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ON SECTORIALITY OF DEGENERATE ELLIPTIC OPERATORS

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Abstract Let $c_{kl} \in W^{1,\infty}(\Omega, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$; and $\Omega \subset \mathbb{R}^d$ be open with uniformly C^2 boundary. We consider the divergence form operator $A_p = -\sum_{k,l=1}^{d} \partial_l(c_{kl}\partial_k)$ in $L_p(\Omega)$ when the coefficient matrix satisfies $(C(x)\xi, \xi) \in \Sigma_\theta$ for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$, where Σ_θ be the sector with vertex 0 and semi-angle θ in the complex plane. We show that a sectorial estimate holds for A_p for all p in a suitable range. We then apply these estimates to prove that the closure of $-A_p$ generates a holomorphic semigroup under further assumptions on the coefficients. The contractivity and consistency properties of these holomorphic semigroups are also considered.

Keywords: degenerate elliptic operator; sectorial operator; holomorphic semigroup; contraction semigroup; consistent semigroup

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1. Introduction

In his book, Kato [11] showed that an m-sectorial operator in a Hilbert space generates a (quasi-)contraction holomorphic semigroup. One can generalize the notion of sectorial operators to L_p-spaces as follows (cf. [10, Definition 1.5.8, 11, Subsection V.3.10, 2, Definition 1]).

Definition 1.1. Let $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be open and $p \in (1, \infty)$. Let A_p be an operator in $L_p(\Omega)$. Then A_p is said to be *sectorial* if there exists a $K > 0$ such that

$$|\mathrm{Im}\,(A_p u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]})| \leq K \mathrm{Re}\,(A_p u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) \tag{1}$$

for all $u \in D(A_p)$.

There are certain interests in showing that an operator is sectorial in this generalized sense. The significance of these estimates lies in the fact that they are useful in showing that the operators under consideration satisfy a necessary condition to generate holomorphic contraction semigroups. In particular, the estimate (1) can be established for certain second-order differential operators in divergence form. In the proof of [16, Theorem 7.3.6], Pazy showed that (1) holds when the operator is strongly elliptic with symmetric real-valued C^1-coefficients, with an explicit constant K which depends on the coefficients, the ellipticity constant and p. Okazawa improved Pazy's result and showed that the estimate also holds for degenerate elliptic operators with symmetric real-valued C^1-coefficients, with $K = (|p-2|/2\sqrt{p-1})$ (cf. [14]). Ouhabaz in [15, Theorem 3.9] proved that (1) is true for generators of sub-Markovian semigroups. It is interesting to note that [15, Theorem 3.9] gives the same constant K in (1) as in [14].

In this paper, we will prove the sectorial estimate (1) for degenerate elliptic second-order differential operators with bounded complex-valued coefficients. The results are generalizations of [14]. In comparison to [15, Theorem 3.9], we note that the operators we consider here are, in general, no longer generators of sub-Markovian semigroups. We will then apply the estimate to show that degenerate elliptic operators with smooth enough coefficients generate contraction holomorphic semigroups.

In order to formulate the main theorem, we need to introduce some notation. Let $d \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ be open with uniformly C^2 boundary and $\theta \in [0, \pi/2)$. Let $c_{kl} \in W^{1,\infty}(\Omega, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$. Define $C = (c_{kl})_{1 \le k,l \le d}$ and

$$\Sigma_\theta = \{re^{i\beta} : r \ge 0 \text{ and } |\beta| \le \theta\}. \tag{2}$$

Assume that

$$(C(x)\xi, \xi) \in \Sigma_\theta \tag{3}$$

for all $x \in \Omega$ and $\xi \in \mathbb{C}^d$. For convenience, we will usually refer to (3) as C *takes values in the sector* Σ_θ.

Let $p \in (1, \infty)$. Consider the operator A_p in $L_p(\Omega)$ defined by

$$A_p u = -\sum_{k,l=1}^{d} \partial_l (c_{kl} \partial_k u)$$

on the domain

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

If $p = 2$ then

$$|\mathrm{Im}\,(A_2 u, u)| \le (\tan\theta) \mathrm{Re}\,(A_2 u, u) \tag{4}$$

for all $u \in D(A_2)$. This follows immediately from integration by parts. If $p \ne 2$, the situation is quite different. Write $C = R + iB$, where R and B are real matrices. Let R_a and B_a be the anti-symmetric parts of R and B, respectively, that is, $R_a = (R - R^T)/2$ and $B_a = (B - B^T)/2$.

The main result of this paper is as follows.

Theorem 1.2. *Let $p \in (1, \infty)$, $\theta \in [0, \pi/2)$, $c_{kl} \in W^{1,\infty}(\Omega, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$ and $C = (c_{kl})_{1 \le k,l \le d}$ take values in the sector Σ_θ. Suppose $|1 - 2/p| < \cos\theta$ and $B_a = 0$. Then*

$$|\mathrm{Im}\,(A_p u, |u|^{p-2} u \mathbb{1}_{[u \ne 0]})| \le K \mathrm{Re}\,(A_p u, |u|^{p-2} u \mathbb{1}_{[u \ne 0]})$$

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for all $u \in D(A_p)$, *where*

$$K = \begin{cases} \tan\left(\frac{\pi}{2} - \phi + \theta\right) & \text{if } R_a = 0, \\ \frac{\left(2/\sin\phi - 1\right)\tan\theta + \cot\phi}{1 - (\tan\theta)\cot\phi} & \text{if } R_a \neq 0 \end{cases} \tag{5}$$

and $\phi = \arccos|1 - 2/p|$.

Note that when the coefficient matrix C consists of real entries and is symmetric, then one can choose $\theta = 0$ and (5) gives

$$K = \tan\left(\frac{\pi}{2} - \phi\right) = \cot\phi = \frac{|p-2|}{2\sqrt{p-1}},$$

which is the constant obtained by Okazawa in [14].

Remark 1.3. The conditions, conclusion and some implications can be rephrased in the recently introduced terminology of Carbonaro and Dragičević [1]. For every $p \in (1, \infty)$ and bounded $d \times d$ matrix valued function $M : \Omega \longrightarrow \mathbb{C}^{d\times d}$ define

$$\Delta_p(M) := ess\,inf_{x\in\Omega} \min_{\substack{\xi\in\mathbb{C}^d \\ \|\xi\|=1}} \operatorname{Re}\left(M(x)\xi, \mathcal{J}_p\xi\right), \tag{6}$$

where $\mathcal{J}_p : \mathbb{C}^d \longrightarrow \mathbb{C}^d$ is defined by

$$\mathcal{J}_p\xi = \xi + \left(1 - \frac{2}{p}\right)\overline{\xi}. \tag{7}$$

Suppose merely $c_{kl} \in L_\infty(\Omega, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$. Then C takes values in the sector Σ_θ if and only if $\Delta_2(e^{\pm i\psi}C) \geq 0$ for all $\psi \in [0, \pi/2 - \theta)$. Also, $|1 - 2/p| < \cos\theta$ if and only if $\Delta_p(e^{i\theta}I) > 0$ (cf. [1, (5.18)]).

Now $\phi > 0$ by assumption and

$$\left(e^{i\gamma}C(x)\xi, \xi\right) \in \Sigma_{\theta+|\gamma|} \subset \Sigma_\phi$$

for all $x \in \Omega$, $\xi \in \mathbb{C}^d$ and $\gamma \in \mathbb{R}$ with $|\gamma| \leq \phi - \theta$.

If both $B_a = 0$ and $R_a = 0$, then [1, Proposition 5.18 (3 $\Rightarrow$ 1)] implies that

$$\Delta_p\left(e^{\pm i(\phi-\theta)}C\right) \geq 0. \tag{8}$$

On the other hand, if $\Delta_2(C) > 0$ (the operator is strongly elliptic) then [1, Theorem 1.3 (a $\Rightarrow$ b)] together with the Lumer–Phillips theorem establishes that (8) implies (1), where $K = \tan(\pi/2 - \phi + \theta)$ which coincides with (5) if $R_a = B_a = 0$.

If $R_a = B_a = 0$ then one has equivalence in [1, Proposition 5.18 (3 $\Leftrightarrow$ 1)] and in the strongly elliptic case one also has equivalence in [1, Theorem 1.3 (a $\Leftrightarrow$ b)]. Hence, the angle of the sector of contractivity in $L_p(\Omega)$, that is $\pi/2 - \arctan K$, is optimal. Consequently, also K is optimal if $R_a = B_a = 0$.

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In Theorem 1.2, we do not require that $R_a = 0$ nor strong ellipticity, but we require Lipschitz continuity of the c_{kl}.

