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NOTES ON SOME ERDŐS–HAJNAL PROBLEMS

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Abstract. We make comments on some problems Erdős and Hajnal posed in their famous problem list. Let X be a graph on ω_1 with the property that every uncountable set A of vertices contains a finite set s such that each element of $A - s$ is joined to one of the elements of s. Does then X contain an uncountable clique? (Problem 69) We prove that both the statement and its negation are consistent. Do there exist circuitfree graphs $\{X_n : n < \omega\}$ on ω_1 such that if $A \in [\omega_1]^{\aleph_1}$, then $\{n < \omega : X_n \cap [A]^2 = \emptyset\}$ is finite? (Problem 61) We show that the answer is yes under CH, and no under Martin's axiom. Does there exist $F : [\omega_1]^2 \to 3$ with all three colors appearing in every uncountable set, and with no triangle of three colors. (Problem 68) We give a different proof of Todorcevic' theorem that the existence of a κ-Suslin tree gives $F : [\kappa]^2 \to \kappa$ establishing $\kappa \not\to [\kappa]^2_\kappa$ with no three-colored triangles. This statement in turn implies the existence of a κ-Aronszajn tree.

In this note we consider three problems of the Erdős–Hajnal collection of unsolved problems in set theory [1].

The first problem is the following.

Problem 69. *Let X be a graph on ω_1. Assume that for every $A \in [\omega_1]^{\aleph_1}$ there is a finite $s \subseteq A$ such that each element of $A - s$ is joined to some element of s. Does X necessarily contain an uncountable clique*?

I slightly modified the formulation by requiring $|A| = \aleph_1$, originally the authors only assumed $A \subseteq \omega_1$. This is, however, problematic, as if there is no uncountable clique, then there is an infinite independent vertex set by the Erdős–Dushnik–Miller theorem, so the statement trivially holds.

For technical reasons we reformulate the problem for the complement of X.

Problem 69. *Let X be a graph on ω_1. Assume that for every $A \in [\omega_1]^{\aleph_1}$ there is a finite $s \subseteq A$ such that no element of $A - s$ is joined to every element of s. Does X necessarily contain an uncountable independent set*?

In this note I prove the consistency of both the statement and its negation (Corollary 2, Theorem 3).

The second problem is the following.

Problem 61. *Do there exist circuitfree graphs $\{X_n : n < \omega\}$ on ω_1 such that if $A \in [\omega_1]^{\aleph_1}$, then $\{n < \omega : X_n \cap [A]^2 = \emptyset\}$ is finite*?

Erdős and Hajnal remarked in [2], CH implies a 'yes' answer. As it remained unpublished, we reprove this result here (Theorem 4).

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For the other direction, we show that a 'no' answer follows from MA_{ω_1} (Theorem 5).

The third problem we address from the Erdős–Hajnal list is the following.

PROBLEM 68. (GCH) *Assume that* $F : [\omega_1]^2 \to 3$ *is such that* F *assumes all values on any uncountable subset of* ω_1. *Do there exist* $\alpha < \beta < \gamma < \omega_1$ *with* $\{F(\alpha, \beta), F(\alpha, \gamma), F(\beta, \gamma)\} = \{0, 1, 2\}$?

In what follows we call a triangle $\{x, y, z\}$ *three-colored*, if $F(x, y)$, $F(x, z)$, and $F(y, z)$ are distinct. A function $F : [\kappa]^2 \to \gamma$ is said to *establish* $\kappa \not\to [\kappa]^2_\gamma$ if F assumes all values on every $A \in [\kappa]^\kappa$. In [3], Shelah proved that $2^\kappa = \kappa^+$ implies the existence of a function establishing $\kappa^+ \not\to [\kappa^+]^2_\kappa$ with no three-colored triangles. We include his proof as it can be considerably simplified using a well known consequence of CH (Theorem 8).

