The intriguing mechanics of a tractrix of cards

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1. *Introduction*

Figure 1 shows the photograph of a tractrix made of cards. This is a simple yet captivating way to make a tractrix — an arrangement that has recently appeared in some engaging pedagogical resources and literature [1, 2]. When closely spaced cards lean like this, one after another, over a horizontal plane, the contour created is a tractrix — a curve of significance in mathematics, since its revolution around its asymptote produces a pseudosphere: a curved surface with constant negative Gaussian curvature [3, 4].

FIGURE 1: A tractrix made of identical cards

One of several fascinating things about the tractrix of cards, which, for the sake of brevity we will just call the tractrix in the rest of the paper, is that a single pulse can travel through the arrangement [1, 2] in either horizontal direction along the tractrix, turning the cards over in the process. This resembles how a pulse propagates through a line of dominoes to tumble them over. However, in the case of the tractrix, the propagating pulse simply carries the cusp of the tractrix with it, while the size and the shape of the contour around the cusp remains a similar tractrix at all times. This got me thinking about the mechanics of the system, the mechanics not only when a pulse is propagating through it, but also when the structure is simply sitting idle. As I delved into the question, layers of subtlety and wonder started to unfold.

First, let us define our system of interest in an unambiguous manner. All the cards are identical uniform, solid, thin (implying zero thickness), frictionless, rectangular plates. For each card, its bottom edge cannot slide over the horizontal plane (so each card is effectively hinged smoothly at the bottom edge). The spacing between any two consecutive bottom edges is the same distance λ, which is small compared to the length l of each card (Figure 2). The smaller the ratio λ / l, the closer the envelope is to a tractrix. The properties that we will derive in this paper pertains to the limiting case of $(\lambda / l) \rightarrow 0$ and the resulting perfect tractrix.

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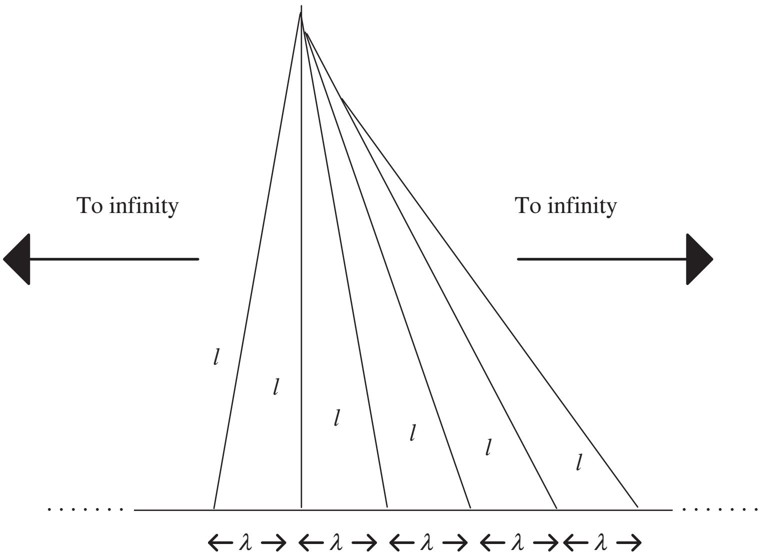

FIGURE 2: Side view of the details of the arrangement of cards in the system

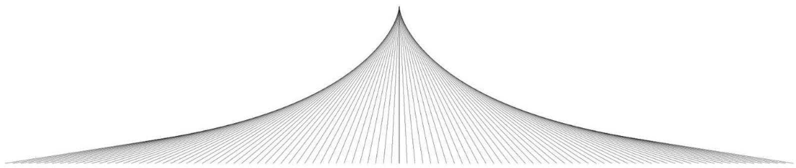

FIGURE 3: Side view of a tractrix formed with a large number of cards, generated by a computer program

We denote the angle a card makes with the vertically downward direction by θ, measured in the anticlockwise sense as shown in Figure 4, and the horizontal coordinate of the hinged bottom end of the card by x.

In Figure 4, the card A_1B_1 makes an angle θ, so that the card A_2B_2 to its right makes an angle $(\theta + \Delta\theta)$. $\Delta\theta$ will be small since $\lambda \ll l$.