It is not difficult to see that A_p is closable. Let $\overline{A_p}$ be the closure of A_p. Under the current conditions imposed on the coefficient matrix C and the domain Ω, we do not know whether $-\overline{A_p}$ is a generator of a C_0-semigroup. If $\Omega = \mathbb{R}^d$ and C consists of twice differentiable entries, then we prove the following generation result for $-\overline{A_p}$ based on Theorem 1.2.

Theorem 1.4. *Let $p \in (1,\infty)$, $\theta \in [0,\pi/2)$, $c_{kl} \in W^{2,\infty}(\mathbb{R}^d,\mathbb{C})$ for all $k,l \in \{1,\ldots,d\}$ and $C = (c_{kl})_{1\le k,l\le d}$ take values in the sector Σ_θ. Suppose $|1-2/p| < \cos\theta$ and $B_a = 0$. Set $\phi = \arccos|1-2/p|$. Then the closure $-\overline{A_p}$ generates a holomorphic semigroup on $L_p(\mathbb{R}^d)$ with angle ψ given by*

$$\psi = \begin{cases} \phi - \theta, & \text{if } R_a = 0,\\ \dfrac{\pi}{2} - \arctan\left(\dfrac{\big(2/\sin\phi - 1\big)\tan\theta + \cot\phi}{1-(\tan\theta)\cot\phi}\right), & \text{if } R_a \neq 0. \end{cases} \tag{9}$$

Note that

$$\psi_1 := \frac{\pi}{2} - \arctan\left(\frac{\big(2/\sin\phi - 1\big)\tan\theta + \cot\phi}{1-(\tan\theta)\cot\phi}\right) \le \phi - \theta$$

since

$$\tan\psi_1 = \frac{1-(\tan\theta)\cot\phi}{\big(2/\sin\phi - 1\big)\tan\theta + \cot\phi} \le \frac{1-(\tan\theta)\cot\phi}{\tan\theta + \cot\phi} = \tan(\phi-\theta). \tag{10}$$

It is also interesting that in the case when $R_a = 0$, Theorem 1.4 provides better angles of holomorphy compared with those of Stein's interpolations [15, Proposition 3.12] and [18, Theorem 1]. In the one-dimensional case, these better angles were also obtained in [5, Corollary 1.3].

Along the same line as our results, [8] considered a type of second-order degenerate elliptic operator in divergence form whose coefficients of the principle part need not satisfy the sectorial condition (3). Other results about angles of holomorphy were considered in [19, Theorem 1, 9, Theorem 1.1, 3, Theorem 1.4.2, 17, Theorem X.55, 12, 15, Theorems 3.12 and 3.13].

The holomorphic semigroup generated by $-\overline{A_p}$ in Theorem 1.4 also possesses nice contractivity and consistency properties.

Theorem 1.5. *Adopt the assumptions and notation as in Theorem 1.4. Let $S^{(p)}$ be the semigroup generated by $-\overline{A_p}$ and S the semigroup generated by $-\overline{A_2}$. Then the following hold.*

(i) *$S^{(p)}$ is contractive on Σ_γ, where*

$$\gamma = \begin{cases} \psi & \text{if } R_a = 0,\\ \psi \wedge \sup\left\{\beta \in \left[0,\dfrac{\pi}{2}\right) : (\tan\theta)\tan\beta < \dfrac{1}{3}\right\} & \text{if } R_a \neq 0. \end{cases} \tag{11}$$

(ii) *$S^{(p)}$ is consistent with S on Σ_ψ.*

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Recently, there is a lot of interest in differential operators with complex coefficients which are accretive on $L_p(\Omega)$ with $p \neq 2$ and then are the minus generator of a C_0-semigroup on $L_p(\Omega)$. Strongly elliptic operators with mixed boundary conditions are considered in [6, 7]. All results in [1] for C_0-semigroups are for strongly elliptic operators. The main emphasis in this paper is to consider degenerate elliptic operators. In [8], the operator is allowed to be degenerate elliptic, but the coefficient matrix cannot degenerate on a set with positive measure. For $W^{1,\infty}$-coefficients in one dimension, the coefficient function cannot vanish at any point in [8]. In contrast, our operators may degenerate on a set with positive measure. The domain of the operator is delicate for proving the range condition for the C_0-semigroup and this is even more delicate for degenerate operators.

The outline of subsequent sections is as follows. In §2, we provide some estimates on the coefficient matrix C. These estimates are used to prove Theorem 1.2 in §3. Theorems 1.4 and 1.5 are proved in §4, in the proof of which we use a density result [4, Proposition 4.9] that is valid if $\Omega = \mathbb{R}^d$. This explains why we require $\Omega = \mathbb{R}^d$ in Theorems 1.4 and 1.5.

2. Estimates on coefficients

Let Ω, θ and C be as in §1. In this section, we provide some preliminary estimates on the coefficient matrix C for later use.

Define

$$\operatorname{Re} C = \frac{C + C^*}{2} \quad \text{and} \quad \operatorname{Im} C = \frac{C - C^*}{2i},$$

where C^* is the conjugate transpose of C. Then $(\operatorname{Re} C)(x)$ and $(\operatorname{Im} C)(x)$ are self-adjoint for all $x \in \Omega$ and

$$C = \operatorname{Re} C + i \operatorname{Im} C. \tag{12}$$

It is important to keep in mind that $\operatorname{Re} C$ and $\operatorname{Im} C$ defined in this manner are not necessarily real-valued.

We will also decompose the coefficient matrix C into

$$C = R + iB, \tag{13}$$

where R and B are matrices with real entries. Write $R = R_s + R_a$, where $R_s = (R + R^T)/2$ is the symmetric part of R and $R_a = (R - R^T)/2$ is the anti-symmetric part of R. Similarly $B = B_s + B_a$, where $B_s = (B + B^T)/2$ and $B_a = (B - B^T)/2$. It follows from (12) and (13) that

$$\operatorname{Re} C = R_s + iB_a \quad \text{and} \quad \operatorname{Im} C = B_s - iR_a.$$

Lemma 2.1. *We have*

$$|(R_s \xi, \eta)| \le \frac{1}{2}\Big((R_s \xi, \xi) + (R_s \eta, \eta)\Big)$$

for all $\xi, \eta \in \mathbb{R}^d$.

Proof. By hypothesis, C takes values in Σ_θ. This implies $((\operatorname{Re} C)\xi, \xi) \ge 0$ for all $\xi \in \mathbb{C}^d$. We deduce that $(R_s \xi, \xi) \ge 0$ for all $\xi \in \mathbb{R}^d$. Finally, we use polarization to obtain the lemma. □

Lemma 2.2. *We have*

$$|(B_s\xi,\eta)| \le \frac{1}{2}(\tan\theta)\Big((R_s\xi,\xi)+(R_s\eta,\eta)\Big)$$

for all $\xi,\eta\in\mathbb{R}^d$.

Proof. Since C takes values in Σ_θ, we have

$$\big|\big((\operatorname{Im}C)\xi,\xi\big)\big| \le (\tan\theta)\big((\operatorname{Re}C)\xi,\xi\big) \tag{14}$$

for all $\xi\in\mathbb{C}^d$. It follows that

$$|(B_s\xi,\xi)| \le (\tan\theta)(R_s\xi,\xi)$$

for all $\xi\in\mathbb{R}^d$. Finally, we use polarization to obtain

$$|(B_s\xi,\eta)| \le (\tan\theta)(R_s\xi,\xi)^{1/2}(R_s\eta,\eta)^{1/2} \le \frac{1}{2}(\tan\theta)\Big((R_s\xi,\xi)+(R_s\eta,\eta)\Big)$$

for all $\xi,\eta\in\mathbb{R}^d$ as required. □

Lemma 2.3. *We have*

$$\big|(B_s\xi,\xi)+(B_s\eta,\eta)-2(R_a\xi,\eta)\big| \le (\tan\theta)\Big((R_s\xi,\xi)+(R_s\eta,\eta)+2(B_a\xi,\eta)\Big)$$

for all $\xi,\eta\in\mathbb{R}^d$.

Proof. Let $\xi,\eta\in\mathbb{R}^d$. Then

$$\big((\operatorname{Im}C)(\xi+i\eta),\xi+i\eta\big) = (B_s\xi,\xi)+(B_s\eta,\eta)-2(R_a\xi,\eta)$$

and

$$\big((\operatorname{Re}C)(\xi+i\eta),\xi+i\eta\big) = (R_s\xi,\xi)+(R_s\eta,\eta)+2(B_a\xi,\eta).$$

The claim is now immediate from (14). □

Lemma 2.4. *Suppose* $B_a=0$. *Then*

$$\big|(R_a\xi,\eta)\big| \le (\tan\theta)\Big((R_s\xi,\xi)+(R_s\eta,\eta)\Big)$$

for all $\xi,\eta\in\mathbb{R}^d$.

