_1 - \frac{43}{2} a_2 a_2' w_1 \\ &+ 130 a_2 w_1^2 + 70 a_1^2 w_2 - \frac{65}{6} a_1 a_2' w_2 - \frac{11}{3} a_2'^2 w_2 + 450 a_1 w_1 w_2 - \frac{45}{2} a_2' w_1 w_2 + 616 w_1^2 w_2 + 6a_2^2 + 20 a_2 w_2 + 118 w_2^2. \end{aligned}$$

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From this, we observe that $t_4$ and $u_4$ are expressible as the sum of a class in

$$\mathbb{Z}[a_1, a_2', a_2, w_1, w_2]_4 \subset A^1(\mathcal{B}_{2,\ell}) \cdot A^3(\mathcal{B}_{2,\ell}) + A^2(\mathcal{B}_{2,\ell})^2,$$

plus a class divisible by $a_2'$. That is, when $\ell = 0$, we can choose generators of

$$A^4(\mathcal{B}_{2,\ell})/(A^1(\mathcal{B}_{2,\ell}) \cdot A^3(\mathcal{B}_{2,\ell}) + A^2(\mathcal{B}_{2,\ell})^2),$$

so that a multiple lies in $A^1(\mathcal{B}_{2,\ell}) \cdot A^3(\mathcal{B}_{2,\ell})$. On the other hand, if $\ell = 1$, then

$$t_4 = \frac{35}{8} a_2^2 + \cdots \qquad \text{and} \qquad u_4 = \frac{63}{8} a_2^2 + \cdots.$$

However, all codimension 2 classes have denominators at most 2. Thus, there is no adjustment of $t_4$ or $u_4$ by integral classes from lower codimension so that the result is divisible by a codimension 1 class. ∎

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