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On the convergence of eigenfunctions to threshold energy states

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We prove the convergence in certain weighted spaces in momentum space of eigenfunctions of $H = T - \lambda V$ as the energy goes to an energy threshold. We do this for three choices of kinetic energy T, namely the non-relativistic Schrödinger operator, the pseudorelativistc operator $\sqrt{-\Delta + m^2} - m$, and the Dirac operator.

1. Introduction

In this paper we consider a family of Hamiltonians

$$H \equiv H(\lambda) = T - \lambda V, \tag{1.1}$$

where $\lambda > 0$ is the coupling constant and $V \geqslant 0$ is a bounded and integrable potential. We will consider different choices of physical kinetic energies T, but, for the moment, to fix ideas, we set $T = -\Delta$, the Laplace operator in three dimensions. The essential spectrum of H is equal to the interval $[0, \infty)$ and (for λ sufficiently large) H has negative discrete eigenvalues $E_i < 0$, $i = 1, 2, \ldots$. We shall henceforth fix an $i \in \mathbb{N}$ and consider the λ-dependence of $E(\lambda) := E_i(\lambda)$. Due to monotonicity, there is a $\lambda_c \in \mathbb{R}$ such that, as $\lambda \downarrow \lambda_c$, $E(\lambda) \uparrow 0$. We call λ_c a *coupling constant threshold*.

Let $\varphi_E = \varphi_{E(\lambda)} \in L_2(\mathbb{R}^3)$ be an eigenfunction of $H(\lambda)$ with eigenvalue $E = E(\lambda)$. A detailed study of the behaviour of E as $\lambda \downarrow \lambda_c$ for various choices of T was carried out in [13–15, 18]. Here, we are interested in the behaviour of φ_E as $E \uparrow 0$ (that is, as $\lambda \downarrow \lambda_c$). It is easy to prove (using closedness of the kinetic energy T) that if φ_E converges in $L_2(\mathbb{R}^3)$, then the limit function φ_0 is an eigenfunction of $H(\lambda_c)$, i.e. a bound state with zero energy. If there is no L_2-convergence, however, we might expect some other kind of convergence of the φ_E. In particular, we are interested in considering the convergence properties of $w(-\mathrm{i}\nabla)\varphi_E$, where w is a suitable function of the kinetic energy. (For the question of existence of zero-energy eigenstates, see, for example, [1], and the above-mentioned papers.)

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Such questions, apart from being of independent interest, are important for problems pertaining to enhanced binding and the Efimov effect (see, for example, [4,24]). (Other papers on enhanced binding, using zero-energy 'eigenfunctions' are [2,3,9]; these, however, do not explicitly use the convergence properties we discuss here.) We shall not comment further on this here. Our work partly uses the techniques used in [12,14,15] for the relativistic case (see also [18]). In these papers the authors investigated the relationship between the analytic properties of the eigenvalues near the threshold energy and the existence of eigenvalues at the threshold.

Let us introduce the three different choices of kinetic energy, T, which we will study in this paper. Let $m > 0$ be the mass of the electron.

Schrödinger case. The free one-particle non-relativistic kinetic energy is given (in units when $\hbar = 1$) by $-\Delta/2m$. Choosing units such that $2m = 1$, the operator is just the Laplace operator in three dimensions mentioned above:

$$T_{\mathrm{S}} := -\Delta. \tag{1.2}$$

Pseudorelativistic case. A naive choice of a free one-particle (pseudo)relativistic kinetic energy is given (in units when $\hbar = c = 1$) by the pseudodifferential operator,

$$T_{\psi\,\mathrm{rel}} := \sqrt{-\Delta + m^2} - m \tag{1.3}$$

(see, for example, [10,25]).

In both of the above cases, assuming that $0 \leqslant V \in L_1(\mathbb{R}^3) \cap L_\infty(\mathbb{R}^3)$, the operators $H_S(\lambda) := T_{\mathrm{S}} - \lambda V$ and $H_{\psi\,\mathrm{rel}}(\lambda) := T_{\psi\,\mathrm{rel}} - \lambda V$ are self-adjoint in $L_2(\mathbb{R}^3)$ with domains $H^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$, respectively; their essential spectrum is $\sigma_{\mathrm{ess}} = [0,\infty)$ and (for sufficiently large λ), they have eigenvalues $E_i(\lambda) < 0$, $i \in \mathbb{N}$ (see [16,20]).

Dirac case. The free one-particle Dirac operator (again, in units when $\hbar = c = 1$) is given by

$$T_{\mathrm{D}} := \boldsymbol{\alpha} \cdot (-\mathrm{i}\nabla) + m\beta - m, \tag{1.4}$$

acting on $L_2(\mathbb{R}^3; \mathbb{C}^4)$. Here $\boldsymbol{\alpha}$ and β are the usual Dirac matrices.

If $0 \leqslant V \in L_1(\mathbb{R}^3; \mathbb{C}^4) \cap L_\infty(\mathbb{R}^3; \mathbb{C}^4)$ is a (diagonal) potential, then $H_{\mathrm{D}}(\lambda) := T_{\mathrm{D}} - \lambda V$ is self-adjoint with domain $H^1(\mathbb{R}^3; \mathbb{C}^4)$, its essential spectrum is $(-\infty, -2m] \cup [0,\infty)$, and it has eigenvalues $E_i(\lambda) \in (-2m, 0)$, $i \in \mathbb{N}$ (see [23]).

We recall that, for $q \geqslant 1$, the Banach space $L_q(\mathbb{R}^3; \mathbb{C}^4)$ consists of four-component vector functions $\phi = (\phi_1, \dots, \phi_4)^{\mathrm{T}}$ with the norm

$$\|\phi\|_{L_q(\mathbb{R}^3;\mathbb{C}^4)} := \biggl(\int_{\mathbb{R}^3} \|\phi(\boldsymbol{x})\|_{\mathbb{C}^4}^q \,\mathrm{d}\boldsymbol{x}\biggr)^{1/q}. \tag{1.5}$$

Here $\|\cdot\|_{\mathbb{C}^4}$ is the usual Euclidean norm. Note that, since all norms in $\mathbb{C}^4$ are equivalent, this norm and

$$|||\phi|||_{L_q(\mathbb{R}^3;\mathbb{C}^4)} := \biggl(\sum_{i=1}^4 \|\phi_i\|_{L_q(\mathbb{R}^3)}^q\biggr)^{1/q} \tag{1.6}$$

are equivalent (for $q = 2$ they are equal).

In order to relax the notation, we denote by $H(\lambda) = T - \lambda V$ a general Hamiltonian, where T corresponds to one of the three kinetic energies defined above. We will also use the symbol L_q for $L_q(\mathbb{R}^3)$ or $L_q(\mathbb{R}^3;\mathbb{C}^4)$ if there is no risk of confusion; the corresponding norm will be denoted $\|\cdot\|_q$. We denote the space of Schwartz functions (with values in $\mathbb{C}$ or $\mathbb{C}^4$) by $\mathcal{S}$, and its dual, the space of tempered distributions, by $\mathcal{S}'$. The $(\mathcal{S}',\mathcal{S})$ pairing is denoted by $\langle\cdot,\cdot\rangle$. We define by

$$\hat{g}(\boldsymbol{p}) := [\mathcal{F}g](\boldsymbol{p}) := \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} g(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \tag{1.7}$$

the Fourier transform of the function $g \in \mathcal{S}(\mathbb{R}^3)$. For four-component vector functions $g = (g_1,\dots,g_4)^{\mathrm{T}}$, $\hat{g}$ is defined componentwise. For $r \in [1,2]$, the Fourier transform extends to a bounded linear mapping from L_r to $L_{r'}$, with $1/r + 1/r' = 1$. On the other hand, by duality, the Fourier transform extends to $\mathcal{S}'$. These two extensions coincide whenever they are both defined.

Consider, for $E \notin \sigma(T)$ and $\|\varphi_E\|_2 = 1$, the eigenvalue equation

$$(T(-\mathrm{i}\nabla) - \lambda V)\varphi_E = E\varphi_E. \tag{1.8}$$

An elementary manipulation shows that this equation can be rewritten as

$$\varphi_E = \lambda(T(-\mathrm{i}\nabla) - E)^{-1} V\varphi_E. \tag{1.9}$$

The latter equation is known (in the physics literature) as the Lipmann–Schwinger equation.

We recall the following. For $E \notin \sigma(T)$ there is a solution φ_E of (1.8) if and only if for

$$\mu_E := V^{1/2}\varphi_E, \tag{1.10}$$

the equation

$$K_E\mu_E = \lambda^{-1}\mu_E \tag{1.11}$$

holds, where

$$K_E = V^{1/2}(T(-\mathrm{i}\nabla) - E)^{-1}V^{1/2} \tag{1.12}$$

is the *Birman–Schwinger operator*.