Shelah also proved, that if V=L, then for every regular cardinal λ, there is a function $F : [\lambda^+]^2 \to \lambda^+$ witnessing $\lambda^+ \not\to [\lambda^+]^2_{\lambda^+}$ with no triangles of three colors. Todorcevic in [5] proved the stronger result that a similar function exists on a cardinal κ for which a κ-Suslin tree exists. We give a different proof to his result (Theorem 10). Following the referee's suggestion, we show that if $\kappa^{<\kappa} = \kappa$ holds then forcing with $\mathrm{Add}(\kappa, 1)$ adds such an example on κ^+ (Theorem 11). Finally, we show that the existence of $F : [\kappa]^2 \to \kappa$ establishing $\kappa \not\to [\kappa]^2_\kappa$ with no three-colored triangles which in turn implies the existence of a κ-Aronszajn tree. By result of Mitchell [7, Theorem 4], this gives that it is consistent (relative to the consistency of a weakly compact) that there is no $F : [\omega_2]^2 \to \omega_2$ establishing $\aleph_2 \not\to [\aleph_2]^2_{\aleph_2}$ with no three-colored triangles.

NOTATION AND DEFINITIONS. We use the notions and definitions of axiomatic set theory. In particular, each ordinal is a von Neumann ordinal, each cardinal is identified with the least ordinal of that cardinality. If f is a function, A a set, then $f[A] = \{f(x) : x \in A\}$. If κ is an infinite cardinal, then κ^+ is its successor cardinal. If $(A, <)$ is an ordered set, then $\mathrm{tp}(A, <)$ or just $\mathrm{tp}(A)$ denotes its order type. If A, B are subsets of the same ordered set, then $A < B$ denotes that $x < y$ holds for any $x \in A$, $y \in B$. If A or B is a singleton, we write $a < B$ instead of $\{a\} < B$, etc. If S is a set, κ a cardinal, then $[S]^\kappa = \{x \subseteq S : |x| = \kappa\}$, $[S]^{<\kappa} = \{x \subseteq S : |x| < \kappa\}$, $[S]^{\leq\kappa} = \{x \subseteq S : |x| \leq \kappa\}$.

A *tree* is a partially ordered set $(T, \leq)$, such that $t{\downarrow} = \{t' \in T : t' < t\}$ is well ordered for each element (or node) $t \in T$. If $(T, \leq)$ is a tree, $t \in T$, then $\mathrm{ht}(t) = \mathrm{tp}((t{\downarrow}))$ is the *height of* t. We also use the piece of notation $t{\uparrow} = \{t' \in T : t < t'\}$. $T_\alpha = \{t \in T : \mathrm{ht}(t) = \alpha\}$ for any ordinal α. The *height* of a tree $(T, \leq)$, $\mathrm{ht}(T, \leq)$ is the least ordinal such that $T_\alpha = \emptyset$.

A *chain* in a tree $(T, \leq)$ is a set of pairwise comparable nodes. A κ-*branch* is a chain $B \subseteq T$, such that $b \cap T_\alpha \neq \emptyset$ $(\alpha < \kappa)$. An *antichain* in a tree $(T, \leq)$ is a set of pairwise incomparable nodes.

A tree is *normal*, if

(1) $|T_0| = 1$,
(2) each $t \in T_\alpha$ has at least two successors in $T_{\alpha+1}$ $(\alpha + 1 < \mathrm{ht}(T, \leq))$,
(3) if $\alpha < \beta < \mathrm{ht}(T)$, $x \in T_\alpha$, then there is $x < y \in T_\beta$, and
(4) if $\alpha < \mathrm{ht}(T)$ is a limit ordinal, x, y are distinct elements of T_α, then $x{\downarrow} \neq y{\downarrow}$.

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If $(T, \leq)$ is normal, then for each $x, y \in T$ there is a largest lower bound denoted by $x \wedge y$.

