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Semigroup algebras of finite ample semigroups

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An adequate semigroup S is called *ample* if $ea = a(ea)^*$ and $ae = (ae)^\dagger a$ for all $a \in S$ and $e \in E(S)$. Inverse semigroups are exactly those ample semigroups that are regular. After obtaining some characterizations of finite ample semigroups, it is proved that semigroup algebras of finite ample semigroups have generalized triangular matrix representations. As applications, the structure of the radicals of semigroup algebras of finite ample semigroups is obtained. In particular, it is determined when semigroup algebras of finite ample semigroup are semiprimitive.

1. Introduction

The relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semigroup S are generalizations of the familiar Green relations $\mathcal{L}$ and $\mathcal{R}$. Two elements a and b in S are said to be $\mathcal{L}^*$*-related* if and only if they are $\mathcal{L}$-related in some oversemigroup of S; the relation $\mathcal{R}^*$ can be defined dually. A semigroup is *abundant* if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains at least one idempotent, and *adequate* if it is an abundant semigroup whose set of idempotents forms a semilattice. In [4], it is pointed out that each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class of an adequate semigroup contains precisely one idempotent. For convenience, we use $a^\dagger$ (a^*) to denote the idempotent in the $\mathcal{R}^*$-class ($\mathcal{L}^*$-class) containing a, and write the set of idempotents of S as $E(S)$. An adequate semigroup S is called *ample* if for every $a \in S$ and $e \in E(S)$, $ea = a(ea)^*$ and $ae = (ae)^\dagger a$. Ample semigroups were formerly called *type-A semigroups*. Inverse semigroups are ample semigroups, and all regular elements of an ample semigroup form an inverse subsemigroup [4]. It is interesting that any (finite) ample semigroup can be embedded into some (finite) inverse semigroup (see the proof of [3, proposition 1.2] and [6]). The study of ample semigroups and their generalizations goes back to the 1960s, and since then a number of papers on this topic have appeared. The structure of ample semigroups and related inverse semigroups were investigated by Lawson in [14, 16]. Recently, it has been observed that these classes of semigroups are closely related to topics in theoretical computing [10].

Inverse semigroup algebras are a class of semigroup algebras that has been widely investigated. In [18], Munn considered the semiprimitivity of combinatorial inverse semigroup algebras. He proved that the semigroup algebra of a combinatorial inverse semigroup over a field is semiprimitive (that is, semisimple in the sense of Jacobson). This shows that the free inverse semigroup must be semiprimitive. In [20], Munn investigated nil ideals in inverse semigroup algebras. It was shown that if S

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is an inverse semigroup and F is a field of characteristic 0 or a prime that is not the order of an element in a subgroup of S, then $F[S]$ has no non-zero nil ideals. More results on inverse semigroup algebras were collected in a survey by Munn [19]. For the related results on semigroup algebras, the reader is referred to [12, 21]. Steinberg [22, 23] used Möbius functions to study finite inverse semigroup algebras. We found that Möbius functions are very useful in the study of finite inverse semigroup algebras. Recently, Guo [7] researched the semigroup algebras of finite locally inverse semigroups, which include finite inverse semigroups as proper subclass, by using Möbius functions.

Ample semigroups are generalizations of inverse semigroups. It is natural to probe ample semigroup algebras. In this paper we shall probe semigroup algebras of finite ample semigroups. We proceed as follows: after listing some known results, we give some properties of finite ample semigroups. In § 4, we prove that any semigroup algebra of finite ample semigroups has generalized triangular matrix representations. Finally, we consider the structure of Jacobson radicals of semigroup algebras of finite ample semigroups. In particular, it is proved that if S is a finite ample semigroup and R is a commutative ring with identity, then $R[S]$ is semiprimitive if and only if

(i) S is an inverse semigroup and

(ii) for all maximum subgroups G of S, $R[G]$ is semiprimitive.

Throughout this paper we use the notation and terminologies from [2, 5]. For other definitions, the reader is referred to [11].

2. Preliminaries

In this section we give some results on semigroups and semigroup algebras.

2.1. Primitive semigroups

We shall provide some known results on semigroups, in particular those about primitive abundant semigroups. These results will be repeatedly used in the following.

To begin with, we recall some known facts about $\mathcal{L}^*$ and the dual for $\mathcal{R}^*$.

Lemma 2.1 (Fountain [5]). *Let S be a semigroup and $a, b \in S$. Then the following statements are equivalent:*

(i) $a \,\mathcal{L}^*\, b$;

(ii) *for all* $x, y \in S^1$, $ax = ay$ *if and only if* $bx = by$.

Corollary 2.2 (Fountain [5]). *Let S be a semigroup and $a, e^2 = e \in S$. Then the following statements are equivalent:*

(i) $a \,\mathcal{L}^*\, e$;

(ii) $ae = a$ *and, for all* $x, y \in S^1$, $ax = ay$ *implies that* $ex = ey$.

It is well known that $\mathcal{L}^*$ is a right congruence, while $\mathcal{R}^*$ is a left congruence. In general, $\mathcal{L} \subseteq L^*$ and $\mathcal{R} \subseteq R^*$. But when a and b are regular, $a \mathcal{L} b$ if and only if $a \mathcal{L}^* b$. Following [5], we define $\mathcal{D}^*$ as the smallest equivalence on S containing $\mathcal{L}^*$ and $\mathcal{R}^*$, and $\mathcal{H}^*$ as the intersection of $\mathcal{L}^*$ and $\mathcal{R}^*$. Note that $\mathcal{L}^*$ and $\mathcal{R}^*$ do not, in general, commute, so that we may not have $\mathcal{D}^* = \mathcal{L}^* \circ \mathcal{R}^*$.

The following lemma, which will be used without proof, is due to Fountain [4].

Lemma 2.3. *Let S be an adequate semigroup and $a, b \in S$. Then $(ab^\dagger)^\dagger = (ab)^\dagger$ and $(a^*b)^* = (ab)^*$. Moreover, if $e \in E(S)$, then $ea^\dagger = (ea)^\dagger$ and $a^*e = (ae)^*$.*

Let S be a semigroup and $a \in S$. The $\mathcal{L}^*$-class containing a will be denoted by L^*_a or $L^*_a(S)$ in case of ambiguity. The corresponding notation will be used for the classes of the other relations. We now define *left (right) $*$-ideal* of S to be a left (right) ideal I of S such that $L^*_a \subseteq I$ ($R^*_a \subseteq I$) for all $a \in I$. A subset I of S is a *$*$-ideal* of S if it is both a left $*$-ideal and a right $*$-ideal. We note that if S is regular, then every left (right, two-sided) ideal of S is a left (right, two-sided) $*$-ideal. As pointed out in [5], there exists a smallest left (right, two-sided) $*$-ideal $L^*(a)$ ($R^*(a)$, $J^*(a)$) containing a. We shall call $L^*(a)$ ($R^*(a), J^*(a)$) the *principal left $*$-ideal (principal right $*$-ideal, principal $*$-ideal)* generated by a. It is clear that $L^*(a) \subseteq J^*(a)$ and $R^*(a) \subseteq J^*(a)$. In [5], Fountain proved that, for $a, b \in S$, $a \mathcal{L}^*(\mathcal{R}^*)\, b$ if and only if $L^*(a) = L^*(b)$ ($R^*(a) = R^*(b)$). We define the relation on S by the rule that $a \mathcal{J}^* b$ if and only if $J^*(a) = J^*(b)$. In general, $\mathcal{L}^*, \mathcal{R}^*, \mathcal{D}^* \subseteq \mathcal{J}^*$.