Proof. Since $B_a=0$, Lemma 2.3 gives

$$\big|(B_s\xi,\xi)+(B_s\eta,\eta)-2(R_a\xi,\eta)\big| \le (\tan\theta)\Big((R_s\xi,\xi)+(R_s\eta,\eta)\Big).$$

The result now follows from the triangle inequality and Lemma 2.2. □

Lemma 2.5. *Let* Q *be a positive matrix and* U *a complex* $d\times d$ *matrix. Then*

$$(QU\xi,U\xi) \le \operatorname{tr}(U^*QU)\|\xi\|^2$$

for all $\xi\in\mathbb{C}^d$.

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Proof. Since Q is a positive matrix, we have $(QU\xi, U\xi) \geq 0$ for all $\xi \in \mathbb{C}^d$. It follows that $U^*QU \geq 0$. Hence $U^*QU \leq \mathrm{tr}\,(U^*QU)I$, where I denotes the identity matrix. This justifies the claim. □

Lemma 2.6. *We have the following.*

(a) $(R_s\xi, \xi) \geq 0$ *for all* $\xi \in \mathbb{C}^d$.

(b) $(((\tan\theta)R_s \pm B_s)\xi, \xi) \geq 0$ *for all* $\xi \in \mathbb{C}^d$.

(c) *Suppose* $B_a = 0$. *Then* $((2(\tan\theta)R_s \pm iR_a)\xi, \xi) \geq 0$ *for all* $\xi \in \mathbb{C}^d$.

Proof. Let $\xi \in \mathbb{C}^d$. Write $\xi = \xi_1 + i\xi_2$, where $\xi_1, \xi_2 \in \mathbb{R}^d$. We note that

$$(R_s\xi, \xi) = (R_s\xi_1, \xi_1) + (R_s\xi_2, \xi_2)$$

and

$$(B_s\xi, \xi) = (B_s\xi_1, \xi_1) + (B_s\xi_2, \xi_2).$$

Also,

$$(R_a\xi, \xi) = -2i(R_a\xi_1, \xi_2).$$

The claim now follows from Lemmas 2.1, 2.2 and 2.4. □

Next, let $\alpha \in (-\pi/2 + \theta, \pi/2 - \theta)$ and write $C_\alpha = e^{i\alpha}C$. In a similar manner as above, we define $\mathrm{Re}\,(C_\alpha)$, $\mathrm{Im}\,(C_\alpha)$, R_α, B_α, $R_{s,\alpha}$, $R_{a,\alpha}$, $B_{s,\alpha}$ and $B_{a,\alpha}$. Note that we also have

$$\mathrm{Re}\,(C_\alpha) = R_{s,\alpha} + iB_{a,\alpha} \quad \text{and} \quad \mathrm{Im}\,(C_\alpha) = B_{s,\alpha} - iR_{a,\alpha}.$$

Lemma 2.7. *Let* $j \in \{1, \ldots, d\}$. *Suppose* U *is a complex* $d \times d$ *matrix with* $U^T = U$. *Then*

$$|\mathrm{tr}\,((\partial_j C_\alpha)U)|^2 \leq M \,\mathrm{tr}\,(UR_{s,\alpha}\overline{U}),$$

where

$$M = 32d\big(1 + \tan(\theta + \alpha)\big)^2 \|\partial_l^2 C\|_\infty.$$

Proof. It follows from [4, Corollary 2.6] that

$$\begin{aligned} |\mathrm{tr}\,((\partial_j C_\alpha)U)|^2 &\leq 32d\big(1 + \tan(\theta + \alpha)\big)^2 \|\partial_l^2 (e^{i\alpha}C)\|_\infty \mathrm{tr}\,(UR_{s,\alpha}\overline{U}) \\ &\leq 32d\big(1 + \tan(\theta + \alpha)\big)^2 \|\partial_l^2 C\|_\infty \mathrm{tr}\,(UR_{s,\alpha}\overline{U}) \end{aligned}$$

as required. □

Lemma 2.8. *Suppose* $B_a = 0$. *Then the following hold.*

(i) $\mathrm{Re}\,(C_\alpha) = R_s\cos\alpha - B_s\sin\alpha + iR_a\sin\alpha$.

(ii) $\mathrm{Im}\,(C_\alpha) = R_s\sin\alpha + B_s\cos\alpha - iR_a\cos\alpha$.

(iii) $R_\alpha = R_s \cos\alpha + R_a \cos\alpha - B_s \sin\alpha$, $R_{s,\alpha} = R_s \cos\alpha - B_s \sin\alpha$, $R_{a,\alpha} = R_a \cos\alpha$.

(iv) $B_\alpha = R_s \sin\alpha + R_a \sin\alpha + B_s \cos\alpha$, $B_{s,\alpha} = R_s \sin\alpha + B_s \cos\alpha$, $B_{a,\alpha} = R_a \sin\alpha$.

Proof. These identities follow directly from the definition of C and C_α. □

3. Sectorial property

Let $p \in (1,\infty)$. Let Ω, θ, C and A_p be as in §1. In this section, we prove Theorem 1.2. A convenient tool that we will use repeatedly is the formula of integration by parts in Sobolev spaces given in the next theorem. The theorem is immediate from the proof of [13, Proposition 3.5]. We emphasize that we do not require $C = C^T$ in this theorem (cf. [13, Theorem 3.1] for the same result but with extra assumption that $C = C^T$).

Theorem 3.1. *Let $u \in D(A_p)$. Then*

$$\begin{aligned}\int_{[u\neq 0]} (A_p u)|u|^{p-2}\overline{u} = \int_{[u\neq 0]} &|u|^{p-2}(C\nabla\overline{u}, \nabla\overline{u}) \\ &+ (p-2)\int_{[u\neq 0]} |u|^{p-4}\big(C\mathrm{Re}\,(u\nabla\overline{u}), \mathrm{Re}\,(u\nabla\overline{u})\big) \\ &- i(p-2)\int_{[u\neq 0]} |u|^{p-4}\big(C\mathrm{Re}\,(u\nabla\overline{u}), \mathrm{Im}\,(u\nabla\overline{u})\big). \qquad (15)\end{aligned}$$

An immediate remark is in order.

Remark 3.2. Recently, [1] introduced the concept of p-ellipticity. Let Δ_p and $\mathcal{J}_p$ be given by (6) and (7). A matrix C is said to be p-elliptic if

$$\Delta_p(C) > 0. \qquad (16)$$

Using the operator $\mathcal{J}_p$, the formula of integration by parts (15) can be rephrased as

$$\int_{[u\neq 0]} (A_p u)|u|^{p-2}\overline{u} = \frac{p}{2}\int_{[u\neq 0]} |u|^{p-4}\big(Cu\nabla\overline{u}, \mathcal{J}_p(u\nabla\overline{u})\big).$$

Following this, the sectorial condition (1) can be rewritten as

$$\big\langle |u|^{p-4}Cu\nabla\overline{u}, \mathcal{J}_p(\overline{u}\nabla u)\mathbb{1}_{[u\neq 0]}\big\rangle_{L_2(\Omega)} \in \Sigma_{\arctan K},$$

and hence can be viewed as a degenerate case of (16).

Using Theorem 3.1, we obtain the following proposition, the first part of which is along the same line as [1, Proposition 7.6 and (5.7)]. Nevertheless, in general, the domain for the accretivity (dissipativity) in [1, Proposition 7.6] on $L_p(\Omega)$ has no relation with our domain $D(A_p)$. Moreover, [1, Proposition 7.6] is only valid for $p \geq 2$.

For the sake of clarity, we present here a proof that holds for all $p \in (1,\infty)$ under our current setting.

Proposition 3.3. *Let $u \in D(A_p)$. Write $u\nabla\overline{u} = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}^d$. Then*

$$\mathrm{Re}\,(A_p u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) = \int_{[u\neq 0]} |u|^{p-4}\Big((p-1)(R_s\xi,\xi) + (R_s\eta,\eta) + (p-2)(B_s\xi,\eta) + p(B_a\xi,\eta)\Big)$$

and

$$\mathrm{Im}\,(A_p u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) = \int_{[u\neq 0]} |u|^{p-4}\Big((p-1)(B_s\xi,\xi) + (B_s\eta,\eta) - (p-2)(R_s\xi,\eta) - p(R_a\xi,\eta)\Big).$$

Proof. We will prove the first inequality only. The second is similar.
Consider (15). We have

$$\begin{aligned}|u|^2(C\nabla\overline{u},\nabla\overline{u}) &= (Cu\nabla\overline{u}, u\nabla\overline{u}) = \big(C(\xi+i\eta),\xi+i\eta\big)\\ &= (R\xi,\xi) + (R\eta,\eta) + (B\xi,\eta) - (B\eta,\xi)\\ &\quad - i\big((R\eta,\xi) - (R\xi,\eta) + (B\xi,\xi) + (B\eta,\eta)\big).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{Re}\,\big(|u|^2(C\nabla\overline{u},\nabla\overline{u})\big) &= (R\xi,\xi) + (R\eta,\eta) + (B\xi,\eta) - (B\eta,\xi)\\ &= (R_s\xi,\xi) + (R_s\eta,\eta) + 2(B_a\xi,\eta).\end{aligned}$$

Also,

$$\mathrm{Re}\,\big(C\mathrm{Re}\,(u\nabla\overline{u}), \mathrm{Re}\,(u\nabla\overline{u})\big) = \mathrm{Re}\,(C\xi,\xi) = (R\xi,\xi) = (R_s\xi,\xi).$$

Similarly

$$\mathrm{Re}\,\Big(i\big(C\mathrm{Re}\,(u\nabla\overline{u}), \mathrm{Im}\,(u\nabla\overline{u})\big)\Big) = \mathrm{Re}\,\big(i(C\xi,\eta)\big) = -(B\xi,\eta) = -(B_s\xi,\eta) - (B_a\xi,\eta).$$

Hence taking the real parts on both sides of (15) yields the result. □

The following lemma is essential in the proof of Theorem 1.2.