REMARK 1.1. Note that $\lambda_{\mathrm{c}} \neq 0$ under the stated assumptions on V. For the Schrödinger and pseudorelativistic cases, this follows from [22, theorem 2.3]. For the Dirac case, see [12, lemma 2.3].

An interesting feature is that, under fairly general assumptions on the potential V, we have the following. If $\lambda_n \downarrow \lambda_{\mathrm{c}}$ as $n \to \infty$, and if $\{\varphi_{E(\lambda_n)}\}_{n\in\mathbb{N}} \subset L_2$ is a sequence of corresponding eigenfunctions of $T - \lambda_n V$, then there exists a *subsequence* $\{\varphi_{E(\lambda_{n_k})}\}_{k\in\mathbb{N}}$ and a $\mu_0 \in L_2$ such that

$$\mu_{E(\lambda_{n_k})} \to \mu_0 \text{ in } L_2 \quad \text{as } k \to \infty, \tag{1.13}$$

where $\mu_{E(\lambda)}$ is given by (1.10).

An analogous result holds for the Dirac operator when $E(\lambda) \downarrow -2m$ as $\lambda \uparrow \lambda_{\mathrm{c}}$, in which case the limiting function is denoted by μ_{-2m}.

The precise statement of the conditions on V is in lemma 5.1 (see § 5, where we also give a proof).

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Throughout this paper '$E \to 0$' ('$E \to -2m$') means to take sequences $\{\lambda_n\}_{n\in\mathbb{N}}$ with $\lambda_n \uparrow \lambda_{\mathrm{c}}$ ($\downarrow \lambda_{\mathrm{c}}$) for which $\{\mu_{E(\lambda_n)}\}_{n\in\mathbb{N}}$ has a limit in L_2.

From (1.13) we can construct what will turn out to be the relevant (generalized) zero-energy solution. We call this a 'threshold energy state'.

Let us now state the condition on the weight functions w. We define $x = |\boldsymbol{x}|$ and $p = |\boldsymbol{p}|$, and $\chi_< := \chi_{[0,1)}$ and $\chi_> := \chi_{[1,\infty)}$, with χ_A the characteristic function of the set A.

Let $w_{\mathrm{S}} : \mathbb{R}^3 \to \mathbb{C}$ (Schrödinger), $w_{\psi\,\mathrm{rel}} : \mathbb{R}^3 \to \mathbb{C}$ (pseudorelativistic) and $w_{\mathrm{D}} : \mathbb{R}^3 \to M_{4\times 4}(\mathbb{C})$ (4×4 matrices over $\mathbb{C}$) (Dirac) satisfy

$$\frac{w_{\mathrm{S}}(\boldsymbol{p})\chi_<(p)}{p^2} \in L_2(\mathbb{R}^3), \qquad \frac{w_{\mathrm{S}}(\boldsymbol{p})\chi_>(p)}{p^2} \in L_\infty(\mathbb{R}^3), \tag{1.14}$$

$$\frac{w_{\psi\,\mathrm{rel}}(\boldsymbol{p})\chi_<(p)}{p^2} \in L_2(\mathbb{R}^3), \qquad \frac{w_{\psi\,\mathrm{rel}}(\boldsymbol{p})\chi_>(p)}{p} \in L_\infty(\mathbb{R}^3), \tag{1.15}$$

$$\frac{|w_{\mathrm{D}}(\boldsymbol{p})|\chi_<(p)}{p^2} \in L_2(\mathbb{R}^3;\mathbb{C}^4), \quad \frac{|w_{\mathrm{D}}(\boldsymbol{p})|\chi_>(p)}{p} \in L_\infty(\mathbb{R}^3;\mathbb{C}^4), \tag{1.16}$$

where in the last expression $|w_{\mathrm{D}}(\boldsymbol{p})|$ denotes any norm of the matrix $w_{\mathrm{D}}(\boldsymbol{p})$ (for instance, its largest eigenvalue, in absolute value). We write in general $w(\boldsymbol{p})$ for one of the three functions defined above. Our main result in this paper is the following.

Theorem 1.2. *Let $H(\lambda) = T - \lambda V$, with T one of the kinetic energy operators mentioned above, and $V \in L_1 \cap L_\infty$. Let λ_{c} be a coupling constant threshold, let $\lambda_n \downarrow \lambda_{\mathrm{c}}$ and $\{\varphi_n\}_{n\in\mathbb{N}} \subset L_2$ such that $H(\lambda_n)\varphi_n = E(\lambda_n)\varphi_n$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be the corresponding Birman–Schwinger eigenfunctions defined by (1.10), and assume that $\mu_n \to \mu_0$ in L_2 as $n \to \infty$. Define*

$$\varphi_0(\boldsymbol{x}) := \lambda_{\mathrm{c}} \int_{\mathbb{R}^3} T^{-1}(\boldsymbol{x},\boldsymbol{y}) V^{1/2}(\boldsymbol{y}) \mu_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{1.17}$$

where $T^{-1}(\boldsymbol{x},\boldsymbol{y}) := \lim_{E\to 0}(T-E)^{-1}(\boldsymbol{x},\boldsymbol{y})$. Finally, let w satisfy the conditions (1.14)–(1.16). Then

$$w\hat{\varphi}_n \to w\hat{\varphi}_0 \ \text{in } L_2 \quad \text{as } n \to \infty. \tag{1.18}$$

Furthermore, φ_0 satisfies

$$H\varphi_0 = 0 \quad \text{in } \mathcal{S}'. \tag{1.19}$$

Remark 1.3.

(i) An analogous theorem holds for the Dirac case when $E \to -2m$. In that case we define

$$\varphi_{-2m}(\boldsymbol{x}) := \lambda_{\mathrm{c}} \int_{\mathbb{R}^3} (T+2m)^{-1}(\boldsymbol{x},\boldsymbol{y}) V^{1/2}(\boldsymbol{y}) \mu_{-2m}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{1.20}$$

This is the limiting object for which (1.18) holds, and which turns out to solve $H\varphi_{-2m} = (-2m)\varphi_{-2m}$ in $\mathcal{S}'$.

(ii) Explicit expressions for $(T-E)^{-1}(\boldsymbol{x},\boldsymbol{y})$ and its limits, for the three choices of kinetic energy T, are given in § 2.3.

(iii) Note that not all solutions of $H\varphi_0 = 0$ in the distributional sense have the form (1.17).

(iv) In contrast to the Laplacian, the pseudorelativistic kinetic energy behaves like p^2 for small (momenta) p and like p for large momenta. The conditions in (1.15) are sufficient to ensure that (see (3.11), below)

$$\left\| \frac{w(\boldsymbol{p})\chi_<(p)}{\sqrt{p^2+m^2}-m} \right\|_2 \quad \text{and} \quad \left\| \frac{w(\boldsymbol{p})\chi_>(p)}{\sqrt{p^2+m^2}-m} \right\|_\infty$$

are finite.

(v) Examples of weight functions are $w_{\mathrm{S}}(\boldsymbol{p}) = p^{2s}$, $w_{\psi\,\mathrm{rel}}(\boldsymbol{p}) = (\sqrt{p^2+m^2}-m)^s$ and $w_{\mathrm{D}}(\boldsymbol{p}) = |\boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta - m|^s$, all for $s \in (\frac{1}{2}, 2]$. Thus, in general we have that $w(\boldsymbol{p}) = |T(\boldsymbol{p})|^s$, $s \in (\frac{1}{2}, 2]$, satisfy conditions (1.14)–(1.16).

(vi) In the Schrödinger case, convergence of $\nabla\varphi_E$ and $\Delta\varphi_E$ is known (see, for example, [24]). The methods there rely on the local properties of the operators ∇ and Δ. The original goal of our work was to prove this convergence for non-local as well as matrix-valued kinetic energies. The insight of the present paper is that studying the problem in momentum-space allows one to prove the convergence not only of $\nabla\varphi_E$ and $\Delta\varphi_E$ for all of these kinetic energies, but also of more general functions of ∇ applied to φ_E.

Remark 1.4. It is important to note that our convergence statements are independent of whether or not there is an eigenvalue at the threshold when $\lambda \to \lambda_{\mathrm{c}}$. Conditions for the limit function φ_0 (or φ_{-2m}) to be in L_2 are well known and we list them here for completeness (we thank A. Jensen for commenting on this to us).

(i) The Schrödinger [11, 14] and pseudorelativistic [18] case: $\varphi_0 \in L_2(\mathbb{R}^3)$ if and only if $\int_{\mathbb{R}^3} V(\boldsymbol{x})\varphi_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$.