An idempotent e of S is called *primitive* if for all $f \in E(S)$, $f \leqslant e$ implies that $e = f$ or $f = 0$ if S has 0. S is called *primitive* if idempotents of S are all primitive.

A semigroup S with a zero element 0 is called *0-$\mathcal{J}^*$-simple* if the only $*$-ideals of S are S, $\{0\}$, and $S^2 \neq \{0\}$. It is easy to see that S is 0-$\mathcal{J}^*$-simple if and only if $S^2 \neq \{0\}$, and $\{0\}$, $S \setminus \{0\}$ are the only $\mathcal{J}^*$-classes.

Let I, Λ be non-empty sets and let Y be a non-empty set indexing partitions $P(I) = \{I_\gamma \colon \gamma \in Y\}$, $P(\Lambda) = \{\Lambda_\gamma ; \gamma \in Y\}$ of I and Λ, respectively. For each pair $(\alpha, \beta) \in Y \times Y$, let $M_{\alpha\beta}$ be a set such that, for each α, $M_{\alpha\alpha} = T_\alpha$ is a monoid, and for $\alpha \neq \beta$ either $M_{\alpha\beta} = \emptyset$ or $M_{\alpha\beta}$ is a (T_α, T_β)-bisystem. We let 0 be a symbol not in any $M_{\alpha\beta}$. By the *(α, β)-block* of an $I \times \Lambda$ matrix, we mean those (i, λ) positions with $i \in I_\alpha$, $\lambda \in \Lambda_\beta$. The (α, α)-blocks are called the *diagonal blocks* of the matrix. We denote by T the set consisting of the zero $I \times \Lambda$ matrix together with all $I \times \Lambda$ matrices with a single non-zero entry, where a non-zero entry in the (α, β)-block is a member of $M_{\alpha\beta}$. Following the usual convention, $A = (a)_{i\lambda}$ will denote the $I \times \Lambda$ matrix with entry a in the (i, λ) position and 0 elsewhere. For any $A = (a)_{i\lambda}, B = (b)_{j\mu} \in T$ we define a multiplication $\circ$ on T by $A \circ B = APB = (ap_{\lambda j}b)_{i\mu}$, where the sandwich matrix $P = (p_{\lambda i})$ is a $\Lambda \times I$ matrix where a non-zero entry in the (α, β)-block of P is a member of $M_{\alpha\beta}$. It is easy to check that T is a semigroup, which we denote by $\mathcal{M}(M_{\alpha\beta}; I, \Lambda, Y; P)$ and call a *blocked Rees matrix semigroup*. By a *primitive abundant (PA) blocked Rees matrix semigroup* we mean one which satisfies the following additional conditions.

(C) If $a, a_1, a_2 \in M_{\alpha\beta}$, $b, b_1, b_2 \in M_{\beta\gamma}$, then $ab_1 = ab_2$ implies $b_1 = b_2$; $a_1 b = a_2 b$ implies $a_1 = a_2$.

(U) For each $\alpha \in Y$ and each $\lambda \in \Lambda_\alpha$ ($i \in I_\alpha$) there is a member i of I_α (λ of Λ_α) such that $p_{\lambda i}$ is a unit in $M_{\alpha\alpha}$.

(R) If $M_{\alpha\beta}, M_{\beta\alpha}$ are both non-empty where $\alpha \neq \beta$, then $aba \neq a$ for all $a \in M_{\alpha\beta}$, $b \in M_{\beta\alpha}$.

We observe that condition (C) forces each T_α to be a cancellative monoid and each $M_{\alpha\beta}$ to be a strongly torsion-free (T_α, T_β)-bisystem. Condition (R) is imposed to ensure that a matrix with a non-zero entry in a non-diagonal block cannot be a regular element of the matrix semigroup. When $|\Gamma| = 1$ and $G := M_{\alpha\alpha}$ is a group, $\mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ is the usual Rees matrix semigroup over the 0-group $M^0_{\alpha\alpha}$ [11]. As usual, we denote this kind of PA blocked Rees matrix semigroup by $\mathcal{M}^0(G; I, \Lambda; P)$.

The following lemma gives the properties of PA blocked Rees matrix semigroups.

Lemma 2.4 (Fountain [5, proposition 2.4]). *Let $S = \mathcal{M}(M_{\alpha\beta}; I, \Lambda, \Gamma; P)$ be a PA blocked Rees matrix semigroup. Then we have the following.*

(i) *A non-zero element $(a)_{i\lambda}$ of S is idempotent if and only if there is an $\alpha \in \Gamma$ such that $(i, \lambda) \in I_\alpha \times \Gamma_\alpha$ and a is a unit in T_α with inverse $p_{\lambda i}$.*

(ii) *The non-zero elements $(a)_{i\lambda}$ and $(b)_{j\mu}$ of S are $\mathcal{R}^*$-related if and only if $i = j$.*

(iii) *The non-zero elements $(a)_{i\lambda}$ and $(b)_{j\mu}$ of S are $\mathcal{L}^*$-related if and only if $\lambda = \mu$.*

(iv) *The non-zero idempotents $e = (a)_{i\lambda}$, $f = (b)_{j\mu}$ of S with $(i, \lambda) \in I_\alpha \times \Lambda_\alpha, (j, \mu) \in I_\beta \times \Lambda_\beta$ are $\mathcal{D}$-related if and only if $\alpha = \beta$.*

(v) *The non-zero element $(a)_{i\lambda}$ of S is regular if and only if there is an $\alpha \in \Gamma$ such that $(i, \lambda) \in I_\alpha \times \Lambda_\alpha$ and a is a unit in T_α.*

Lemma 2.5 (Fountain [5, proposition 5.5]). *For a semigroup S with a zero element, the following conditions are equivalent:*

(i) *S is a primitive adequate semigroup;*

(ii) *S is isomorphic to $\mathcal{M}(M_{\alpha\beta}; I, I, \Gamma; P)$, a PA blocked Rees matrix $I \times I$ matrix semigroup, where the sandwich matrix P is diagonal and $p_{ii} = e_\alpha$ for all $i \in I_\alpha$, $\alpha \in \Gamma$.*

A semigroup S with a zero element 0 is called a *weak Brandt semigroup* when the following conditions are satisfied:

(B1) if $a, b, c \in S$ such that $ac = bc \neq 0$, then $a = b$;

(B2) if $a, b, c \in S$ such that $ab \neq 0$ and $bc \neq 0$, then $abc \neq 0$;

(B3) for each $a \in S$ there is $e \in S$ such that $ea = a$ and $f \in S$ such that $af = a$;

(B4) if e and f are non-zero idempotents of S, there are non-zero idempotents $e_1, e_2, \ldots, e_n \in S$ with $e_1 = e$, $e_n = f$ such that, for each $i = 1, 2, \ldots, n-1$, one of $e_i S e_{i+1}$ and $e_{i+1} S e_i$ is non-zero.

