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The method of generalised conditional symmetries and its various implementations

JOANNA M. GOARD

School of Mathematics, University of Wollongong, Wollongong NSW 2522, Australia

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We relate Kaptsov's method of B-determining equations for finding invariant solutions of PDEs to the nonclassical method and to the method of generalised conditional symmetries. An extension of Kaptsov's method is then used to find new solutions of degenerate diffusion equations.

1 Introduction

The technique for finding exact solutions of the widest variety of Differential Equations (DE) comes from Lie group analysis of differential equations, initiated by Sophus Lie (1881) over 100 years ago. Since then, there have been many advances in the area of symmetry analysis of DEs, particularly in the area of higher-order symmetries.

In this paper, we look at how one such very recent advance, namely the method of B-determining equations for finding invariant solutions (Kaptsov, 1995), relates to the nonclassical method (Bluman & Cole, 1969) and the method of Generalised Conditional Symmetries (GCS) (Fokas & Liu, 1994), for a scalar Partial Differential Equation (PDE):

$$\Omega(\mathbf{x}, u, u^1, u^2, \ldots) = 0, \tag{1.1}$$

where $\mathbf{x} = (x_1, x_2, \ldots x_n)$ denotes n independent variables, u denotes the dependent variable, and

$$u^k = \frac{\partial^k u}{\partial x_{i_1} \partial x_{i_2} \ldots \partial x_{i_k}}$$

denotes the set of co-ordinates corresponding to all kth order partial derivatives of u with respect to $\mathbf{x}$. In Kaptsov's method, a useful extension of the classical symmetry determining relations is made by incorporating an additional factor B. One aspect that has not been mentioned before is that a non-zero B-factor may occur when a classical symmetry is simply multiplied by a scalar function to form an equivalent nonclassical symmetry.

We begin by providing a brief conceptual background to this paper. Readers who require further details may refer to the comprehensive accounts of Bluman & Kumei (1989), Olver (1986) and Ibragimov (1994).

1.1 Nonclassical method

Bluman & Cole (1969) proposed a generalisation of Lie's method for finding group-invariant solutions, which they called the nonclassical method. In this method, also known as 'the method of conditional symmetries', our PDE (1.1) is augmented with its Invariant Surface Condition (ISC)

$$\sum_i X_i u_{x_i} + N = 0, \tag{1.2}$$

where X_i and N are functions of $\mathbf{x}$ and u. Then the requirement that this *system* be invariant under

$$\begin{aligned} x_i^* &= x_i + \epsilon X_i(\mathbf{x}, u) + O(\epsilon^2) \\ u^* &= u - \epsilon N(\mathbf{x}, u) + O(\epsilon^2), \end{aligned} \tag{1.3}$$

or equivalently,

$$\begin{aligned} x_i^* &= x_i \\ u^* &= u - \epsilon(N(\mathbf{x}, u) + X_i u_{x_i}) + O(\epsilon^2) \end{aligned} \tag{1.4}$$

yields an overdetermined *nonlinear* system of equations for the infinitesimals X_i and N, the solutions of which are the nonclassical symmetries. Given a nonclassical symmetry, one may construct invariant solutions that satisfy not only the governing PDE (1.1), but also the invariance condition (1.2). Hence the set of such invariant solutions includes all those to be found by the classical method and thus, in general, is a larger set. By the nonclassical method, it is possible to find further types of explicit solutions by the same reduction technique that is commonly used in the classical method. Indeed, it is known that there do exist PDEs which possess symmetry reductions not obtainable via the classical Lie group method (e.g. see Goard & Broadbridge (1996), Arrigo, Hill & Broadbridge (1994) and Clarkson & Mansfield (1994)).

We will say that a nonclassical symmetry (X_i, N) is *equivalent* to some classical symmetry vector field with co-ordinates $(\bar{X}_i, \bar{N})$ if

$$(\bar{X}_i, \bar{N}) = \psi(\mathbf{x}, u)(X_i, N) \tag{1.5}$$

for some function ψ. This is a practical definition of equivalence, since both of these symmetries have the same invariant surface, leading to the same invariant reductions of the governing PDE. We will use the term 'strictly nonclassical' for the nonclassical symmetries which are not equivalent to any classical symmetry.

