Journal of the Inst. of Math. Jussieu (2011) **10**(3), 449–461
doi:10.1017/S1474748011000016 © Cambridge University Press 2011

THE TIAN–YAU–ZELDITCH THEOREM AND TOEPLITZ OPERATORS

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(Received 12 July 2010; revised 5 December 2010; accepted 5 December 2010)

À Professeur Louis Boutet de Monvel, à l'occasion de son 70e anniversaire: À la tienne, Louis!

Abstract Zelditch's proof of the Tian–Yau–Zelditch Theorem makes use of the Boutet de Monvel–Sjöstrand results on the asymptotic properties of Szegő projectors for strictly pseudoconvex domains. However, as we will show below, the theorem is also closely related to another theorem of Boutet de Monvel's, namely his wave trace formula for Toeplitz operators. Finally, we will derive, for the pseudoconvex manifolds considered by Zelditch in his proof of the Tian–Yau–Zelditch Theorem, an analogue of another result of Boutet de Monvel's, the extendability theorem of Berndtsson for holomorphic functions on Grauert tubes.

Keywords: holomorphic extension; Toeplitz operator; homogeneous complex Monge–Ampère equation

AMS 2010 *Mathematics subject classification:* Primary 32D15
Secondary 58J40

1. Introduction

The topic of this paper is a beautiful theorem of Steve Zelditch [**24**] which refined and generalized earlier results of Tian and Yau. Let (M, L) be a polarized Kähler manifold* of complex dimension n and let $s_{k,i}$, $i = 1, \dots, N_k$, be an orthonormal basis of the space of sections of $L^{\otimes k}$. Zelditch's theorem asserts that as $k \to +\infty$, the expression

$$\sum_{i-1}^{N_k} |s_{k,i}(x)|^2, \quad x \in M, \tag{1.1}$$

* That is, L is an ample line bundle equipped with an inner product $\langle \cdot\,, \cdot \rangle$ and Kähler form $\omega = \mathrm{i}\,\mathrm{dd}^c \log\langle \cdot\,, \cdot \rangle$.

https://doi.org/10.1017/S1474748011000016 Published online by Cambridge University Press

has an asymptotic expansion in k:

$$\sum_{N=0}^{\infty} k^{n-N} a_N(x), \tag{1.2}$$

with $a_N \in \mathcal{C}^\infty(M)$. An application of this result is the following: let $\mathcal{H}$ be the set of Kähler forms on M which are cohomologous to ω, that is, the Kähler forms of the form

$$\omega_\phi = \omega + \mathrm{i}\partial\bar{\partial}\phi, \quad \phi \in \mathcal{C}^\infty(M), \tag{1.3}$$

and let B_k be the set of Hermitian inner products on $H^0(M, \mathcal{O}(L^{\otimes k}))$. Define a map

$$\mathrm{Hilb}_k : \mathcal{H} \to B_k \tag{1.4}$$

by setting

$$\mathrm{Hilb}^k(\omega_\phi)(s) = \int_M |s|_\phi^2 \, \mathrm{d}\mu_\phi, \tag{1.5}$$

where $\langle \cdot\,, \cdot \rangle_\phi := e^{-\phi}\langle \cdot\,, \cdot \rangle$, and $\mu_\phi := \omega_\phi^n/n!$, and define a map

$$\mathrm{FS}_k : B_k \to \mathcal{H} \tag{1.6}$$

by setting

$$\mathrm{FS}_k(B) = \omega + \mathrm{i}\partial\bar{\partial}\phi_B, \tag{1.7}$$

where

$$\phi_B(x) = k^{-1} \log \sum_{i=1}^{N_k} |s_{k,i}^B(x)|^2 \tag{1.8}$$

and the $s_{k,i}^B$ are an orthonormal basis of $H^0(M, \mathcal{O}(L^{\otimes k}))$ with respect to B.*

Define $R_k = \mathrm{FS}_k \circ \mathrm{Hilb}_k$, so that if $R_k(\omega_\phi) = \omega_{\phi_k}$, we have

$$\phi_k = \phi + \log \sum_{i=1}^{N_k} |s_{k,i}|^2 + \log \frac{\omega_\phi^n}{\omega^n}, \tag{1.9}$$

and therefore the asymptotics of (1.1) can be viewed as the asymptotics of the composite map

$$R_k : \mathcal{H} \to \mathcal{H}.$$

Using this map, Ruan [**18**], Berndtsson [**1**], Phong and Sturm [**16**], Song and Zelditch [**21**], Chen and Sun [**6**], Rubinstein and Zelditch [**19**] and others have amassed a great deal of corroborating evidence for a conjecture which roughly states that the Kähler geometry of $\mathcal{H}$ with its natural metric [**15**] is well approximated by the Kähler geometry of the symmetric spaces

$$B_k = \mathrm{GL}(N_k, \mathbb{C})/\,\mathrm{U}(N_k).$$

* The form (1.7) is the Kähler form induced on M by pulling back the Fubini–Study form on $\mathbb{CP}^{N_k-1}$ by the imbedding of M into $\mathbb{CP}^{N_k-1}$ defined by the $s_{k,i}$, hence the name FS_k.