A tree $(T, \leq)$ of height κ is a κ-*Suslin tree* if there are no chains or antichains of size κ in it. We freely use the facts that if there is a κ-Suslin tree, then there is a normal κ-Suslin tree, and that for a normal tree to be κ-Suslin it suffices to assume that it does not contain antichains of cardinality κ. An ω_1-Suslin tree is simply called a *Suslin-tree*.

A *graph* is any pair (V, X) where $X \subseteq [V]^2$. If κ, λ are cardinals, then $K_{\kappa,\lambda}$ is the complete bipartite graph with bipartition classes of size κ, λ.

If $(T, \leq)$ is a tree, then a graph $X \subseteq [T]^2$ *obeys* $(T, \leq)$, if $\{t, t'\} \in X$ implies t' or $t' < t$.

THEOREM 1. *It is consistent that there exist a Suslin tree $(T, \leq)$ and a graph X on T such that*

(1) *X obeys $(T, \leq)$,*
(2) *there is no uncountable independent set in X, and*
(3) *if $t_0, t_1 \in T$ are incomparable, then*

$$N(t_0, t_1) = \{t \in T : \{t, t_0\}, \{t, t_1\} \in X\}$$

is finite.

Notice that if t_0, t_1 are as in (3), $t \in N(t_0, t_1)$, then $t < t_0, t_1$. Indeed, both $t_0 < t < t_1$ and $t_0, t_1 < t$ are ruled out as they would imply that t_0, t_1 are comparable.

PROOF. Let $(T, \leq)$ be a Suslin tree.

Define the notion of forcing $(P, \leq)$ as follows. $p \in P$ if $p = (s, g)$ where $s \in [T]^{<\omega}$, $g \subseteq [s]^2$, g obeys $(T, \leq)$. $(s', g') \leq (s, g)$ iff $s' \supseteq s$, $g = g' \cap [s]^2$, and there are no $t_0, t_1 \in s$ incomparable, $t \in s' - s$ such that $\{t, t_0\}, \{t, t_1\} \in g'$. ⊣

CLAIM 1. $\leq$ is transitive.

PROOF. Straightforward. ⊣

CLAIM 2. If $t \in T$, then $D_t = \{(s, g) : t \in s\}$ is dense.

PROOF. If $t \in T$, $(s, g) \in P$, then $(s \cup \{t\}, g)$ is a condition and $(s \cup \{t\}, g) \leq (s, g)$. ⊣

CLAIM 3. $(P, \leq)$ has the Knaster property.

PROOF. Assume that $p_\xi \in P$ $(\xi < \omega_1)$. Using the pigeon hole principle and the Δ-system lemma, we can assume that $p_\xi = (s \cup s_\xi, g_\xi)$ with

$$\mathrm{ht}[s] < \mathrm{ht}[s_0] < \mathrm{ht}[s_1] < \cdots < \mathrm{ht}[s_\xi] < \cdots,$$

$g_\xi \cap [s]^2 = g$. If now $\xi < \eta$, then $p' = (s', g')$ is a condition where $s' = s \cup s_\xi \cup s_\eta$, $g' = g_\xi \cup g_\eta$. The only possibility that $p' \leq p_\xi, p_\eta$ does not hold is that there are incomparable $t_0, t_1 \in s_\eta$, $t \in s_\xi$, with $\{t, t_0\}, \{t, t_1\} \in g'$, which is not the case. ⊣

Let $G \subseteq P$ be generic.

CLAIM 4. $(T, \leq)$ remains Suslin in $V[G]$.

PROOF. Immediate from Claim 3. ⊣

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In $V[G]$, define $X = \bigcup\{g : (s, g) \in G\}$.

CLAIM 5. If $t_0, t_1 \in T$ are incomparable, then $N(t_0, t_1)$ is finite.