Since 1969 there have been various modifications to the nonclassical method. In particular, Clarkson & Kruskal (1989) developed a direct, algorithmic and non-group theoretic method called the *direct method* for finding symmetry reductions. However Levi & Winternitz (1989) established, using a group-theoretic explanation, that all new solutions obtained by the direct method can also be obtained using the nonclassical method.

1.2 Generalised Conditional Symmetries (GCS)

Fokas & Liu (1994) define GCS as follows:

'The function $\sigma(u)$ is a GCS of the equation $u_t = K(u)$ iff

$$K'\sigma - \sigma' K = F(u, \sigma), \qquad F(u, 0) = 0, \tag{1.6}$$

where $K(u), \sigma(u)$ are differentiable functions of $u, u_x, u_{xx}, \ldots$, while $F(u, \sigma)$ is a differentiable function of $u, u_x, u_{xx}, \ldots$ and $\sigma, \sigma_x, \sigma_{xx}, \ldots$ and prime denotes the Frechét derivative.'

This definition can easily be extended to any PDE, so that a GCS of (1.1) is defined as a function σ of $\mathbf{x}$, $u, u^1, u^2, \ldots$, which satisfies

$$\Gamma(\sigma)\Omega + F(u, \sigma)|_{\Omega=0} = 0, \qquad F(u, 0) = 0, \tag{1.7}$$

where

$$\Gamma(\sigma) = \sigma \frac{\partial}{\partial u} + \sum_{|\alpha| \geqslant 1} D^{\alpha}(\sigma) \frac{\partial}{\partial u_{\alpha}}, \tag{1.8}$$

and where $D^{\alpha} = D_{x_1}^{\alpha_1} \ldots D_{x_n}^{\alpha_n}$ and D_{x_i} is the total x_i derivative.

1.3 B-determining equations

Kaptsov (1995) introduces the concept of B-determining equations of a system of PDEs that generalise the defining equations of the symmetry groups. For a scalar PDE (1.1), Kaptsov defined equations of the form

$$\Gamma(\sigma)\Omega + B\sigma|_{\Omega=0} = 0, \tag{1.9}$$

where $\Gamma(\sigma)$ is defined in equation (1.8), as a 'B-determining equation', where B is a function of $\mathbf{x}$,$u, u^1, \ldots$, and the function σ of $\mathbf{x}$, $u, u^1, u^2, \ldots$, generates a manifold $\sigma = 0$ in the graph space of ($\mathbf{x}$, u).

2 Relationship between B-determining equations and the nonclassical method

If we set σ to be the ISC

$$\sigma = \sum_i X_i u_{x_i} + N, \tag{2.1}$$

where X_i and N are functions of $\mathbf{x}$ and u, then the nonclassical method on (1.1) requires

$$\Gamma(\sigma)\Omega|_{\Omega=0;\sigma=0} = 0 \tag{2.2}$$

i.e $\sigma = 0$ and its differential consequences are substituted into

$$\Gamma(\sigma)\Omega|_{\Omega=0}. \tag{2.3}$$

We can now easily compare this with (1.9). The term $B\sigma$ in equation (1.9) is equivalent to substituting the ISC $\sigma = 0$, but not its differential consequences, into (2.3). Hence the symmetries σ of the form (2.1) found by the B-determining equations method are a subset of the full class of nonclassical symmetries. In practice, this is often a very convenient method for finding interesting nonclassical solutions, without needing to resort to the method of nonclassical symmetries in its full generality.

Kaptsov (1994) considered the nonlinear Poisson equation

$$u_{xx} + u_{yy} = f(u), \tag{2.4}$$

for which the corresponding B-determining equation is

$$D_x^2\sigma + D_y^2\sigma - (f_u + b)\sigma = 0, \tag{2.5}$$

subject to (2.4). Here he chose b to be a function of x, y and u, and σ to be the ISC:

$$\sigma = Xu_x + Yu_y + N. \tag{2.6}$$

He suggests that a solution (X, Y, N) to (2.5) for $b \neq 0$ leads to a strictly nonclassical symmetry.