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While inspired by earlier work of Mabuchi [**15**], Semmes [**20**], Donaldson [**8**] and others, such approximation goes back to [**16**]. The recent survey [**17**] and the sample references above will help guide the reader through these very interesting developments.

As for the proof of (1.2) in [**24**], its main ingredients are the following. Let L^* be the line bundle dual to L, and $\langle\cdot,\cdot\rangle^*$ the dual Hermitian metric. To derive the expansion (1.2), Zelditch transforms (1.2) into an assertion about the geometry of the domain

$$D = \{(p,\ell) \mid p \in M,\ \ell \in L^*_p,\ \langle \ell,\ell\rangle^* \leqslant 1\} \tag{1.10}$$

and its boundary, the unit circle bundle

$$X = \{(p,\ell) \in D \mid \langle \ell,\ell\rangle^* = 1\}. \tag{1.11}$$

More explicitly, he shows that the expressions (1.1) are, up to a constant multiple, the Fourier coefficients of Π, where $\Pi : L^2(X) \to H^2(X)$ is the Szegő projector, and observes that if one interprets them this way, then (1.2) can be read off from the parametrix construction of Π given by Boutet de Monvel and Sjöstrand in [**4**].

In the next section we will outline another proof of (1.2), using Fourier–Toeplitz operators, which has some advantages over that of [**24**] (it extends to a larger class of examples) and a serious disadvantage (it does not give the pointwise $\mathcal{C}^k$ convergence for all k). In § 3, we go back to [**9**] for a result on extension similar to Boutet de Monvel's extension theorem, but for a strictly pseudoconvex domain, in preparation for § 4. In passing we note that there is an interesting class of domains with a characterization of extension domains in terms of *iterated* Fourier–Toeplitz operators. In § 4 we analyse again the domains considered by Zelditch in [**24**] and apply the extension theorems of [**9**] and § 3 to these domains. One gets more precision here than generally because the disk bundle domains are invariant under a holomorphic circle action. Sections 3 and 4 end with open questions and a comment on the limits of the extension theorem of [**2**].

2. Toeplitz operators

As above, let D be the unit disk bundle in L^*, and X its boundary. Let $\mathcal{O}(\bar{D})$ be the space of holomorphic functions smooth on $\bar{D}$, and $H^2(X)$ the L^2-closure of $\mathcal{O}(\bar{D}) \subset L^2(X)$. Let Π denote the Szegő projector $L^2(X) \to H^2(X)$. The algebra of Toeplitz operators is the compression of the algebra of pseudodifferential operators to $H^2(X)$, that is, operators of the form $\Pi \circ P \circ \Pi$, where $P \in \Psi(X)$, the algebra of pseudodifferential operators on X. Examples of such operators are given by

$$Q = \frac{1}{\sqrt{-1}} \frac{\partial}{\partial\theta}\bigg|_{H^2(X)}, \tag{2.1}$$

where $\partial/\partial\theta$ is the infinitesimal generator of the holomorphic circle action on D and X, as well as the example

$$T_\phi = \Pi \circ M_\phi \circ \Pi, \quad \phi \in \mathcal{C}^\infty(X), \tag{2.2}$$

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where $M_\phi : L^2(X) \to L^2(X)$ is multiplication by ϕ. If $\phi \in \mathcal{C}^\infty(M)$, i.e. in $\mathcal{C}^\infty(X)^{S^1}$, then M_ϕ and Q commute, and Boutet de Monvel's wave trace formula for Toeplitz operators (see, for example, [**3**, § 13]) asserts that

$$\operatorname{trace}(\exp(\mathrm{i}tQ)) \circ T_\phi \sim \sum_{r=n}^{-\infty} a_r(\phi)\chi_r(t), \tag{2.3}$$

where the χ_r are the periodic conormal distributions

$$\chi_r(t) = \sum_{k=1}^{\infty} k^{r-1} \exp(\mathrm{i}kt). \tag{2.4}$$

On the other hand, letting Π_k be the projection onto the k-eigenspace of Q, we get