PROOF. Assume that $t_0, t_1 \in T$ are incomparable. By Claim 2, there is $p = (s, g) \in G$ with $t_0, t_1 \in s$. By the definition of extension of conditions,

$$\{t \in s' : \{t, t_0\}, \{t, t_1\} \in g'\} = \{t \in s : \{t, t_0\}, \{t, t_1\} \in g\}$$

holds for every $p' = (s', g') \leq p$. But then, $N(t_0, t_1) \subseteq s$, therefore $N(t_0, t_1)$ is finite. $\dashv$

CLAIM 6. X has no uncountable independent subset.

PROOF. Assume that p forces that $A \subseteq T$ is an uncountable independent set of X. For an uncountable set $B \subseteq T$ there are $p \geq p_t \Vdash t \in A$. Using again the pigeon hole principle and the Δ-system lemma, there is an uncountable $B' \subseteq B$, such that $p_t = (s \cup s_t, g_t)$ where $g_t \cap [s]^2 = g$, $t \in s_t$, if $t', t'' \in B'$ then $\mathrm{ht}(t') \neq \mathrm{ht}(t'')$ and if $\mathrm{ht}(t') < \mathrm{ht}(t'')$, then $\mathrm{ht}[s] < \mathrm{ht}[s_{t'}] < \mathrm{ht}[s_{t''}]$.

As $(T, \leq)$ is a Suslin tree, there are $t', t'' \in B'$ with $t' < t''$. Define $p^* = (s^*, g^*)$ where

$$s^* = s \cup s_{t'} \cup s_{t''}$$

and

$$g^* = g_{t'} \cup g_{t''} \cup \{\{t', t''\}\}.$$

It is clear that p^* is a condition. The only trouble with $p^* \leq p_{t'}, p_{t''}$ could be that there are incomparable $t_0, t_1 \in s_{t''}$, $t \in s_{t'}$ with $\{t, t_0\}, \{t, t_1\} \in g^*$. As there are no such elements, we have $p^* \leq p_{t'}, p_{t''}$ and so

$$p^* \Vdash t', t'' \in A, \{t', t''\} \in X,$$

a contradiction.

By Claims 5 and 6, the proof of the Theorem is finished. $\dashv$

COROLLARY 2. *It is consistent that there is a graph X on ω_1 such that*

(A) *if $A \in [\omega_1]^{\aleph_1}$, then there is some $\emptyset \neq s \in [A]^{<\omega}$, such that no $x \in A - s$ is joined to every element of s, and*

(B) *there is no uncountable independent set in X.*

PROOF. Let $(T, \leq)$ be a Suslin tree and X be a graph on T, as in Theorem 1. We claim that X is as required.

Assume that A is an uncountable subset of T. As $(T, \leq)$ is Suslin, there are incomparable $t_0, t_1 \in A$. Let

$$s = \{t_0, t_1\} \cup (N(t_0, t_1) \cap A).$$

The finite set s satisfies the above property (A): if $t \in A - s$, $\{t_0, t\}, \{t_1, t\} \in X$, then $t \in N(t_0, t_1) \cap A \subseteq s$, a contradiction. As property (B) obviously holds, we are done. $\dashv$

We will apply the following result of Todorcevic.

THEOREM A (Todorcevic, [6]). *It is consistent that* $\mathfrak{c} = \aleph_2$, MA_{ω_1} *holds, and if* X *is a graph on* ω_1 *then either* X *contains an uncountable independent set or it has two sets,* $A, B \subseteq \omega_1$, $|A| = \aleph_0$, $|B| = \aleph_1$, *such that* $[A]^2 \subseteq X$ *and each vertex of* A *is joined to each vertex in* B.

In fact, the result follows from PFA.