(ii) The Dirac case [13]: $\varphi_0 \in L_2(\mathbb{R}^3)$ if and only if $\int_{\mathbb{R}^3} V(\boldsymbol{x})\beta_+\varphi_0(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$ (or $\int_{\mathbb{R}^3} V(\boldsymbol{x})\beta_-\varphi_{-2m}(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 0$ for φ_{-2m}). Here, $\beta_\pm := \frac{1}{2}(1 \pm \beta)$.

In the case when $\varphi_0 \notin L_2$, φ_0 is called a zero-resonance or a half-bound state (see, for example, [11]). (In the physics literature this is sometimes called a 'virtual level' or 'virtual state'; these appear in the context of scattering theory; see, for example, [5, 17, 19] and references therein.)

2. Preliminaries

2.1. Additional tools for the Dirac operator

We define

$$T_{\mathrm{D}}(\boldsymbol{p}) := \mathcal{F}T_{\mathrm{D}}\mathcal{F}^{-1} = \boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta - m. \tag{2.1}$$

To study the Dirac case, we introduce the Foldy–Wouthuysen transformation [6, 23] $U_{\mathrm{FW}} : L_2(\mathbb{R}^3;\mathbb{C}^4) \to L_2(\mathbb{R}^3;\mathbb{C}^4)$, which has the property that

$$U_{\mathrm{FW}}T_{\mathrm{D}}U_{\mathrm{FW}}^{-1} = \beta\sqrt{-\Delta+m^2} - m. \tag{2.2}$$

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In momentum space, $\hat{U}_{\mathrm{FW}} := \mathcal{F} U_{\mathrm{FW}} \mathcal{F}^{-1}$ is given by the matrix-valued multiplication operator

$$\hat{U}_{\mathrm{FW}}(\boldsymbol{p}) := a_+(p) + \beta \boldsymbol{\alpha} \cdot \frac{\boldsymbol{p}}{p} a_-(p), \tag{2.3}$$

where

$$a_\pm(p) = \sqrt{\frac{1}{2}\biggl(1 \pm \frac{m}{\sqrt{p^2+m^2}}\biggr)}. \tag{2.4}$$

Noting that

$$\hat{U}_{\mathrm{FW}}(\boldsymbol{p})^{-1} = a_+(p) - \beta \boldsymbol{\alpha} \cdot \frac{\boldsymbol{p}}{p} a_-(p), \tag{2.5}$$

we see that $\hat{U}_{\mathrm{FW}}(\boldsymbol{p})$ is an orthogonal matrix for every $\boldsymbol{p} \in \mathbb{R}^3$. Therefore, by the definition (1.5) we have the following lemma.

Lemma 2.1. *For $q \geqslant 1$, the mapping $\hat{U}_{\mathrm{FW}} : L_q(\mathbb{R}^3; \mathbb{C}^4) \to L_q(\mathbb{R}^3; \mathbb{C}^4)$ with $\hat{U}_{\mathrm{FW}}(\boldsymbol{p})$ given in (2.3) is an isometry.*

Also note that, from (2.1) and (2.2), it follows that

$$\hat{U}_{\mathrm{FW}}(\boldsymbol{p}) T_{\mathrm{D}}(\boldsymbol{p}) \hat{U}_{\mathrm{FW}}^{-1}(\boldsymbol{p}) = \mathcal{F} U_{\mathrm{FW}} T_{\mathrm{D}} U_{\mathrm{FW}}^{-1} \mathcal{F}^{-1} = \beta\sqrt{p^2+m^2} - m, \tag{2.6}$$

and so, by the spectral theorem (for matrices),

$$\begin{aligned}
&\hat{U}_{\mathrm{FW}}(\boldsymbol{p})(T_{\mathrm{D}}(\boldsymbol{p}) - E)^{-1} \hat{U}_{\mathrm{FW}}^{-1}(\boldsymbol{p}) \\
&\quad = (\beta\sqrt{p^2+m^2} - m - E)^{-1} \\
&\quad = \begin{pmatrix} (\sqrt{p^2+m^2} - m - E)^{-1} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & (-\sqrt{p^2+m^2} - m - E)^{-1} I_{2\times 2} \end{pmatrix} \\
&\quad = \beta_+ (\sqrt{p^2+m^2} - m - E)^{-1} + \beta_- (-\sqrt{p^2+m^2} - m - E)^{-1} \\
&\quad \equiv \beta_+ h_E^+(p) + \beta_- h_E^-(p),
\end{aligned} \tag{2.7}$$

where $\beta_\pm := \frac{1}{2}(1 \pm \beta)$. Equation (2.7) makes manifest the fact that the problems $E \to 0$ and $E \to -2m$ are symmetric.

In order to perform L_q-estimates in the Dirac case we need the following lemma, which is a Hölder inequality for matrix-valued functions.

Lemma 2.2. *Let $A : \mathbb{R}^3 \to M_{4\times 4}(\mathbb{C})$, $g : \mathbb{R}^3 \to \mathbb{C}^4$. Then, for $1/q = 1/r + 1/s$,*

$$\|Ag\|_{L_q(\mathbb{R}^3;\mathbb{C}^4)} \leqslant \|\lambda_{\max}(A)\|_{L_r(\mathbb{R}^3)} \|g\|_{L_s(\mathbb{R}^3;\mathbb{C}^4)}, \tag{2.8}$$

where $\lambda_{\max}(A)(\boldsymbol{x}) := \|A(\boldsymbol{x})\|_{\mathcal{B}(\mathbb{C}^4)}$ is the largest eigenvalue (in absolute value) of the matrix $A(\boldsymbol{x})$.

Proof. Let $\mathcal{G}(\boldsymbol{x}) = \|A(\boldsymbol{x})g(\boldsymbol{x})\|_{\mathbb{C}^4}$, $\mathcal{A}(\boldsymbol{x}) = \|A(\boldsymbol{x})\|_{\mathcal{B}(\mathbb{C}^4)}$ and $\mathfrak{g}(\boldsymbol{x}) = \|g(\boldsymbol{x})\|_{\mathbb{C}^4}$. Then

$$\mathcal{G}(\boldsymbol{x}) \leqslant \mathcal{A}(\boldsymbol{x}) \mathfrak{g}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in \mathbb{R}^3, \tag{2.9}$$

and so this inequality, (1.5) and Hölder's inequality imply that

$$\begin{aligned}\|Ag\|_{L_q(\mathbb{R}^3;\mathbb{C}^4)} &= \left(\int_{\mathbb{R}^3} \|A(\boldsymbol{x})g(\boldsymbol{x})\|^q_{\mathbb{C}^4}\,\mathrm{d}x\right)^{1/q} \\ &= \|\mathcal{G}\|_{L_q(\mathbb{R}^3)} \\ &\leqslant \|\mathcal{A}\mathfrak{g}\|_{L_q(\mathbb{R}^3)} \leqslant \|\mathcal{A}\|_{L_r(\mathbb{R}^3)}\|\mathfrak{g}\|_{L_s(\mathbb{R}^3)} \\ &= \|\lambda_{\max}(A)\|_{L_r(\mathbb{R}^3)}\|g\|_{L_s(\mathbb{R}^3;\mathbb{C}^4)}. \end{aligned} \tag{2.10}$$

□

2.2. Preliminaries of the proof

For $E \notin \sigma(T)$ we define $f_E := V^{1/2}\mu_E$ (see also (1.10)) and if (1.13) holds, we set $f_0 := V^{1/2}\mu_0$ and $f_{-2m} := V^{1/2}\mu_{-2m}$, respectively. We rewrite the Lipmann–Schwinger equation (1.9) as

$$\varphi_E = \lambda(T(-\mathrm{i}\nabla) - E)^{-1} f_E. \tag{2.11}$$

The following properties of f_E and its Fourier transform, $\hat{f}_E$, will be important.

LEMMA 2.3. *If $V \in L_1 \cap L_\infty$, then $f_E \in L_1 \cap L_2$. Moreover, $f_E \to f_0$ in L_q for any $q \in [1, 2]$. Consequently, also $\hat{f}_E \to \hat{f}_0$ in L_r for any $r \in [2, \infty]$.*

REMARK 2.4. An analogous result holds when $E \to -2m$, with f_0 replaced by f_{-2m}.