To obtain all the possible nonclassical symmetries of (2.4), we would need to consider the following more general version of (2.5):

$$\begin{aligned} D_x^2\sigma + D_y^2\sigma - f_u &+ [B_1(x,y,u)u_{xx} + B_2(x,y,u)u_x^2 + B_3(x,y,u)u_x + B_4(x,y,u)]\sigma \\ &+ [B_5(x,y,u)u_x + B_6(x,y,u)]\sigma^2 \\ &+ [B_7(x,y,u)u_x + B_8(x,y,u)]\sigma_x \\ &+ B_9(x,y,u)\sigma\sigma_x \\ &+ B_{10}(x,y,u)\sigma_y + B_{11}(x,y,u)u_x\sigma_y \\ &= 0, \end{aligned} \tag{2.7}$$

subject to (2.4). However, we will demonstrate that a non-zero function B_i does not necessarily imply a strictly nonclassical symmetry.

Letting $B_{10} = B_{11} = 0$, substitution of (2.6) into (2.7) yields the determining equations (A.1) (see the Appendix), where on eliminating the B_i terms, we find the determining equations (A.2), with the B_i as specified in (A.3). We consider two solutions of (A.2):

Solution 1

$$\underline{X = 0, Y = e^{cx}, N = e^{ax+by},} \qquad \underline{f = (a^2 + b^2 + c^2 - 2ac)u + d, \quad a, b, c, d \quad \text{constants.}} \tag{2.8}$$

In this case, the non-zero functions B_i are

$$B_4(x,y,u) = c^2$$
$$B_8(x,y,u) = -2c.$$

However, (2.8) does not correspond to a strictly nonclassical symmetry as (2.4) is invariant under

$$(\bar{X}, \bar{Y}, \bar{N}) = e^{-cx}(X, Y, N),$$

which can be chosen to be the classical symmetry $(0, 1, e^{(a-c)x+by})$.

Solution 2

$$\underline{X = \tan(cx/2 + d) + \beta \sec(cx/2 + d)e^{\frac{-cy}{2}}, Y = 1, N = a,} \qquad \underline{f = Be^{\frac{cu}{a}}, \quad a, c, d \quad \text{constants.}} \tag{2.9}$$

In this case, the non-zero function B_i is

$$B_8 = -\beta c \ \sec\left(\frac{cx}{2} + d\right) e^{\frac{-cy}{2}}$$

However, (2.9) again does not correspond to a strictly nonclassical symmetry as (2.4) is invariant under

$$(\bar{X}, \bar{Y}, \bar{N}) = \cos\left(\frac{cx}{2} + d\right) e^{\frac{cy}{2}} (X, Y, N),$$

which is the classical symmetry $(\sin(cx/2 + d)e^{\frac{cy}{2}} + \beta, \cos(cx/2 + d)e^{\frac{cy}{2}}, a\cos(cx/2 + d)e^{\frac{cy}{2}})$. Hence, as nonclassical symmetries can be equivalent to classical symmetries by relationship (1.5), a non-zero function B_i in (2.7) does not ensure a strictly nonclassical symmetry. That is, if all the functions B_i can be effectively set to zero, by multiplying each infinitesimal by some non-zero function of $\mathbf{x}$ and u, then the symmetry is not strictly nonclassical. The usual simplification of setting $Y = 1$ in finding nonclassical symmetries would not be appropriate here if we wanted to check if all the functions B_i could be set to zero in this manner. We note that Kaptsov's method will not always require as many functions B_i as in (2.7).

Consider the reaction-diffusion equation

$$u_t = u_{xx} + f(u), \tag{2.10}$$

for which the corresponding B-determining equation is

$$D_{xx}\sigma - D_t\sigma + \sigma f_u + b\sigma|_{u_t = u_{xx} + f(u)} = 0. \tag{2.11}$$

To find all the possible nonclassical symmetries of (2.10), we would need the following extension of (2.11):

$$D_{xx}\sigma - D_t\sigma + \sigma f_u + [B_1 u_x^2 + B_2 u_x + B_3]\sigma + [B_4 u_x + B_5]\sigma_x = 0,$$

subject to (2.10), and where $B_1, .., B_5$ are functions of x, y and u and σ is the ISC.[1]