$$\operatorname{trace}(\exp(\mathrm{i}tQ)) \circ T_\phi = \sum \exp(\mathrm{i}kt)\Pi_k M_\phi \Pi_k, \tag{2.5}$$

so, by comparing (2.3) and (2.5) one gets

$$\int_X \Pi_k(x,x)\phi(x)\,\mathrm{d}x = \sum a_r(\phi)k^{r-1}. \tag{2.6}$$

Moreover, the functional $\phi \mapsto a_r(\phi)$ is, as we will show below, just the pairing of ϕ with a $\mathcal{C}^\infty$ function a_r, that is,

$$a_r(\phi) = \int_X a_r(x)\phi(x)\,\mathrm{d}x, \tag{2.7}$$

so (2.6) is an integrated form of an expansion of the form

$$\Pi_k(x,x) \sim \sum_{r=n}^{-\infty} a_r(x)k^{r-1}. \tag{2.8}$$

To show that the $a_r(\phi)$ can be expressed as integrals of the form (2.7) we will first show that they can be interpreted as 'residue traces'. More explicitly, from (2.6) one gets for $z \in \mathbb{C}$,

$$\operatorname{trace} \Pi_k \circ Q^{-z} \circ T_\phi \circ \Pi_k \sim \sum_{r=n}^{-\infty} a_r(\phi)k^{r-1-z}, \tag{2.9}$$

and hence for $\operatorname{Re}(z) \gg 0$ we have

$$\operatorname{trace} Q^{-z} \circ T_\phi \sim \sum_{r=n}^{-\infty} a_r(\phi)\zeta(z-r-1), \tag{2.10}$$

where $\zeta(z)$ is the Riemann zeta function.

Thus $\operatorname{trace} Q^{-z} \circ T_\phi$ is meromorphic with at worst simple poles at $z = n-1, n-2, \dots,$ and the $a_r(\phi)$ are the residues at these poles. To compute these residues, we note that for every point x of X there exists a neighbourhood U of x so that locally on U the Toeplitz algebra is isomorphic to the algebra $\Psi(\mathbb{R}^n)$ of pseudodifferential operators on

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$\mathbb{R}^n$ (see [**3**, § 2]). Hence to compute these residues for $\phi \in \mathcal{C}_0^\infty(U)$, it suffices to compute them for the operators corresponding to Q and T_ϕ in $\Psi(\mathbb{R}^n)$. For those operators, however, the residues are indeed given by integrals of the form (2.7) (cf. [**10**, **23**]).

The results we have just described can be extended to a much larger class of operators. If X is a compact manifold and $\Sigma \subset T^*X$ a symplectic cone, one can define on X a generalized Szegő projector of Hermite type having micro-support on the diagonal of $\Sigma \times \Sigma$. One can use this projector to define an algebra of generalized Toeplitz operators $\Pi\Psi(X)\Pi$ with many of the same properties as the 'classical' Toeplitz operators considered above. For example, to take an extreme case, let $\Sigma = T^*X \setminus 0$, $\Pi = I$ and $\Pi\Psi(X)\Pi = \Psi(X)$. Then if $Q \in \Psi^1(X)$ is a positive, self-adjoint elliptic operator whose spectrum is the non-negative integers, and $A \in \Psi^0(X)$, then

$$\operatorname{trace}(\mathrm{i}tQA) \sim \sum_{r=n}^{-\infty} a_r(A)\chi_r(t), \tag{2.11}$$

and if Q commutes with A, this translates into a formula of type (2.6). Colin de Verdière [**7**] and others have used this formula to analyse clustering phenomena for the spectrum of the Laplacian on Zoll manifolds, i.e. Riemannian manifolds for which the geodesic flow on $T^*X \setminus 0$ is periodic and gives a free $\mathbb{R}/2\pi\mathbb{Z}$ action.

3. Toeplitz operators and extendability theorems

One of the most widely quoted of Boutet de Monvel's results is the note [**2**]. In this note he considers the following extension problem. Let Y^n be an n-dimensional real analytic manifold and W a complex 'thickening' of Y, i.e. a complex manifold in which Y sits as a maximal totally real submanifold. Given a real analytic metric g on Y, one gets, for each $p \in Y$, an exponential map $\exp_p : T_pY \to Y$ which can be extended holomorphically to a map $\gamma_p : \mathcal{U}_p \to W$ of a neighbourhood $\mathcal{U}_p$ of the origin in $T_p \otimes \mathbb{C}$ into W. Let $\mathcal{U}$ be a neighbourhood of the zero section in TY with the property that for every $p \in Y$, and $(p, v) \in \mathcal{U}$ implies $\sqrt{-1}v \in \mathcal{U}_p$. Then, shrinking the neighbourhood $\mathcal{U}$ if necessary, we can arrange that the map