Proof. By lemma 5.1, below, we have that $\mu_E \to \mu_0$ in L_2 as $E \to 0$ for our choice of the potential V. Using the fact that $V^{1/2} \in L_2 \cap L_\infty$, we have, for $E \leqslant 0$, that $f_E \in L_1 \cap L_2$, since

$$\|f_E\|_1 \leqslant \|V^{1/2}\|_2\|\mu_E\|_2, \qquad \|f_E\|_2 \leqslant \|V^{1/2}\|_\infty\|\mu_E\|_2. \tag{2.12}$$

In particular, for $r \in [1, 2]$ and $q = 2r/(2 - r)$, we have that

$$\|f_E - f_0\|_r \leqslant \|V^{1/2}\|_q\|\mu_E - \mu_0\|_2 \to 0 \quad \text{as } E \to 0, \tag{2.13}$$

i.e. $\|f_E - f_0\|_r \to 0$ for any $r \in [1, 2]$. Finally, using the Hausdorff–Young inequality (see, for example, [16, theorem 5.7]) we get the desired result. In the Dirac case, the Hölder inequalities used in (2.12) and (2.13) should be understood in the sense explained in lemma 2.2. □

2.3. The kernels of $(T - E)^{-1}$ and the eigenfunctions in coordinate space

In order to have explicit expressions for (2.11) in coordinates we need to recall the kernels in $\boldsymbol{x}$-space of the operators $(T - E)^{-1}$ for $E \notin \sigma(T)$.

For the Schrödinger case we have the well-known expression (see, for example, [20])

$$(T_{\mathrm{S}} - E)^{-1}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{4\pi}\frac{\exp(-\sqrt{|E|}|\boldsymbol{x} - \boldsymbol{y}|)}{|\boldsymbol{x} - \boldsymbol{y}|}. \tag{2.14}$$

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For the pseudorelativistic case the kernel can be found in [18]; for completeness we also derive it in § 4.1. For $\nu_E = \sqrt{|m^2 - (E+m)^2|}$, we have

$$(T_{\psi\,\mathrm{rel}} - E)^{-1}(\boldsymbol{x},\boldsymbol{y}) = \frac{(E+m)\exp(-\nu_E|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} + \frac{m}{2\pi^2}\frac{K_1(m|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|} + (m^2-\nu_E^2)\left[\frac{m}{2\pi^2}\frac{K_1(m|\cdot|)}{|\cdot|} * \frac{\exp(-\nu_E|\cdot|)}{4\pi|\cdot|}\right](\boldsymbol{x}-\boldsymbol{y}), \tag{2.15}$$

where K_1 is a modified Bessel function of the second kind.

In the Dirac case the kernel is computed in [23]; it is given by

$$(T_{\mathrm{D}} - E)^{-1}(\boldsymbol{x},\boldsymbol{y}) = \frac{\exp(-\nu_E|x-y|)}{4\pi} \times \left(\frac{m\beta+m+E}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{\mathrm{i}\nu_E\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^2} + \frac{\mathrm{i}\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) \tag{2.16}$$

with ν_E as before ($\nu_E = \sqrt{m^2-(E+m)^2}$ since $E \in (-2m, 0)$).

Thus, for $E \notin \sigma(T)$, in coordinate space we write in general (see (2.11))

$$\varphi_E(\boldsymbol{x}) = \lambda \int_{\mathbb{R}^3} (T-E)^{-1}(\boldsymbol{x},\boldsymbol{y}) f_E(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}, \tag{2.17}$$

where as usual T is one of our choices of kinetic energy.

In order to make the connection to the threshold energy states we have the following lemma.

Lemma 2.5. *For $E \notin \sigma(T)$ let φ_E be given pointwise by (2.17) with one of the choices of kernels of $(T-E)^{-1}$ given in (2.14)–(2.16), and let φ_0 be given by (1.17).*

Then, as $E \to 0$, we have that $\varphi_E \to \varphi_0$ in $\mathcal{S}'$. Moreover, $V\varphi_E \to V\varphi_0$ in $\mathcal{S}'$. Case by case, φ_0 is given explicitly by the following.

For the Schrödinger case:

$$\varphi_0(\boldsymbol{x}) = \frac{\lambda_{\mathrm{c}}}{4\pi}\int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|} f_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{2.18}$$

For the pseudorelativistic case:

$$\varphi_0(\boldsymbol{x}) = \lambda_{\mathrm{c}}\int_{\mathbb{R}^3}\left\{\frac{m}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} + \frac{m}{2\pi^2}\frac{K_1(m|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|} + m^2\left[\frac{m}{2\pi^2}\frac{K_1(m|\cdot|)}{|\cdot|} * \frac{1}{4\pi|\cdot|}\right](\boldsymbol{x}-\boldsymbol{y})\right\} f_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{2.19}$$

For the Dirac case:

$$\varphi_0(\boldsymbol{x}) = \frac{\lambda_{\mathrm{c}}}{4\pi}\int_{\mathbb{R}^3}\left(\frac{2m\beta_+}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{\mathrm{i}\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) f_0(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{2.20}$$

Remark 2.6. In the case $E \to -2m$ the limit function φ_{-2m} is given by

$$\varphi_{-2m}(\boldsymbol{x}) = \frac{\lambda_{\mathrm{c}}}{4\pi}\int_{\mathbb{R}^3}\left(\frac{-2m\beta_-}{|\boldsymbol{x}-\boldsymbol{y}|} + \frac{\mathrm{i}\boldsymbol{\alpha}\cdot(\boldsymbol{x}-\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^3}\right) f_{-2m}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{2.21}$$

Proof. By lemma 4.2, below, the functions φ_0 in (2.18)–(2.20) are well defined in $L_1 + L_\infty \subset \mathcal{S}'$ since $f_0 \in L_1 \cap L_2$. The statement on the convergence follows from lemma 4.3, below, using lemma 2.3. In the pseudorelativistic case, the conditions of lemmas 4.2 and 4.3 are satisfied by lemma 4.4. □

2.4. The eigenfunctions in momentum space

Since $\varphi_E \in L_2$ for $E \notin \sigma(T)$, the expressions in momentum space for φ_E in (2.11) are straightforward to derive. In general they are given by

$$[\mathcal{F}\varphi_E](\boldsymbol{p}) = \hat{\varphi}_E(\boldsymbol{p}) = \lambda(T(\boldsymbol{p}) - E)^{-1}\hat{f}_E(\boldsymbol{p}), \tag{2.22}$$

where $T(\boldsymbol{p})$ can be

$$T_{\mathrm{S}}(\boldsymbol{p}) = p^2, \quad T_{\psi\,\mathrm{rel}}(\boldsymbol{p}) = \sqrt{p^2 + m^2} - m \quad \text{or} \quad T_{\mathrm{D}}(\boldsymbol{p}) = \boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta - m, \tag{2.23}$$

for the Schrödinger, pseudorelativistic and Dirac case, respectively. In general, the functions φ_0 are not in L_2.

Lemma 2.7. *For $E \notin \sigma(T)$ let $\hat{\varphi}_E$ be given pointwise by (2.22) with $T(\boldsymbol{p})$ one of the choices given in (2.23). Then, as $E \to 0$, we have that $\hat{\varphi}_E \to \tilde{\varphi}_0 := \lambda_{\mathrm{c}} T(\boldsymbol{p})^{-1}\hat{f}_0$ in $\mathcal{S}'$. Case by case, $\tilde{\varphi}_0$ is given explicitly as follows.*

For the Schrödinger case:

$$\tilde{\varphi}_0(\boldsymbol{p}) = \frac{\lambda_{\mathrm{c}}}{p^2}\hat{f}_0(\boldsymbol{p}). \tag{2.24}$$

For the pseudorelativistic case:

$$\tilde{\varphi}_0(\boldsymbol{p}) = \frac{\lambda_{\mathrm{c}}}{\sqrt{p^2 + m^2} - m}\hat{f}_0(\boldsymbol{p}). \tag{2.25}$$

For the Dirac case:

$$\tilde{\varphi}_0(\boldsymbol{p}) = \lambda_{\mathrm{c}}(\boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta - m)^{-1}\hat{f}_0(\boldsymbol{p}). \tag{2.26}$$

Remark 2.8.

(i) In the case when $E \to -2m$, the limit function $\tilde{\varphi}_{-2m}$ is given by

$$\tilde{\varphi}_{-2m}(\boldsymbol{p}) = \lambda_{\mathrm{c}}(\boldsymbol{\alpha}\cdot\boldsymbol{p} + m\beta + m)^{-1}\hat{f}_{-2m}(\boldsymbol{p}). \tag{2.27}$$

(ii) The limit function denoted by $\tilde{\varphi}_0$ is in fact the Fourier transform of the function φ_0 defined in lemma 2.5. This is proved in the next section (see (3.4)).

The proof of lemma 2.7 is given in § 4.

3. Proof of theorem 1.2

Now we are ready to prove theorem 1.2.