Lemma 3.4. *Suppose $|1-2/p| < \cos\theta$. Let $\phi = \arccos|1-2/p|$. Then*

$$\begin{aligned}&\left(\tan\left(\frac{\pi}{2}-\phi\right) + \tan\theta\right)\big((R_s\xi,\xi) + (R_s\eta,\eta)\big)\\ &\quad\leq \tan\left(\frac{\pi}{2}-\phi+\theta\right)\left((R_s\xi,\xi) + (R_s\eta,\eta) + \frac{p-2}{\sqrt{p-1}}(B_s\xi,\eta)\right)\end{aligned}$$

for all $\xi, \eta \in \mathbb{R}^d$.

Proof. First, note that

$$\tan\left(\frac{\pi}{2}-\phi\right)(\tan\theta)\big((R_s\xi,\xi) + (R_s\eta,\eta)\big) + \frac{p-2}{\sqrt{p-1}}(B_s\xi',\eta)$$

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$$\begin{aligned}&\geq \tan\left(\frac{\pi}{2}-\phi\right)(\tan\theta)\big((R_s\xi,\xi)+(R_s\eta,\eta)\big)-\frac{|p-2|}{\sqrt{p-1}}|(B_s\xi,\eta)|\\&=\tan\left(\frac{\pi}{2}-\phi\right)\Big((\tan\theta)\big((R_s\xi,\xi)+(R_s\eta,\eta)\big)-2|(B_s\xi,\eta)|\Big)\geq 0 \qquad (17)\end{aligned}$$

as $\tan(\pi/2-\phi)=\cot(\phi)=|p-2|/2\sqrt{p-1}$ and we used Lemma 2.2 in the last step. We also deduce from the hypotheses that $\tan(\pi/2-\phi+\theta)\geq 0$. Therefore,

$$\begin{aligned}&\left(\tan\left(\frac{\pi}{2}-\phi\right)+\tan\theta\right)\big((R_s\xi,\xi)+(R_s\eta,\eta)\big)\\&\quad\leq\left(\tan\left(\frac{\pi}{2}-\phi\right)+\tan\theta\right)\big((R_s\xi,\xi)+(R_s\eta,\eta)\big)\\&\qquad+\tan\left(\frac{\pi}{2}-\phi+\theta\right)\left(\tan\left(\frac{\pi}{2}-\phi\right)(\tan\theta)\big((R_s\xi,\xi)+(R_s\eta,\eta)\big)+\frac{p-2}{\sqrt{p-1}}(B_s\xi,\eta)\right)\\&\quad=\tan\left(\frac{\pi}{2}-\phi+\theta\right)\Big((R_s\xi,\xi)+(R_s\eta,\eta)+\frac{p-2}{\sqrt{p-1}}(B_s\xi,\eta)\Big),\end{aligned}$$

where we used (17) in the first step. □

Next, we prove Theorem 1.2.

Proof of Theorem 1.2. Let $u\in D(A_p)$. Write $u\nabla\overline{u}=\xi+i\eta$, where $\xi,\eta\in\mathbb{R}^d$. By Proposition 3.3, it suffices to show that

$$\begin{aligned}&\big|(p-1)(B_s\xi,\xi)+(B_s\eta,\eta)-(p-2)(R_s\xi,\eta)-p(R_a\xi,\eta)\big|\\&\quad\leq K\big((p-1)(R_s\xi,\xi)+(R_s\eta,\eta)+(p-2)(B_s\xi,\eta)\big), \qquad (18)\end{aligned}$$

where K is defined by (5). Set $\xi'=\sqrt{p-1}\xi$. Then (18) is equivalent to

$$\begin{aligned}&\left|(B_s\xi',\xi')+(B_s\eta,\eta)-\frac{p-2}{\sqrt{p-1}}(R_s\xi',\eta)-\frac{p}{\sqrt{p-1}}(R_a\xi',\eta)\right|\\&\quad\leq K\left((R_s\xi',\xi')+(R_s\eta,\eta)+\frac{p-2}{\sqrt{p-1}}(B_s\xi',\eta)\right). \qquad (19)\end{aligned}$$

Note that by Lemma 2.1, we have

$$\frac{|p-2|}{\sqrt{p-1}}\big|(R_s\xi',\eta)\big|\leq\tan\left(\frac{\pi}{2}-\phi\right)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big) \qquad (20)$$

as $\tan(\pi/2-\phi)=\cot(\phi)=|p-2|/2\sqrt{p-1}$.

Now we consider two cases.

Case 3.5. *Suppose $R_a=0$. Using Lemma 2.2 again, we obtain*

$$\big|(B_s\xi',\xi')+(B_s\eta,\eta)\big|\leq(\tan\theta)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big). \qquad (21)$$

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It follows that

$$\begin{aligned}
&\big|(B_s\xi',\xi')+(B_s\eta,\eta)-\frac{p-2}{\sqrt{p-1}}(R_s\xi',\eta)-\frac{p}{\sqrt{p-1}}(R_a\xi',\eta)\big| \\
&\quad=\big|(B_s\xi',\xi')+(B_s\eta,\eta)-\frac{p-2}{\sqrt{p-1}}(R_s\xi',\eta)\big| \\
&\quad\le\Big(\tan\Big(\frac{\pi}{2}-\phi\Big)+\tan\theta\Big)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big) \\
&\quad\le\tan\Big(\frac{\pi}{2}-\phi+\theta\Big)\Big((R_s\xi',\xi')+(R_s\eta,\eta)+\frac{p-2}{\sqrt{p-1}}(B_s\xi',\eta)\Big),
\end{aligned}$$

where we used $R_a=0$ in the first step, (21) and (20) in the second step and Lemma 3.4 in the last step.

Hence, (*19*) is valid and the result follows in this case. □

Case 3.6. *Suppose $R_a\neq 0$. We rewrite the left-hand side of (19) as*

$$\begin{aligned}
L:=\Big|&\Big((B_s\xi',\xi')+(B_s\eta,\eta)-2(R_a\xi',\eta)\Big)-\frac{p-2}{\sqrt{p-1}}(R_s\xi',\eta) \\
&-\Big(\frac{p}{\sqrt{p-1}}-2\Big)(R_a\xi',\eta)\Big|.
\end{aligned}$$

(Note that $p/\sqrt{p-1}\ge 2$ for all $p\in(1,\infty)$.) Since $B_a=0$, it follows from Lemma 2.3 that

$$\big|(B_s\xi',\xi')+(B_s\eta,\eta)-2(R_a\xi',\eta)\big|\le(\tan\theta)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big). \tag{22}$$

Next, we deduce from Lemma 2.4 that

$$\Big(\frac{p}{\sqrt{p-1}}-2\Big)\big|(R_a\xi',\eta)\big|\le\Big(\frac{2}{\sin\phi}-2\Big)(\tan\theta)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big) \tag{23}$$

as $\sin\phi=2\sqrt{p-1}/p$. Now it follows from (20), (22) and (23) that

$$\begin{aligned}
L&\le\Big(\Big(\frac{2}{\sin\phi}-1\Big)\tan\theta+\tan\Big(\frac{\pi}{2}-\phi\Big)\Big)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big) \\
&=\frac{\big(2/\sin\phi-1\big)\tan\theta+\tan\left(\pi/2-\phi\right)}{\tan\theta+\tan\left(\pi/2-\phi\right)}\Big(\tan\theta+\tan\Big(\frac{\pi}{2}-\phi\Big)\Big)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big) \\
&\le\frac{\big(2/\sin\phi-1\big)\tan\theta+\tan\left(\pi/2-\phi\right)}{\tan\theta+\tan\left(\pi/2-\phi\right)}\tan\Big(\frac{\pi}{2}-\phi+\theta\Big) \\
&\quad\times\Big((R_s\xi',\xi')+(R_s\eta,\eta)+\frac{p-2}{\sqrt{p-1}}(B_s\xi',\eta)\Big) \\
&=\frac{\big(2/\sin\phi-1\big)\tan\theta+\tan\left(\pi/2-\phi\right)}{1-(\tan\theta)\tan\left(\pi/2-\phi\right)}\Big((R_s\xi',\xi')+(R_s\eta,\eta)+\frac{p-2}{\sqrt{p-1}}(B_s\xi',\eta)\Big),
\end{aligned}$$

where we used Lemma 3.4 in the second step.