It is evident that weak Brandt semigroups are generalizations of Brandt semigroups. In precis, a Brandt semigroup is just a weak Brandt semigroup which is regular. In [4], it is pointed out that a semigroup is a weak Brandt semigroup if and only if it is isomorphic to some PA blocked Rees matrix semigroup $\mathcal{M}(M_{\alpha\beta}; I, I, Y; P)$ in which

1. the sandwich matrix P is diagonal,
2. for every $i \in I_\alpha$, $\alpha \in \Gamma$, we have $p_{ii} = e_\alpha$ (here e_α is the identity of M_α) and
3. for each pair $(\alpha, \beta) \in \Gamma \times \Gamma$, there is a finite sequence $\alpha(1), \dots, \alpha(n)$ of members of Γ such that $\alpha = \alpha(1)$, $\beta = \alpha(n)$ and for each $i = 1, 2, \dots, n-1$ at least one of $M_{\alpha(i),\alpha(i+1)}$, $M_{\alpha(i+1),\alpha(i)}$ is non-empty.

In fact, S is a weak Brandt semigroup if and only if it is a 0-$\mathcal{J}^*$-simple primitive adequate semigroup [5, corollary 5.6].

For our purpose, we need the following known result due to [17].

Lemma 2.6 (Liu and Yao [17, theorem 2.2]). *A weak Brandt semigroup S is finite if and only if S is isomorphic to $\mathcal{M}(M_{\alpha\beta}; I, I, Y; P)$, a PA blocked Rees matrix semigroup in which I is finite, Γ is a finite well-ordered set and, for any $\alpha, \beta \in Y$,*

(i) *$M_{\alpha\alpha}$ is a finite group,*

(ii) *$|M_{\alpha\beta}| \leqslant +\infty$,*

(iii) *if $M_{\alpha\beta} \neq \emptyset$, then $\alpha \leqslant \beta$,*

(iv) *the sandwich matrix P is diagonal and $p_{ii} = e_\alpha$ for all $i \in I_\alpha, \alpha \in \Gamma$.*

Assume $\{S_\alpha : \alpha \in A\}$ is a family of semigroups each with a zero element. We denote all zero elements by the same symbol, '0', and form the set S, which consists of 0 together with the disjoint union of all $S_\alpha \setminus \{0\}$. Clearly, S becomes a semigroup if we define the product of x and y in S to be their product in S_α if they are from the same semigroup S_α and to be 0 otherwise. Thus, $S_\alpha S_\beta = \{0\}$ if $\alpha \neq \beta$. We call the semigroup S the 0-*direct union* of the semigroups S_α with $\alpha \in A$.

Fountain pointed out that a primitive abundant semigroup with a zero element is a 0-direct union of primitive abundant 0-$\mathcal{J}^*$-simple semigroups (see the arguments after theorem 4.4 in [5, p. 19]). This shows that a primitive adequate semigroup with a zero element is a 0-direct union of 0-$\mathcal{J}^*$-simple adequate semigroups. Associated with [5, corollary 5.6], we immediately have the following lemma.

Lemma 2.7. *A semigroup is a primitive adequate semigroup if and only if it is the 0-direct union of weak Brandt semigroups.*

Based on lemmas 2.5–2.7, the following corollary is immediate.

Corollary 2.8. *A semigroup S is a finite primitive adequate semigroup if and only if S is isomorphic to $\mathcal{M}(M_{\alpha\beta}; I, I, Y; P)$, a PA blocked Rees matrix semigroup in which I is finite, Γ is a finite well-ordered set and, for any $\alpha, \beta \in \Gamma$,*

(i) *$M_{\alpha\alpha}$ is a finite group,*

(ii) *$|M_{\alpha\beta}| \leqslant +\infty$,*

(iii) *if $M_{\alpha\beta} \neq \emptyset$, then $\alpha \leqslant \beta$,*

(iv) *the sandwich matrix P is diagonal and $p_{ii} = e_\alpha$ for all $i \in I_\alpha$, $\alpha \in \Gamma$.*

2.2. Semigroup algebras

Now, we introduce some notation related to semigroup algebras.

Let S be a semigroup. We use $R[S]$ to denote the semigroup algebra of the semigroup S over R. In general, if I is a subset of S, we shall denote by $R[I]$ the set of R-linear combinations of elements in I, that is, $R[I]$ is a free R-module with the I as a basis, so each element of $R[I]$ is a finite summation of the form

$$\sum_{x\in I} r_x x, \quad r_x \in R.$$

In particular, if I_1 and I_2 are subsets of S, then $R[I_1 \cap I_2] = R[I_1] \cap R[I_2]$. If S is a semigroup with a zero element θ, then $R[\theta]$ is an ideal of $R[S]$. We set $R_0[S] = R[S]/R[\theta]$ and call it the *contracted semigroup algebra* of S over R. If S has no zero, we define $R_0[S] = R[S]$. Clearly, an element a of $R_0[S]$ can be represented by finite linear combinations $a = \sum r_s s$ of elements $s \in S \setminus \{\theta\}$. The support of $a \in R_0[S]$, denoted by $\operatorname{supp}(a)$, is the set $\{s \in S \setminus \{\theta\} \mid r_s \neq 0\}$.

3. Finite ample semigroups

In this section we investigate finite ample semigroups.

Let S be an abundant semigroup and $a, b \in S$. Define

$$a \leqslant b \iff \text{there exist } e, f \in E(S) \text{ such that } a = eb = bf.$$

Then $\leqslant$ is a partial order on S (see [15]).

Lemma 3.1. *Let S be an ample semigroup and $a, b \in S$.*

(i) *$a \leqslant b$ if and only if $a = ba^*$ if and only if $a = a^\dagger b$.*

(ii) *For any $u \leqslant ab$, there exist $x \leqslant a$ and $y \leqslant b$ such that $u = xy, x \in R^*_u$ and $y \in L^*_u$.*

Proof. (i) We only prove the first part because the proof of the second part is dual to the first. If $a \leqslant b$, then there exists $e \in E(S)$ such that $a = eb$; hence, $a = b(eb)^* = ba^*$ since S is ample. For the converse, we assume $a = ba^*$. Then $a = ba^* = (ba^*)^\dagger b$, since S is ample. By definition, $a \leqslant b$.