Let $\phi \in \mathcal{S}$. Then (1.8) implies that

$$\langle T(-\mathrm{i}\nabla)\varphi_E, \phi\rangle - \lambda\langle V\varphi_E, \phi\rangle = E\langle\varphi_E, \phi\rangle. \tag{3.1}$$

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Here $\langle\cdot,\cdot\rangle$ is the $(\mathcal{S}',\mathcal{S})$-pairing. Firstly, note that $T(-\mathrm{i}\nabla)\phi \in \mathcal{S}$. Secondly, due to lemma 2.5, we have $\varphi_E \to \varphi_0$ and $V\varphi_E \to V\varphi_0$ in $\mathcal{S}'$ as $E \to 0$. Therefore, taking the limit in (3.1), we get

$$\langle T(-\mathrm{i}\nabla)\varphi_0, \phi\rangle - \lambda_{\mathrm{c}}\langle V\varphi_0, \phi\rangle = 0, \tag{3.2}$$

which proves that φ_0 satisfies $H(\lambda_{\mathrm{c}})\varphi_0 = 0$ in $\mathcal{S}'$. This argument holds for all three choices of T.

Consider the fact that

$$\langle \mathcal{F}\varphi_E, \phi\rangle := \langle \varphi_E, \mathcal{F}\phi\rangle. \tag{3.3}$$

The function $\mathcal{F}\varphi_E$ satisfies (2.22) and, by lemma 2.7, converges in $\mathcal{S}'$ to the function $\tilde{\varphi}_0$ defined in (2.24)–(2.26). On the other hand, by lemma 2.5, the right-hand side of (3.3) converges to $\langle\varphi_0, \mathcal{F}\phi\rangle$ as $E \to 0$. Therefore, taking the limit $E \to 0$ in (3.3) we get

$$\hat{\varphi}_0 = \mathcal{F}\varphi_0 = \tilde{\varphi}_0 \quad \text{in } \mathcal{S}'. \tag{3.4}$$

It remains to prove that, for w satisfying the conditions (1.14)–(1.16), we have

$$w\hat{\varphi}_E \to w\hat{\varphi}_0 \text{ in } L_2 \quad \text{as } E \to 0. \tag{3.5}$$

This is now carried out in detail. We start by working with the general expressions. The specific cases are left to the end. The main object of interest is the difference $w\hat{\varphi}_E - w\hat{\varphi}_0$. This we rewrite using (2.22) and its counterpart for $E = 0$ (now $\hat{\varphi}_0 = \tilde{\varphi}_0 = T(\boldsymbol{p})^{-1}\hat{f}_0$ from (2.24)–(2.26)). We have

$$\begin{aligned}\|w(\hat{\varphi}_E - \hat{\varphi}_0)\|_2 &= \|w(\lambda(T(\boldsymbol{p}) - E)^{-1}\hat{f}_E - \lambda_{\mathrm{c}}T(\boldsymbol{p})^{-1}\hat{f}_0)\|_2 \\ &\leqslant \|w(\lambda(T(\boldsymbol{p}) - E)^{-1} - \lambda_{\mathrm{c}}T(\boldsymbol{p})^{-1})\hat{f}_0\|_2 \\ &\quad + \lambda\|w(T(\boldsymbol{p}) - E)^{-1}(\hat{f}_E - \hat{f}_0)\|_2.\end{aligned}$$

Since $\lambda \to \lambda_{\mathrm{c}}$ as $E \to 0$, it is sufficient to prove that

$$\|w((T(\boldsymbol{p}) - E)^{-1} - T(\boldsymbol{p})^{-1})\hat{f}_0\|_2 \to 0 \quad \text{as } E \to 0 \tag{3.6}$$

and

$$\|w(T(\boldsymbol{p}) - E)^{-1}(\hat{f}_E - \hat{f}_0)\|_2 \to 0 \quad \text{as } E \to 0. \tag{3.7}$$

The term in (3.7) can be estimated by

$$\begin{aligned}\|w(T(\boldsymbol{p}) - E)^{-1}(\hat{f}_E - \hat{f}_0)\|_2 &\leqslant \|w\chi_<(T(\boldsymbol{p}) - E)^{-1}\|_2\|\hat{f}_E - \hat{f}_0\|_\infty \\ &\quad + \|w\chi_>(T(\boldsymbol{p}) - E)^{-1}\|_\infty\|\hat{f}_E - \hat{f}_0\|_2.\end{aligned} \tag{3.8}$$

Owing to lemma 2.3, it is sufficient to show that the first factors in the two terms on the right-hand side of (3.8) stay finite as $E \to 0$. Then (3.7) follows.

Now we prove the convergence statement case by case.

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Schrödinger case. We have that (for $E \leqslant 0$)

$$(T_{\mathrm{S}}(\boldsymbol{p}) - E)^{-1} = (p^2 - E)^{-1} \leqslant p^{-2}. \tag{3.9}$$

Therefore the first two factors on the right-hand side of (3.8) are finite by the condition (1.14). This proves (3.7).

To prove (3.6) we use Lebesgue's theorem of dominated convergence, with the function $2|w(\boldsymbol{p})|\hat{f}_0|/p^2$ as a dominant (see (3.9)); this is in L_2 since we can again split it into large and small p as in (3.8) and use the condition (1.14). Hence, we have proved (3.5) for the Schrödinger case.

Pseudorelativistic case. We here use the fact that, for $E < 0$,

$$(T_{\psi\,\mathrm{rel}}(\boldsymbol{p}) - E)^{-1} = \frac{1}{\sqrt{p^2 + m^2} - m - E} \leqslant \frac{1}{\sqrt{p^2 + m^2} - m}. \tag{3.10}$$

Additionally, there exist constants c_1 and c_2 such that

$$\frac{1}{\sqrt{p^2 + m^2} - m}\chi_<(p) \leqslant c_1 \frac{\chi_<(p)}{p^2} \quad \text{and} \quad \frac{1}{\sqrt{p^2 + m^2} - m}\chi_>(p) \leqslant c_2 \frac{\chi_>(p)}{p}. \tag{3.11}$$

The finiteness of the first two factors on the right-hand side of (3.8) follows from (3.10) by using the estimates (3.11) and the condition (1.15). This proves (3.7).

As before, to prove (3.6) we use Lebesgue's theorem of dominated convergence, with $2|w(\boldsymbol{p})||\hat{f}_0|/(\sqrt{p^2 + m^2} - m)$ as dominant.

Dirac case. We prove the case when $E \to 0$ and comment on the case $E \to -2m$ at the end. Here the general strategy is the same as in the two cases considered above, i.e. we use (3.8) to prove (3.7), and Lebesgue's theorem with the dominant given by the zero-energy expression to prove (3.6). The Hölder estimate in (3.8) should be understood in the sense of lemma 2.2. In order to work with diagonal matrices, we use the Foldy–Wouthuysen transformation $\hat{U}_{\mathrm{FW}}$ defined in (2.3). Using (2.7) we have (with $\tilde{w} = \hat{U}_{\mathrm{FW}} w \hat{U}_{\mathrm{FW}}^{-1}$)

$$\|w\chi_<(T_{\mathrm{D}}(\boldsymbol{p}) - E)^{-1}\|_2 \leqslant \|\tilde{w}\chi_<\beta_+ h_E^+(p)\|_2 + \|\tilde{w}\chi_<\beta_- h_E^-(p)\|_2, \tag{3.12}$$

where we used lemma 2.1 and the fact that $\chi_<$ and $\hat{U}_{\mathrm{FW}}(\boldsymbol{p})$ commute. Analogously, we get

$$\|w\chi_>(T_{\mathrm{D}}(\boldsymbol{p}) - E)^{-1}\|_\infty \leqslant \|\tilde{w}\chi_>\beta_+ h_E^+(p)\|_\infty + \|\tilde{w}\chi_>\beta_- h_E^-(p)\|_\infty. \tag{3.13}$$

The terms with $h_E^+(p)$ are completely analogous to the pseudorelativistic case (see (2.7) and (3.10), (3.11)), except for the fact that the conditions needed for convergence are

$$\left.\begin{aligned} \left\|\tilde{w}\chi_<\beta_+\frac{1}{p^2}\right\|_2 &= \left\|\|\tilde{w}\beta_+\|_{\mathcal{B}(\mathbb{C}^4)}\chi_<\frac{1}{p^2}\right\|_2 < \infty, \\ \left\|\tilde{w}\chi_>\beta_+\frac{1}{p}\right\|_\infty &= \left\|\|\tilde{w}\beta_+\|_{\mathcal{B}(\mathbb{C}^4)}\chi_>\frac{1}{p}\right\|_\infty < \infty. \end{aligned}\right\} \tag{3.14}$$

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The terms with $h_E^-(p)$ are not critical; in fact, for $0 \geqslant E \geqslant -m$ we have

$$|h_E^-(p)| = \frac{1}{\sqrt{p^2+m^2}+m+E} \leqslant \frac{1}{\sqrt{p^2+m^2}}, \tag{3.15}$$

which implies the estimates

$$|h_E^-(p)| \leqslant \frac{1}{m} \quad \text{and} \quad |h_E^-(p)| \leqslant \frac{1}{p}, \tag{3.16}$$

and therefore gives us the following conditions for convergence:

$$\left.\begin{aligned} \|\tilde{w}\chi_<\beta_-\|_2 &= \|\|\tilde{w}\beta_-\|_{\mathcal{B}(\mathbb{C}^4)}\chi_<\|_2 < \infty, \\ \left\|\tilde{w}\chi_>\beta_-\frac{1}{p}\right\|_\infty &= \left\|\|\tilde{w}\beta_-\|_{\mathcal{B}(\mathbb{C}^4)}\chi_>\frac{1}{p}\right\|_\infty < \infty. \end{aligned}\right\} \tag{3.17}$$

Since $\beta_\pm$ are projections and $\hat{U}_{\mathrm{FW}}(\boldsymbol{p})$ is an orthogonal matrix, (3.14) and (3.17) are fulfilled by (1.16).