THEOREM 3. *It is consistent that if* X *is a graph on* ω_1 *such that for every* $A \in [\omega_1]^{\aleph_1}$ *there is* $\emptyset \neq s \in [A]^{<\omega}$ *with the property that every* $x \in A - s$ *is not joined to at least one* $y \in s$, *then* X *has an uncountable independent vertex set.*

PROOF. We refer to a model of Theorem A. In that model, if X is a graph on ω_1 not containing an uncountable independent set, then there are $A \in [\omega_1]^{\omega}$, $B \in [\omega_1]^{\omega_1}$ with $[A]^2 \subseteq X$ and

$$\{\{x, y\} : x \in A, y \in B\} \subseteq X.$$

We claim that $A \cup B$ does not satisfy the condition in the Problem. For that, let $s \in [A \cup B]^{<\omega}$. If now $x \in A - s$, then x is joined to each element of s. ⊣

THEOREM 4. (Erdős–Hajnal) *If CH holds, then there are disjoint circuitfree graphs* $\{X_n : n < \omega\}$ *such that if* $A \in [\omega_1]^{\aleph_1}$, *then* $\{n < \omega : X_n \cap [A]^2 = \emptyset\}$ *is finite.*

PROOF. Enumerate $[\omega_1]^{\omega} \times [\omega]^{\omega}$ as $\{\langle A_\alpha, B_\alpha\rangle : \alpha < \omega_1\}$. Then define the partial function F with $\mathrm{Dom}(F) \subseteq [\omega_1]^2$, $\mathrm{Ran}(F) \subseteq \omega$ such that

(a) $F(\beta, \alpha) \neq F(\beta', \alpha)$ $(\beta < \beta' < \alpha)$ and

(b) if $\beta < \alpha$ is such that $A_\beta \subseteq \alpha$, then there are $\gamma \in A_\beta$, $n \in B_\beta$ with $F(\gamma, \alpha) = n$.

One can immediately see that $X_n = F^{-1}(n)$ $(n < \omega)$ are as required. ⊣

One cannot change "finite" to "empty" in the Theorem, as for every $n < \omega$ there is $A \in [\omega_1]^{\omega_1}$ such that $X_n \cap [A]^2 = \emptyset$. Namely, any uncountable color class in the two-coloring of the graph X_n.

Forcing with $P = \mathrm{Add}(\omega, \kappa)$ over a model of CH gives a model of the statement in the Theorem with large continuum. In fact, the ground model construction retains its property via forcing with any P with the Knaster property.

THEOREM 5. (MA_{ω_1}) *There are no graphs* $\{X_n : n < \omega\}$ *on* ω_1 *such that if* $A \in [\omega_1]^{\aleph_1}$, *then* $\{n < \omega : X_n \cap [A]^2 = \emptyset\}$ *is finite.*

PROOF. Assume that $X_0, X_1, \ldots$ are circuitfree graphs on ω_1. Let D be a nonprincipal ultrafilter on ω. Set $e \in X$ iff $\{n < \omega : e \in X_n\} \in D$. X is circuitfree: should $\{e_0, \ldots, e_k\}$ be a circuit, then we had $e_0, \ldots, e_k \in X_n$ for some n, a contradiction. X is a bipartite graph on ω_1, let V be one of the uncountable bipartition classes.

If we now restrict to V and redefine it to ω_1, then we obtain that $X_0, \ldots$ are circuitfree graphs on ω_1 and

(*) $\{n : e \in X_n\} \notin D$ for every $e \in [\omega_1]^2$.

Define $(s, t) \in P$ if $s \in [\omega_1]^{<\omega}$, $t \in [\omega]^{<\omega}$, and s is independent in X_n $(n \in t)$. $(s', t') \leq (s, t)$ if $s' \supseteq s$, $t' \supseteq t$.

By (*), for each $n < \omega$, the set $D_n = \{(s, t) : |t| > n\}$ is dense.

In order to show ccc, assume that $p_\alpha \in P$ $(\alpha < \omega_1)$. Without loss of generality, $p_\alpha = (s \cup s_\alpha, t)$ where $\{s, s_\alpha : \alpha < \omega_1\}$ are disjoint. If $(s \cup s_\alpha, t)$, $(s \cup s_\beta, t)$ are

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incompatible, then there is an edge of $Y = \bigcup\{X_k : k \in t\}$ between s_α and s_β. Using this, one obtains that Y contains a K_{n+1,n^2} where $n = |t|$. If the edges of this K_{n+1,n^2} are colored with n colors, then there is a monocolored C_4, i.e., some X_i contains a C_4, a contradiction.