Now B_1N is a perpendicular dropped on A_2B_2. Hence angle NB_1B_2 is also $(\theta + \Delta\theta)$. Hence

$$NB_1 = B_1B_2 \cos(\theta + \Delta\theta) \approx B_1B_2 \cos\theta. \tag{1}$$

At the same time, $NB_1 = A_2B_1 \sin(\Delta\theta)$. However, making use of $\sin(\Delta\theta) \approx \Delta\theta$ and $A_2B_1 \approx A_1B_1$, we write

$$NB_1 \approx A_1B_1(\Delta\theta). \tag{2}$$

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Now using the facts $A_1B_1 = l$, and $B_1B_2 = \lambda = \Delta x$, from (1) and (2) we can write

$$l(\Delta\theta) \approx (\Delta x)\cos\theta. \tag{3}$$

At this point, we approximate (3) in the form of a differential equation:

$$l\frac{d\theta}{dx} = \cos\theta \tag{4}$$

or

$$l\frac{d\theta}{\cos\theta} = dx. \tag{5}$$

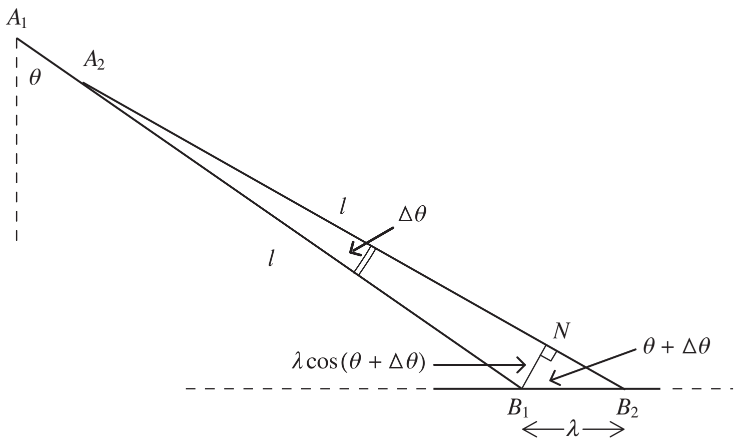

FIGURE 4: Two consecutive cards in contact, making an angle of $\Delta\theta$ between them

At some given instant of time, if the position of the erect card, i.e. the one with $\theta = 0$, is $x = x_0$, then integrating (5) we obtain, after some algebraic simplification,

$$\cos\theta = \operatorname{sech}\left(\frac{x - x_0}{l}\right),$$

which gives the relation between θ and x at that instant.

For the full tractrix extending to infinity in both directions, θ ranges from $-\pi/2$ to $\pi/2$ as x ranges from $-\infty$ to $+\infty$. During the propagation of a pulse, every card in the system still has a unique fixed value of x, while at a given instant of time, a unique value of θ as well.

Now the contact force f between any two consecutive cards is a 'normal force', since the cards are frictionless. To start with, one interesting question to ask is *how the magnitude of the contact force varies with the location*, i.e. how f varies as a function of θ, even when the tractrix is static.

While it is possible to obtain the answer by considering a quasistatic evolution of the tractrix, we will save this question for later. In due course the answer will be borne out as part of the analysis of the dynamic scenario when a pulse propagates through the tractrix.

2. *The gravitational potential energy of the tractrix*

The height of the centre of mass of a single card at angular position θ, measured from the horizontal plane, is given by $(l/2)\cos\theta$. Hence, if the mass of the card is m, then the gravitational potential energy of that card is equal to $mg(l/2)\cos\theta$, taking the horizontal plane as the zero level for measuring gravitational potential energy.

Now since the number of cards within the interval x and $(x + dx)$ is given by (dx/λ), the potential energy contribution from the cards located within that interval is given by

$$dp = \left(mg\frac{l}{2}\right)\cos\theta\,(dx/\lambda). \tag{6}$$

However, using (4), we can write (6) as

$$dp = \left(mg\frac{l^2}{2}\right)(d\theta/\lambda), \tag{7}$$

which is the potential energy contribution from the cards spanning the angular interval between θ and $(\theta + d\theta)$.

From (8), we write for the potential energy contribution from the angular range between θ_1 and θ_2 to be

$$p = \frac{mgl^2}{2\lambda}\int_{\theta_1}^{\theta_2} d\theta. \tag{8}$$

At this point we define

$$\sigma = \frac{m}{\lambda} \tag{9}$$

which is effectively the linear mass density of the system when all the cards happen to be down on the horizontal plane.

Performing the simple integration in (8), and then using (9), we obtain the crisp expression

$$p = \frac{\sigma g l^2}{2}(\theta_2 - \theta_1), \tag{10}$$

which is the expression for the gravitational potential energy for the segment of the tractrix with cards spanning the angular range $\theta_1 \leqslant \theta \leqslant \theta_2$.