$$\gamma : \mathcal{U} \to W, \quad (p, v) \to \gamma_p(v), \tag{3.1}$$

is a diffeomorphism of $\mathcal{U}$ onto a neighbourhood of Y in W. For $\epsilon > 0$ let $T_\epsilon Y = \{(p, v) \in TY \mid g_p(v, v) < \epsilon^2\}$ be the tangent ball bundle of radius ϵ, and for ϵ small, let Ω_ϵ be the diffeomorphic image $\gamma(T_\epsilon Y)$ of $T_\epsilon Y$ in W by γ. The Ω_ϵ are known as *Grauert tubes* [**11**], or *adapted domains* [**14**], and are strictly pseudoconvex for ϵ sufficiently small. The push-forward under γ of $r = \sqrt{g_p(v, v)}$ on Ω_ϵ is a smooth solution of the homogeneous complex Monge–Ampère equation (HCMA) outside the image of the zero section $Y \subset T_\epsilon Y$ [**11**, **14**]. Boutet de Monvel's theorem is that a function f on Y extends holomorphically to the tube $\{r < \epsilon\}$ if and only if f is in the domain of the operators $\exp(t\sqrt{\triangle})$ for all $t \in (0, \epsilon)$, or equivalently, if f is in the image of $\exp(-t\sqrt{\triangle})$, for all $t \in (0, \epsilon)$.

We will be concerned in this section with a variant of this last result on the HCMA equation. Let W^n be, as above, an n-dimensional complex manifold and $\Omega \subset\subset W$ a

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strictly pseudoconvex domain in W with $\mathcal{C}^\omega$ boundary X. Let N^*X be the conormal bundle to X, and N^*_+X the outward pointing non-zero conormals, a component of $N^*X\setminus\{0\}$. We recall the following initial value theorem of Jack Lee for HCMA (see [**9**]).

Theorem 3.1. *Let u be a $\mathcal{C}^\omega$ section of N^*_+X. Then there exists a neighbourhood $\mathcal{U}=\mathcal{U}(\phi)$ of X in W, and a solution $\phi\in\mathcal{C}^\omega(\mathcal{U})$ of the following problem*

$$\left.\begin{aligned} \phi &= 0 \quad \text{on } X,\\ \mathrm{d}\phi &= u \quad \text{on } X,\\ (\mathrm{i}\partial\bar\partial\phi)^n &\equiv 0 \quad \text{on } \Omega. \end{aligned}\right\} \tag{3.2}$$

Furthermore, if $(\phi_i,\mathcal{U}_i)$, $i=1,2$, are two such solutions, then $\phi_1\equiv\phi_2$ on the connected component of $\mathcal{U}_1\cap\mathcal{U}_2$ containing X.

Since X is strictly pseudoconvex, we have the non-degeneracy condition

$$\partial\phi\wedge\bar\partial\phi\wedge(\partial\bar\partial\phi)^{n-1}\neq 0 \quad \text{on } \mathcal{U} \tag{3.3}$$

if $\mathcal{U}$ is small enough about X. Let α be the restriction to X of

$$\beta = 2\,\mathrm{Im}(\bar\partial\phi) = J\cdot\mathrm{d}\phi = J\cdot u. \tag{3.4}$$

Then α is a contact form on X and hence there exists a Reeb vector field ξ on X with the defining properties

$$i(\xi)\alpha = 1 \tag{3.5}$$

and

$$i(\xi)\,\mathrm{d}\alpha = 0. \tag{3.6}$$

As in § 2 above, let $H^2(X)$ be the L^2 closure of the space of functions $\{f|_X \mid f\in\mathcal{O}(\bar\Omega)\}$, and let T_ξ be the first order Toeplitz operator on $H^2(X)$ given by $\Pi\circ(1/\mathrm{i})\mathcal{L}_\xi\circ\Pi$. From [**9**] we have the following theorem.

Theorem 3.2. *For $\epsilon>0$ sufficiently small, and $f\in H^2(X)\cap\mathcal{C}^\omega$, the following are equivalent.*

(i) *The equation*

$$\frac{\partial u}{\partial t}(x,t) = T_\xi u, \quad u(x,0)=f(x),\ x\in X, \tag{3.7}$$

can be solved backwards over the interval $-\epsilon<t\leqslant 0$.