Hence, (19) is also valid in this case.

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4. Generation of contraction holomorphic semigroup

Let $\Omega = \mathbb{R}^d$ and $\theta \in [0, \pi/2)$. We assume $c_{kl} \in W^{2,\infty}(\mathbb{R}^d, \mathbb{C})$ for all $k, l \in \{1, \ldots, d\}$. Assume further that $(C(x)\xi, \xi) \in \Sigma_\theta$ for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{C}^d$, where $C = (c_{kl})_{1 \le k,l \le d}$ and Σ_θ is defined by (2).

Let $p \in (1, \infty)$. We will prove in Proposition 4.1 that A_p is closable. Let $\overline{A_p}$ be the closure of A_p. We will show in this section that $-\overline{A_p}$ generates a holomorphic semigroup on $L_p(\mathbb{R}^d)$ which is contractive on a sector. This is the content of Theorems 1.4 and 1.5.

First, we introduce some more definitions. Let q be such that $1/p + 1/q = 1$. Define

$$H_q u = -\sum_{k,l=1}^{d} \partial_k (\overline{c_{kl}} \partial_l u) \tag{24}$$

on the domain

$$D(H_q) = C_c^\infty(\mathbb{R}^d).$$

Define

$$X_p = (H_q)^*,$$

which is the dual of H_q. Then X_p is closed by [11, Theorem III.5.29]. Also note that $W^{2,p}(\mathbb{R}^d) \subset D(X_p)$ and

$$X_p u = -\sum_{k,l=1}^{d} \partial_l (c_{kl} \partial_k u)$$

for all $u \in W^{2,p}(\mathbb{R}^d)$.

Proposition 4.1. *The operator A_p is closable.*

Proof. Since $A_p \subset X_p$ and X_p are closed, the operator A_p is closable. □

It turns out that $X_p = \overline{A_p}$ under certain conditions, as shown in the following proposition.

Proposition 4.2. *Suppose $|1 - 2/p| \le \cos\theta$ and $B_a = 0$. Then $\overline{A_p} = X_p$. Moreover, $\overline{A_p}$ is m-accretive.*

Proof. By [4, Proposition 4.9] the operator X_p is m-accretive and the space $C_c^\infty(\mathbb{R}^d)$ of test functions is a core for X_p. It follows that $\overline{A_p} = X_p$ and A_p is m-accretive as claimed. □

Using Theorem 1.2, we are now able to prove the generation result in Theorem 1.4.

Proof of Theorem 1.4. It follows from Theorem 1.2 that

$$|\mathrm{Im}\,(\overline{A_p}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]})| \le K \mathrm{Re}\,(\overline{A_p}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]})$$

for all $u \in D(\overline{A_p})$, where K is defined by (5). Therefore, the interior $\Sigma^\circ_{\pi - \arctan(K)} \subset \rho(-\overline{A_p})$ by [16, Theorem 1.3.9] and Proposition 4.2, where $\rho(-\overline{A_p})$ denotes the resolvent set of $-\overline{A_p}$. Moreover,

$$\|(\lambda + \overline{A_p})^{-1}\|_{p\to p} \le \frac{1}{\operatorname{dist}(\lambda, S(-\overline{A_p}))} \tag{25}$$

for all $\lambda \in \Sigma^\circ_{\pi-\arctan(K)}$, where $S(-\overline{A_p})$ is the numerical range of $-\overline{A_p}$ defined by

$$S(-\overline{A_p}) = \big\{ -\big(\overline{A_p}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}\big) : u \in D(\overline{A_p}) \text{ and } \|u\|_p = 1\big\}.$$

Let $\varepsilon \in (0, \pi - \arctan(K))$. Then $\operatorname{dist}(\lambda, S(-\overline{A_p})) \ge (\sin\varepsilon)|\lambda|$ for all $\lambda \in \Sigma_{\pi-\arctan(K)-\varepsilon}$. Therefore, (25) implies

$$\|(\lambda + \overline{A_p})^{-1}\|_{p\to p} \le \frac{1}{(\sin\varepsilon)|\lambda|}$$

for all $\lambda \in \Sigma_{\pi-\arctan(K)-\varepsilon}$. Hence we deduce from [16, Theorem 2.5.2(c)] that $-\overline{A_p}$ generates a holomorphic semigroup on $L_p(\mathbb{R}^d)$ with angle $\psi = \pi/2 - \arctan(K)$. □

Our next aim is to show Theorem 1.5. We will do this by first showing that $-X_p$ generates a holomorphic semigroup which is contractive on a sector. This together with Proposition 4.2 implies the theorem. We first obtain some preliminary results.

In what follows we let $X_{p,\alpha} = e^{i\alpha}X_p$ for all $\alpha \in (-\pi/2 + \theta, \pi/2 - \theta)$ and adopt the notation used in Lemmas 2.7 and 2.8. We aim to show that $X_{p,\alpha}$ is an m-accretive operator for all α in a suitable range. Following [4, 20], we need two crucial inequalities for $X_{p,\alpha}$ in order to do this which are given in Propositions 4.3 and 4.5, respectively.

The first inequality is as follows.

Proposition 4.3. *Suppose $B_a = 0$. Let $p \in (1,\infty)$ be such that $|1 - 2/p| < \cos\theta$. Let $\alpha \in (-\psi, \psi)$, where ψ is given by (9). Then*

$$\operatorname{Re}\,(X_{p,\alpha}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) \ge 0$$

for all $u \in W^{2,p}(\mathbb{R}^d)$.

Proof. Let $u \in W^{2,p}(\mathbb{R}^d)$. It follows from Theorem 3.1 that

$$\begin{aligned}(X_{p,\alpha}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) &= \int_{[u\neq 0]} |u|^{p-2}(C_\alpha\nabla\overline{u}, \nabla\overline{u}) \\ &\quad + (p-2)\int_{[u\neq 0]} |u|^{p-4}\big(C_\alpha \operatorname{Re}(u\nabla\overline{u}), \operatorname{Re}(u\nabla\overline{u})\big) \\ &\quad - i(p-2)\int_{[u\neq 0]} |u|^{p-4}\big(C_\alpha \operatorname{Re}(u\nabla\overline{u}), \operatorname{Im}(u\nabla\overline{u})\big). \end{aligned} \tag{26}$$

Write $u\nabla\overline{u} = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}^d$. Then

$$\begin{aligned}|u|^2(C_\alpha\nabla\overline{u}, \nabla\overline{u}) &= (C_\alpha u\nabla\overline{u}, u\nabla\overline{u}) = \big(C_\alpha(\xi + i\eta), \xi + i\eta\big) \\ &= (R_\alpha\xi, \xi) + (R_\alpha\eta, \eta) + (B_\alpha\xi, \eta) - (B_\alpha\eta, \xi) \\ &\quad + i\big((R_\alpha\eta, \xi) - (R_\alpha\xi, \eta) + (B_\alpha\xi, \xi) + (B_\alpha\eta, \eta)\big).\end{aligned}$$

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Therefore,

$$\begin{aligned}\operatorname{Re}\big(|u|^2(C_\alpha\nabla\overline{u},\nabla\overline{u})\big) &= (R_\alpha\xi,\xi)+(R_\alpha\eta,\eta)+(B_\alpha\xi,\eta)-(B_\alpha\eta,\xi)\\ &= (R_{s,\alpha}\xi,\xi)+(R_{s,\alpha}\eta,\eta)+2(B_{a,\alpha}\xi,\eta).\end{aligned}$$

We also have

$$\operatorname{Re}\big(C_\alpha\operatorname{Re}(u\nabla\overline{u}),\operatorname{Re}(u\nabla\overline{u})\big)=\operatorname{Re}(C_\alpha\xi,\xi)=(R_\alpha\xi,\xi)=(R_{s,\alpha}\xi,\xi).$$

Similarly

$$\begin{aligned}\operatorname{Re}\big(i\big(C_\alpha\operatorname{Re}(u\nabla\overline{u}),\operatorname{Im}(u\nabla\overline{u})\big)\big) &= \operatorname{Re}\big(i(C_\alpha\xi,\eta)\big)=-(B_\alpha\xi,\eta)\\ &= -(B_{s,\alpha}\xi,\eta)-(B_{a,\alpha}\xi,\eta).\end{aligned}$$