For the full tractrix, the total gravitational potential energy becomes

$$p_{\text{full}} = \frac{\pi\sigma g l^2}{2}. \tag{11}$$

3. *Propagation of a pulse and kinetic energy of the tractrix*

When a pulse propagates through the tractrix, all the cards are in a state of rotation (albeit at different angular speeds) in such a way that the envelope remains a similar tractrix of the same shape and size at all times, while the position of the cusp containing the erect card ($\theta = 0$) keeps drifting with the pulse. If our system of cards happens to be truncated at one or both ends, its envelope simply becomes a different segment of the same tractrix shape at different times in the course of the propagation of a pulse.

Since the spatial separation of the fixed bottom edges of the cards is uniform, at any given point in time the rate of propagation of the pulse can be characterised by a single quantity, namely the linear speed of propagation given by

$$V = \frac{dx}{dt}, \tag{12}$$

where dx is the horizontal displacement of, say, the cusp of the tractrix during a time interval dt.

At this point, combining (4) and (12), for the angular velocity of an individual card, we obtain

$$\frac{d\theta}{dt} = \frac{V}{l}\left(\cos\theta\right). \tag{13}$$

Now, since the rotation axis for every card is its hinged bottom edge, the kinetic energy of a single rotating card will be given by $\frac{1}{6}ml^2(d\theta/dt)^2$, which, making use of (13), can be written as $\frac{1}{6}mV^2\cos^2\theta$.

Once again, since the number of cards within the interval x and $(x + dx)$ is given by dx/λ, the kinetic energy contribution from the cards located within that interval is given by

$$dk = \frac{1}{6}mV^2\cos^2\theta\,(dx/\lambda),$$

which, using (4), can be written as

$$dk = \frac{1}{6}mV^2\cos\theta\,(d\theta/\lambda).$$

Hence the kinetic energy contribution from the cards within the angular range between θ_1 and θ_2 will be

$$k = \frac{1}{6\lambda}mlV^2\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta,$$

Evaluating the above integral, and using (9), we obtain

$$k = \frac{1}{6}\sigma lV^2\left(\sin\theta_2 - \sin\theta_1\right), \tag{14}$$

while the total kinetic energy for the full tractrix becomes

$$k_{\text{full}} = \frac{1}{3}\sigma lV^2. \tag{15}$$

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4. *Propagation of a pulse through the full tractrix and distribution of the contact force*

The normal force pair at every contact point between any two consecutive pair of cards does zero net work and thus dissipates no energy, and the hinge forces at the bottom edges do zero work as well. Therefore, if the work done by gravity is incorporated in the gravitational potential energy term, then the net mechanical energy, namely the sum of the kinetic energy and the gravitational potential energy of the full tractrix will be conserved. However, since the net potential energy stays the same value given by (11) irrespective of the instantaneous position of the cusp of the tractrix, the net kinetic energy given by (15) in itself will remain a constant, implying that a pulse can propagate forever with a constant speed V through the infinite system. Furthermore, that constant speed V can have any arbitrary value.

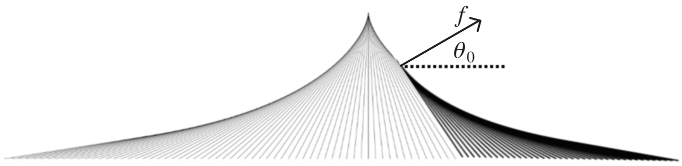

FIGURE 5: Applying the work-energy theorem to a part of the tractrix shown with a darker shade

At this point, we intend to find the distribution of the normal contact force between consecutive cards along the tractrix during the propagation of a pulse. For that purpose, we focus our attention on the part of the infinite tractrix which extends from an arbitrarily chosen card all the way to infinity towards the right, shown with darker shading in Figure 5. We refer to this system as $\mathscr{F}_0$. We denote the angle of the card at the left boundary of $\mathscr{F}_0$ by θ_0. Without losing generality, let us take a situation where a pulse travels towards, say, the left, through the infinite tractrix with a constant speed V. Consequently, all cards will be rotating anticlockwise.