(ii) If $u \leqslant ab$, then, by (i), $u = u^\dagger(ab) = (ab)u^* = (u^\dagger a)(bu^*)$. By the first equality, $u = u^\dagger a^\dagger (ab) = a^\dagger u$ and so, by lemma 2.3, $u^\dagger = (a^\dagger u)^\dagger = a^\dagger u^\dagger$. Thus, $u^\dagger a \,\mathcal{R}^*\, u^\dagger a^\dagger = u^\dagger \,\mathcal{R}^*\, u$. On the other hand, by (i), $u^\dagger a \leqslant a$. Dually, $bu^* \leqslant b$ and $bu^* \,\mathcal{L}^*\, u$. □

Proposition 3.2. *Let S be an ample semigroup. Then $\leqslant$ is compatible with the multiplication.*

Proof. Let $a, b, c \in S$ and $a \leqslant b$. Since S is an ample semigroup, we have $a = a^\dagger b$, so

$$ac = a^\dagger bc = a^\dagger (bc)^\dagger bc = (a^\dagger bc)^\dagger bc = (ac)^\dagger bc,$$

thereby $ac \leqslant bc$; similarly, $ca \leqslant cb$. Thus, $\leqslant$ is compatible with the multiplication. □

PROPOSITION 3.3. *If S is a finite ample semigroup, then $\mathcal{D}^* = J^*$.*

Proof. Since S is finite, it is clear that there do not exist infinite chains in S with respect to $\leqslant$. Note that a semigroup is ample if and only if it is an adequate semigroup which is idempotent-connected. Again, by [9, corollary 3.13], $\mathcal{D}^* = J^*$. □

Define the relations of the set of $\mathcal{J}^*$-classes of S by

$$J_x^* \leqslant J_y^* \iff J^*(x) \subseteq J^*(y).$$

It can be easily seen that the above relation is a partial order on the set of $\mathcal{J}^*$-classes. Let $a \in S$ and form the set $I^*(a) = \{b \in J^*(a) \colon J_b^* < J_a^*\}$. By routine computation, $I^*(a)$ is a $*$-ideal of S. We now call the Rees quotient $J^*(a)/I^*(a) = J_a^* \cup \{0\}$ of $J^*(a)$ over $I^*(a)$ the *principal $*$-factor of S containing a*. By [8, lemma 2.2] and its proof, it is indeed proved that if S is an abundant semigroup, then every principal $*$-factor of S is a 0-$\mathcal{J}^*$-simple semigroup which is abundant. Moreover, we can show the following.

PROPOSITION 3.4. *Let S be a finite ample semigroup and $a \in S$. Then $J^*(a)/I^*(a)$ is a weak Brandt semigroup.*

Proof. Let $a \in S$. Then $J^*(a)/I^*(a)$ is 0-$\mathcal{J}^*$-simple. Note that S is adequate; it is easy to see that $J^*(a)/I^*(a)$ is adequate.

It remains to verify that each idempotent of $J^*(a)/I^*(a)$ is primitive. For this, we need only to show that, for all $e, f \in E(S)$ such that $e \mathrel{\mathcal{J}^*} f$, $e \leqslant f$ implies that $e = f$. This follows from [9, theorem 4.4]. □

Assume S is a finite ample semigroup. Now we define a multiplication $\odot$ on $S^0 = S \cup \{0\}$ by

$$x \odot y = \begin{cases} xy & \text{if } x \neq 0,\ y \neq 0 \text{ and } y, xy \in J_x^*, \\ 0 & \text{otherwise}, \end{cases} \tag{3.1}$$

where xy is the product of x and y in S. Clearly, $(S^0, \odot)$ is a semigroup. We denote by $S^{\odot}$ the semigroup $(S^0, \odot)$. For any $J^* \in S/\mathcal{J}^*$, we define $J^{*0} = J^* \cup \{0\}$. It is easy to check that $(J^{*0}, \odot)$ is a subsemigroup of $S^{\odot}$, which is isomorphic to the principal $*$-factor of S determined by J^*. We will denote the semigroup $(J^{*0}, \odot)$ by $J^{*\odot}$. By the definition of $\odot$, it is easy to see that, in the semigroup $S^{\odot}$,

(i) $J_x^{*\odot} \odot J_x^{*\odot} \subseteq J_x^{*\odot}$ for all $x \in S$,

(ii) $J_x^{*\odot} \odot J_y^{*\odot} = 0$ for all $x, y \in S$ such that $x \notin J_y^*$.

We have now proved that the semigroup $S^{\odot}$ is the 0-direct union of the weak Brandt semigroups $J^{*\odot}$ with $J^* \in S/\mathcal{J}^*$. By lemma 2.7 and proposition 3.4, we have the following.

PROPOSITION 3.5. *The above semigroup $S^{\odot}$ is a primitive adequate semigroup.*

We shall call the primitive adequate semigroup $S^{\odot}$ *the associate semigroup* of S.

4. Triangular matrix representations

Let $R_1, R_2, \dots, R_n$ be associative rings (algebras) with identity and let R_{ij} be a left R_i- right R_j-bimodule for $i, j = 1, 2, \dots, n$ and $i < j$. We call the formal $n \times n$ matrix

$$\begin{pmatrix} a_1 & a_{12} & \cdots & a_{1n} \\ 0 & a_2 & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$$

with $a_i \in R_i$, $a_{ij} \in R_{ij}$ for $i, j = 1, 2, \dots, n$ a *generalized upper triangular* $n \times n$ *matrix*. Denote the set of all generalized upper triangular matrices by

$$\begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}.$$

As in [1], a ring (an algebra) A has a *generalized triangular matrix representation* if there exists a ring (an algebraic) isomorphism

$$\phi\colon A \to \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix}$$

in which the matrices obey the usual rules for matrix addition and multiplication.

The main aim of this section is to prove that any semigroup algebra of any finite ample semigroup has a generalized triangular matrix representation.

We need the Möbius inversion theorem [22].

LEMMA 4.1. *Let $(P, \leqslant)$ be a locally finite partially ordered set (i.e. intervals are finite) in which each principal ideal has a maximum and let G be an Abelian group. Suppose that $f\colon P \to G$ is a function and define $g\colon P \to G$ by $g(x) = \sum_{y \leqslant x} f(y)$. Then $f(x) = \sum_{y \leqslant x} g(y)\mu(x, y)$, where μ is a Möbius function (a function from $P \times P$ into R).*

THEOREM 4.2. *Let S be a finite ample semigroup and R a commutative ring. Then $R[S]$ is isomorphic to $R_0[S^{\odot}]$, where $S^{\odot}$ is the associate semigroup of S.*

Proof. For convenience, we introduce the semigroup $\bar{S}$. Set $\bar{S} = \{\bar{x} \mid x \in S\} \cup \{0\}$. Define a multiplication on $\bar{S}$ as follows:

$$\bar{x} \star \bar{y} = \overline{x \odot y}, \tag{4.1}$$

where we identify $\bar{0}$ with 0. It is easy to see that $\bar{S}$ is isomorphic to $S^{\odot}$. Hence, the contracted semigroup algebra $R_0[\bar{S}]$ is isomorphic to the contracted semigroup algebra $R_0[S^{\odot}]$. For $J^* \in S/\mathcal{J}^*$, we define

$$\overline{J^*} = \{\bar{x} \mid x \in J^*\} \cup \{0\}.$$

It is easy to check that $(\overline{J^*}, \star)$ is a subsemigroup of $\bar{S}$ isomorphic to the semigroup $J^{*\odot}$. So for any $J^*, K^* \in S/\mathcal{J}^*$, we have

$$\left.\begin{aligned} \overline{J^*} \star \overline{K^*} &\subseteq \overline{J^*} \quad \text{if } J^* = K^*, \\ \overline{J^*} \star \overline{K^*} &= 0 \qquad \text{if } J^* \neq K^*. \end{aligned}\right\} \tag{4.2}$$