Hence taking the real parts on both sides of (26) yields

$$\begin{aligned}&\operatorname{Re}(X_{p,\alpha}u,|u|^{p-2}u\mathbb{1}_{[u\neq0]})\\ &= \int_{[u\neq0]}|u|^{p-4}\Big((p-1)(R_{s,\alpha}\xi,\xi)+(R_{s,\alpha}\eta,\eta)+p(B_{a,\alpha}\xi,\eta)+(p-2)(B_{s,\alpha}\xi,\eta)\Big)\\ &= \int_{[u\neq0]}|u|^{p-4}\Big((R_{s,\alpha}\xi',\xi')+(R_{s,\alpha}\eta,\eta)+\frac{p}{\sqrt{p-1}}(B_{a,\alpha}\xi',\eta)+\frac{p-2}{\sqrt{p-1}}(B_{s,\alpha}\xi',\eta)\Big),\end{aligned}\tag{27}$$

where $\xi'=\sqrt{p-1}\xi$. Set

$$P=(R_{s,\alpha}\xi',\xi')+(R_{s,\alpha}\eta,\eta)+\frac{p}{\sqrt{p-1}}(B_{a,\alpha}\xi',\eta)+\frac{p-2}{\sqrt{p-1}}(B_{s,\alpha}\xi',\eta).\tag{28}$$

We will show that $P\geq 0$. We consider 2 cases.

Case 4.4. *Suppose $R_a=0$. Note that $\cot\phi=|p-2|/2\sqrt{p-1}$. We have*

$$\big|(\sin\alpha)\big((B_s\xi',\xi')+(B_s\eta,\eta)\big)\big|\leq\sin(|\alpha|)(\tan\theta)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big)\tag{29}$$

and

$$\Big|\frac{p-2}{\sqrt{p-1}}(\cos\alpha)(B_s\xi',\eta)\Big|\leq(\cot\phi)(\cos\alpha)(\tan\theta)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big).\tag{30}$$

by Lemma 2.2. Also,

$$\Big|\frac{p-2}{\sqrt{p-1}}(\sin\alpha)(R_s\xi',\eta)\Big|\leq(\cot\phi)\sin(|\alpha|)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big)\tag{31}$$

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by Lemma 2.1. Since $R_a = 0$, Lemma 2.8(iv) gives $B_{a,\alpha} = (\sin\alpha)R_a = 0$. It follows from Lemma 2.8, (28)–(31) that

$$
\begin{aligned}
P &= (R_{s,\alpha}\xi',\xi') + (R_{s,\alpha}\eta,\eta) + \frac{p-2}{\sqrt{p-1}}(B_{s,\alpha}\xi',\eta) \\
&= (\cos\alpha)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big) - (\sin\alpha)\big((B_s\xi',\xi') + (B_s\eta,\eta)\big) \\
&\quad + \frac{p-2}{\sqrt{p-1}}(\sin\alpha)(R_s\xi',\eta) + \frac{p-2}{\sqrt{p-1}}(\cos\alpha)(B_s\xi',\eta) \\
&\geq \big(\cos\alpha - \sin(|\alpha|)\tan\theta - (\cot\phi)\sin(|\alpha|) - (\cot\phi)(\cos\alpha)\tan\theta\big)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big) \\
&\geq 0,
\end{aligned}
$$

where we used the fact that $\alpha \in (-\psi,\psi)$ in the last step. Hence, we deduce from (27) that $\operatorname{Re}(X_{p,\alpha}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) \geq 0$ in this case.

Case 4.5. *Suppose $R_a \neq 0$. Expanding (28) using Lemma 2.8 gives*

$$
\begin{aligned}
P &= (R_{s,\alpha}\xi',\xi') + (R_{s,\alpha}\eta,\eta) + \frac{p}{\sqrt{p-1}}(B_{a,\alpha}\xi',\eta) + \frac{p-2}{\sqrt{p-1}}(B_{s,\alpha}\xi',\eta) \\
&= (\cos\alpha)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big) - (\sin\alpha)\big((B_s\xi',\xi') + (B_s\eta,\eta)\big) \\
&\quad + \frac{p}{\sqrt{p-1}}(\sin\alpha)(R_a\xi',\eta) + \frac{p-2}{\sqrt{p-1}}(\sin\alpha)(R_s\xi',\eta) + \frac{p-2}{\sqrt{p-1}}(\cos\alpha)(B_s\xi',\eta) \\
&= (\cos\alpha)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big) - (\sin\alpha)\big((B_s\xi',\xi') + (B_s\eta,\eta) - 2(R_a\xi',\eta)\big) \\
&\quad + \left(\frac{p}{\sqrt{p-1}} - 2\right)(\sin\alpha)(R_a\xi',\eta) + \frac{p-2}{\sqrt{p-1}}(\sin\alpha)(R_s\xi',\eta) \\
&\quad + \frac{p-2}{\sqrt{p-1}}(\cos\alpha)(B_s\xi',\eta),
\end{aligned} \tag{32}
$$

where we used Lemma 2.8(iii) and (iv) in the second step. Next, we estimate the terms in (32). By Lemma 2.3, we have

$$
\Big|(\sin\alpha)\big((B_s\xi',\xi') + (B_s\eta,\eta) - 2(R_a\xi',\eta)\big)\Big| \leq \sin(|\alpha|)(\tan\theta)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big) \tag{33}
$$

since $B_a = 0$ by hypothesis. Using Lemma 2.4 and the fact that $\sin\phi = \frac{2\sqrt{p-1}}{p}$, we deduce that

$$
\left|\left(\frac{p}{\sqrt{p-1}} - 2\right)(\sin\alpha)(R_a\xi',\eta)\right| \leq \left(\frac{2}{\sin\phi} - 2\right)\sin(|\alpha|)(\tan\theta)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big). \tag{34}
$$

Next note that $\cot\phi = |p-2|/2\sqrt{p-1}$. Therefore,

$$
\left|\frac{p-2}{\sqrt{p-1}}(\sin\alpha)(R_s\xi',\eta)\right| \leq (\cot\phi)\sin(|\alpha|)\big((R_s\xi',\xi') + (R_s\eta,\eta)\big) \tag{35}
$$

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by Lemma 2.1. It follows from Lemma 2.2 that

$$\left|\frac{p-2}{\sqrt{p-1}}(\cos\alpha)(B_s\xi',\eta)\right| \le (\cot\phi)(\cos\alpha)(\tan\theta)\big((R_s\xi',\xi')+(R_s\eta,\eta)\big). \tag{36}$$

Next, (32)–(36) together imply

$$\begin{aligned} P &\ge \left(\big(1-(\tan\theta)\cot\phi\big)\cos\alpha - \left(\left(\frac{2}{\sin\phi}-1\right)\tan\theta+\cot\phi\right)\sin(|\alpha|)\right) \\ &\quad \times \big((R_s\xi',\xi')+(R_s\eta,\eta)\big) \\ &\ge 0, \end{aligned} \tag{37}$$

where we used that fact that $\alpha \in (-\psi,\psi)$ and Lemma 2.1 in the last step. Combining (27) and (37) yields $\operatorname{Re}(X_{p,\alpha}u, |u|^{p-2}u\mathbb{1}_{[u\neq 0]}) \ge 0$ in this case.

Next, we prove the second inequality for $X_{p,\alpha}$. We need the following density result.

Proposition 4.4. *Let $\alpha \in (-\psi,\psi)$, where ψ is given by (9). Then the space $C_c^\infty(\mathbb{R}^d)$ is dense in $(D(X_{p,\alpha}) \cap W^{1,p}(\mathbb{R}^d), \|\cdot\|_{D(X_{p,\alpha})})$.*

Proof. The claim follows from [4, Proposition 4.7]. □

The second inequality is as follows (see [20, proposition 6.1] for the case when $\alpha = 0$ and X_p has real symmetric coefficients as well as [4, Proposition 4.8] for the case when $\alpha = 0$ and X_p has complex coefficients).

Proposition 4.5. *Suppose $B_a = 0$. Let $p \in (1,\infty)$ be such that $|1-2/p| < \cos\theta$. Let $\alpha \in (-\gamma,\gamma)$, where γ is given by (11). Then there exists an $M > 0$ such that*

$$\operatorname{Re}(\nabla(X_{p,\alpha}u), |\nabla u|^{p-2}\nabla u\mathbb{1}_{[\nabla u\neq 0]}) \ge -M\|\nabla u\|_p^p$$

for all $u \in W^{2,p}(\mathbb{R}^d)$ such that $\nabla(X_{p,\alpha}u) \in (L_p(\mathbb{R}^d))^d$.