By ccc, there is $\gamma < \omega_1$, such that if $(s,t) \in P$, $s \cap \gamma = \emptyset$, then for every $\gamma < \alpha < \omega_1$ there is $(s',t') \leq (s,t)$, $s' \cap \gamma = \emptyset$, $s' \not\subseteq \alpha$. We can now redefine P by adding the condition $s \cap \gamma = \emptyset$ to the definition of $(s,t) \in P$.

Finally, a standard application of MA_{ω_1} gives the result. ⊣

Lemma 6. *If $f : [\kappa]^2 \to \lambda$ is a coloring with no three-colored triangle, $g : \lambda \to \mu$ is arbitrary, then $g \circ f : [\kappa]^2 \to \mu$ is a coloring with no three-colored triangle.*

Proof. Immediate. ⊣

Theorem 7. (Shelah [3]) *If $\kappa^{<\kappa} = \kappa$ and $2^\kappa = \kappa^+$, then there is a coloring $F : [\kappa^+]^2 \to \kappa$ establishing $\kappa^+ \not\to [\kappa^+]^2_\kappa$ with no three-colored triangle.*

Proof. Let $A \subseteq {}^\kappa 2$ be a Lusin set, i.e., $|A| = \kappa^+$ and every $B \subseteq [A]^{\kappa^+}$ is somewhere dense in the lexicographically ordered ${}^\kappa 2$. Define $H(f,g) < \kappa$ as the place of the least difference of $f, g \in A$. If $B \in [A]^{\kappa^+}$, then the range of H on $[B]^2$ contains an end-segment of κ, by the Lusin property. If $g : \kappa \to \kappa$ is such that $g^{-1}(\alpha)$ is cofinal in κ for each $\alpha < \kappa$, then $F = g \circ H$ is as required by Lemma 6 and the above observation. ⊣

Lemma 8. *If $(T, \leq)$ is a normal tree, $\mathrm{ht}(T) = \kappa$, $F(t_0,t_1) = \mathrm{ht}(t_0 \wedge t_1)$, then $F : [T]^2 \to \kappa$ is a coloring with no three-colored triangle.*

Proof. Let t_0, t_1, and t_2 be three distinct elements of T. It suffices to show that some two of $t_0 \wedge t_1, t_0 \wedge t_2, t_1 \wedge t_2$ are equal. As we have $t_0 \wedge t_1, t_0 \wedge t_2 \leq t_0$, the nodes $t_0 \wedge t_1$ and $t_0 \wedge t_2$ are comparable. As we are done if they are equal, we can assume that $t_0 \wedge t_1 < t_0 \wedge t_2$. If $t' = t_0 \wedge t_1$, then $t' \leq t_1$, $t' < t_0 \wedge t_2 \leq t_2$. If x, y are the immediate successors of t' with $x \leq t_0$, $y \leq t_1$, then $t_1 \wedge t_2 = t'$, and we are finished. ⊣

Lemma 9. *If $(T, \leq)$ is a κ-Suslin tree and $H \in [T]^\kappa$, then H is somewhere dense in $(T, \leq)$.*

Proof. Assume indirectly that H is nowhere dense. Let $A \subseteq T$ be a maximal antichain consisting of elements $t \in T$ such that $t\!\uparrow \cap H = \emptyset$. As $(T, \leq)$ is κ-Suslin, $|A| < \kappa$, consequently, $A \subseteq T_{<\alpha}$ for some $\alpha < \kappa$. ⊣

Claim. $H \subseteq T_{<\alpha}$.