3 Relationship between B-determining equations and the method of GCS

Comparing equations (1.7) and (1.9) we see that the B-determining equations are one special case of the method of GCS where

$$F(u, \sigma) = B\sigma.$$

As such, many of the GCS so far found in the literature can be found by using B-determining equations. As an example, Qu (1997) found that the diffusion equation

$$u_t = u_{xx} + c_2 u \log\ u + c_3 u - c_1^2 u(\log\ u)^2$$

admits the GCS

$$\sigma = u_{xx} - \frac{u_x^2}{u} + c_1 u_x; \qquad c_1 \neq 0.$$

Using Kaptsov's method, this symmetry corresponds to

$$b = (2c_3 - 2c_1^2(\log\ u)^2 + 2c_2 \log\ u) - \frac{2u_t}{u} + \frac{2c_1}{u}u_x + \frac{2}{u^2}u_x^2.$$

[1] We note that if the infinitesimal Y = 1 in the ISC, then we only require the extension

$$(B_1(x, y, u)u_x + B_2(x, y, u))\sigma,$$

to get the most general nonclassical determining equations as found by Arrigo, Hill & Broadbridge (1994).

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We now state the results of an investigation into finding GCS of the degenerate diffusion equation

$$u_t = \frac{\partial}{\partial x}(f(u)g(u_x)) \tag{3.1}$$

in the cases $g(u_x) = e^{-u_x}$ and $g(u_x) = u_x^{-3}$, by using an extension of the method of B-determining equations. Both of these examples were chosen because the diffusivities decrease at a rate faster than u_x^{-2} as u_x tends to infinity. When $f = 1$, this is the case of strongly degenerate diffusion, within which non-parabolic behaviour may be apparent in the solutions (see Bertsch & Dal Passo, 1992).

We looked for σ of the form

$$\sigma = u_{xx} + P(u)u_x^2 + Q(u)u_x + R(u) \tag{3.2}$$

satisfying

$$\begin{aligned} \Gamma(\sigma)\Omega - g(u_x) \ & \{\sigma[b_{11} + b_{12}u_x + b_{13}u_x^2] \\ & +\sigma\sigma_x[b_{21} + b_{22}u_x + b_{23}u_x^2] \\ & +\sigma^2[b_{31} + b_{32}u_x + b_{33}u_x^2] \\ & +\sigma^3[b_{41} + b_{42}u_x + b_{43}u_x^2] \\ & +\sigma_x[b_{51} + b_{52}u_x + b_{53}u_x^2]\} = 0, \end{aligned} \tag{3.3}$$

subject to

$$\Omega = u_t - \frac{\partial}{\partial x}(f(u)g(u_x)) = 0.$$

The GCS (3.2) reduces to a nonclassical symmetry in the case $g(u_x) = e^{-u_x}$ when

$$P = 0, \qquad Q = -\frac{f'}{f}, \qquad R = 0,$$

and in the case $g(u_x) = u_x^{-3}$ when

$$P = -\frac{f'}{3f}, \qquad Q = 0, \qquad R = 0.$$

We ignore those cases here, as it has already been shown by Goard (1997), that in the case $T = 1$, no strictly nonclassical symmetries exist for these equations.

The symmetries thus found are listed in Tables 1 and 2, together with corresponding solutions. As an example of the functions b_{ij} which were needed, for the case

$$f = cu^2 + du + e, \quad g(u_x) = e^{-u_x},$$

the GCS was found with $\sigma = u_{xx}$ and

$b_{11} = 0$	$b_{12} = 4c$	$b_{13} = -6c$
$b_{21} = 2du + 2e + 2cu^2$	$b_{22} = 0$	$b_{23} = 0$
$b_{31} = -2d - 4cu$	$b_{32} = 3 + 6cu$	$b_{33} = 0$
$b_{41} = -e - cu^2 - du$	$b_{42} = 0$	$b_{43} = 0$
$b_{51} = 0$	$b_{52} = -4cu - 2d$	$b_{53} = 0.$

Table 1. *Solutions for* (*3.1*) *with* $g(u_x) = e^{-u_x}$, *where* $li(x) = \int^x \frac{1}{\ln u} du$ *and* $A,\ B, \alpha,\ \beta,\ c,\ d,\ \epsilon,\ p,\ \gamma$ *are constants.*