For the gravitational potential energy and kinetic energy of $\mathscr{F}_0$, it follows from (10) and (14) respectively that

$$p_0 = \frac{\sigma g l^2}{2}\left(\frac{\pi}{2} - \theta_0\right), \qquad (16)$$

and

$$k_0 = \frac{1}{6}\sigma l V^2\left(1 - \sin\theta_0\right). \qquad (17)$$

Now, for the system $\mathscr{F}_0$, once again the hinge forces exerted at the bottom edges of the cards still do no work. The internal normal force pairs at all contact points between consecutive cards do net work of zero as well. On the other hand, the work done by gravity is incorporated in the gravitational potential energy term given by (16). However, as shown in Figure 5, there is an external force f on the card at the left boundary of $\mathcal{F}_0$, exerted by the card immediately to its left. This 'normal force' acts in a direction normal to the card, thus making an angle of θ_0 with the horizontal, as shown. Furthermore, since the contact point is the top edge of the card, that is where the force is exerted as well. What is important to realise is that this force f does non-zero external work on $\mathcal{F}_0$.

Now, during a time interval dt, if the corresponding change in θ_0 is $d\theta_0$, then the displacement of the top edge of the card is given by $l\,d\theta_0$ in the direction opposite to f, since the card rotates anticlockwise about its bottom edge. Hence the work done by f is given by

$$dW = -fl\,d\theta_0. \tag{18}$$

At this point we recall the work-energy theorem, namely the property that the change in the total mechanical energy of a physical system equals the net external work done, provided there are no internal dissipative forces.

Applying this to our system $\mathcal{F}_0$, we can write

$$dW = dp_0 + dk_0.$$

Hence, using (16), (17) and (18), we obtain

$$-fl\,d\theta_0 = -\frac{\sigma g l^2}{2}d\theta_0 - \frac{1}{6}\sigma l V^2 \cos\theta_0\,d\theta_0,$$

which, upon simplification, yields

$$f = \frac{\sigma g l}{2} + \frac{1}{6}\sigma V^2 \cos\theta_0.$$

Since the boundary of our system $\mathcal{F}_0$ was arbitrarily chosen, there is no reason why the above expression will not apply to any point in the whole tractrix at any instant of time. Moreover, it is not difficult to justify that the expression holds no matter if the pulse is travelling left or right.

Hence we write the final mathematical relation depicting how the magnitude of the normal contact force varies with position along the tractrix as

$$f = \frac{\sigma g l}{2} + \frac{1}{6}\sigma V^2 \cos\theta. \tag{19}$$

At this point we are finally ready to derive the dependence of the force for a static tractrix. All we need to do for that purpose is to put $V = 0$ in (19), which gives us

$$f_{\text{static}} = \frac{\sigma g l}{2}.$$

This simple expression actually bears out an intriguing, and perhaps somewhat surprising, fact. It tells us that for the infinite tractrix, when sitting idle, the magnitude of the contact force between any two consecutive cards has the *same value everywhere*, given by $\frac{1}{2}\sigma g l$, no matter the position (and thereby the angular tilt) of the card!

5. *The half-tractrix: free fall*

Finally, we consider the dynamics of the tractrix truncated at one end. We take the card at the free end, referred to here as the terminal card, to be initially erect (so that initially the system looks like a full tractrix cut in half at the centre), as shown in Figure 6, and release the structure from rest. It is not hard to predict that all the cards will start rotating, speeding up due to gravity in the anticlockwise sense. We are interested in the dynamics of the whole structure falling, as part of which we can also find the speed with which of the top edge of the terminal card will strike the horizontal plane.

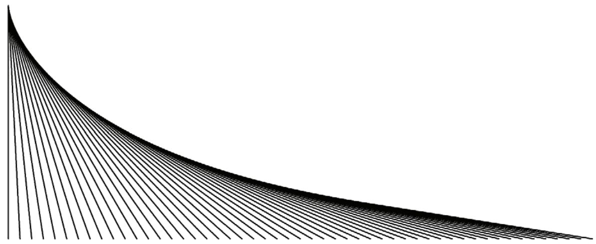

FIGURE 6. The half-tractrix

It is worthwhile to define a natural speed for the system:

$$V^* = \sqrt{gl},$$

so that we have a benchmark for comparison. For example, a single card, aligned horizontally, when released from rest from a height l, strikes the horizontal plane with a speed of $\sqrt{2}V^* \approx 1.41V^*$.

In this section, let us denote the angle of the terminal card by θ_0. Since the structure is released from rest with the terminal card erect, we have $V = 0$ and $\theta_0 = 0$ at the initial instant, where, once again, V is the physical speed of pulse propagation.