(ii) *f can be extended holomorphically to the domain*

$$\Omega_\epsilon := \{p\in W \mid \phi(p)<\epsilon\}. \tag{3.8}$$

Remark 3.3. It is not immediately clear that this result is related to Boutet de Monvel's result [**2**]. To see that it is, we note that if the Ω_ϵ are Grauert tubes, and $X_\epsilon=\partial\Omega_\epsilon$ are the corresponding boundaries, then there exists a function ϕ on Ω_ϵ with $\phi=\epsilon$ on

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X_ϵ and 0 on Y, and $(\partial\bar{\partial}\phi)^n = 0$ outside Y. This is the same as the function $\sqrt{g_p(v,v)}$ transported by γ to Ω_ϵ above (see [**11**,**12**,**14**]). From the diffeomorphism $\gamma^{-1} : \Omega_\epsilon \to T_\epsilon Y$ one gets a fibration $\rho : X_\epsilon \to Y$, and hence a fibre integration operator,

$$\rho_* : \mathcal{C}^\infty(X_\epsilon) \to \mathcal{C}^\infty(Y), \tag{3.9}$$

and one can show that its restriction to the intersection $\mathcal{C}^\infty(X_\epsilon) \cap H^2(X_\epsilon)$ is an elliptic Fourier integral operator of Toeplitz type, and that for ϵ small, it is bijective and satisfies

$$\rho_* \circ T_\xi = (\sqrt{\triangle} + \mathcal{Q}_\epsilon) \circ \rho_*, \tag{3.10}$$

for a $\mathcal{Q}_\epsilon \in \Psi^0(Y)$. Hence, the extendability criterion of [**2**] is intertwined by ρ_* with the extendability criterion in Theorem 3.2.

We remark further that the condition of small ϵ is necessary in these theorems (see Remark 4.6).

Theorem 3.2 is a corollary of a sharper result involving an analogue of the intertwining identity (3.10). Let X_ϵ be the boundary of the domain (3.8), let W_ϵ be the annulus $\{0 \leqslant \phi \leqslant \epsilon\}$ and let v be the vector field on this annulus defined by the identities

$$i(v)\,\mathrm{d}\phi = 1, \tag{3.11}$$

$$i(v)\beta = 0, \tag{3.12}$$

$$i(v)\,\mathrm{d}\beta = 0, \tag{3.13}$$

where $\beta = J \cdot \mathrm{d}\phi$, as in (3.4), and the equations (3.11)–(3.13) have a unique solution because of the non-degeneracy condition (3.3). By (3.11), $\exp(\epsilon v)$ maps X onto X_ϵ and by (3.12), (3.13), it is a contact isomorphism. Thus, if we let Π_ϵ be the Szegő projector on X_ϵ, we get an operator

$$F_\epsilon = \Pi \circ (\exp \epsilon v)^* \circ \Pi_\epsilon \tag{3.14}$$

mapping $H^2(X_\epsilon)$ into $H^2(X)$ and one can show the following theorem (see [**9**, Proposition B, § 5] and also [**3**]).

Theorem 3.4. *F_ϵ is an elliptic Fourier–Toeplitz operator of order* 0 *quantizing the canonical transformation* $\exp(-\epsilon v)$. *Moreover, for ϵ small enough it is invertible.*

The sharpened version of Theorem 3.2 makes the following assertion [**9**, Theorem 4].

Theorem 3.5. *Let R_ϵ be the restriction map*

$$R_\epsilon : H^2(X_\epsilon) \to \mathcal{O}(\Omega_\epsilon) \to H^2(X),$$

where the first arrow denotes holomorphic extension. Then there exists an invertible Toeplitz operator of order 0,

$$G_\epsilon : H^2(X) \to H^2(X),$$

such that

$$R_\epsilon = \exp(-\epsilon T_\xi) \circ G_\epsilon \circ F_\epsilon, \tag{3.15}$$

where T_ξ is the operator in Theorem 3.2 above.

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Using this result we will prove an extendibility theorem for holomorphic functions on X to a larger class of domains, by concatenating extensions as above via Monge–Ampère solutions. For convenience, we will set $G_\epsilon \circ F_\epsilon$ in the theorem equal to $\tilde{F}_\epsilon$. Setting $\epsilon = \epsilon_1$, let u_2 be a real analytic section of $N_+^* X_{\epsilon_1}$ and let $\phi_2 \in \mathcal{C}^\omega(\mathcal{U}_2)$, $X_{\epsilon_1} \subset \mathcal{U}_2$, a solution of the HCMA satisfying $\phi_2|_{X_{\epsilon_1}} = 0$, $\mathrm{d}\phi_2|_{X_{\epsilon_1}} = u_2$, and let us denote by $\Omega_{\epsilon_1,\epsilon_2}$ the domain $\phi_2 < \epsilon_2$, with boundary $X_{\epsilon_1,\epsilon_2}$. By Theorem 3.5 there exists a zeroth-order invertible Fourier–Toeplitz operator