Symmetry	$f(u)$	Solution
$u_{xx} + \frac{u_x}{u+c}$	1	$\alpha\, li\left(\frac{u+c}{\alpha}\right) = -x - \frac{1}{\alpha}t + p$
$u_{xx} - \frac{u_x}{1+u}$	$u(1+u)$	$li(\beta(1+u)) = \beta\left(x + \frac{1}{\beta}t + p\right)$
$u_{xx} + \left(\frac{-f'}{f} + \frac{B}{A+Bu}\right)u_x$	$(A+Bu)e^u$	$u = \alpha e^{Be^{-\gamma}t} e^x - \gamma$
$u_{xx} + \left(\frac{-f'}{f} + \frac{B}{A+Bu}\right)u_x$	$(A+Bu)e^{e^u}$	$u = -\ln\left\{\frac{1}{\beta}[e^{-\beta(x+Be^{-\beta}t+\gamma)} - 1]\right\}$
$u_{xx} - u_x + 1$	e^u	$u = c + (e^{c-1}t + \alpha)e^x + x$
u_{xx}	$cu^2 + du + \epsilon$	$u = xh(t) + j(t)$ where h, j satisfy $-\frac{1}{h(t)}e^{h(t)} + li(e^h) = 2ct + \alpha$ and $j'(t) - 2ce^{-h}hj = 2cde^{-h}h$

4 Conclusion

The B-determining equations provide a neat formalism for determining not only nonclassical point symmetries but also higher-order generalised conditional symmetries of partial differential equations. In general, if a simple form is assumed for the B ansatz, then the nonclassical determining relations may be considerably simplified. However, to recapture all nonclassical symmetries, a single B-factor might not be sufficient. Provided a sufficient number of B-factors are incorporated to multiply many possible derivatives of a higher-order invariance condition $\sigma(\mathbf{x}, u, u^1, u^2, \ldots)$, generalised conditional symmetries might be recovered.

Appendix A

This section provides details of the nonclassical determining equations for (2.4) using the extension of (2.6).

With $B_{10} = B_{11} = 0$, substitution of (2.6) into (2.7) yields the following determining equations:

$$
\begin{aligned}
&2X_u + B_1X + B_7X + B_9X^2 = 0\\
&-2Y_u + B_1Y + B_9YX = 0\\
&2Y_u + B_7Y + B_9XY = 0\\
&2Y_{xu} + 2X_{yu} + 2YNB_5 + 2XYB_6 + B_3Y + B_7Y_x + B_8Y_u + B_9XY_x + B_9YX_x\\
&\quad +B_9YN_u + B_9NY_u = 0\\
&X_{uu} + B_5Y^2 + B_9YY_u = 0
\end{aligned}
$$

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Table 2. *Solutions for (3.1) with $g(u_x) = u_x^{-3}$, where $A,\ B,\ C,\ K,\ \gamma,\ \epsilon,\ c,\ \alpha, \beta$ are constants.*

Symmetry	$f(u)$	Solution
$u_{xx} + \dfrac{u_x^2}{u+c}$	1	$u = [\pm(48t+\gamma)^{\frac{1}{2}} x + \epsilon(48t+\gamma)^{\frac{1}{2}}]^{\frac{1}{2}} - c$
$u_{xx} + \dfrac{u_x^2}{3u+c}$	1	$u = \left(\alpha x + \dfrac{64}{27\alpha^2} t + \beta\right)^{\frac{3}{4}} - c$
u_{xx}	$A + Bu + Cu^2$	$u = (4Ct+\gamma)^{\frac{1}{2}} x + \epsilon(4Ct+\gamma)^{\frac{1}{2}} - \dfrac{B}{2C}$
$u_{xx} + K u_x^2$	$e^{-3Ku}(A+Bu)$	$u = \dfrac{1}{K} \ln\Big[(2K^4Ct+2\epsilon)^{\frac{1}{2}} x - \dfrac{B}{KC} + Ce^{-2Ku}\alpha(2K^4Ct+2\epsilon)^{\frac{1}{2}}\Big] K \neq 0,\ C \neq 0$
$u_{xx} + K u_x^2$	$e^{-3Ku}(A+Bu)$	$u = \dfrac{1}{K} \ln\left[(2\epsilon)^{\frac{1}{2}} x + \dfrac{K^3B}{2\epsilon} t + \gamma\right] K \neq 0$
u_{xx}	$A + Bu$	$u = \alpha x + \dfrac{Bt}{\alpha^2} + \beta$
$u_{xx} - \dfrac{u_x^2}{3u}$	u^2	$u = \left(cx + \dfrac{8}{27c^2} t + \beta\right)^{\frac{3}{2}}$