Now, in this situation, there is no external work done on the system and the net mechanical energy will be conserved. Using the expressions (16) and (17) for the potential and kinetic energies respectively, we can write

$$\frac{\sigma g l^2}{2}\left(\frac{\pi}{2} - \theta_0\right) + \frac{1}{6}\sigma l V_2\left(1 - \sin\theta_0\right) = \frac{\sigma g l^2}{2}\frac{\pi}{2},$$

where the left-hand side represents the net mechanical energy at some later instant, while the right-hand side represents the net mechanical energy at the initial instant when $\theta_0 = 0$.

Simplifying, we obtain

$$V = \sqrt{3gl}\sqrt{\frac{\theta_0}{1 - \sin\theta_0}}. \tag{20}$$

At this point, combining (13) and (20), we derive:

$$\frac{d\theta_0}{dt} = \sqrt{\frac{3g}{l}}\sqrt{\frac{\theta_0}{1 - \sin\theta_0}}\cos\theta_0. \tag{21}$$

The above relation is an informative one, capturing how the angular speed of the terminal card varies with its angular position during the fall.

We leave it to the reader as a little challenge on the side to prove that

$$\lim_{\theta_0 \to \pi/2} \frac{\cos\theta_0}{\sqrt{1 - \sin\theta_0}} = \sqrt{2}. \tag{22}$$

Using (22) in (21), we obtain

$$\lim_{\theta_0 \to \pi/2} \left(\frac{d\theta_0}{dt}\right) = \sqrt{\frac{3\pi g}{l}}. \tag{23}$$

Now let us remember that the speed of the top edge of the card is given by $l(d\theta_0/dt)$, since the bottom edge is the effective rotation axis. Combining this fact with (33), we obtain for the speed of the top edge of the card just before it strikes the horizontal plane:

$$V_{\text{strike}} = \sqrt{3\pi gl} \approx 3.07V^*. \tag{24}$$

Out of curiosity, we can compare the above with the terminal speed of the free end of a single card if it was falling on its own, purely under gravity but still rotating about its bottom edge. That speed happens to be $\sqrt{3gl} \approx 1.73V^*$. The speed given in (24) is greater, as expected.

6. *The half-tractrix: escape speed*

One way to release the half-tractrix so that it does not fall down is to give the top edge of the terminal card an initial velocity in the opposite direction (which would be to the right in Figures 6 and 7), so that a pulse starts propagating away from the free end deeper into the system. However, not just any initial velocity would work. To ensure that the pulse never stops to turn round and return to the free end, the initial velocity must have a certain minimum magnitude, which we call the escape speed.

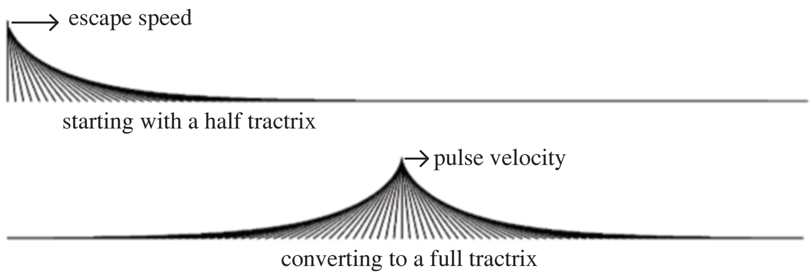

FIGURE 7: A half-tractrix turning into a full-tractrix

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One simple but beautiful realisation about the process is that when the initial speed is equal to the escape speed or greater, as the pulse carries the vertex of the tractrix indefinitely deeper into the system, the structure approaches a full tractrix, having started from only a half-tractrix. This evolution is depicted in Figure 7. Apparently, we end up with two copies of what we started with. This is yet another humble manifestation of the magic of infinities.

However, when the initial speed is exactly equal to the escape speed, the later speed of the pulse approaches zero as it travels indefinitely deeper. Once again, applying the principle of conservation of mechanical energy to this system, we write, using (16) and (17),

$$\frac{\sigma g l^2}{2}\frac{\pi}{2} + \frac{1}{6}\sigma l V_{\text{escape}}^2 = \frac{\sigma g l^2}{2}\pi, \tag{25}$$

where the left-hand side represents the net mechanical energy of the half-tractrix at the start, while the right-hand side represents the net mechanical energy in the limit when the pulse speed has died off to zero and the structure has become a full tractrix.

Simplifying (25), we obtain

$$V_{\text{escape}} = \sqrt{\frac{3\pi g l}{2}} \approx 2.17V^*,$$

which is the minimum speed required to be imparted to the top of the terminal card so that the structure never falls, and retains one erect card somewhere within it at all later times!

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