Consider the mapping $\psi\colon R[S] \to R_0[\bar{S}]$ given on the basis by $\psi(s) = \sum_{t \leqslant s} \bar{t}$ and the mapping $\bar{\cdot}\colon S \to \bar{S}$ given by $\bar{\cdot}(s) = \bar{s}$. Clearly, ψ and $\bar{\cdot}$ are well defined. Of course, ψ and $\bar{\cdot}$ may be regarded as the mapping of the ordered set $(S, \leqslant)$ into the additive group of $R_0[\bar{S}]$. Now, by applying the Möbius inversion theorem (lemma 4.1) to the mappings ψ and $\bar{\cdot}$, we have

$$\bar{s} = \sum_{t \leqslant s} \psi(t)\mu(s,t) = \psi\biggl(\sum_{t \leqslant s} t\mu(s,t)\biggr), \tag{4.3}$$

where μ is the Möbius function for $(S, \leqslant)$. Hence, ψ is surjective.

In the following we will prove that ψ is injective. For any $\alpha_0 = \sum_{x \in S} p_x^0 x \in R[S]$, we denote by $\operatorname{supp}(\alpha_0)$ the set $\{x \in S \mid p_x^0 \neq 0\}$ and by $M(\alpha_0)$ the set of maximal elements in the set $\operatorname{supp}(\alpha_0)$ with respect to the partial order $\leqslant$. We define inductively

$$\alpha_n = \alpha_{n-1} - \sum_{x \in M(\alpha_{n-1})} p_x^{n-1} x, \quad \text{where } \alpha_n = \sum_{x \in \operatorname{supp}(\alpha_n)} p_x^n x.$$

Let

$$\beta_n = \sum_{x \in \operatorname{supp}(\beta_n)} q_x^n x \quad \text{with } n = 0, 1, 2, \dots.$$

If $\psi(\alpha_n) = \psi(\beta_n)$, then, by the definition of ψ,

$$\sum_{x \in M(\alpha_n)} p_x^n \bar{x} + \Gamma_{\alpha_n} = \psi(\alpha_n) = \psi(\beta_n) = \sum_{y \in M(\beta_n)} q_y^n \bar{y} + \Gamma_{\beta_n},$$

where

$$\Gamma_{\alpha_n} = \sum_{x \in M(\alpha_n)} \sum_{y \in S, y < x} p_y^n \bar{y} \quad \text{and} \quad \Gamma_{\beta_n} = \sum_{x \in M(\beta_n)} \sum_{y \in S, y < x} q_y^n \bar{y},$$

and hence

$$\sum_{x \in M(\alpha_n)} p_x^n \bar{x} = \sum_{x \in M(\beta_n)} q_x^n \bar{x};$$

thus, $M(\alpha_n) = M(\beta_n)$ and $p_x^n = q_x^n$ for any $x \in M(\alpha_n)$, which implies the following.

Fact 1. If $\psi(\alpha_n) = \psi(\beta_n)$, then $M(\alpha_n) = M(\beta_n)$ and by the definition of ψ, $\psi(\alpha_{n+1}) = \psi(\beta_{n+1})$.

By the definition of ψ, the following facts are immediate.

Fact 2. $\alpha_n = \beta_n$ if and only if $M(\alpha_n) = M(\beta_n)$ and $\alpha_{n+1} = \beta_{n+1}$.

Fact 3. If $\psi(\alpha_n) = \psi(\beta_n)$ and $M(\alpha_n) = \operatorname{supp}(\alpha_n)$, $M(\beta_n) = \operatorname{supp}(\beta_n)$, then $\alpha_n = \beta_n$.

Note that $|\mathrm{supp}(\alpha_0)| < +\infty$ and $\mathrm{supp}(\alpha_{n+1}) \subseteq \mathrm{supp}(\alpha_n)$. We thus have a smallest integer k such that $M(\alpha_k) = \mathrm{supp}(\alpha_k)$. Clearly, $\alpha_{k+1} = 0$. This means that k is the smallest integer t such that $\alpha_{t+1} = 0$. Similarly, there exists the smallest integer l such that $\beta_{l+1} = 0$ and $M(\beta_l) = \mathrm{supp}(\beta_l)$. Now, assume $\psi(\alpha_0) = \psi(\beta_0)$. By using fact 1 repeatedly, we obtain

$$\psi(\alpha_1) = \psi(\beta_1),\ \psi(\alpha_2) = \psi(\beta_2),\ \dots,\ \psi(\alpha_{k+1}) = \psi(\beta_{k+1}). \tag{4.4}$$

But $\psi(\alpha_{k+1}) = 0$, we have $\psi(\beta_{k+1}) = 0$ and, by the definition of ψ, $\beta_{k+1} = 0$. Thus, $k+1 \geqslant l+1$ by the minimality of l, and so $k \geqslant l$. Similarly, $l \geqslant k$. Therefore, $k = l$. Since $\psi(\alpha_k) = \psi(\beta_k)$, by fact 3, we have $\alpha_k = \beta_k$ since $M(\alpha_k) = \mathrm{supp}(\alpha_k)$, $M(\beta_l) = \mathrm{supp}(\beta_l)$. Again by the hypothesis $\psi(\alpha_0) = \psi(\beta_0)$ and by fact 1, $M(\alpha_0) = M(\beta_0)$ and, by (4.4),

$$M(\alpha_1) = M(\beta_1),\ M(\alpha_2) = M(\beta_2),\ \dots,\ M(\alpha_k) = M(\beta_k).$$

By fact 2, $M(\alpha_{k-1}) = M(\beta_{k-1})$ and $\alpha_k = \beta_k$ imply $\alpha_{k-1} = \beta_{k-1}$; moreover, by using fact 2 repeatedly, $\alpha_{k-2} = \beta_{k-2}, \dots, \alpha_1 = \beta_1$ and $\alpha_0 = \beta_0$. We have now proved that ψ is injective.

Finally, for any $s, t \in S$, by (4.2), we have

$$\bar{s} \star \bar{t} = \begin{cases} \overline{st} & \text{if } s, t \in J^*_{st}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.5}$$

and, by (4.2) and lemma 3.1(ii),

$$\begin{aligned} \psi(s) \star \psi(t) &= \Big(\sum_{x \leqslant s} \bar{x}\Big) \star \Big(\sum_{y \leqslant t} \bar{y}\Big) \\ &= \sum_{\substack{x \in J^*_{st},\\ x \leqslant s}} \sum_{\substack{y \in J^*_{st},\\ y \leqslant t}} \bar{x} \star \bar{y} \\ &= \sum_{\substack{x \in J^*_{st},\\ x \leqslant s}} \sum_{\substack{y \in J^*_{st},\\ y \leqslant t}} \overline{xy}. \end{aligned}$$

Moreover, by (4.5) and lemma 3.1(ii), we have

$$\psi(st) = \sum_{u \leqslant st} \bar{u} = \sum_{\substack{x \in J^*_{st},\\ x \leqslant s}} \sum_{\substack{y \in J^*_{st},\\ y \leqslant t}} \overline{xy} = \Big(\sum_{x \leqslant s} \bar{x}\Big) \star \Big(\sum_{y \leqslant t} \bar{y}\Big) = \psi(s) \star \psi(t).$$

Thus, ψ is a homomorphism of $R[S]$ into $R_0[\bar{S}]$. Consequently, ψ is an isomorphism of $R[S]$ onto $R_0[\bar{S}]$. □

By theorem 4.2, we have the following.