$$
\begin{aligned}
&2X_u + Y^2 B_9 = 0\\
&Y_{uu} + 2B_5 XY + B_2 Y + B_7 Y_u + B_9 X Y_u + B_9 Y X_u = 0\\
&2X_x - 2Y_y + B_1 N + B_8 X + B_9 N X = 0\\
&X_{uu} + B_5 X^2 + B_2 X + B_7 X_u + B_9 X X_u = 0\\
&2X_{xu} + N_{uu} + 2XNB_5 + B_6 X^2 + B_2 N + B_3 X + B_7 X_x + B_7 N_u\\
&\quad + B_8 X_u + B_9 X X_x + X N_u B_9 + B_9 N X_u = 0\\
&2Y_{yu} + N_{uu} + B_6 Y^2 + B_9 Y Y_x = 0\\
&Y_{uu} = 0\\
&2Y_x + 2X_y + B_8 Y + B_9 N Y = 0\\
&X_u f + 2N_{xu} + X_{xx} + X_{yy} + N^2 B_5 + 2XNB_6 + B_3 N + B_4 X + B_7 N_x + B_8 X_x\\
&\quad + B_8 N_u + X N_x B_9 + B_9 N X_x + B_9 N N_u = 0\\
&3Y_u f + 2N_{yu} + Y_{yy} + Y_{xx} + 2YNB_6 + B_4 Y + B_8 Y_x + B_9 Y N_x + B_9 N Y_x = 0\\
&N_{xx} + 2Y_y f + N_u f + N_{yy} - N f_u + B_6 N^2 + B_4 N + B_8 N_x + B_9 N N_x = 0
\end{aligned}
\tag{A 1}
$$

where on eliminating the B_i terms we find the determining equations:

$$
\begin{aligned}
&2X_u + \frac{2X^2 X_u}{Y^2} = 0\\
&2X_x - 2Y_y + \frac{2NY_u}{Y} - \frac{2XY_x}{Y} - \frac{2XX_y}{Y} + \frac{2NXX_u}{Y^2} = 0\\
&X_{uu} + \frac{X^2 X_{uu}}{Y^2} - \frac{XY_{uu}}{Y} + \frac{2XY_u^2}{Y^2} + \frac{2X_u^2 X}{Y^2} - \frac{2X_u Y_u}{Y} = 0
\end{aligned}
$$