Proof. Assume not, and so $s \in H \cap T_{\geq\alpha}$. As $s \in H$, we cannot have $t < s$ for some $t \in A$. Also, $s < t$ is impossible by height considerations. Let $s' > s$ be such that $(s'\!\uparrow) \cap H = \emptyset$ (exists by the indirect assumption). As s' is incomparable by any element of A, $A \cup \{s'\}$ would properly extend A, a contradiction.

As by the Claim $|H| < \kappa$ holds, we have reached the desired contradiction. ⊣

A different argument is the following. Let G be generic for $(T, \leq)$. As we force with a κ-cc forcing, some $t \in T$ forces that $|H \cap G| = \kappa$. Obviously, H is dense above t.

THEOREM 10. (Todorcevic [5]) *If there is a κ-Suslin tree, then there is a coloring $F : [\kappa]^2 \to \kappa$ establishing $\kappa \not\to [\kappa]^2_\kappa$ with no three-colored triangles.*

PROOF. Let $(T, \leq)$ be a normal κ-Suslin tree. Set $F_0(t_0, t_1) = \mathrm{ht}(t_0 \wedge t_1)$. Let $g : \kappa \to \kappa$ be such that $|g^{-1}(\alpha)| = \kappa$ $(\alpha < \kappa)$. We claim the $F = g \circ F_0$ is as required. F has no three-colored triangle by Lemmas 6 and 8.

In order to prove that F establishes $T \not\to [\kappa]^2_\kappa$, let $H \in [T]^\kappa$. ⊣

CLAIM. $\mathrm{Ran}(F_0|[H]^2)$ contains an end segment of κ.

PROOF. By Lemma 9, H is somewhere dense, say, above $t \in T$. Set $\alpha = \mathrm{ht}(t)$. As T is normal, if $\gamma > \alpha$, there is $t' > t$, $\mathrm{ht}(t') = \gamma$. Let t_0, t_1 be two immediate successors of t' (exist, as T is normal). As H is dense above t, there are $x_0 > t_0$, $x_1 > t_1$, $x_0, x_1 \in H$. Now $F_0(x_0, x_1) = \mathrm{ht}(x_0 \wedge x_1) = \mathrm{ht}(t') = \gamma$.

As we proved that $(\alpha, \kappa) \subseteq \mathrm{Ran}(F_0|[H]^2)$, we are done.

By the definition of g, g assumes every value of (α, κ), we are finished. ⊣

THEOREM 11. $(\kappa^{<\kappa} = \kappa)$ *After forcing with* $\mathrm{Add}(\kappa, 1)$, *there is a three-colored triangle free coloring $F : [\kappa^+]^2 \to \kappa^+$ establishing $\kappa^+ \not\to [\kappa^+]^2_{\kappa^+}$.*

PROOF. By Shelah's theorem [4], in the forced model there is a κ^+-Suslin tree and so we can apply Theorem 10. ⊣

THEOREM 12. *If $\kappa > \omega$ is regular, there is a coloring $F : [\kappa]^2 \to \kappa$ establishing $\kappa \not\to [\kappa]^2_\kappa$ with no three-colored triangles, then there is a κ-Aronszajn tree.*

PROOF. Let F be as in the Theorem. Set $h(\alpha) = \{F(\alpha, \beta) : \alpha < \beta < \kappa\}$ for $\alpha < \kappa$. ⊣

CLAIM 1. $|h(\alpha)| < \kappa(\alpha < \kappa)$.

PROOF. Otherwise pick $\alpha < \beta_i$ for $0 < i \in h(\alpha)$ such that $F(\alpha, \beta_i) = i$. If $i \neq j$, then, as $F(\alpha, \beta_i) = i$ and $F(\alpha, \beta_j) = j$, necessarily $F(\beta_i, \beta_j) \in \{i, j\}$ (as otherwise $\{\alpha, \beta_i, \beta_j\}$ formed a three-colored triangle). This implies that F restricted to $\{\beta_i : i \in h(\alpha) - \{0\}\}$ misses color 0, a contradiction. ⊣

We now describe the tree $(T, \leq)$. Its elements are of the form $t \in {}^{<\kappa}\kappa$ with $t < t'$ iff t' extends t. If $t \in {}^\alpha\kappa$ is in T and $\beta < \alpha$, then $t|\beta \in T$, i.e., $(T, \leq)$ is a subtree of ${}^{<\kappa}\kappa$.