Proof. We consider two cases. □

Case 4.8. *Suppose $R_a = 0$. Then it follows from Lemma 2.8 that $B_{a,\alpha} = R_a\sin\alpha = 0$. Moreover, the condition $\alpha \in (-\psi,\psi)$ implies $\tan(\theta+|\alpha|) < \tan\phi$. Therefore, [4, Proposition 4.8] still applies to yield the result.*

Case 4.9. *Suppose $R_a \neq 0$. If $\alpha = 0$, the claim follows from [4, Proposition 4.8]. Therefore, we may assume that $\alpha \neq 0$ for the rest of the proof. Note that $\alpha \in (-\gamma,\gamma)$ implies $(\tan\theta)\tan(|\alpha|) < \frac{1}{3}$ and $K\tan(|\alpha|) < 1$, where K is defined by (5). Let $\varepsilon_0 \in$*

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$(0, 1 \wedge (p-1))$ *be such that*

$$(\tan\theta)\tan(|\alpha|) \le \frac{1-\varepsilon}{3-\varepsilon} \tag{38}$$

and

$$\left(\left(\frac{p}{\sqrt{\left(1-\varepsilon\right)(p-1-\varepsilon)}} - 1 \right) \tan\theta + \frac{|p-2|}{2\sqrt{(1-\varepsilon)(p-1-\varepsilon)}} \right) \tan(|\alpha|)$$
$$\le 1 - (\tan\theta)\frac{|p-2|}{2\sqrt{(1-\varepsilon)(p-1-\varepsilon)}} \tag{39}$$

for all $\varepsilon \in (0, \varepsilon_0)$. *Let* $\varepsilon \in (0, \varepsilon_0)$ *be such that*

$$\varepsilon < \frac{\varepsilon_0}{32d\big(1+\tan(\theta+|\alpha|)\big)^2 \sup_{1\le l\le d} \|\partial_l^2 C\|_\infty}. \tag{40}$$

Let $u \in W^{2,p}(\mathbb{R}^d)$. *By Lemma 4.4, we can assume without loss of generality that* u *has a compact support. For the rest of the proof, all integrations are over the set* $\{x \in \mathbb{R}^d : |(\nabla u)(x)| \neq 0\}$. *We have*

$$\begin{aligned}
(\nabla(X_{p,\alpha}u), |\nabla u|^{p-2}\nabla u) &= -\sum_{k,l,j=1}^{d} \int \Big(\partial_j \partial_l (e^{i\alpha} c_{kl} \partial_k u)\Big) |\nabla u|^{p-2} \partial_j \overline{u} \\
&= -\sum_{k,l,j=1}^{d} \int e^{i\alpha} \Big(\partial_l \big((\partial_j c_{kl})(\partial_k u) + c_{kl}(\partial_j \partial_k u)\big)\Big) |\nabla u|^{p-2} \partial_j \overline{u} \\
&= -\sum_{k,l,j=1}^{d} \int e^{i\alpha} \Big(\partial_l \big((\partial_j c_{kl})(\partial_k u)\big)\Big) |\nabla u|^{p-2} \partial_j \overline{u} \\
&\quad + \sum_{k,l,j=1}^{d} \int e^{i\alpha} c_{kl} (\partial_j \partial_k u) \partial_l \big(|\nabla u|^{p-2} \partial_j \overline{u}\big) \\
&= (I) + (II).
\end{aligned}$$

We first consider the real part of (I). We have

$$\begin{aligned}
&-\mathrm{Re} \sum_{k,l,j=1}^{d} \int e^{i\alpha} \Big(\partial_l \big((\partial_j c_{kl})(\partial_k u)\big)\Big) |\nabla u|^{p-2} \partial_j \overline{u} \\
&\quad = -\mathrm{Re} \sum_{k,l,j=1}^{d} \int e^{i\alpha} (\partial_l \partial_j c_{kl})(\partial_k u)(\partial_j \overline{u}) |\nabla u|^{p-2}
\end{aligned}$$

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$$- \operatorname{Re} \sum_{k,l,j=1}^{d} \int e^{i\alpha} (\partial_j c_{kl})(\partial_l \partial_k u)(\partial_j \overline{u}) |\nabla u|^{p-2}$$
$$= (Ia) + (Ib).$$

For (Ia), we have

$$(Ia) \geq -\frac{1}{2} \sum_{k,l,j=1}^{d} \|c_{kl}\|_{W^{2,\infty}} \int (|\partial_k u|^2 + |\partial_j u|^2) |\nabla u|^{p-2} \geq -M_1 \|\nabla u\|_p^p,$$

where $M_1 = d^2 \sup\{\|c_{kl}\|_{W^{2,\infty}} : 1 \leq k, l \leq d\}$. *Let* $U = (\partial_l \partial_k u)_{1 \leq k,l \leq d}$. For (Ib), we estimate

$$\begin{aligned}
(Ib) &= -\operatorname{Re} \sum_{j=1}^{d} \int \operatorname{tr}\left((\partial_j C_\alpha) U\right)(\partial_j \overline{u}) |\nabla u|^{p-2} \\
&\geq -\sum_{j=1}^{d} \int \left(\varepsilon |\operatorname{tr}\left((\partial_j C_\alpha) U\right)|^2 |\nabla u|^{p-2} + \frac{1}{4\varepsilon} |\partial_j \overline{u}|^2 |\nabla u|^{p-2} \right) \\
&\geq -\varepsilon' \int \operatorname{tr}\left(U R_{s,\alpha} \overline{U}\right) |\nabla u|^{p-2} - M_2 \|\nabla u\|_p^p \\
&= -\varepsilon' \int \operatorname{tr}\left(\overline{U} R_{s,\alpha} U\right) |\nabla u|^{p-2} - M_2 \|\nabla u\|_p^p,
\end{aligned}$$

where we used Lemma 2.7 in the third step with

$$\varepsilon' = 32 \varepsilon d \big(1 + \tan(\theta + |\alpha|)\big)^2 \sup_{1 \leq l \leq d} \|\partial_l^2 C\|_\infty$$

and $M_2 = 1/4\varepsilon$. Note that $\varepsilon' \in (0, \varepsilon_0)$ by (40).

Next, we consider the real part of (II). Note that

$$\begin{aligned}
&\operatorname{Re} \sum_{k,l,j=1}^{d} \int e^{i\alpha} c_{kl} (\partial_j \partial_k u) \partial_l \big(|\nabla u|^{p-2} \partial_j \overline{u}\big) \\
&\quad = \operatorname{Re} \sum_{k,l,j=1}^{d} \int e^{i\alpha} c_{kl} (\partial_j \partial_k u)(\partial_l \partial_j \overline{u}) |\nabla u|^{p-2} \\
&\qquad + \operatorname{Re} \sum_{k,l,j=1}^{d} \int e^{i\alpha} c_{kl} (\partial_j \partial_k u)(\partial_j \overline{u}) \partial_l (|\nabla u|^{p-2}) \\
&\quad = (IIa) + (IIb).
\end{aligned}$$

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In what follows we let $U\nabla\overline{u} = \xi + i\eta$, where $\xi, \eta \in \mathbb{R}^d$. For (IIa), we have

$$\text{(IIa)} = \int \mathrm{tr}\,(\overline{U}\mathrm{Re}\,(C_\alpha)U)|\nabla u|^{p-2} = \int \mathrm{tr}\,(\overline{U}R_{s,\alpha}U)|\nabla u|^{p-2} + i\int \mathrm{tr}\,(\overline{U}B_{a,\alpha}U)|\nabla u|^{p-2}.$$

For (IIb), we have

$$\begin{aligned}
\text{(IIb)} &= \mathrm{Re} \sum_{k,l,i,j=1}^{d} \frac{p-2}{2} \int e^{i\alpha} c_{kl}(\partial_j\partial_k u)(\partial_j\overline{u})\Big((\partial_l\partial_i u)(\partial_i\overline{u}) + (\partial_l\partial_i\overline{u})(\partial_i u)\Big)|\nabla u|^{p-4} \\
&= \frac{p-2}{2}\int \mathrm{Re}\,\Big((C_\alpha U\nabla\overline{u}, \overline{U\nabla\overline{u}}) + (C_\alpha U\nabla\overline{u}, U\nabla\overline{u})\Big)|\nabla u|^{p-4} \\
&= (p-2)\int \Big((R_\alpha\xi,\xi) - (B_\alpha\eta,\xi)\Big)|\nabla u|^{p-4} \\
&= (p-2)\int \Big((R_{s,\alpha}\xi,\xi) - (B_{s,\alpha}\xi,\eta) + (B_{a,\alpha}\xi,\eta)\Big)|\nabla u|^{p-4},
\end{aligned}$$

where $\xi, \eta \in \mathbb{R}^d$ and $U\nabla\overline{u} = \xi + i\eta$.