Corollary 4.3. *Any semigroup algebra of a finite ample semigroup has an identity.*

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Proof. Let S be a finite ample semigroup and let R be a commutative ring with identity. Then by theorem 4.2, $R[S]$ is isomorphic to $R_0[S^{\odot}]$. By corollary 2.8, $S^{\odot}$ is isomorphic to some PA blocked Rees matrix semigroup $\mathcal{M}(M_{\alpha\beta}; I, I, Y; P)$, where each P is a diagonal matrix, each of whose non-zero elements is the identity of $M_{\alpha\alpha}$. It is easy to check that

$$\sum_{e\in E(\mathcal{M})} e = \sum_{\alpha\in Y} \sum_{i\in I_\alpha} (e_\alpha)_{ii}$$

is the identity of the contracted semigroup algebra of $\mathcal{M}(M_{\alpha\beta}; I, I, Y; P)$, where e_α is the identity of $M_{\alpha\alpha}$. This shows that $R_0[S^{\odot}]$ has an identity, whereby $R[S]$ has an identity. □

Corollary 4.4. *Let S be a finite ample semigroup and R a commutative ring with identity. Then $R[S]$ is isomorphic to $\bigoplus_{J^*\in S/\mathcal{J}^*} R_0[J^{*\odot}]$.*

Proof. By the arguments before proposition 3.5, $S^{\odot}$ is the 0-direct union of the weak Brandt semigroups $J^{*\odot}$ with $J^* \in S/\mathcal{J}^*$. This shows easily that

$$R_0[S^{\odot}] = \bigoplus_{J^*\in S/\mathcal{J}^*} R_0[J^{*\odot}],$$

which completes the proof. □

We now arrive at the main theorem of this section.

Theorem 4.5. *Any semigroup algebra of a finite ample semigroup has a generalized triangular matrix representation.*

Proof. By theorem 4.2, we need only to prove that any contracted semigroup algebra of a finite primitive adequate semigroup has a generalized triangular matrix representation. To this end, we let S be a finite primitive adequate semigroup and R be a commutative ring with identity. By corollary 2.8, we can assume that S is a PA blocked Rees matrix semigroup $\mathcal{M}(M_{\alpha\beta}; I, I, \Gamma; P)$ in which I is finite, Γ is a finite well-ordered set and, for any $\alpha, \beta \in \Gamma$,

(i) $M_{\alpha\alpha}$ is a finite group,

(ii) $|M_{\alpha\beta}| \leqslant +\infty$,

(iii) if $M_{\alpha\beta} \neq \emptyset$, then $\alpha \leqslant \beta$,

(iv) the sandwich matrix P is diagonal and $p_{ii} = e_\alpha$ for all $i \in I_\alpha$, $\alpha \in \Gamma$.

Since I and Γ are both finite, without loss of generality we suppose that $\Gamma = \{1, 2, \dots, s\}$,

$$\begin{aligned} I_1 &= \{1, 2, \dots, n_1\}, \\ I_2 &= \{n_1 + 1, \dots, n_1 + n_2\}, \\ &\vdots \\ I_s &= \left\{\sum_{i=1}^{s-1} n_{s_i} + 1, \dots, \sum_{i=1}^{s} n_i\right\}. \end{aligned}$$

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Hence,

$$I = \left\{1, 2, \dots, \sum_{i=1}^{s} n_i\right\}.$$

Obviously, with respect to the matrix addition and the matrix scalar multiplication, the set $M_{n_i n_j}(R[M_{ij}])$ of $n_i \times n_j$ matrices over $R[M_{ij}]$ is an R-algebra. Note that $M_{ii}M_{ij}, M_{ij}M_{jj} \subseteq M_{ij}$ for $i, j \in \Gamma$. Thus, we observe that $R[M_{ij}]$ is a left $R[M_{ii}]$- right $R[M_{jj}]$-bimodule. From these, it is not difficult to check that $M_{n_i n_j}(R[M_{ij}])$ is a left $M_n(R[M_{ii}])$- right $M_{n_j}(R[M_{jj}])$-bimodule under matrix multiplication.

For $(a)_{ij} \in S$, there exist $k, l \in \Gamma$ such that $a \in M_{kl}$ and

$$\sum_{\alpha=1}^{n_{k-1}} n_\alpha < i \leqslant \sum_{\alpha=1}^{n_k} n_\alpha \quad \text{and} \quad \sum_{\alpha=1}^{n_{l-1}} n_\alpha < j \leqslant \sum_{\alpha=1}^{n_l} n_\alpha.$$

By (4.2), we have $k \leqslant l$; if $J^* \neq K^*$, $M_{kl} = \emptyset$, contrary to $a \in M_{kl}$. Set

$$\varphi((a)_{ij}) = \begin{pmatrix} A_{11} & \cdots & A_{1k} & \cdots & A_{1l} & \cdots & A_{1s} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{k1} & \cdots & A_{kk} & \cdots & A_{kl} & \cdots & A_{ks} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{l1} & \cdots & A_{lk} & \cdots & A_{ll} & \cdots & A_{ls} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{s1} & \cdots & A_{sk} & \cdots & A_{sl} & \cdots & A_{ss} \end{pmatrix},$$

where A_{pq} is an $n_p \times n_q$ zero matrix over $R[M_{pq}]$ for $p \neq k$, $q \neq l$ and A_{kl} is the $n_k \times n_l$ matrix over $R[M_{kl}]$ in which the

$$\left(i - \sum_{\alpha=1}^{n_{k-1}} n_\alpha, j - \sum_{\alpha=1}^{n_{k-1}} n_\alpha\right)$$

position is a and all other positions are 0. Obviously, $\phi\colon (a)_{ij} \mapsto \varphi((a)_{ij})$ is a mapping of S into

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & M_{n_1 n_2}(R[M_{12}]) & \cdots & M_{n_1 n_s}(R[M_{1s}]) \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & M_{n_2 m_n}(R[M_{2s}]) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}.$$

We now extend the mapping φ R-linearly to the R-algebra

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & M_{n_1 n_2}(R[M_{12}]) & \cdots & M_{n_1 n_s}(R[M_{1s}]) \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & M_{n_2 m_n}(R[M_{2s}]) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}.$$

By routine computing, we can prove that φ is an algebraic isomorphism. The proof is finished. $\square$

Now, we re-prove the following known result.