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$$
\begin{aligned}
&2X_{xu} + N_{uu} - \frac{2NXX_uY_u}{Y^3} + \frac{2X^2Y_{uy}}{Y^2} + \frac{X^2N_{uu}}{Y^2} \\
&\quad - \frac{2X^2X_uY_x}{Y^3} - \frac{NY_{uu}}{Y} + \frac{2NY_u^2}{Y^2} + \frac{2NX_u^2}{Y^2} - \frac{2XY_{xu}}{Y} \\
&\quad - \frac{2XX_{yu}}{Y} + \frac{2NXX_{uu}}{Y^2} + \frac{4Y_xY_uX}{Y^2} + \frac{2Y_uX_yX}{Y^2} \\
&\quad + \frac{2XX_uX_x}{Y^2} + \frac{2XN_uX_u}{Y^2} - \frac{2Y_uX_x}{Y} - \frac{2Y_uN_u}{Y} \\
&\quad - \frac{2X_uY_x}{Y} - \frac{2X_uX_y}{Y} = 0 \\
&X_uf + 2N_{xu} + X_{xx} + X_{yy} + \frac{N^2X_{uu}}{Y^2} - \frac{2N^2X_uY_u}{Y^3} + \frac{4XNY_{uy}}{Y^2} \\
&\quad + \frac{2XNN_{uu}}{Y^2} - \frac{6XNX_uY_x}{Y^2} - \frac{2NY_{xu}}{Y} - \frac{2NX_{yu}}{Y} + \frac{4NY_xY_u}{Y^2} \\
&\quad + \frac{2NY_uX_y}{Y^2} + \frac{2XNY_xX_u}{Y^3} + \frac{2NX_xX_u}{Y^2} + \frac{2N_uX_uN}{Y^2} \\
&\quad - \frac{3Y_ufX}{Y} - \frac{2XN_{yu}}{Y} - \frac{Y_{yy}X}{Y} - \frac{Y_{xx}X}{Y} + \frac{2XY_x^2}{Y^2} \\
&\quad + \frac{2XX_yY_x}{Y^2} + \frac{2N_xXX_u}{Y^2} - \frac{2N_xY_u}{Y} - \frac{2X_xY_x}{Y} - \frac{2X_yX_x}{Y} \\
&\quad - \frac{2N_uY_x}{Y} - \frac{2N_uX_y}{Y} = 0 \\
&N_{xx} + 2Y_yf + N_uf + N_{yy} - Nf_u + N^2\left(\frac{-2Y_{uy}}{Y^2} - \frac{N_{uu}}{Y^2} + \frac{2X_uY_x}{Y^3}\right) \\
&\quad + N\left(\frac{-3Y_uf}{Y} - \frac{2N_{uy}}{Y} - \frac{Y_{yy}}{Y} - \frac{Y_{xx}}{Y} + \frac{4NY_{yu}}{Y^2} + \frac{2NN_{uu}}{Y^2}\right. \\
&\quad \left. - \frac{4NX_uY_x}{Y^3} + \frac{2X_yY_x}{Y^2} + \frac{2N_xX_u}{Y^2} + \frac{2Y_x^2}{Y^2}\right) \\
&\quad + N_x\left(\frac{-2Y_x}{Y} - \frac{2X_y}{Y} + \frac{2NX_u}{Y^2}\right) + NN_x\left(\frac{-2X_u}{Y^2}\right) = 0 \qquad \text{(A 2)}
\end{aligned}
$$

where

$$
\begin{aligned}
B_1 &= \frac{2Y_u}{Y} + \frac{2XX_u}{Y^2} \\
B_2 &= -\frac{Y_{uu}}{Y} + \frac{2XX_{uu}}{Y^2} - \frac{2XX_uY_u}{Y^3} + \frac{2Y_u^2}{Y^2} + \frac{2X_u^2}{Y^2} \\
B_3 &= \frac{-2Y_{xu}}{Y} - \frac{2X_{yu}}{Y} + \frac{2NX_{uu}}{Y^2} - \frac{6NX_uY_u}{Y^3} + \frac{4XY_{uy}}{Y^2} \\
&\quad + \frac{2XN_{uu}}{Y^2} - \frac{6XX_uY_x}{Y^3} + \frac{4Y_xY_u}{Y^2} + \frac{2Y_uX_y}{Y^2} + \frac{2XY_xX_u}{Y^3} \\
&\quad + \frac{2X_uX_x}{Y^2} + \frac{2N_uX_u}{Y^2} + \frac{2NX_uY_u}{Y^3} \\
B_4 &= \frac{-3Y_uf}{Y} - \frac{2N_{yu}}{Y} - \frac{Y_{yy}}{Y} - \frac{Y_{xx}}{Y} + \frac{4NY_{yu}}{Y^2} + \frac{2NN_{uu}}{Y^2} \\
&\quad - \frac{4NX_uY_x}{Y^3} + \frac{2Y_x^2}{Y^2} + \frac{2X_yY_x}{Y^2} + \frac{2N_xX_u}{Y^2}
\end{aligned}
$$

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$$
\begin{aligned}
B_5 &= -\frac{X_{uu}}{Y^2} + \frac{2X_u Y_u}{Y^3} \\
B_6 &= -\frac{2Y_{uy}}{Y^2} - \frac{N_{uu}}{Y^2} + \frac{2X_u Y_x}{Y^3} \\
B_7 &= -\frac{2Y_u}{Y} + \frac{2XX_u}{Y^2} \\
B_8 &= -\frac{2Y_x}{Y} - \frac{2X_y}{Y} + \frac{2NX_u}{Y^2} \\
B_9 &= -\frac{2X_u}{Y^2}
\end{aligned} \tag{A 3}
$$

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