For each $t \in T$ we define $V(t) \subseteq \kappa$ and $p(t) < \kappa$.

Set $T_0 = \{\emptyset\}$, $V(\emptyset) = \kappa$, $p(\emptyset) = 0$. If $t \in T_\alpha$, define

$$V(t^\frown i) = \{x \in V(t) - \{p(t)\} : F(p(t), x) = i\}$$

for $i \in h(p(t))$ and put $t^\frown i$ into T if $V(t^\frown i) \neq \emptyset$. In this case, define $p(t^\frown i) = \min(V(t^\frown i))$.

If $\alpha < \kappa$ is limit, $t \in {}^\alpha\kappa$, define $t \in T_\alpha$ if $V(t) = \bigcap\{V(t|\beta) : \beta < \alpha\} \neq \emptyset$. In this case set $p(t) = \min(V(t))$.

CLAIM 2. $\bigcup\{V(t) : t \in T_\alpha\} = \kappa - \{p(t) : t \in T_{<\alpha}\}$.

PROOF. By induction on α, immediately from the definition. ⊣

CLAIM 3. If $t \in T_\alpha$, then $|\{t' < T_{\alpha+1} : t < t'\}| < \kappa$.

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PROOF. As $|h(p(t))| < \kappa$. ⊣

CLAIM 4. $|T_\alpha| < \kappa$ $(\alpha < \kappa)$.

PROOF. By induction on α. If this holds for α, then $T_{\alpha+1}$ is the union of $< \kappa$ sets each of size $< \kappa$ by Claim 3 and we are finished as κ is regular.

Assume that $\alpha < \kappa$ is limit and we have the Claim for all $\beta < \alpha$. Then $|T_{<\alpha}| < \kappa$ as κ is regular. Assume indirectly that $|T_\alpha| = \kappa$. Consider the set $P = \{p(t) : t \in T_\alpha\}$. If $t_0, t_1 \in T_\alpha$, then there are a $\beta < \alpha$ and $t \in T_\beta$ and there are $t'_0, t'_1 \in T_{\beta+1}$ such that $t < t'_0 < t_0$, $t < t'_1 < t_1$ and so by the construction

$$F(p(t_0), p(t_1)) \in \{F(p(t), p(t'_0)), F(p(t), p(t'_1))\}.$$

Consequently, the values of F on $[P]^2$ are of the form $F(p(t), p(t'))$ for some $t < t' \in T_{<\alpha}$ and there are $< \kappa$ such possibilities by the inductive hypothesis, i.e., F misses some values on $[P]^2$, contradicting our condition on F. ⊣

CLAIM 5. $(T, \leq)$ has no κ-branch.

PROOF. Assume indirectly that $b = \{t_\alpha : \alpha < \kappa\}$ is a κ-branch with $t_\alpha \in T_\alpha$ $(\alpha < \kappa)$. By the way we constructed T, we have $t_\alpha = g|\alpha$ for some function $g : \kappa \to \kappa$. Further, $F(p(t_\beta), p(t_\alpha)) = g(\beta)$ for $\beta < \alpha$, i.e., the branch is end-homogeneous. There is some $i < \kappa$ such that $|\kappa - g^{-1}(i)| = \kappa$ and then the set $\{p(t_\alpha) : g(\alpha) \neq i\}$ has cardinality κ and it misses color i, a contradiction.

As by Claims 2, 4, and 5 $(T, \leq)$ is a κ-Aronszajn tree, the proof is complete. ⊣

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