THEOREM 4.6. *Let S be a finite inverse semigroup and R be a commutative ring. Then $R[S]$ is isomorphic to the direct sum of full matrix algebras over the group algebras of the maximum subgroups of S.*

Proof. Assume S is a finite inverse semigroup. Then, in S, $\mathcal{L} = L^*$ and $\mathcal{R} = R^*$. Note that $\mathcal{D}$ is the smallest equivalence containing $\mathcal{L}$ and $\mathcal{R}$, and that $\mathcal{D}^*$ is the smallest equivalence containing $\mathcal{L}^*$ and $\mathcal{R}^*$. Then, in S, $\mathcal{D} = D^*$. Keeping the above notation, $J^{*\odot}$ is a regular semigroup. This shows that the associate semigroup $S^{\odot}$ is a regular semigroup. Thus, $S^{\odot}$ is a regular primitive adequate semigroup. By corollary 2.8, we assume $S^{\odot} = \mathcal{M}(M_{\alpha\beta}, I, I, \varGamma; P)$ with $\varGamma = \{1, 2, \dots, s\}$ and in which $M_{\alpha\alpha}$ is a finite group, and P is diagonal with $p_{ii} = e_{\alpha}$ for $\alpha \in \varGamma$, $i \in I_{\alpha}$. Now, since $S^{\odot}$ is regular, by lemma 2.4(v), we have $M_{\alpha\beta} = \emptyset$ for $\alpha \neq \beta$. By the proof of theorem 4.5, $R[S]$ is isomorphic to the algebra

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & 0 & \cdots & 0 \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}.$$

But the latter is isomorphic to the direct sum of $M_{n_i}(R[M_{ii}])$ with $i = 1, 2, \dots, s$.

It remains to verify that each M_{ii} is isomorphic to some maximum subgroup of S. But S and $S^{\odot}$ have the same maximum subgroups. Note that any maximum subgroup of $S^{\odot}$ is an $\mathcal{H}$-class of $S^{\odot}$ containing an idempotent. Thus, we observe that any maximum subgroup of $S^{\odot}$ is indeed an $\mathcal{H}^*$-class of $S^{\odot}$ containing an identity, since $S^{\odot}$ is regular, giving $\mathcal{H}^* = \mathcal{H}$ in $S^{\odot}$. By parts (ii) and (iii) of lemma 2.4, $S^{\odot}$ has exactly $\mathcal{H}^*$-classes containing an idempotent:

$$H_{ii} = \{(m)_{ii} : m \in M_{\alpha\alpha}\}, \quad i \in I_{\alpha},\ \alpha \in \varGamma,$$

where i can run over I. It is not difficult to see that H_{ii} is isomorphic to $M_{\alpha\alpha}$. Consequently, $M_{\alpha\alpha}$ is isomorphic to some maximum subgroup of S, as required. $\square$

5. Radicals

In this section we consider the (Jacobson) radicals of semigroup algebras of finite ample semigroups.

Let S be an ample semigroup and $a, b \in S$. We say *a covers b* (in notation, $b \prec a$) if $b < a$ and there is no $x \in S$ such that $b < x < a$.

For convenience, we always assume that S is a finite ample semigroup and R is a commutative ring with identity. Denote by $S^{\odot}$ the associate semigroup of S. If no other assumptions hold, we always suppose that $S^{\odot}$ is isomorphic to the PA blocked Rees matrix semigroup PA $= \mathcal{M}(M_{\alpha\beta}; I, I, \varGamma; P)$. For convenience, we identify $S^{\odot}$ with PA. Let us return to the proof of theorem 4.2 and define, using the notation

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therein,

$$\hat{a} = a - \sum_{b \prec a} b.$$

Since S is finite, $\hat{a}$ is well defined. For $x = \sum_{s\in S} r_s s \in R[S]$, we define $\hat{x} = \sum_{s\in S} r_s \hat{s}$. And, we shall define $\hat{X} = \{\hat{x}\colon x \in X\}$ for $X \subseteq R[S]$. It is not difficult to see that the mapping $\theta\colon R[\bar{S}] \to R[S]$, defined as the linear span of the mapping $\bar{s} \mapsto \hat{s}$, is the reverse mapping ψ. By the proof of corollary 4.3, we have that $\sum_{e\in E(S)} \bar{e}$ is the identity of $R[\bar{S}]$. Hence,

$$\sum_{e\in E(S)} \hat{e} = \theta\biggl(\sum_{e\in E(S)} \bar{e}\biggr)$$

is the identity of $R[S]$.

Proposition 5.1. *Let S be a finite ample semigroup and let R be a commutative ring with identity. Then $\sum_{e\in E(S)} \hat{e}$ is the identity of $R[S]$.*

Lemma 5.2. *Let $T := \mathcal{M}(M_{\alpha\beta}, I, I, \Gamma; P)$ be a finite PA blocked Rees matrix semigroup and let R be a commutative ring with identity. If $\Gamma = \{1, 2, \dots, s\}$,*

$$\begin{aligned} I_1 &= \{1, 2, \dots, n_1\},\\ I_2 &= \{n_1 + 1, \dots, n_1 + n_2\},\\ &\vdots\\ I_s &= \biggl\{\sum_{i=1}^{s-1} n_{s_i} + 1, \dots, \sum_{i=1}^{s} n_i\biggr\} \end{aligned}$$

and, for any $\alpha, \beta \in \Gamma$,

(i) *$M_{\alpha\alpha}$ is a finite group,*

(ii) *$|M_{\alpha\beta}| \leqslant +\infty$,*

(iii) *if $M_{\alpha\beta} \neq \emptyset$, then $\alpha \leqslant \beta$ and*

(iv) *the sandwich matrix P is diagonal and $p_{ii} = e_\alpha$ for all $i \in I_\alpha, \alpha \in \Gamma$,*

then the radical of the generalized triangular matrix algebra

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & M_{n_1 n_2}(R[M_{12}]) & \cdots & M_{n_1 n_s}(R[M_{1s}])\\ 0 & M_{n_2}(R[M_{22}]) & \cdots & M_{n_2 n_s}(R[M_{2s}])\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}$$

is

$$\begin{pmatrix} \overline{M_{n_1}(R[M_{11}])} & M_{n_1 n_2}(R[M_{12}]) & \cdots & M_{n_1 n_s}(R[M_{1s}])\\ 0 & \overline{M_{n_2}(R[M_{22}])} & \cdots & M_{n_2 n_s}(R[M_{2s}])\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \overline{M_{n_s}(R[M_{ss}])} \end{pmatrix},$$

where $\overline{M_{n_i}(R[M_{ii}])}$ is the radical of $M_{n_i}(R[M_{ii}])$.