$$\tilde{F}_{\epsilon_2} : H^2(X_{\epsilon_1,\epsilon_2}) \to H^2(X_{\epsilon_1})$$

such that

$$R_{\epsilon_2} = \exp(-\epsilon_2 T_{\xi_2}) \circ \tilde{F}_{\epsilon_2}, \tag{3.16}$$

where R_{ϵ_2} is the restriction map

$$H^2(X_{\epsilon_1,\epsilon_2}) \to \mathcal{O}(\Omega_{\epsilon_1,\epsilon_2}) \to H^2(X_{\epsilon_1}),$$

and ξ_2 is a real analytic vector field on X_{ϵ_1} satisfying

$$i(\xi_2) 2 \operatorname{Im}(\bar{\partial}\phi_2) = 1$$

and

$$i(\xi_2)\partial\bar{\partial}\phi_2 = 0.$$

Hence, with $T_1 = T_\xi$, as in Theorem 3.5, we obtain

$$R_{\epsilon_1} \circ R_{\epsilon_2} = \exp(-\epsilon_1 T_1) \circ \tilde{F}_{\epsilon_1} \circ \exp(-\epsilon_2 T_{\xi_2}) \circ \tilde{F}_{\epsilon_2}. \tag{3.17}$$

However, $\tilde{F}_{\epsilon_1} \circ T_{\xi_2} \circ \tilde{F}_{\epsilon_1}^{-1}$ is a first-order Toeplitz operator on X of the form

$$T_2 = T_{w_2} + A(\epsilon_1),$$

where $w_2 = (\exp(\epsilon_1 v))^* \xi_2$ and $A(\epsilon_1)$ is a zeroth-order Toeplitz operator which depends analytically on ϵ_1, and vanishes at $\epsilon_1 = 0$. Thus,

$$R_{\epsilon_1} \circ R_{\epsilon_2} = \exp(-\epsilon_1 T_1) \circ \exp(-\epsilon_2 T_2) \circ \tilde{F}_{\epsilon_1} \circ \tilde{F}_{\epsilon_2}. \tag{3.18}$$

Iterating this procedure one can construct sequences of domains $\Omega_{\epsilon_1,\dots,\epsilon_k}$ with $\Omega_{\epsilon_1,\dots,\epsilon_{k-1}} \subset \Omega_{\epsilon_1,\dots,\epsilon_k}$ such that the extendibility problem for Ω relative to $\Omega_{\epsilon_1,\dots,\epsilon_k}$ is controlled by a sequence of Toeplitz semi-groups

$$\exp(-\epsilon_1 T_1) \circ \cdots \circ \exp(-\epsilon_k T_k),$$

where $T_k(\epsilon_1, \dots, \epsilon_{k-1})$ depends analytically on the epsilons and at $\epsilon = 0$ is of the form $T_k(0) = T_{\xi_k}$, where ξ_k is a $\mathcal{C}^\omega$ vector field on X satisfying $\alpha(\xi_k) > 0$, which is the condition in Toeplitz theory for T_{ξ_k} to be elliptic and positive. To summarize, we have shown the following theorem.

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Theorem 3.6. *Let $\Omega_{\epsilon_1,\dots,\epsilon_k}$ be an iterated extension domain as above, where the $\epsilon_1,\dots,\epsilon_k$ are sufficiently small that Theorem 3.5 holds for each extension $\Omega_{\epsilon_1,\dots,\epsilon_{i-1}}$ to $\Omega_{\epsilon_1,\dots,\epsilon_i}$, for $i=2,\dots,k$ (as well as Ω to Ω_{ϵ_1}), and let $X_{\epsilon_1,\dots,\epsilon_k}=\partial\Omega_{\epsilon_1,\dots,\epsilon_k}$. Then there exist positive, analytic, elliptic Toeplitz operators of order 1, $T_1,\dots,T_k$, on X such that the image of the restriction map*

$$R: H^2(X_{\epsilon_1,\dots,\epsilon_k}) \to H^2(X)$$

is equal to the range of $\exp(-\epsilon_1 T_1)\circ\cdots\circ\exp(-\epsilon_k T_k)$ in $H^2(X)$.