In the case when $E \to -2m$, we have the following estimates for $-2m \leqslant E \leqslant -m$:

$$|h_E^-(\boldsymbol{p})| \leqslant \frac{1}{\sqrt{p^2+m^2}-m} \quad \text{and} \quad h_E^+(\boldsymbol{p}) \leqslant \frac{1}{\sqrt{p^2+m^2}}, \tag{3.18}$$

i.e. in this case the terms with h_E^- are those analogous to the pseudorelativistic case, and the terms with h_E^+ are non-critical. The conditions (3.14) and (3.17) are the same with the substitution $\beta_\pm \mapsto \beta_\mp$.

Remark 3.1. Note that (3.14) and (3.17) are slightly more general than (1.16), but that (1.16) covers both $E \to 0$ and $E \to -2m$.

4. Useful lemmas

In this section we prove some technical lemmas; these are not optimal and can easily be further generalized, but they are sufficient for our purposes.

The following lemma is a special case of the Hardy–Littlewood–Sobolev inequality in three dimensions [16, theorem 4.3].

Lemma 4.1. *Let $\varepsilon \in (0, \frac{1}{2})$, $\gamma \in [1, 2]$, $g \in L_{1+\varepsilon}(\mathbb{R}^3)$ and $f \in L_q(\mathbb{R}^3)$, $q = (2 - \frac{1}{3}\gamma - 1/(1+\varepsilon))^{-1}$.*

Then

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f(\boldsymbol{x})\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\right| \leqslant C_\gamma \|f\|_{L_q(\mathbb{R}^3)}\|g\|_{L_{1+\varepsilon}(\mathbb{R}^3)}. \tag{4.1}$$

Lemma 4.2. *Let $\varepsilon \in (0, \frac{1}{2})$, $\gamma \in [1, 2]$, and $g \in L_1(\mathbb{R}^3) \cap L_{1+\varepsilon}(\mathbb{R}^3)$, and define*

$$(I^\gamma g)(\boldsymbol{x}) := \int_{\mathbb{R}^3} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}. \tag{4.2}$$

Then $I^\gamma g \in L_1^{\mathrm{loc}}(\mathbb{R}^3)$, and $\|I^\gamma g\|_{L_1(K)} \leqslant C(\gamma, K)\|g\|_{L_{1+\varepsilon}(\mathbb{R}^3)}$ for any compact $K \subset \mathbb{R}^3$. Furthermore, $I^\gamma g = I_1^\gamma g + I_2^\gamma g$ with $I_1^\gamma g \in L_1(\mathbb{R}^3)$, $I_2^\gamma g \in L_\infty(\mathbb{R}^3)$.

Proof. Firstly, multiply (4.2) by the characteristic function χ_K and integrate in $\boldsymbol{x}$. The first statement, and the estimate, follow from Fubini's theorem and lemma 4.1.

Secondly, for $R > 0$, split the integral:

$$\begin{aligned}(I^\gamma g)(\boldsymbol{x}) &= \int_{B_R(\boldsymbol{x})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} + \int_{\mathbb{R}^3\setminus B_R(\boldsymbol{x})} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^\gamma} g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \\ &= (I_1^\gamma g)(\boldsymbol{x}) + (I_2^\gamma g)(\boldsymbol{x}). \end{aligned} \tag{4.3}$$

For the first term in (4.3) use [7, lemma 7.12], which says that, for $q \in [1,\infty]$ and $0 \leqslant 1/p - 1/q < 1 - \gamma/3$, I_1^γ maps $L_p(\mathbb{R}^3)$ continuously into $L_q(\mathbb{R}^3)$ with

$$\|I_1^\gamma g\|_q \leqslant C_{\gamma,p,q}\|g\|_p.$$

Use this with $p = q = 1$. Then $I_1^\gamma g \in L_1(\mathbb{R}^3)$.

For the second term in (4.3),

$$|(I_2^\gamma g)(\boldsymbol{x})| \leqslant \int_{\mathbb{R}^3\setminus B_R(\boldsymbol{x})} \frac{1}{R^\gamma}|g(\boldsymbol{y})|\,\mathrm{d}\boldsymbol{y} \leqslant \frac{1}{R^\gamma}\|g\|_1,$$

so $I_2^\gamma g \in L_\infty(\mathbb{R}^3)$. □

Lemma 4.3. *Let $\varepsilon \in (0, \frac{1}{2})$, $\gamma \in [1,2]$, and let $G_n, G : \mathbb{R}^3 \to \mathbb{R}^3$, $n \in \mathbb{N}$, satisfy $G_n(\boldsymbol{x}) \to G(\boldsymbol{x})$ as $n \to \infty$. Assume there exist $c_1, c_2 \in \mathbb{R}_+$ such that*

$$|G_n(\boldsymbol{x})| \leqslant \frac{c_1}{|\boldsymbol{x}|^\gamma} \quad \text{and} \quad |G(\boldsymbol{x})| \leqslant \frac{c_2}{|\boldsymbol{x}|^\gamma}. \tag{4.4}$$

Let $\{g_n\}_{n\in\mathbb{N}} \subset L_{1+\varepsilon}(\mathbb{R}^3)$ satisfy $g_n \to g$ in $L_{1+\varepsilon}(\mathbb{R}^3)$ as $n \to \infty$. Define the functions

$$(T_n^\gamma g_n)(\boldsymbol{x}) := \int_{\mathbb{R}^3} G_n(\boldsymbol{x}-\boldsymbol{y}) g_n(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$$

and

$$(T^\gamma g)(\boldsymbol{x}) := \int_{\mathbb{R}^3} G(\boldsymbol{x}-\boldsymbol{y}) g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}.$$

Then $VT_n^\gamma g_n \to VT^\gamma g$ in $\mathcal{S}'(\mathbb{R}^3)$ for all $V \in L_\infty(\mathbb{R}^3)$. In particular, $T_n^\gamma g_n \to T^\gamma g$ in $\mathcal{S}'(\mathbb{R}^3)$.

Proof. It follows from lemma 4.2 and (4.4) that

$$VT_n^\gamma g_n, VT^\gamma g \in L_1(\mathbb{R}^3) + L_\infty(\mathbb{R}^3) \subset \mathcal{S}'(\mathbb{R}^3).$$

For $\phi \in \mathcal{S}(\mathbb{R}^3) \subset L_q(\mathbb{R}^3)$, $q > 1$, we have, using Fubini's theorem, that

$$\begin{aligned}\langle V(T_n^\gamma g_n - T^\gamma g), \phi\rangle &= \int_{\mathbb{R}^6} \phi(\boldsymbol{x})V(\boldsymbol{x})[G_n(\boldsymbol{x}-\boldsymbol{y}) - G(\boldsymbol{x}-\boldsymbol{y})]g(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\ &\quad + \int_{\mathbb{R}^6} \phi(\boldsymbol{x})V(\boldsymbol{x})G_n(\boldsymbol{x}-\boldsymbol{y})(g_n - g)(\boldsymbol{y})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \\ &\equiv I_1(n) + I_2(n).\end{aligned}$$

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We will use Lebesgue's theorem of dominated convergence for $I_1(n)$. Lemma 4.1 shows that the inequality

$$|G_n(\boldsymbol{x}-\boldsymbol{y}) - G(\boldsymbol{x}-\boldsymbol{y})| \leqslant \frac{c_1+c_2}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma}}$$

provides a dominant, so that $I_1(n) \to 0$ as $n \to \infty$, since $G_n(\boldsymbol{x}) \to G(\boldsymbol{x})$ as $n \to \infty$.