Proof. Denote by E the identity of $R[T]$. Let

$$X = \begin{pmatrix} x_1 & x_{12} & \cdots & x_{1s} \\ 0 & x_2 & \cdots & x_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_s \end{pmatrix}.$$

Obviously, if X is left invertible, then $x_1, x_2, \dots, x_s$ are all left invertible. Conversely, assume $x_1, x_2, \dots, x_s$ are all left invertible. Then the matrix

$$X_1 := \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_s \end{pmatrix}$$

is left invertible. If X_2 is a left inverse element of X_1, then $X_2 X = E - Y$, where Y is of the form

$$\begin{pmatrix} 0 & y_{12} & \cdots & y_{1s} \\ 0 & 0 & \cdots & y_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $Y^s = 0$. Thus,

$$\left(E + \sum_{k=1}^{s-1} Y^k\right) X_2 X = E$$

and X is left invertible. We have now proved that X is left invertible if and only if $x_1, x_2, \dots, x_s$ are all left invertible. By this and [13, lemma 4.1, p. 53], the rest of the proof is a routine computation. □

Lemma 5.3. *If T is the inverse subsemigroup of S consisting of all regular elements of S, then*

(i) $R[T]$ *is isomorphic to*

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & 0 & \cdots & 0 \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}.$$

(ii) $s \in S \setminus T$ *if and only if*

$$\psi(s) \in \begin{pmatrix} 0 & M_{n_1 n_2}(R[M_{12}]) & \cdots & M_{n_1 n_{s-1}}(R[M_{1,s-1}]) & M_{n_1 n_s}(R[M_{1s}]) \\ 0 & 0 & \cdots & M_{n_2 n_{s-1}}(R[M_{2,s-1}]) & M_{n_2 n_s}(R[M_{2s}]) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & M_{n_{s-1} n_s}(R[M_{s-1,s}]) \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where ψ has the same meaning as in the proof of theorem 4.2.

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Proof. By lemma 2.4(v), for $(a)_{i\lambda} \in \mathrm{PA}$, we have that $(a)_{i\lambda}$ is regular if and only if there exists $p \in \Gamma$ such that $i, \lambda \in I_p$ and if and only if

$$\varphi((a)_{i\lambda}) \in \begin{pmatrix} M_{n_1}(R[M_{11}]) & 0 & \cdots & 0 \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix},$$

where φ has the same meaning as in the proof of theorem 4.5. On the other hand, by the definition of $S^{\odot}$, for any $s \in S$, s is regular in S if and only if s is regular in $S^{\odot}$; that is, $\bar{s}$ is regular in $\bar{S}$. This proves the lemma. □

We now arrive at the main result of this section.

THEOREM 5.4. *Let S be a finite ample semigroup. Denote by T the inverse subsemigroup of regular elements of S. Then the radical of $R[S]$ is a sum of the radical of $R[T]$ and $\widehat{R[S \setminus T]}$.*

Proof. Assume $s \in S \setminus T$. Then by lemmas 5.2 and 5.3(ii), $\psi(s)$ belongs to the radial of $R_0[S^{\odot}]$, thereby $\hat{s}$ is contained in the radical of $R[S]$. Thus, $\widehat{R[S \setminus T]}$ is contained in the radical of $R[S]$. On the other hand, by lemma 5.3(i), we see that $R[T]$ is isomorphic to

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & 0 & \cdots & 0 \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}.$$

Note that the radical of

$$\begin{pmatrix} M_{n_1}(R[M_{11}]) & 0 & \cdots & 0 \\ 0 & M_{n_2}(R[M_{22}]) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{n_s}(R[M_{ss}]) \end{pmatrix}$$

is

$$\begin{pmatrix} \overline{M_{n_1}(R[M_{11}])} & 0 & \cdots & 0 \\ 0 & \overline{M_{n_2}(R[M_{22}])} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{M_{n_s}(R[M_{ss}])} \end{pmatrix}.$$

Therefore, the image of the radical of $R[T]$ under ψ is

$$\begin{pmatrix} \overline{M_{n_1}(R[M_{11}])} & 0 & \cdots & 0 \\ 0 & \overline{M_{n_2}(R[M_{22}])} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{M_{n_s}(R[M_{ss}])} \end{pmatrix}.$$

Again by lemma 5.2, we obtain that the radical of $R[S]$ is a sum of the radical of $R[T]$ and $R\widehat{[S \setminus T]}$. □

As an application of theorem 5.4, we have the following.

THEOREM 5.5. *Let S be a finite ample semigroup. Then $R[S]$ is semiprimitive if and only if the following two conditions hold:*

(i) *S is an inverse semigroup;*

(ii) *for every maximum subgroup G of S, $R[G]$ is semiprimitive.*

Proof. By theorem 4.6, we need only to prove the direct part. To the end, assume that $R[S]$ is semiprimitive. If T is the inverse subsemigroup of S consisting of the regular elements of S, then, by theorem 5.4, $R\widehat{[S \setminus T]} = \{0\}$ so that $S \setminus T = \emptyset$. It follows that $S = T$. In other words, S is an inverse semigroup. By the arguments before theorem 4.6,

$$R[S] \cong \bigoplus_{i=1}^{s} M_{n-i}(R[M_{ii}])$$

if $S^{\odot} = \mathcal{M}(M_{\alpha\beta}, I, I, \varGamma; P)$. But S is finite; each M_{ii} is a finite subgroup of S. In fact, M_{ii} is a maximum subgroup of S. By hypothesis, the fact that

$$R[S] \cong \bigoplus_{i=1}^{s} M_{n_i}(R[M_{ii}])$$

implies that $M_{n_i}(R[M_{ii}])$ is semiprimitive for $i = 1, \ldots, s$. This shows that $R[M_{ii}]$ is semiprimitive. □

The following theorem provides the description of the radicals of finite inverse semigroup algebras.

THEOREM 5.6. *Let S be a finite inverse semigroup. Denote by X the set of maximum subgroups of S. Then the radical of $R[S]$ is the sum of ideals of $R[S]$ generated by $J(\widehat{R[G]})$ with $G \in X$, where $J(R[G])$ is the radical of $R[G]$.*

Proof. By theorem 4.6, $R[S]$ is isomorphic to the direct sum D of $M_{n_i}(R[M_{ii}])$ with $i = 1, 2, \ldots, s$. It is well known that the radical of $M_{n_i}(R[M_{ii}])$ is equal to $M_{n_i}(J(R[M_{ii}]))$. Since $(e_{ii})_{k1}(a)_{11}(e_{ii})_{1l} = (a)_{kl}$, we easily see that $M_{n_i}(J(R[M_{ii}]))$ is the ideal of $M_{n_i}(R[M_{ii}])$ generated by

$$\begin{pmatrix} J(R[M_{ii}]) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Set $G_i = \{(x)_{11} \colon x \in M_{ii}\}$. Then G_i is a maximum subgroup of S. Note that

$$\begin{pmatrix} J(R[M_{ii}]) & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = J(R[G_i]).$$

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Thus, the radical of D is the direct sum of ideals of D ($D = \psi(R[S])$) generated by $J(R[G_i])$ with $i = 1, 2, \ldots, s$. That is, the radical of D is $\sum_{i=1}^{s} DJ(R[G_i])D$. Therefore, the radical of $R[S]$ is

$$\theta\biggl(\sum_{i=1}^{s} DJ(R[G_i])D\biggr)\biggl(\sum_{i=1}^{s} R[S]J(\widehat{R[G_i]})R[S]\biggr).$$

□

By Theorems 5.4 and 5.6, we can give a more precise description of semigroup algebras of finite ample semigroups and omit the detail.

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