Remark 3.7. Given any domain $\tilde{\Omega}\supset\supset\Omega$, it is clear that one can construct a sequence of domains

$$\Omega=\Omega_0\subsetneq\Omega_1\subsetneq\Omega_2\subsetneq\cdots\subset\tilde{\Omega}$$

defined successively by sub-level sets of $\mathcal{C}^\omega$ solutions of the HCMA as above. If Ω is real analytic and $\mathcal{C}^k$ close for large enough k, is it possible to exhaust $\tilde{\Omega}$, i.e. achieve

$$\bigcup_i \Omega_i=\tilde{\Omega}? \tag{3.19}$$

4. We come full circle

To come full circle we will apply the extendibility results in § 3 to the domain considered by Zelditch in his proof of the Tian–Yau–Zelditch Theorem. As in that proof, let D be the domain

$$D=\{(p,\ell)\in L^* \mid \langle\ell,\ell\rangle\leqslant 1\}. \tag{4.1}$$

To apply those results to D, assume that $X=\partial D$ is real analytic, i.e. that the metric in (4.1) is $\mathcal{C}^\omega$. Then uniqueness in the Cauchy–Kowalevski Theorem gives the following lemma.

Lemma 4.1. *Let u be a $\mathcal{C}^\omega$ section of N_+^*X and let ψ be a real analytic solution of the homogenous complex Monge–Ampère equation which vanishes on X and has $\mathrm{d}\psi=u$ on X. Then if u is S^1-invariant, so is ψ.*

Let r be the function $r(p,\ell)=\sqrt{\langle\ell,\ell\rangle_p}$, and let $\phi\in\mathcal{C}^\omega(X)$ be the function given by

$$\phi(x)=\left(\frac{d\log r}{u}\right)(x)>0. \tag{4.2}$$

We will show that for the domain D the extendibility criterion in Theorem 3.2 can be formulated in terms of the operators

$$Q=\Pi\circ\frac{1}{\mathrm{i}}\frac{\partial}{\partial\theta}\circ\Pi$$

and T_ϕ in (2.2) figuring in the proof we sketched of the Tian–Yau–Zelditch Theorem in § 2. Let u be an S^1-invariant section of N_+^*, and let ψ be the S^1-invariant solution of HCMA with $\psi=0$ and $\mathrm{d}\psi=u$ on X. Finally, let $D_\epsilon=\{\psi<\epsilon\}$.

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Theorem 4.2. *A function $f \in \mathcal{C}^\omega \cap H^2(X)$ extends holomorphically to D_ϵ if and only if f is in the range of the operator*

$$\exp(-tT_\phi \circ Q) : H^2(X) \to H^2(X) \cap \mathcal{C}^\omega(X) \tag{4.3}$$

for all $t < \epsilon$.

This theorem may be viewed as an extension result to disk bundles with respect to deformations of the Hermitian metric $\langle \ell, \ell \rangle$.

Proof. Let

$$\alpha = d^c \log r|_X \tag{4.4}$$

be the connection form for the canonical connection on the circle bundle $\pi : X \to M$ associated to the Kähler form ω on M, i.e. $i(\partial/\partial\theta)\alpha = 1$, and $d\alpha = \pi^*\omega$.

Lemma 4.3. *If w is a vector field on X, the associated Toeplitz operator*

$$T_w = \Pi \circ \frac{1}{\mathrm{i}} \mathcal{L}_w \circ \Pi$$

is a Toeplitz operator of order 0 if and only if $i(w)\alpha = 0$.

Proof of Lemma 4.3. It suffices by [**3**, Proposition 2.12] to show that the principal symbol of T_w is zero. Let $\Sigma \subset T^*X$ be the symplectic cone

$$\Sigma = \{(p, \xi) \mid p \in X;\ \xi = \lambda\alpha_p,\ \lambda > 0\}.$$

Then the symbol of T_w is just the restriction to Σ of the symbol of the differential operator $(1/\mathrm{i})\mathcal{L}_w$, and hence at (p, ξ), $\xi = \lambda\alpha_p$, $\lambda > 0$ is just $\lambda \cdot i(w)\alpha_p$. □

Corollary 4.4. *Let $\phi = i(w)\alpha$ and as above let*

$$Q_\phi = \Pi \circ \frac{1}{\mathrm{i}} \phi \frac{\partial}{\partial\theta} \circ \Pi.$$

Then $T_w - Q_\phi$ is a zeroth-order Toeplitz operator.

To exploit this result we will need the following theorem, a proof of which may be found in § 6 of [**9**]. (This theorem is due to Boutet de Monvel, and is a key ingredient in the proof of his extendibility theorem.)