For $I_2(n)$, the first inequality in (4.4) and lemma 4.1 give that $I_2(n) \to 0$ as $n \to \infty$, since $g_n \to g$ in $L_{1+\varepsilon}(\mathbb{R}^3)$ by assumption. □

4.1. The pseudorelativistic kernel

Although (2.15) is given in [18] we want to sketch its proof. Let us start by noting that (see [16, 7.11 (11)])

$$\begin{aligned}\left(\frac{1}{\sqrt{-\Delta+m^2}}\right)(\boldsymbol{x},\boldsymbol{y}) &= \frac{m^2}{2\pi^2}\int_0^\infty \frac{t}{t^2+|\boldsymbol{x}-\boldsymbol{y}|^2} K_2(m(t^2+|\boldsymbol{x}-\boldsymbol{y}|^2)^{1/2})\,\mathrm{d}t \\ &= \frac{m^2}{2\pi^2}\int_{m|\boldsymbol{x}-\boldsymbol{y}|}^\infty \frac{K_2(s)}{s}\,\mathrm{d}s \\ &= \frac{m}{2\pi^2}\frac{K_1(m|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|}, \end{aligned} \tag{4.5}$$

where in the latter step we used the fact that $K_2(x)/x = -(K_1(x)/x)'$ and that $K_1(s)/s \to 0$ as $s \to \infty$ (see [8, (8.486.15)] and (4.9) below). On the other hand, with $\nu_E = \sqrt{m^2-(E+m)^2}$ and $E<0$, we have the operator identity

$$\begin{aligned}\frac{1}{\sqrt{-\Delta+m^2}-m-E} &= \frac{E+m}{-\Delta+\nu_E^2} + \frac{1}{\sqrt{-\Delta+m^2}} \\ &\quad + (m^2-\nu_E^2)\frac{1}{\sqrt{-\Delta+m^2}}\frac{1}{-\Delta+\nu_E^2}. \end{aligned} \tag{4.6}$$

The expression in (2.15) follows by computing the kernel of each summand of (4.6) separately, using (2.14) and (4.5).

Next we have the following convergence statement for the third summand in equation (2.15).

Lemma 4.4. *For* $\nu_E = \sqrt{m^2-(E+m)^2}$*,* $E<0$*, and* $\boldsymbol{x} \in \mathbb{R}^3$*, we have*

$$\left[\frac{K_1(m|\cdot|)}{|\cdot|} * \frac{\exp(-\nu_E|\cdot|)}{|\cdot|}\right](\boldsymbol{x}) \to \left[\frac{K_1(m|\cdot|)}{|\cdot|} * \frac{1}{|\cdot|}\right](\boldsymbol{x}) \quad \text{as } E \to 0. \tag{4.7}$$

Moreover, there exists a constant $c_1 > 0$ *such that*

$$\left[\frac{K_1(m|\cdot|)}{|\cdot|} * \frac{\exp(-\nu_E|\cdot|)}{|\cdot|}\right](\boldsymbol{x}) \leqslant \left[\frac{K_1(m|\cdot|)}{|\cdot|} * \frac{1}{|\cdot|}\right](\boldsymbol{x}) \leqslant \frac{c_1}{|\boldsymbol{x}|}. \tag{4.8}$$

Proof. The following properties of the Bessel function K_1 (see [8, (8.446), (8.451.6)]) will be useful: there exist constants c and ρ such that

$$K_1(x) \leqslant c\frac{\mathrm{e}^{-x}}{\sqrt{x}} \quad \text{for } x > \rho; \tag{4.9}$$

moreover, for $x > 0$,

$$K_1(x) \leqslant \frac{1}{x}. \tag{4.10}$$

Then, by Newton's theorem (see, for example, [16]),

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{\exp(-\nu_E|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|}\frac{K_1(m|\boldsymbol{y}|)}{|\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} &\leqslant \int_{\mathbb{R}^3}\frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\frac{K_1(m|\boldsymbol{y}|)}{|\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y} \\ &\leqslant \frac{1}{|\boldsymbol{x}|}\int_{\mathbb{R}^3}\frac{K_1(m|\boldsymbol{y}|)}{|\boldsymbol{y}|}\,\mathrm{d}\boldsymbol{y}. \end{aligned}\tag{4.11}$$

The last integral is finite by (4.9) and (4.10); this proves (4.8). The convergence in (4.7) follows from Lebesgue's monotone convergence theorem. □

4.2. Proof of lemma 2.7

Let $\phi \in \mathcal{S} \subset L_q$, $q \geqslant 1$. Then

$$\begin{aligned}\langle(\mathcal{F}\varphi_E - \tilde{\varphi}_0), \phi\rangle &= \int_{\mathbb{R}^3}[(T(\boldsymbol{p})-E)^{-1}\hat{f}_E(\boldsymbol{p}) - T(\boldsymbol{p})^{-1}\hat{f}_0(\boldsymbol{p})]\cdot\phi(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} \\ &= \int_{\mathbb{R}^3}((T(\boldsymbol{p})-E)^{-1} - T(\boldsymbol{p})^{-1})\hat{f}_0(\boldsymbol{p})\cdot\phi(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} \\ &\quad + \int_{\mathbb{R}^3}(T(\boldsymbol{p})-E)^{-1}(\hat{f}_E - \hat{f}_0)(\boldsymbol{p})\cdot\phi(\boldsymbol{p})\,\mathrm{d}\boldsymbol{p} \\ &\equiv I_1(E) + I_2(E).\end{aligned}$$

(In the Dirac case, the dot denotes the scalar product in $\mathbb{C}^4$.)

We first consider the Schrödinger and the pseudorelativistic cases.

Note that, in both cases, there exist positive constants $c_<$, $c_>$ such that, for all $\boldsymbol{p} \in \mathbb{R}^3$ and $E \leqslant 0$ (for the pseudorelativistic case, use (3.11)),

$$|(T(\boldsymbol{p})-E)^{-1}\phi(\boldsymbol{p})| \leqslant \frac{c_<}{p^2}\chi_<(\boldsymbol{p})\phi(\boldsymbol{p}) + c_>\chi_2(\boldsymbol{p})\phi(\boldsymbol{p}).$$

By Hölder's inequality, this implies that

$$|I_2(E)| \leqslant C\|\hat{f}_E - \hat{f}_0\|_\infty\biggl(\biggl\|\frac{\chi_<\phi}{p^2}\biggr\|_1 + \|\phi\|_1\biggr).$$

The last factor is finite since $\phi \in \mathcal{S}(\mathbb{R}^3) \subset L_1(\mathbb{R}^3)$, and by lemma 2.3 the first one goes to zero as E goes to zero, so $I_2(E) \to 0, E \to 0$.

For I_1, we use Lebesgue's theorem of dominated convergence. By arguments similar to that above, the function $c(\chi_</p^2 + \chi_>)\phi$ is a dominant (for some $c > 0$) and therefore also $I_1(E) \to 0$, $E \to 0$.

For the Dirac case,

$$|I_1(E)| \leqslant \int_{\mathbb{R}^3}\|(T(\boldsymbol{p})-E)^{-1} - T(\boldsymbol{p})^{-1}\|_{\mathcal{B}(\mathbb{C}^4)}\|\hat{f}_0(\boldsymbol{p})\|_{\mathbb{C}^4}\|\phi(\boldsymbol{p})\|_{\mathbb{C}^4}\,\mathrm{d}\boldsymbol{p}.$$

Using the fact that $\hat{U}_{\mathrm{FW}}(\boldsymbol{p})$ is an orthogonal matrix for all $\boldsymbol{p} \in \mathbb{R}^3$, and (2.7), for $-m \leqslant E \leqslant 0$ we have that

$$
\begin{aligned}
&\|(T(\boldsymbol{p}) - E)^{-1}\|_{\mathcal{B}(\mathbb{C}^4)} \\
&\quad = \left\| \begin{pmatrix} (\sqrt{p^2+m^2} - m - E)^{-1} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & (-\sqrt{p^2+m^2} - m - E)^{-1} I_{2\times 2} \end{pmatrix} \right\|_{\mathcal{B}(\mathbb{C}^4)} \\
&\quad = (\sqrt{p^2+m^2} - m - E)^{-1} \leqslant (\sqrt{p^2+m^2} - m)^{-1}. \qquad (4.12)
\end{aligned}
$$

By an argument as above (in the pseudorelativistic case), Lebesgue's theorem on dominated convergence gives that $I_1(E) \to 0$, $E \to 0$ also in this case. Also by arguments as above, (4.12) and the fact that (by lemma 2.3) $\hat{f}_E \to \hat{f}_0$ in L_∞ gives that also $I_2(E) \to 0, E \to 0$.