In total, we obtain

$$\begin{aligned}
\mathrm{Re}\,(\nabla(X_{p,\alpha}u), |\nabla u|^{p-2}\nabla u) \geq -(M_1+M_2)\|\nabla u\|_p^p &+ (1-\varepsilon')\int \mathrm{tr}\,(UR_{s,\alpha}\overline{U})|\nabla u|^{p-2} \\
&+ i\int \mathrm{tr}\,(UB_{a,\alpha}\overline{U})|\nabla u|^{p-2} \\
&+ (p-2)\int \Big((R_{s,\alpha}\xi,\xi) - (B_{s,\alpha}\xi,\eta) + (B_{a,\alpha}\xi,\eta)\Big)|\nabla u|^{p-4} \\
= -(M_1+M_2)\|\nabla u\|_p^p + P, &\qquad (41)
\end{aligned}$$

where

$$\begin{aligned}
P = (1-\varepsilon')\int \mathrm{tr}\,(UR_{s,\alpha}\overline{U})|\nabla u|^{p-2} &+ i\int \mathrm{tr}\,(UB_{a,\alpha}\overline{U})|\nabla u|^{p-2} \\
+ (p-2)\int \Big((R_{s,\alpha}\xi,\xi) &- (B_{s,\alpha}\xi,\eta) + (B_{a,\alpha}\xi,\eta)\Big)|\nabla u|^{p-4}.
\end{aligned}$$

Next, we will show that $P \geq 0$. First note that $(1-\varepsilon')(\cos\alpha) - (3-\varepsilon')\sin(|\alpha|)\tan\theta \geq 0$ due to (38). It follows that

$$\begin{aligned}
&(1-\varepsilon')\mathrm{tr}\,(\overline{U}R_{s,\alpha}U)|\nabla u|^2 + i\mathrm{tr}\,(\overline{U}B_{a,\alpha}U)|\nabla u|^2 \\
&\quad = (1-\varepsilon')(\cos\alpha)\mathrm{tr}\,(\overline{U}R_sU)|\nabla u|^2 - (1-\varepsilon')(\sin\alpha)\mathrm{tr}\,(\overline{U}B_sU)|\nabla u|^2 \\
&\qquad + i(\sin\alpha)\mathrm{tr}\,(\overline{U}R_aU)|\nabla u|^2 \\
&\quad = \Big((1-\varepsilon')\cos\alpha - (3-\varepsilon')\sin(|\alpha|)\tan\theta\Big)\mathrm{tr}\,(\overline{U}R_sU)|\nabla u|^2
\end{aligned}$$

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$$
\begin{aligned}
&+ (1-\varepsilon')\sin(|\alpha|)\mathrm{tr}\,\left(\overline{U}\left((\tan\theta)R_s - \frac{\sin\alpha}{\sin(|\alpha|)}B_s\right)U\right)|\nabla u|^2\\
&+ \sin(|\alpha|)\mathrm{tr}\,\left(\overline{U}\left(2(\tan\theta)R_s + i\frac{\sin\alpha}{\sin(|\alpha|)}R_a\right)U\right)|\nabla u|^2\\
\geq &\Big((1-\varepsilon')(\cos\alpha) - (3-\varepsilon')\sin(|\alpha|)\tan\theta\Big)(R_sU\nabla\overline{u}, U\nabla\overline{u})\\
&+ (1-\varepsilon')\sin(|\alpha|)\left(\left((\tan\theta)R_s - \frac{\sin\alpha}{\sin(|\alpha|)}B_s\right)U\nabla\overline{u}, U\nabla\overline{u}\right)\\
&+ \sin(|\alpha|)\left(\left(2(\tan\theta)R_s + i\frac{\sin\alpha}{\sin(|\alpha|)}R_a\right)U\nabla\overline{u}, U\nabla\overline{u}\right)\\
= &(1-\varepsilon')(\cos\alpha)(R_sU\nabla\overline{u}, U\nabla\overline{u}) - (1-\varepsilon')(\sin\alpha)(B_sU\nabla\overline{u}, U\nabla\overline{u})\\
&+ i(\sin\alpha)(R_aU\nabla\overline{u}, U\nabla\overline{u})\\
= &(1-\varepsilon')(\cos\alpha)\Big((R_s\xi,\xi) + (R_s\eta,\eta)\Big) - (1-\varepsilon')(\sin\alpha)\Big((B_s\xi,\xi) + (B_s\eta,\eta)\Big)\\
&+ 2(\sin\alpha)(R_a\xi,\eta),
\end{aligned}
$$

where we used Lemmas 2.5 and 2.6 in the third step. Hence we obtain

$$
\begin{aligned}
P \geq &\int\bigg((1-\varepsilon')(\cos\alpha)\Big((R_s\xi,\xi) + (R_s\eta,\eta)\Big) - (1-\varepsilon')(\sin\alpha)\Big((B_s\xi,\xi) + (B_s\eta,\eta)\Big)\\
&+ 2(\sin\alpha)(R_a\xi,\eta)\bigg)|\nabla u|^{p-4}\\
&+ (p-2)\int\Big((R_{s,\alpha}\xi,\xi) - (B_{s,\alpha}\xi,\eta) + (B_{a,\alpha}\xi,\eta)\Big)|\nabla u|^{p-4}\\
= &\int\bigg((\cos\alpha)\big((p-1-\varepsilon')(R_s\xi,\xi) + (1-\varepsilon')(R_s\eta,\eta)\big)\\
&- (\sin\alpha)\big((p-1-\varepsilon')(B_s\xi,\xi) + (1-\varepsilon')(B_s\eta,\eta)\big)\\
&+ p(\sin\alpha)(R_a\xi,\eta) - (p-2)(\sin\alpha)(R_s\xi,\eta) - (p-2)(\cos\alpha)(B_s\xi,\eta)\bigg)|\nabla u|^{p-4}\\
= &\int\bigg((\cos\alpha)\big((R_s\xi',\xi') + (R_s\eta',\eta')\big) - (\sin\alpha)\big((B_s\xi',\xi') + (B_s\eta',\eta')\big)\\
&+ \frac{p}{\sqrt{(1-\varepsilon')(p-1-\varepsilon')}}(\sin\alpha)(R_a\xi',\eta') - \frac{p-2}{\sqrt{(1-\varepsilon')(p-1-\varepsilon')}}(\sin\alpha)(R_s\xi',\eta')\\
&- \frac{p-2}{\sqrt{(1-\varepsilon')(p-1-\varepsilon')}}(\cos\alpha)(B_s\xi',\eta')\bigg)|\nabla u|^{p-4}, \qquad (42)
\end{aligned}
$$

where we used Lemma 2.8(iii) and (iv) in the second step, $\xi' = \sqrt{p-1-\varepsilon'}\xi$ and $\eta' = \sqrt{1-\varepsilon'}\eta$. Finally, using (39), we argue in a similar manner to that used in Case 2 of the proof of Proposition 4.3 to derive $P \geq 0$. Thus, it follows from (41) that

$$\mathrm{Re}\,(\nabla(X_{p,\alpha}u), |\nabla u|^{p-2}\nabla u) \geq -(M_1 + M_2)\|\nabla u\|_p^p$$

as claimed. □

Next, we use the two inequalities obtained in Propositions 4.3 and 4.5 to show that $X_{p,\alpha}$ is m-accretive for all α in a suitable range.

Proposition 4.6. *Suppose $B_a = 0$. Let $p \in (1, \infty)$ be such that $|1 - 2/p| < \cos\theta$. Let $\alpha \in (-\gamma, \gamma)$, where γ is given by (11). Then $X_{p,\alpha}$ is m-accretive.*

Proof. The result follows from the arguments used in the proof of [4, Proposition 4.9]. Note that [4, Propositions 4.1, 4.7 and 4.8] used in the proof of [4, Proposition 4.9] are now replaced by Propositions 4.3, 4.4 and 4.5 respectively.

We are now ready to prove Theorem 1.5. □

Proof of Theorem 1.5. We consider two parts.

(i) Contractivity: Using Proposition 4.6 and [11, Theorem IX.1.23], we deduce that $-X_p$ generates a holomorphic semigroup with angle ψ given by (9) which is contractive on the sector Σ_γ, where γ is given by (11). Note that $X_p = \overline{A_p}$ by Proposition 4.2. Hence $S^{(p)}$ is contractive on Σ_γ.

(ii) Consistency: It suffices to show that $S^{(p)}$ is consistent with S. It follows from [4, Propositions 1.1 and 5.1] that the C_0-semigroup generated by $-B_2$ is consistent with the C_0-semigroup generated by $-X_p$. Since $B_2 = \overline{A_2}$ and $X_p = \overline{A_p}$ by Proposition 4.2, the semigroup $S^{(p)}$ is consistent with S as required. □

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