Proposition 4.5. *Let P be a real analytic elliptic Toeplitz operator on X with the same symbol as Q_ϕ. Then there exists a smooth zeroth-order Toeplitz operator $U(t)$ which is invertible and depends real analytically on t such that*

$$\exp(-tP) = \exp(-tQ_\phi) \circ U(t). \tag{4.5}$$

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To prove Theorem 4.2 we apply Theorem 3.1 to the section u of $N_+^* X$. Since the vector field ξ in Theorem 3.2 has the defining properties $i(\xi)\beta = 1$, $i(\xi)\,\mathrm{d}\beta = 0$, for $\beta = 2\,\mathrm{Im}(\bar\partial\psi) = J\,\mathrm{d}\psi$ and

$$\begin{aligned} J\,\mathrm{d}\psi &= Ju && \text{by the initial condition} \\ &= \phi^{-1} J \frac{\mathrm{d}t}{t} && \text{by (4.2)} \\ &= \phi^{-1}\alpha && \text{by (4.4),} \end{aligned}$$

the interior product $i(\xi)\alpha$ is equal to ϕ and hence

$$i(\xi)\alpha = \mathrm{i}\left(\phi\frac{\partial}{\partial\theta}\right)\alpha.$$

Therefore, in view of Corollary 4.4 and Proposition 4.5, we can replace T_ξ by Q_ϕ in Theorem 3.2. □

Remark 4.6. We close by noting that a question remains in Theorems 3.5 and 4.2 concerning the globalization, or long-term behaviour, of the problem. That is, suppose we have a complex manifold $\tilde\Omega \supset M$, where M is a compact strongly pseudoconvex manifold with boundary $\partial M = X$, and that the $\mathcal{C}^\omega$ solution ψ of HCMA in Theorem 4.2 is defined and proper

$$\psi : \tilde\Omega \setminus M \to [0, +\infty).$$

Suppose also that ψ is non-degenerate on $\tilde\Omega \setminus M$, that is, $\partial\psi \wedge \bar\partial \wedge (\partial\bar\partial\psi)^{n-1} \neq 0$ there. Let $X_s = \psi^{-1}(s)$. Then one has a restriction operator $R_s : H^2(X_s) \to H^2(X)$. Is the characterization of the image of R_s in Theorem 4.2 valid for all values of $s \in [0, +\infty)$? To illustrate, if $\tilde\Omega = L^*$ as above, and $\psi = \log r(p, \ell)$, these conditions are satisfied. In this case the vector field v generates a holomorphic flow $\ell \mapsto \exp(\mathrm{i}t) \cdot \ell$. Let $\delta_t : L^* \to L^*$ be this flow in the imaginary direction, i.e.

$$F_t = \delta_t^* : H^2(X_t) \to H^2(X),$$

and one sees by checking term-by-term the effect on the Fourier expansion of a function in $H^2(X_t)$ that one has an equality

$$R_t = \exp(-tQ) \circ \delta_t^* : H^2(X_t) \to H^2(X) \tag{4.6}$$

for *all* $t > 0$. In particular, there is no need here for the correction operator G_t as in Theorem 3.5. The same reasoning works for $\tilde\Omega = \mathbb{C}^n$ and M a $\mathcal{C}^\omega$ circled strictly pseudoconvex domain $\subset\subset \mathbb{C}^n$ containing the origin. Here we take $\psi = \log G(z)$, where G is the gauge function of M.

A more complex example is to take a Riemannian symmetric space $Y = G/K$, where G, K are a compact Cartan pair, and let $G_{\mathbb{C}}$, $K_{\mathbb{C}}$ denote the corresponding complex groups. Then Lassalle [**13**], which was an inspiration for [**2**], determined the extendibility properties of functions on G/K to $G_{\mathbb{C}}/K_{\mathbb{C}}$ using the representation theory of G. This can

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be cast into the framework of [**2**] and this paper by means of a theorem of Szőke's [**22**, Theorem 2.5] and is valid for all levels of the Monge–Ampère solution on $G_{\mathbb{C}}/K_{\mathbb{C}}$.

Finally, however, for some examples of non-symmetric metrics on S^2, also due to Szőke [**22**], the extendibility theorems are not valid for all levels X_t of the Monge–Ampère exhaustion ψ [**5**].

Acknowledgements. This work was supported in part by NSF grant DMS-0514070, DMS-0805877 (DB) and DMS-1005696 (VG). D.B. thanks the Institut des Hautes Études Scientifiques and the Université de Paris-Sud for support during the preparation of this paper. The authors thank an anonymous referee for several suggestions which improved this paper considerably.

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