Note that a similar argument works for the Dirac case when $E \to -2m$; in this case, for $-2m \leqslant E \leqslant -m$,

$$
\begin{aligned}
&\|(T(\boldsymbol{p}) - E)^{-1}\|_{\mathcal{B}(\mathbb{C}^4)} \\
&\quad = \left\| \begin{pmatrix} (\sqrt{p^2+m^2} - m - E)^{-1} I_{2\times 2} & 0_{2\times 2} \\ 0_{2\times 2} & (-\sqrt{p^2+m^2} - m - E)^{-1} I_{2\times 2} \end{pmatrix} \right\|_{\mathcal{B}(\mathbb{C}^4)} \\
&\quad = (\sqrt{p^2+m^2} + m + E)^{-1} \leqslant (\sqrt{p^2+m^2} - m)^{-1}. \qquad (4.13)
\end{aligned}
$$

5. Convergence of Birman–Schwinger operators and eigenfunctions

We denote the compact operators by $\mathcal{S}_\infty$. For $r \geqslant 1$, we denote by $\mathcal{S}_r$ the r'th Schatten-class of compact operators (which is a norm-closed two-sided ideal in $\mathcal{S}_\infty$) and $\|\cdot\|_{\mathcal{S}_r}$ its norm.

Lemma 5.1. *Let $\varepsilon > 0$ and assume that $V \geqslant 0$ satisfies*

$$V \in L_{3/2+\varepsilon}(\mathbb{R}^3) \cap L_{3/2-\varepsilon}(\mathbb{R}^3) \text{ and } E < 0 \qquad \text{(Schrödinger case)}, \qquad (5.1)$$

$$V \in L_{3+\varepsilon}(\mathbb{R}^3) \cap L_{3/2-\varepsilon}(\mathbb{R}^3) \text{ and } E < 0 \qquad \text{(pseudorelativistic case)}, \qquad (5.2)$$

$$V \in L_{3+\varepsilon}(\mathbb{R}^3; \mathbb{C}^4) \cap L_{3-\varepsilon}(\mathbb{R}^3; \mathbb{C}^4) \text{ and } E \in (-2m, 0) \qquad \text{(Dirac case)}. \qquad (5.3)$$

Let λ_{c} be a coupling constant threshold, and let $\lambda_n, E_n, \varphi_{E_n}$ satisfy $(T - \lambda_n V)\varphi_{E_n} = E_n \varphi_{E_n}$, $\|\varphi_{E_n}\|_2 = 1$, $\lambda_n \downarrow \lambda_{\mathrm{c}}$ as $E_n \uparrow 0$ (or $\lambda_n \uparrow \lambda_{\mathrm{c}}$ when $E_n \downarrow -2m$ in the Dirac case). Finally, let

$$K_E = V^{1/2}(T(-\mathrm{i}\nabla) - E)^{-1} V^{1/2} \qquad (5.4)$$

be the Birman–Schwinger operator and $\mu_{E_n} = V^{1/2}\varphi_{E_n}$ be the Birman–Schwinger eigenfunctions associated with φ_{E_n}.

Then

(i) *K_{E_n} is a compact operator,*

(ii) *the norm-limit $K_0 := \lim_{n\to\infty} K_{E_n}$ exists (and, in the Dirac case, $K_{-2m} := \lim_{n\to\infty} K_{E_n}$ exists),*

(iii) *K_0 (and in the Dirac case, K_{-2m}) is compact,*

(iv) *there exists a subsequence* $\{\mu_{E_{n_k}}\}_{k\in\mathbb{N}}$ *and* $\mu_0 \in L_2$ *such that* $\mu_{E_{n_k}} \to \mu_0$ *as* $k \to \infty$ *and* $K_0\mu_0 = (1/\lambda_c)\mu_0$.

Proof. In the Schrödinger and pseudorelativistic cases, it is enough to show that $V^{1/2}(T(-\mathrm{i}\nabla) - E_n)^{-1/2}$ is compact (since S is compact if and only if S^*S is compact). For this, we will use the fact that operators of the form $f(x)g(-\mathrm{i}\nabla)$ belong to $\mathcal{S}_r$ if $f, g \in L_r$, $r \in [2,\infty)$, and that furthermore

$$\|f(x)g(-\mathrm{i}\nabla)\|_{\mathcal{S}_r} \leqslant (2\pi)^{-3/r}\|f\|_r\|g\|_r \tag{5.5}$$

(see [21, theorem XI.20]). Note that, for $E < 0$, the function $(p^2 - E)^{-1/2}$ belongs to $L_{3+\varepsilon}(\mathbb{R}^3)$, and $(\sqrt{p^2+m^2} - m - E)^{-1/2}$ belongs to $L_{6+\varepsilon}(\mathbb{R}^3)$. By (5.5) and the assumptions (5.1) and (5.2) on the potential V, this implies that the Birman–Schwinger operator K_{E_n} is compact in both cases.

To show the statement on convergence, write

$$\begin{aligned} S_{E_n} &:= V^{1/2}(T(-\mathrm{i}\nabla) - E_n)^{-1/2} \\ &= V^{1/2}\mathcal{F}^{-1}(T(\boldsymbol{p}) - E_n)^{-1/2}\chi_<(p)\mathcal{F} + V^{1/2}\mathcal{F}^{-1}(T(\boldsymbol{p}) - E_n)^{-1/2}\chi_>(p)\mathcal{F} \\ &\equiv S_{n,<} + S_{n,>}. \end{aligned} \tag{5.6}$$

Again using (5.5), the assumptions (5.1) and (5.2) on the potential, and Lebesgue's theorem on dominated convergence, $\{S_{n,<}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the $\mathcal{S}_r$-norm for $r \in [2,3)$ (in both cases), and $\{S_{n,>}\}_{n\in\mathbb{N}}$ in the $\mathcal{S}_r$-norm for $r \in (3,\infty)$ in the Schrödinger case, and for $r \in (6,\infty)$ in the pseudorelativistic case. Therefore, both sequences are Cauchy sequences in the operator norm. Since the set of compact operators is norm-closed, $\lim_{n\to\infty} S_{n,\gtrless}$ exist, and are compact operators. Therefore, $K_0 := \lim_{n\to\infty} K_{E_n}$ exists and is compact, in both the Schrödinger and the pseudorelativistic cases.

The proof in the Dirac case is essentially the same, only slightly more involved due to the fact that $T_\mathrm{D} - E$ is not positive. Note that, using the Foldy–Wouthuysen transformation U_FW, we have (see (2.7))

$$\begin{aligned} V^{1/2}(T_\mathrm{D} - E)^{-1}V^{1/2} &= V^{1/2}U_\mathrm{FW}^{-1}\mathcal{F}^{-1}(\beta\sqrt{p^2+m^2} - m - E)^{-1}\mathcal{F}U_\mathrm{FW}V^{1/2} \\ &= S_+^*S_+ - S_-^*S_-, \end{aligned}$$

with

$$S_+ = \begin{pmatrix} (\sqrt{p^2+m^2} - m - E)^{-1/2}I_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} \end{pmatrix}\mathcal{F}U_\mathrm{FW}V^{1/2}, \tag{5.7}$$

$$S_- = \begin{pmatrix} 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & (\sqrt{p^2+m^2} + m + E)^{-1/2}I_{2\times2} \end{pmatrix}\mathcal{F}U_\mathrm{FW}V^{1/2}. \tag{5.8}$$

As before, it suffices to prove that S_+ and S_- are compact. Note that U_FW is bounded, and that both of the functions

$$(\sqrt{p^2+m^2} - m - E)^{-1/2} \quad \text{and} \quad (\sqrt{p^2+m^2} + m + E)^{-1/2}$$

belong to $L_{6+\varepsilon}$ (since $E \in (-2m, 0)$), and so the same argument as above implies that S_+ and S_- are compact. It follows that K_{E_n} is compact also in the Dirac case. The convergence follows by similar arguments as above.

It remains to prove (iv). Note that $\|\mu_{E_n}\|_2 \leqslant C$ since $V \in L_\infty$ and $\|\varphi_{E_n}\|_2 = 1$. Since K_0 is compact, there exists a subsequence $\{\mu_{E_{n_k}}\}_{k\in\mathbb{N}}$ such that $\psi := \lim_{k\to\infty} K_0\mu_{E_{n_k}}$ exists. Using (ii) we get that $\|K_{E_{n_k}}\mu_{E_{n_k}} - \psi\|_2 \to 0$ as $k \to \infty$. Since

$$K_{E_n}\mu_{E_n} = \frac{1}{\lambda_n}\mu_{E_n} \quad \text{and} \quad \lambda_n \to \lambda_{\mathrm{c}} \quad \text{as } n \to \infty,$$

it follows that $\mu_0 := \lim_{k\to\infty} \mu_{E_{n_k}}$ exists, and satisfies $K_0\mu_0 = (1/\lambda_{\mathrm{c}})\mu_0$. □

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